HKR-TYPE INVARIANTS OF FLAT BUNDLES OVER 3-MANIFOLDS

ALEXIS VIRELIZIER

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Introduction

In view of studying quantum invariants for maps, Turaev [17] introduced the notion of a *Homotopy quantum field theory* (HQFT). Such a theory in dimension n with target a topological space X consists in associating a vector space $V(g)$ to any map $g\colon N \to X$, where N is a closed $(n-1)$-manifold, and a linear map $L(f)\colon V(f_{|\partial_- M}) \to V(f_{|\partial_+ M})$ to any map $f\colon M \to X$, where M is a n-cobordism with $\partial M = \partial_+ M \cup (-\partial_- M)$. In particular, this assignment must only depend on the homotopy classes of the maps and must satisfy that to compose two such cobordisms amounts composing their associated linear maps. When X is reduced to a single point, one recovers the notion of a topological quantum field theory, as described in [1].

Fix a group π. A HQFT with target an Eilenberg-Mac Lane space of type $K(\pi,1)$ gives rise to invariants of flat π-bundles. Turaev [18] introduced the notion of a modular crossed π-category and showed that such a category gives rise to a 3-dimensional homotopy quantum field theory with target space $K(\pi,1)$. Examples of π-categories can be constructed from so-called Hopf π-coalgebras (see [18, 19]).

Briefly speaking, a Hopf π-coalgebra is a family $H=\{H_\alpha\}_{\alpha\in\pi}$ of algebras (over a field $\Bbbk$) endowed with a comultiplication $\Delta=\{\Delta_{\alpha,\beta} : H_{\alpha\beta} \to H_\alpha \otimes H_\beta\}_{\alpha,\beta\in\pi}$, a counit $\varepsilon : H_1 \to \Bbbk$, and an antipode $S=\{S_\alpha : H_\alpha \to H_{\alpha^{-1}}\}_{\alpha\in\pi}$ which verify some compatibility conditions. A Hopf π-coalgebra H is quasitriangular (resp. ribbon) when it is endowed with an R-matrix $R=\{R_{\alpha,\beta} \in H_\alpha \otimes H_\beta\}_{\alpha,\beta\in\pi}$ (resp. an R-matrix and a twist $\theta=\{\theta_\alpha \in H_\alpha\}_{\alpha\in\pi}$) verifying some axioms which generalize the classical ones given in [3] (resp. [15]). The case $\pi=1$ is the standard setting of Hopf algebras.

Recall that Hennings [7, 8] constructed invariants of links and 3-manifolds in terms of right integrals on certain Hopf algebras. Kauffman and Radford [10] clarified the relationships between these invariants and Hopf algebras and simplified Hennings' construction.

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The purpose of this paper is to generalize Hennings-Kauffman-Radford method to the setting of HQFT: starting from a ribbon Hopf π-coalgebra, we construct invariants of principal π-bundles over link complements and over 3-manifolds.

More precisely, from a ribbon Hopf π-coalgebra $H = \{H_\alpha\}_{\alpha\in\pi}$ endowed with a trace $\mathrm{tr} = \{\mathrm{tr}_\alpha : H_\alpha \to \Bbbk\}_{\alpha\in\pi}$, we give an improved version of the Kauffman-Radford method of [10] in order to construct an invariant $\mathrm{Inv}_{\{H,\mathrm{tr}\}}(L, g)$ of framed links L endowed with a group homomorphism $g : \pi_1(S^3 \setminus L) \to \pi$ (called *π-links*). This construction is made by coloring the vertical segments of a generic diagram of L with π via the homomorphism, by decorating the crossings with the R-matrix, by concentrating this algebraic decoration with the structure morphisms of H, and then by evaluating the result with the trace. We show that the Reidemeister moves colored in some sense by π report the equivalence of the pairs (L, g), and we verify the invariance under these moves by using properties of quasitriangular and ribbon Hopf π-coalgebras and of the trace. We give examples of computations (by using Hopf π-coalgebras constructed from bicharacters of π) which shows that this invariant is not trivial.

When a trace constructed from an integral of H is used, the invariant $\mathrm{Inv}_{\{H,\mathrm{tr}\}}$ can be normalized to an invariant $\tau_H(M, \xi)$ of principal π-bundles ξ over 3-manifolds M (called *π-manifolds*). This construction is made by presenting M by surgery along a framed link L, by defining $g : \pi_1(S^3 \setminus L) \to \pi$ by means of the monodromy of the π-bundle, and then by normalizing $\mathrm{Inv}_{\{H,\mathrm{tr}^\lambda\}}(L, g)$. We show that the Kirby moves colored in some sense by π report the equivalence of principal π-bundles over 3-manifolds, and we verify the invariance under these moves by using the properties of integrals. This invariant is not trivial (we give an example of computation for some $\mathbb{Z}/n\mathbb{Z}$-bundles over lens spaces, starting from the Hopf $\mathbb{Z}/n\mathbb{Z}$-coalgebras of [13]) and coincides with the Hennings-Kauffman-Radford invariant of 3-manifolds when $\pi = 1$.

The paper is organized as follows. In Section 1, we review the main properties of Hopf π-coalgebras. In Section 2, we construct an invariant of π-links. In Section 3, we normalize it to an invariant of π-manifolds. Finally, in Appendix A, detailed computations using quantum groups are performed.

1. HOPF GROUP-COALGEBRAS

In this section, we review definitions and properties concerning Hopf group-coalgebras. For a detailed treatment, we refer to [19].

1.1. **Hopf π-coalgebras.** Following [18], a *Hopf π-coalgebra* (over $\Bbbk$) is a family $H = \{H_\alpha\}_{\alpha\in\pi}$ of $\Bbbk$-algebras endowed with a family $\Delta = \{\Delta_{\alpha,\beta} \colon H_{\alpha\beta} \to H_\alpha \otimes H_\beta\}_{\alpha,\beta\in\pi}$ of algebra homomorphisms (the *comultiplication*) and an algebra homomorphism $\varepsilon \colon H_1 \to \Bbbk$ (the *counit*) such that, for all $\alpha, \beta, \gamma \in \pi$,

$$(\Delta_{\alpha,\beta} \otimes \mathrm{id}_{H_\gamma})\Delta_{\alpha\beta,\gamma} = (\mathrm{id}_{H_\alpha} \otimes \Delta_{\beta,\gamma})\Delta_{\alpha,\beta\gamma} \tag{1.1}$$

and

$$(\mathrm{id}_{H_\alpha} \otimes \varepsilon)\Delta_{\alpha,1} = \mathrm{id}_{H_\alpha} = (\varepsilon \otimes \mathrm{id}_{H_\alpha})\Delta_{1,\alpha}, \tag{1.2}$$

and with a family $S = \{S_\alpha \colon H_\alpha \to H_{\alpha^{-1}}\}_{\alpha\in\pi}$ of $\Bbbk$-linear maps (the *antipode*) which verifies that, for all $\alpha \in \pi$,

$$m_\alpha(S_{\alpha^{-1}} \otimes \mathrm{id}_{H_\alpha})\Delta_{\alpha^{-1},\alpha} = \varepsilon\, 1_\alpha = m_\alpha(\mathrm{id}_{H_\alpha} \otimes S_{\alpha^{-1}})\Delta_{\alpha,\alpha^{-1}}, \tag{1.3}$$

where $m_\alpha \colon H_\alpha \otimes H_\alpha \to H_\alpha$ and $1_\alpha \in H_\alpha$ denote respectively the multiplication and unit element of H_α.

When $\pi = 1$, one recovers the usual notion of a Hopf algebra. In particular $(H_1, m_1, 1_1, \Delta_{1,1}, \varepsilon, S_1)$ is a Hopf algebra.

Remark that the notion of a Hopf π-coalgebra is not self-dual, and that if $H = \{H_\alpha\}_{\alpha\in\pi}$ is a Hopf π-coalgebra, then $\{\alpha \in \pi \,|\, H_\alpha \neq 0\}$ is a subgroup of π.

A Hopf π-coalgebra $H = \{H_\alpha\}_{\alpha\in\pi}$ is said to be of *finite type* if, for all $\alpha \in \pi$, H_α is finite-dimensional (over $\Bbbk$). Note that it does not mean that $\oplus_{\alpha\in\pi} H_\alpha$ is finite-dimensional (unless $H_\alpha = 0$ for all but a finite number of $\alpha \in \pi$).

The antipode of a Hopf π-coalgebra $H = \{H_\alpha\}_{\alpha\in\pi}$ is anti-multiplicative: each $S_\alpha\colon H_\alpha \to H_{\alpha^{-1}}$ is an anti-homomorphism of algebras, and anti-comultiplicative: $\varepsilon S_1 = \varepsilon$ and $\Delta_{\beta^{-1},\alpha^{-1}} S_{\alpha\beta} = \sigma_{H_{\alpha^{-1}},H_{\beta^{-1}}}(S_\alpha \otimes S_\beta)\Delta_{\alpha,\beta}$ for any $\alpha, \beta \in \pi$, see [19, Lemma 1.1].

The antipode $S = \{S_\alpha\}_{\alpha\in\pi}$ of $H = \{H_\alpha\}_{\alpha\in\pi}$ is said to be *bijective* if each S_α is bijective. As for Hopf algebras, the antipode of a finite type Hopf π-coalgebra is always bijective (see [19, Corollary 3.7(a)]).

We extend the Sweedler notation for a comultiplication to the setting of a Hopf π-coalgebra $H = \{H_\alpha\}_{\alpha\in\pi}$ in the following way: for any $\alpha, \beta \in \pi$ and $h \in H_{\alpha\beta}$, we write $\Delta_{\alpha,\beta}(h) = \sum_{(h)} h_{(1,\alpha)} \otimes h_{(2,\beta)} \in H_\alpha \otimes H_\beta$, or shortly, if we leave the summation implicit, $\Delta_{\alpha,\beta}(h) = h_{(1,\alpha)} \otimes h_{(2,\beta)}$. The coassociativity of Δ gives that, for any $\alpha, \beta, \gamma \in \pi$ and $h \in H_{\alpha\beta\gamma}$,

$$h_{(1,\alpha\beta)(1,\alpha)} \otimes h_{(1,\alpha\beta)(2,\beta)} \otimes h_{(2,\gamma)} = h_{(1,\alpha)} \otimes h_{(2,\beta\gamma)(1,\beta)} \otimes h_{(2,\beta\gamma)(2,\gamma)}.$$

This element of $H_\alpha \otimes H_\beta \otimes H_\gamma$ is written as $h_{(1,\alpha)} \otimes h_{(2,\beta)} \otimes h_{(3,\gamma)}$. By iterating the procedure, we define inductively $h_{(1,\alpha_1)} \otimes \cdots \otimes h_{(n,\alpha_n)}$ for any $h \in H_{\alpha_1\cdots\alpha_n}$.

1.2. π-integrals.

Let us recall that a left (resp. right) integral for a Hopf algebra $(A, \Delta, \varepsilon, S)$ is an element $\Lambda \in A$ such that $x\Lambda = \varepsilon(x)\Lambda$ (resp. $\Lambda x = \varepsilon(x)\Lambda$) for all $x \in A$. A left (resp. right) integral for the dual Hopf algebra A^* is a $\Bbbk$-linear form $\lambda \in A^*$ verifying $(\mathrm{id}_A \otimes \lambda)\Delta(x) = \lambda(x)1_A$ (resp. $(\lambda \otimes \mathrm{id}_A)\Delta(x) = \lambda(x)1_A$) for all $x \in A$.

By a *left* (resp. *right*) *π-integral* for a Hopf π-coalgebra $H = \{H_\alpha\}_{\alpha\in\pi}$, we shall mean a family of $\Bbbk$-linear forms $\lambda = (\lambda_\alpha)_{\alpha\in\pi} \in \Pi_{\alpha\in\pi} H_\alpha^*$ such that

$$(\mathrm{id}_{H_\alpha} \otimes \lambda_\beta)\Delta_{\alpha,\beta}(x) = \lambda_{\alpha\beta}(x)1_\alpha \quad (\text{resp.} \quad (\lambda_\alpha \otimes \mathrm{id}_{H_\beta})\Delta_{\alpha,\beta}(x) = \lambda_{\alpha\beta}(x)1_\beta\,) \tag{1.4}$$

for all $\alpha, \beta \in \pi$ and $x \in H_{\alpha\beta}$.

Note that λ_1 is a usual left (resp. right) integral for the Hopf algebra H_1^*.

A π-integral $\lambda = (\lambda_\alpha)_{\alpha\in\pi}$ is said to be *non-zero* if $\lambda_\beta \neq 0$ for some $\beta \in \pi$. Note that a non-zero π-integral $\lambda = (\lambda_\alpha)_{\alpha\in\pi}$ verifies that $\lambda_\alpha \neq 0$ for all $\alpha \in \pi$ such that $H_\alpha \neq 0$ (and in particular $\lambda_1 \neq 0$).

It is known that the space of left (resp. right) integrals for a finite-dimensional Hopf algebra is one-dimensional. In the setting of Hopf π-coalgebras, we also have that the space of left (resp. right) π-integrals for a finite type Hopf π-coalgebra is one-dimensional (even when π is infinite), see [19, Theorem 3.6].

1.3. Crossed Hopf π-coalgebras.

The notion of a crossing for a Hopf π-coalgebra is needed to define the quasitriangularity of a Hopf π-coalgebra (see [18, 19]). A Hopf π-coalgebra $H = \{H_\alpha\}_{\alpha\in\pi}$ is said to be *crossed* if it is endowed with a family $\varphi = \{\varphi_\beta\colon H_\alpha \to H_{\beta\alpha\beta^{-1}}\}_{\alpha,\beta\in\pi}$ of algebra isomorphisms (the *crossing*) such that each φ_β preserves the comultiplication and the counit and φ is *multiplicative*, that is, for all $\alpha, \beta, \gamma \in \pi$,

$$(\varphi_\beta \otimes \varphi_\beta)\Delta_{\alpha,\gamma} = \Delta_{\beta\alpha\beta^{-1},\beta\gamma\beta^{-1}}\varphi_\beta; \tag{1.5}$$

$$\varepsilon\varphi_\beta = \varepsilon; \tag{1.6}$$

$$\varphi_{\alpha\beta} = \varphi_\alpha\varphi_\beta. \tag{1.7}$$

One easily verifies that a crossing preserves the antipode, that is, $\varphi_\beta S_\alpha = S_{\beta\alpha\beta^{-1}}\varphi_\beta$ for all $\alpha, \beta \in \pi$.

A particular class of crossed Hopf π-coalgebras is that of Hopf π-coalgebras with π abelian: if π is an abelian group, then a Hopf π-coalgebra $H=\{H_\alpha\}_{\alpha\in\pi}$ is always crossed (e.g., by taking $\varphi_\beta|_{H_\alpha}=\mathrm{id}_{H_\alpha}$).

From the uniqueness of π-integrals, we have that if $H=\{H_\alpha\}_{\alpha\in\pi}$ be a finite type crossed Hopf π-coalgebra with crossing φ. Then there exists a unique group homomorphism $\widehat{\varphi}:\pi\to\Bbbk^*$ such that if $\lambda=(\lambda_\alpha)_{\alpha\in\pi}$ is a left or right π-integral for H, then $\lambda_{\beta\alpha\beta^{-1}}\varphi_\beta=\widehat{\varphi}(\beta)\lambda_\alpha$ for all $\alpha,\beta\in\pi$.

1.4. **Quasitriangular Hopf π-coalgebras.** Following [18, §11.3], a *quasitriangular* Hopf π-coalgebra is a crossed Hopf π-coalgebra $H=\{H_\alpha\}_{\alpha\in\pi}$ endowed with a family $R=\{R_{\alpha,\beta}\in H_\alpha\otimes H_\beta\}_{\alpha,\beta\in\pi}$ of invertible elements (the *R-matrix*) such that:

$$R_{\alpha,\beta}\cdot\Delta_{\alpha,\beta}(x)=\sigma_{\beta,\alpha}(\varphi_{\alpha^{-1}}\otimes\mathrm{id}_{H_\alpha})\Delta_{\alpha\beta\alpha^{-1},\alpha}(x)\cdot R_{\alpha,\beta};\tag{1.8}$$

$$(\mathrm{id}_{H_\alpha}\otimes\Delta_{\beta,\gamma})(R_{\alpha,\beta\gamma})=(R_{\alpha,\gamma})_{1\beta3}\cdot(R_{\alpha,\beta})_{12\gamma};\tag{1.9}$$

$$(\Delta_{\alpha,\beta}\otimes\mathrm{id}_{H_\gamma})(R_{\alpha\beta,\gamma})=[(\mathrm{id}_{H_\alpha}\otimes\varphi_{\beta^{-1}})(R_{\alpha,\beta\gamma\beta^{-1}})]_{1\beta3}\cdot(R_{\beta,\gamma})_{\alpha23};\tag{1.10}$$

$$(\varphi_\beta\otimes\varphi_\beta)(R_{\alpha,\gamma})=R_{\beta\alpha\beta^{-1},\beta\gamma\beta^{-1}}\tag{1.11}$$

for all $\alpha,\beta,\gamma\in\pi$ and $x\in H_{\alpha\beta}$, where $\sigma_{\beta,\alpha}$ denotes the flip map $H_\beta\otimes H_\alpha\to H_\alpha\otimes H_\beta$ and, for $\Bbbk$-spaces P,Q and $r=\sum_j p_j\otimes q_j\in P\otimes Q$, we set $r_{12\gamma}=r\otimes 1_\gamma\in P\otimes Q\otimes H_\gamma$, $r_{\alpha23}=1_\alpha\otimes r\in H_\alpha\otimes P\otimes Q$, and $r_{1\beta3}=\sum_j p_j\otimes 1_\beta\otimes q_j\in P\otimes H_\beta\otimes Q$

Note that $R_{1,1}$ is a (classical) R-matrix for the Hopf algebra H_1.

When π is abelian and φ is *trivial* (that is, $\varphi_\beta|_{H_\alpha}=\mathrm{id}_{H_\alpha}$ for all $\alpha,\beta\in\pi$), one recovers the definition of a quasitriangular π-colored Hopf algebra given by Ohtsuki in [13].

Notation. In the proofs, when we write a component $R_{\alpha,\beta}$ of an R-matrix as $R_{\alpha,\beta}=a_\alpha\otimes b_\beta$, it is to signify that $R_{\alpha,\beta}=\sum_j a_j\otimes b_j$ for some $a_j\in H_\alpha$ and $b_j\in H_\beta$, where j runs over a finite set of indices.

The main properties of quasitriangular Hopf algebras are still true in the setting of quasitriangular Hopf π-coalgebras, see [19]. In particular, if $H=\{H_\alpha\}_{\alpha\in\pi}$ is a quasitriangular Hopf π-coalgebra then, for any $\alpha,\beta,\gamma\in\pi$,

$$(\varepsilon\otimes\mathrm{id}_{H_\alpha})(R_{1,\alpha})=1_\alpha=(\mathrm{id}_{H_\alpha}\otimes\varepsilon)(R_{\alpha,1});\tag{1.12}$$

$$S_{\alpha^{-1}}\varphi_\alpha\otimes\mathrm{id}_{H_\beta})(R_{\alpha^{-1},\beta})=R^{-1}_{\alpha,\beta}\text{ and }(\mathrm{id}_{H_\alpha}\otimes S_\beta)(R^{-1}_{\alpha,\beta})=R_{\alpha,\beta^{-1}};\tag{1.13}$$

$$(S_\alpha\otimes S_\beta)(R_{\alpha,\beta})=(\varphi_\alpha\otimes\mathrm{id}_{H_{\beta^{-1}}})(R_{\alpha^{-1},\beta^{-1}});\tag{1.14}$$

$$\begin{aligned}(R_{\beta,\gamma})_{\alpha23}\cdot(R_{\alpha,\gamma})_{1\beta3}\cdot(R_{\alpha,\beta})_{12\gamma}&\\ =(R_{\alpha,\beta})_{12\gamma}\cdot[(\mathrm{id}_{H_\alpha}\otimes\varphi_{\beta^{-1}})&(R_{\alpha,\beta\gamma\beta^{-1}})]_{1\beta3}\cdot(R_{\beta,\gamma})_{\alpha23}.\end{aligned}\tag{1.15}$$

1.5. **The Drinfeld elements.** Let $H=\{H_\alpha\}_{\alpha\in\pi}$ be a quasitriangular Hopf π-coalgebra. The *Drinfeld elements* of H are defined by

$$u_\alpha=m_\alpha(S_{\alpha^{-1}}\varphi_\alpha\otimes\mathrm{id}_{H_\alpha})\sigma_{\alpha,\alpha^{-1}}(R_{\alpha,\alpha^{-1}})\in H_\alpha.$$

Note that u_1 is the Drinfeld element of the quasitriangular Hopf algebra H_1 (see [4]). For all $\alpha,\beta\in\pi$ and $x\in H_\alpha$, we have that

(1.16) u_α is invertible and $u_\alpha^{-1}=m_\alpha(\mathrm{id}_{H_\alpha}\otimes S_{\alpha^{-1}}S_\alpha)\sigma_{\alpha,\alpha}(R_{\alpha,\alpha})$;

(1.17) $S_{\alpha^{-1}}S_\alpha(\varphi_\alpha(x))=u_\alpha x u_\alpha^{-1}$;

(1.18) The antipode of H is bijective;

(1.19) $\varphi_\beta(u_\alpha)=u_{\beta\alpha\beta^{-1}}$;

(1.20) $S_{\alpha^{-1}}(u_{\alpha^{-1}})u_\alpha=u_\alpha S_{\alpha^{-1}}(u_{\alpha^{-1}})$ and this element, noted c_α, verifies

(1.21) $c_\alpha\varphi_{\alpha^{-1}}(x)=\varphi_\alpha(x)c_\alpha$;

(1.22) $$\begin{aligned}\Delta_{\alpha,\beta}(u_{\alpha\beta})&=[\sigma_{\beta,\alpha}(\mathrm{id}_{H_\beta}\otimes\varphi_\alpha)(R_{\beta,\alpha})\cdot R_{\alpha,\beta}]^{-1}\cdot(u_\alpha\otimes u_\beta)\\&=(u_\alpha\otimes u_\beta)\cdot[\sigma_{\beta,\alpha}(\varphi_{\beta^{-1}}\otimes\mathrm{id}_{H_\alpha})(R_{\beta,\alpha})\cdot(\varphi_{\alpha^{-1}}\otimes\varphi_{\beta^{-1}})(R_{\alpha,\beta})]^{-1};\end{aligned}$$

(1.23) $\varepsilon(u_1)=1$.

1.6. **Ribbon Hopf π-coalgebras.** Following [18, §11.4], a quasitriangular Hopf π-coalgebra $H=\{H_\alpha\}_{\alpha\in\pi}$ is said to be *ribbon* if it is endowed with a family $\theta=\{\theta_\alpha\in H_\alpha\}_{\alpha\in\pi}$ of invertible elements (the *twist*) such that, for all $\alpha,\beta\in\pi$ and $x\in H_\alpha$,

$$\varphi_\alpha(x)=\theta_\alpha^{-1}x\theta_\alpha; \tag{1.24}$$

$$S_\alpha(\theta_\alpha)=\theta_{\alpha^{-1}}; \tag{1.25}$$

$$\varphi_\beta(\theta_\alpha)=\theta_{\beta\alpha\beta^{-1}}; \tag{1.26}$$

$$\Delta_{\alpha,\beta}(\theta_{\alpha\beta})=(\theta_\alpha\otimes\theta_\beta)\cdot\sigma_{\beta,\alpha}((\varphi_{\alpha^{-1}}\otimes\mathrm{id}_{H_\alpha})(R_{\alpha\beta\alpha^{-1},\alpha}))\cdot R_{\alpha,\beta}. \tag{1.27}$$

Note that θ_1 is a (classical) twist of the quasitriangular Hopf algebra H_1 and that $\varphi_{\alpha^{-1}}(x)=\theta_\alpha x\theta_\alpha^{-1}$ for all $\alpha\in\pi$ and $x\in H_\alpha$.

1.7. **The spherical π-grouplike element.** Let $H=\{H_\alpha\}_{\alpha\in\pi}$ be a ribbon Hopf π-coalgebra. For any $\alpha\in\pi$, set

$$G_\alpha=\theta_\alpha u_\alpha=u_\alpha\theta_\alpha\in H_\alpha.$$

Then $G=(G_\alpha)_{\alpha\in\pi}$ is a π-grouplike element, called the *spherical π-grouplike element of H*. It verifies that, for all $\alpha,\beta\in\pi$ and $x\in H_\alpha$,

$$\varphi_\beta(G_\alpha)=G_{\beta\alpha\beta^{-1}}; \tag{1.28}$$

$$S_\alpha(G_\alpha)=G_{\alpha^{-1}}^{-1}; \tag{1.29}$$

$$S_\alpha(u_\alpha)=G_{\alpha^{-1}}^{-1}u_{\alpha^{-1}}G_{\alpha^{-1}}^{-1}; \tag{1.30}$$

$$S_{\alpha^{-1}}S_\alpha(x)=G_\alpha xG_\alpha^{-1}. \tag{1.31}$$

1.8. **π-traces.** A Hopf π-coalgebra $H=\{H_\alpha\}_{\alpha\in\pi}$ is said to be *unimodular* if the Hopf algebra H_1 is unimodular (it means that the spaces of left and right integrals for H_1 coincide). If H_1 is finite-dimensional, then H is unimodular if and only if $\nu=\varepsilon$, where ν is the distinguished grouplike element of H_1^*.

Let $H=\{H_\alpha\}_{\alpha\in\pi}$ be a crossed Hopf π-coalgebra. A *π-trace* for H is a family of $\Bbbk$-linear forms $\mathrm{tr}=(\mathrm{tr}_\alpha)_{\alpha\in\pi}\in\Pi_{\alpha\in\pi}H_\alpha^*$ such that, for any $\alpha,\beta\in\pi$ and $x,y\in H_\alpha$,

$$\mathrm{tr}_\alpha(xy)=\mathrm{tr}_\alpha(yx); \tag{1.32}$$

$$\mathrm{tr}_{\alpha^{-1}}(S_\alpha(x))=\mathrm{tr}_\alpha(x); \tag{1.33}$$

$$\mathrm{tr}_{\beta\alpha\beta^{-1}}(\varphi_\beta(x))=\mathrm{tr}_\alpha(x). \tag{1.34}$$

Note that tr_1 is a (usual) trace for the Hopf algebra H_1, invariant under the action φ of π.

In the next lemma, generalizing [8, Proposition 4.2], we give a characterization of the π-traces.

Lemma 1.1. *Let $H=\{H_\alpha\}_{\alpha\in\pi}$ be a finite type unimodular ribbon Hopf π-coalgebra with crossing φ. Let $\lambda=(\lambda_\alpha)_{\alpha\in\pi}$ be a non-zero right π-integral for H, $G=(G_\alpha)_{\alpha\in\pi}$ be the spherical π-grouplike element of H, and $\widehat{\varphi}$ be as in Section 1.3. Let $\mathrm{tr}=(\mathrm{tr}_\alpha)_{\alpha\in\pi}\in\Pi_{\alpha\in\pi}H_\alpha^*$. Then tr is a π-trace for H if and only if there exists a family $z=(z_\alpha)_{\alpha\in\pi}\in\Pi_{\alpha\in\pi}H_\alpha$ satisfying, for all $\alpha,\beta\in\pi$,*

(a) $\mathrm{tr}_\alpha(x)=\lambda_\alpha(G_\alpha z_\alpha x)$ *for all* $x\in H_\alpha$*;*
(b) z_α *is central in* H_α*;*
(c) $S_\alpha(z_\alpha)=\widehat{\varphi}(\alpha)^{-1}z_{\alpha^{-1}}$*;*
(d) $\varphi_\beta(z_\alpha)=\widehat{\varphi}(\beta)z_{\beta\alpha\beta^{-1}}$.

In the setting of Lemma 1.1, constructing a π-trace from a right π-integral $\lambda=(\lambda_\alpha)_{\alpha\in\pi}$ reduces to finding a family $z=(z_\alpha)_{\alpha\in\pi}$ which satisfies Conditions (b)-(d) of Lemma 1.1. Let us give two possible choices of the family z.

Let Λ be a left integral for H_1 such that $\lambda_1(\Lambda)=1$. Set $z_1=\Lambda$ and $z_\alpha=0$ if $\alpha\neq 1$. This family $z=(z_\alpha)_{\alpha\in\pi}$ verifies Conditions (b)-(d) since H is unimodular (and so Λ is central and $S_1(\Lambda)=\Lambda$). The π-trace obtained is given by $\mathrm{tr}_1=\varepsilon$ and $\mathrm{tr}_\alpha=0$ if $\alpha\neq 1$.

If the homomorphism $\widehat{\varphi}$ is trivial (that is, $\widehat{\varphi}(\alpha)=1$ for all $\alpha\in\pi$), then another possible choice is $z_\alpha=1_\alpha$. In the two next lemmas, we give sufficient conditions for the homomorphism $\widehat{\varphi}$ to be trivial.

We conclude with the following theorem, which follows directly from Lemma 1.1 (by choosing $z_\alpha=1_\alpha$ for all $\alpha\in\pi$).

Theorem 1.2. *Let $H=\{H_\alpha\}_{\alpha\in\pi}$ be a finite type unimodular ribbon Hopf π-coalgebra with crossing φ and twist $\theta=\{\theta_\alpha\}_{\alpha\in\pi}$. Let $\lambda=(\lambda_\alpha)_{\alpha\in\pi}$ be a right π-integral for H and $G=(G_\alpha)_{\alpha\in\pi}$ be the spherical π-grouplike element of H. Suppose that $\lambda_1(\theta_1)\neq 0$ or that $\varphi_\beta|_{H_1}=\mathrm{id}_{H_1}$ for all $\beta\in\pi$. Then $\mathrm{tr}=(\mathrm{tr}_\alpha)_{\alpha\in\pi}$, defined by $\mathrm{tr}_\alpha(x)=\lambda_\alpha(G_\alpha x)$ for all $\alpha\in\pi$ and $x\in H_\alpha$, is a π-trace for H.*

1.9. **Examples of Hopf π-coalgebras.**

Example 1.3. Let π be a group and $c:\pi\times\pi\to\Bbbk^*$ be a bicharacter of π, that is, verifying $c(\alpha,\beta\gamma)=c(\alpha,\beta)c(\alpha,\gamma)$ and $c(\alpha\beta,\gamma)=c(\alpha,\gamma)c(\beta,\gamma)$ for all $\alpha,\beta,\gamma\in\pi$.

Consider the crossed Hopf algebra $\Bbbk^c$ defined as follows: for any $\alpha,\beta\in\pi$, set $\Bbbk^c_\alpha=\Bbbk$ (as algebra), $\Delta_{\alpha,\beta}(1_\Bbbk)=1_\Bbbk\otimes 1_\Bbbk$, $\epsilon(1_\Bbbk)=1_\Bbbk$, $S_\alpha(1_\Bbbk)=1_\Bbbk$ and $\varphi_\beta(1_\Bbbk)=1_\Bbbk$. Then it is a ribbon Hopf π-coalgebra with R-matrix and twist given by $R_{\alpha,\beta}=c(\alpha,\beta)\,1_\Bbbk\otimes 1_\Bbbk$ and $\theta_\alpha=c(\alpha,\alpha)$. The Drinfeld elements of $\Bbbk^c$ are $u_\alpha=c(\alpha,\alpha)^{-1}$. Moreover $\Bbbk^c$ is finite dimensional and unimodular and $(\mathrm{id}_\Bbbk)_{\alpha\in\pi}$ is a two-sided π-integral and a π-trace for $\Bbbk^c$. This Hopf π-coalgebra is used in Section 2.7.

Example 1.4. Recall that, when π is an abelian group, a ribbon Hopf π-coalgebra with trivial crossing is a ribbon π-colored Hopf algebra in the sense of [13]. Following [14], we give an example of a ribbon Hopf $(\frac{1}{N}\mathbb{Z})/\mathbb{Z}$-coalgebra, where N is a fixed positive integer, which is derived from finite dimensional quotients of $U_q(sl_2)$.

Fix an integer $r\geq 2$. Set $t=\exp(\frac{i\pi}{2r})$ and $q=t^2=\exp(\frac{i\pi}{r})$. For any $x\in\mathbb{R}$, t^x will denote the scalar $\exp(\frac{i\pi x}{2r})$. In particular, $q^x=t^{2x}=\exp(\frac{i\pi x}{r})$. Note that if $x'\equiv x \mod 4r$, then $t^{x'}=t^x$.

For each $\alpha\in(\frac{1}{N}\mathbb{Z})/\mathbb{Z}$, let A_α be the associative algebra over $\mathbb{C}$ with generators $a^{\frac{1}{N}}$, e, and f, subject to the following relations:

$$\begin{aligned}
&a^{\frac{1}{N}}e=q^{\frac{1}{N}}\,ea^{\frac{1}{N}}, && a^{\frac{1}{N}}f=q^{-\frac{1}{N}}\,fa^{\frac{1}{N}}, && ef-fe=\tfrac{a^2-a^{-2}}{q-q^{-1}},\\
&e^r=0, && f^r=0, && a^{4r}=t^{-4r\alpha}.
\end{aligned}$$

The family $A = \{A_\alpha\}_{\alpha\in\pi}$ is a Hopf $(\frac{1}{N}\mathbb{Z})/\mathbb{Z}$-coalgebra by setting:

$$\begin{array}{lll}
\Delta_{\alpha,\beta}(a^{\frac{1}{N}}) = a^{\frac{1}{N}} \otimes a^{\frac{1}{N}}, & \Delta_{\alpha,\beta}(e) = e \otimes a^{-1} + a \otimes e, & \Delta_{\alpha,\beta}(f) = f \otimes a^{-1} + a \otimes f, \\
\epsilon(a) = 1, & \epsilon(e) = 0, & \epsilon(f) = 0, \\
S_\alpha(a^{\frac{1}{N}}) = a^{-\frac{1}{N}}, & S_\alpha(e) = -q^{-1}\, e, & S_\alpha(f) = -q\, f.
\end{array}$$

We endow A with the trivial crossing, that is, $\varphi_\beta|_{A_\alpha} = \mathrm{id}_{A_\alpha}$. The crossed Hopf $(\frac{1}{N}\mathbb{Z})/\mathbb{Z}$-coalgebra $A = \{A_\alpha\}_{\alpha\in(\frac{1}{N}\mathbb{Z})/\mathbb{Z}}$ is ribbon with R-matrix

$$R_{\alpha,\beta} = \frac{1}{4r}\sum_{n=0}^{r-1}\sum_{k,l\in\mathbb{Z}/4r\mathbb{Z}} \frac{(q-q^{-1})^n}{[n]!} t^{-(l+\alpha)n+(k-\beta)(l+\alpha-n)-n} f^n a^{k-\beta} \otimes e^n a^{-(l+\alpha)}$$

and twist $\theta_\alpha = a^{2(r-1)}u_\alpha^{-1}$, where the u_α are the Drinfeld elements of A. Here $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$, $[n]! = [n][n-1]\cdots[1]$, and $[0]! = 1$.

Some results concerning this Hopf $(\frac{1}{N}\mathbb{Z})/\mathbb{Z}$-coalgebra (used in Example 3.4) are established in Appendix A.

2. Invariants of π-links

In this section, we generalize the Kauffman-Radford method to construct HKR-type invariants of framed links endowed with a morphism from their fundamental group to π, by using a ribbon Hopf π-coalgebra.

2.1. π-links.

Following [18], a *π-link* in S^3 is a triple (L, z, g) where L is a framed link in S^3, $z \in S^3 \setminus L$ (the *base point*), and $g : \pi_1(S^3 \setminus L, z) \to \pi$ is a group homomorphism. Recall that a link $L = L_1 \cup \cdots \cup L_m$ is *framed* if each of its components L_i is provided with a *longitude* $\widetilde{L}_i \subset S^3 \setminus L$ which goes very closely along L_i (or equivalently with an integer n_i, called *framing number*, which is related to $\widetilde{L}_i$ by $n_i = \mathrm{lk}(\widetilde{L}_i, L_i)$ where a parallel orientation for L_i and $\widetilde{L}_i$ is chosen). The framing of a framed link L will be denoted by $\widetilde{L} = \widetilde{L}_1 \cup \cdots \cup \widetilde{L}_m$.

Two π-links (L, z, g) and (L', z', g') are said to be *equivalent* if there exists an orientation-preserving homeomorphism $h : S^3 \to S^3$ such that $h(L) = L'$, $h(\widetilde{L}) = \widetilde{L}'$, $h(z) = z'$, and $g' \circ h_* = g$ where $h_* : \pi_1(S^3 \setminus L, z) \to \pi_1(S^3 \setminus L', z')$ is the group isomorphism induced by h in homotopy.

2.2. π-colored link diagrams.

By a *generic diagram* of a framed link we shall mean a diagram of the link, arranged with respect to a vertical direction and with blackboard framing, such that the only critical points of the height function are crossings and extrema and the height function is non-degenerate in all extremal points (i.e., in a neighborhood of any extremal point, the diagram looks like a cap or a cup). The segments of a generic diagram delimited by extremal points and under-crossings are called the *vertical segments* of the diagram.

A *π-colored link diagram* is a generic diagram of a framed link such that each of its vertical segments is provided with an element of π, called the *color* of the vertical segment, in such a way that for crossings and extrema the colors are related as in Figure 1.

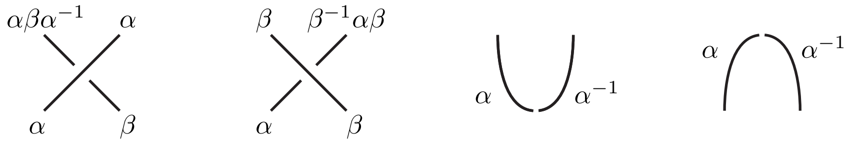


Figure 1

Two π-colored link diagrams are said to be *equivalent* if one can be obtained from the other by a finite sequence of isotopies (in the class of generic link diagrams) which preserve the colors of the vertical segments and of moves of Figure 2.

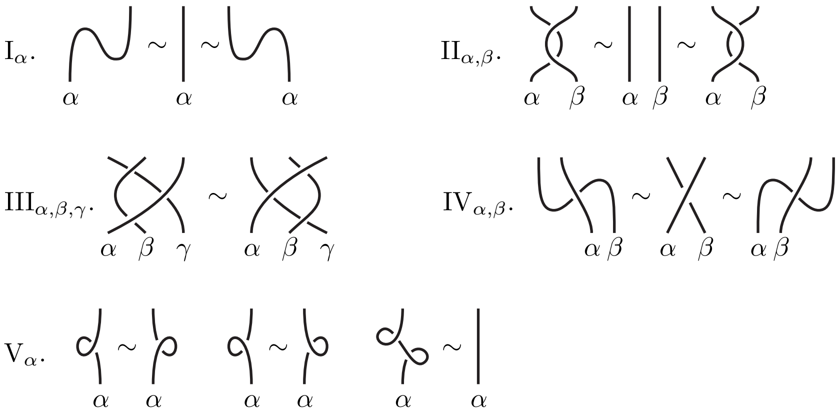


Figure 2. Equivalence moves for π-colored link diagrams

Note that π-colored link diagrams can be associated to a π-link (L, z, g) by the following procedure: regularly project the framed link L onto a plane from the base point, i.e., consider a generic diagram of L such that the base point z corresponds to the eyes of the reader. Color then the vertical segments in the following way: a vertical segment is colored by $\alpha = g([\mu]) \in \pi$ where μ represents a loop that, starting from the base point z (the eyes of the reader) above the diagram, goes straight to the segment, encircles it from left to right (i.e., in such a way that its linking number with the segment oriented downwards is 1), and returns immediately to the base point as shown in Figure 3.

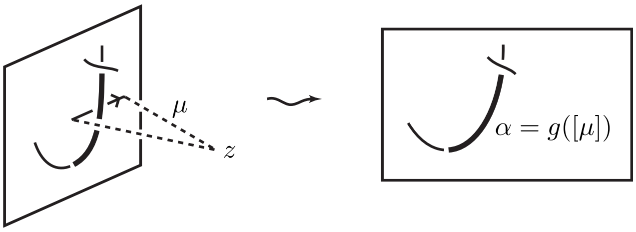


Figure 3. Coloration of diagrams of π-links

Reciprocally, using the Wirtinger presentation of knot groups (see, e.g., [12]), one easily verifies that a π-colored link diagram determines (up to equivalence) an unique π-link. Moreover the operations defining the equivalence of π-colored link diagrams can be realized by an ambient isotopy and thus by an equivalence between the π-links they determine. Hence equivalent π-colored link diagrams define equivalent π-links. We show in the next lemma that the converse is also true.

Lemma 2.1. *Two π-links are equivalent if and only if all their π-colored link diagrams are equivalent.*

Proof. Let us first verify that two π-colored diagrams D and D' of a same π-link (L, z, g) are equivalent. Let p and p' be two directed projections which leads to

D and D' respectively. Think of the set of directed projection as points on a unit sphere $S^2 \subset S^3$, centered in the base point z, endowed with the induced topology. A standard argument (general position) shows that singular projections (those that not lead to generic diagrams) are represented on S^2 by a finite number of curves (see [2]). Then choose on S^2 a path s from p to p' in general position with respect to the curves of singular projections. When such a curve is crossed, the π-colored link diagram will be changed by a move $I_\alpha, \dots, V_\alpha$, depending on the type of singularity corresponding to the singular curve that is, crossed. Moreover parts of s between the singular curves correspond to isotopies (in the class of generic link diagrams) which preserve the colors of the vertical segments.

It remains to show that for a fixed projection, the π-colored diagrams obtained from two equivalent π-links are equivalent. Let (L, z, g) and (L', z', g') be two equivalent π-links and fix a directed projection onto a plane P which leads to generic diagrams of L and L'. Since (L, z, g) and (L', z', g') are equivalent, there exists an orientation-preserving homeomorphism $h : S^3 \to S^3$ such that $h(L) = L'$, $h(\widetilde{L}) = \widetilde{L}'$, $h(z) = z'$, and $g' \circ h_* = g$. Since h is an orientation-preserving homeomorphism of S^3, it is isotopic to the identity, i.e., there exists a family $(h_t)_{t \in [0,1]}$ of homeomorphisms of S^3 such that $h_0 = \mathrm{id}_{S^3}$ and $h_1 = h$. By translating the plane P (with respect to the direction of the projection), we can assume that all the $h_t(z)$ remains in the same half-space delimited by P and, by general position argument, we can suppose that the projection onto P of the framed link $h_t(L)$ is a generic diagram for all but a finite number of $t \in [0, 1]$ which correspond to Reidemeister moves for framed links. Using this finite sequence of transformations and the coloring homomorphisms $g \circ (h_t^{-1})_* : \pi_1(h_t(L), h_t(z)) \to \pi$, one easily deduces that the π-colored diagrams obtained by projecting (L, z, g) and (L', z', g') onto P are equivalent. □

2.3. π-links compatible with a crossed Hopf π-coalgebra.

Let $H = \{H_\alpha\}_{\alpha \in \pi}$ be a crossed Hopf π-coalgebra with crossing φ. A π-link (L, z, g) is said to be *compatible with* H or, shortly, *H-compatible* if, for any component C of L, for any path $\gamma : [0, 1] \to S^3 \setminus L$ connecting the base point $z \in S^3 \setminus L$ to a point $\gamma(1) \in \widetilde{C}$, and for any orientation ν of $\widetilde{C}$, the following conditions are satisfied:

(2.1) $g(\lambda_{(\gamma,\nu)})$ belongs to the center $Z(\pi)$ of π;

(2.2) $\varphi_{g(\lambda_{(\gamma,\nu)})|H_\beta} = \mathrm{id}_{H_\beta}$ for all $\beta \in \pi$;

where $\lambda_{(\gamma,\nu)} = [\gamma^{-1}\widetilde{C}\gamma] \in \pi_1(S^3 \setminus L, z)$ is the homotopy class of the loop $\gamma^{-1}\widetilde{C}\gamma$ (here the oriented circle $\widetilde{C}$ is viewed as a loop based on the point $\gamma(1)$).

Lemma 2.2. *Let $H = \{H_\alpha\}_{\alpha \in \pi}$ be a crossed Hopf π-coalgebra and (L, z, g) be a π-link.*

(a) *If, for any component C of L, there exist a path $\gamma : [0, 1] \to S^3 \setminus L$ connecting the base point $z \in S^3 \setminus L$ to a point $\gamma(1) \in \widetilde{C}$ and an orientation ν of $\widetilde{C}$ such that* (2.1) *and* (2.2) *hold, then (L, z, g) is H-compatible.*

(b) *If (L, z, g) is H-compatible and if ρ is a homeomorphism of S^3 (preserving or reversing the orientation), then the π-link $(\rho(L), \rho(z), g \circ \rho_*^{-1})$ is H-compatible. In particular H-compatibility is preserved under equivalence of π-links.*

(c) *(L, z, g) is H-compatible if and only if it is H^{cop}-compatible, where H^{cop} is the crossed Hopf π-coalgebra coopposite to H.*

Proof. Let us show Part (a). Suppose first that the opposite orientation $-\nu$ for $\widetilde{C}$ is chosen. Then $\lambda_{(\gamma,-\nu)} = \lambda_{(\gamma,\nu)}^{-1}$ and so $\lambda_{(\gamma,-\nu)} \in Z(\pi)$ and $\varphi_{g(\lambda_{(\gamma,-\nu)})} = \varphi^{-1}_{g(\lambda_{(\gamma,\nu)})} = \mathrm{id}$. Suppose secondly that γ' is another path in $S^3 \setminus L$ connecting the base point z

to $\widetilde{C}$. Then there exists a loop ℓ in $S^3 \setminus L$ based on z such that γ' is homotopic to $\gamma\ell$ in $(S^3 \setminus L, z)$. Set $\xi = [\ell] \in \pi_1(S^3 \setminus L, z)$. We have that $\lambda_{(\gamma',\nu)} = [{\gamma'}^{-1}\widetilde{C}\gamma'] = [\ell^{-1}\gamma^{-1}\widetilde{C}\gamma\ell] = \xi^{-1}\lambda_{(\gamma,\nu)}\xi$ and so

$$g(\lambda_{(\gamma',\nu)}) = g(\xi^{-1}\lambda_{(\gamma,\nu)}\xi) = g(\xi)^{-1}\, g(\lambda_{(\gamma,\nu)})\, g(\xi) = g(\xi)^{-1}\, g(\xi)\, g(\lambda_{(\gamma,\nu)}) = g(\lambda_{(\gamma,\nu)}).$$

Hence $g(\lambda_{(\gamma',\nu)}) \in Z(\pi)$ and $\varphi_{g(\lambda_{(\gamma',\nu)})} = \varphi_{g(\lambda_{(\gamma,\nu)})} = \mathrm{id}$.

To show Part (b), fix a component C of L. Let $\gamma : [0,1] \to S^3 \setminus L$ be a path connecting the base point $\rho(z) \in S^3 \setminus \rho(L)$ to a point $\gamma(1) \in \widetilde{\rho(C)} = \rho(\widetilde{C})$ and ν be an orientation of $\rho(\widetilde{C})$. Then

$$\lambda_{(\gamma,\nu)} = [\gamma^{-1}\rho(\widetilde{C})\gamma] = \rho_*[\rho^{-1}(\gamma)\widetilde{C}\rho^{-1}(\gamma)] = \rho_*(\lambda_{(\rho^{-1}(\gamma),\rho^{-1}(\nu))}),$$

where $\rho^{-1}(\nu)$ is the orientation of $\widetilde{C}$ induced by ρ^{-1} from the orientation ν of $\rho(\widetilde{C})$. Therefore we have that $(g \circ \rho_*^{-1})(\lambda_{(\gamma,\nu)}) = g(\lambda_{(\rho^{-1}(\gamma),\rho^{-1}(\nu))})$. Hence (2.1) and (2.2) are satisfied since (L, z, g) is H-compatible.

Part (c) follows directly from the fact that $\varphi^{\mathrm{cop}}_{\alpha|H^{\mathrm{cop}}_\beta} = \varphi_{\alpha|H_{\beta^{-1}}}$ for all $\alpha, \beta \in \pi$. □

2.4. **Invariants of π-links.** Fix a ribbon Hopf π-coalgebra $H = (\{H_\alpha\}, \Delta, \varepsilon, S, \varphi, R, \theta)$ with bijective antipode, endowed with a π-trace $\mathrm{tr} = (\mathrm{tr}_\alpha)_{\alpha\in\pi}$. We now give a method to define an invariant of H-compatible π-links, which generalizes that of Kauffman-Radford [10] for computing Hennings' invariants.

Let $(L = L_1 \cup \cdots \cup L_m, z, g)$ be a H-compatible π-link.

(A). Present the π-link (L, z, g) by a π-colored link diagram (as explained in Section 2.2).

(B). Each crossing of the π-colored link diagram is decorated with elements of the Hopf π-coalgebra $H = \{H_\alpha\}_{\alpha\in\pi}$ and with discs labelled by elements of π (which represent the action of φ) as shown in Figure 4, where $R_{\alpha,\beta} = a_\alpha \otimes b_\beta$ and $R_{\beta^{-1},\alpha} = c_{\beta^{-1}} \otimes d_\alpha$. Recall that it is implicit in this formalism that there is a summation over all the pairs a_α, b_β and $S_{\beta^{-1}}(c_{\beta^{-1}})$, d_α. The diagram obtained after this step is called the *flat diagram of* L. Note that the flat diagram of L is composed by m closed plane curves (possibly endowed with labelled discs), each of them arising from a component of L. These closed plane curves are called the *components* of the flat diagram of L. The component of the flat diagram of L arising from the component L_i of L is called the *flat diagram of* L_i. The *algebraic decoration* of the flat diagram of L consists in the points decorated by elements of H.

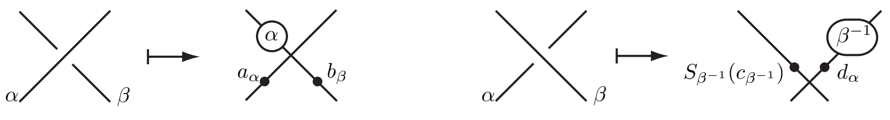


FIGURE 4. Algebraization of a π-colored link diagram

(C). On each component of the flat diagram of L, the algebraic decoration is concentrated in a point other than extrema and labelled discs, according to the rules of Figure 5, where $\alpha, \beta \in \pi$ and $a, b \in H_\alpha$.

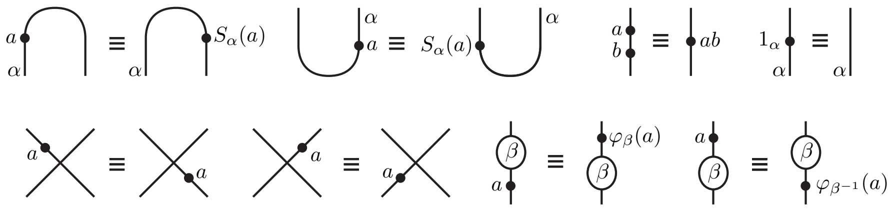
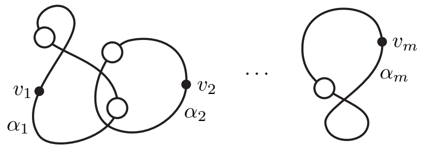


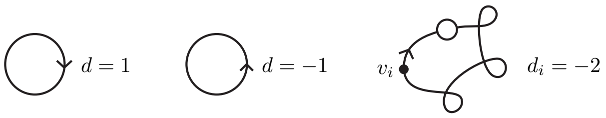
FIGURE 5. Rules for concentrating the algebraic decoration

In that way we get elements $v_1 \in H_{\alpha_1}, \dots, v_m \in H_{\alpha_m}$:



Note that $v_i = 1_{\alpha_i}$ if the flat diagram of L_i is free of algebraic decoration.

(D). For $1 \le i \le m$, let d_i be the Whitney degree of the flat diagram of L_i obtained by traversing it upwards from the vertical segment where the algebraic decoration have been concentrated. The Whitney degree is the total turn of the tangent vector to the curve when one traverses it in the given direction. For example:



Finally set

$$\mathrm{Inv}_{\{H,\mathrm{tr}\}}(L,z,g) = \mathrm{tr}_{\alpha_1}(G_{\alpha_1}^{d_1} v_1) \cdots \mathrm{tr}_{\alpha_m}(G_{\alpha_m}^{d_m} v_m), \tag{2.3}$$

where $G = (G_\alpha)_{\alpha\in\pi}$ is the spherical π-grouplike element of H.

Recall that H-compatibility is preserved under equivalence of π-links.

Theorem 2.3. *Let $H = \{H_\alpha\}_{\alpha\in\pi}$ be a ribbon Hopf π-coalgebra endowed with a π-trace $\mathrm{tr} = (\mathrm{tr}_\alpha)_{\alpha\in\pi}$. Then $\mathrm{Inv}_{\{H,\mathrm{tr}\}}$ is an invariant of H-compatible π-links.*

The theorem is proven in the next subsection.

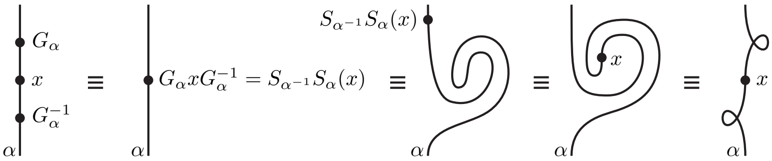
This invariant is not trivial (we give explicit computations in Examples 2.7 and 2.8).

When $\pi = 1$, $\mathrm{Inv}_{\{H,\mathrm{tr}\}}$ equals the Hennings' invariant of framed links (in the Kauffman-Radford formulation of [10]) calculated from the ribbon Hopf algebra H_1^{op} (endowed with the R-matrix $R_{1,1}^{-1}$ and the twist θ_1^{-1}) and the trace tr_1.

2.5. **Proof of Theorem 2.3.** We first remark that, when concentrating the algebraic decoration as explained in Step (B), we can identify the curls, in a compatible way with normalization of the invariant by the Whitney degree, as in Figure 6.

FIGURE 6. Identification of the curls

Indeed, since $S_{\alpha^{-1}}S_\alpha(x)=G_\alpha x G_\alpha^{-1}$ for all $\alpha\in\pi$ and $x\in H_\alpha$, the identification is justified by:



Moreover, since $\varphi_\alpha\varphi_\beta=\varphi_{\alpha\beta}$, $\varphi_1|_{H_\alpha}=\mathrm{id}_{H_\alpha}$, $\varphi_\beta S_\alpha=S_{\beta\alpha\beta^{-1}}\varphi_\beta$, and an element $a\in H_\alpha$ is replaced by $\varphi_\beta(a)$ (resp. $\varphi_{\beta^{-1}}(a)$) when it crosses upwards (resp. downwards) a disc labelled by β (see Figure 5), the labelled discs can be moved, gathered, or collapsed as in Figure 7.

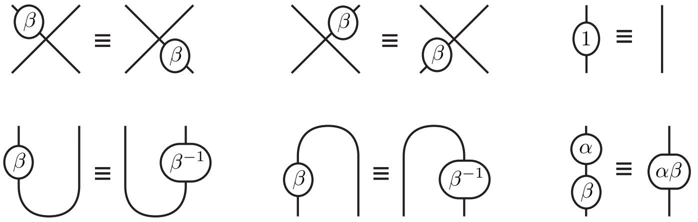


FIGURE 7. Rules for concentrating labelled discs

To demonstrate Theorem 2.3, we have to show that:

(i) for a given π-colored diagram of a π-link, the scalar obtained by performing Steps (B), (C), and (D) is well-defined (that is, independent of the manner of applying the these steps);

(ii) the scalar $\mathrm{Inv}_{\{H,\mathrm{tr}\}}(L,z,g)$ does not depend on the choice of a π-colored diagram for the π-link (L,z,g);

(iii) two equivalent π-links give rise to the same scalar.

Proof of (i). Consider a π-colored diagram of a π-link $(L=\cup_{i=1}^m L_i,z,g)$ and apply Step (B) (note that there is only one way to apply it). Recall that the obtained diagram is called the flat diagram of L. Fix $1\leq i\leq m$ and choose a point p_i on the flat diagram of L_i other than extrema, labelled discs, and points decorated by algebraic elements. Denote by α_i the color of the (vertical) segment of p_i and by d_i the Whitney degree of the flat diagram of L_i obtained by traversing it upwards from p_i. Let $v_i\in H_{\alpha_i}$ be a result of concentrating the algebraic decoration on p_i. We have to verify that the scalar $\mathrm{tr}_{\alpha_i}(G_{\alpha_i}^{d_i}v_i)$ is independent of the manner of concentrating the algebraic decoration on the point p_i and that it does not depend on the choice of the point p_i.

To show that the scalar $\mathrm{tr}_{\alpha_i}(G_{\alpha_i}^{d_i}v_i)$ is independent of the manner of concentrating the algebraic decoration on the point p_i, we choose another point q_i on the flat diagram of L_i (other than extrema, labelled discs, and points decorated by algebraic elements). The couple of points (p_i,q_i) divides the flat diagram of L_i into two arcs. Following the rules of Figure 5 and since the H_β are associative, the φ_β are isomorphisms of algebras, and the S_β are anti-isomorphisms of algebras, there is a unique manner to concentrate the algebraic decoration of each arc on a point located just above p_i (resp. below p_i). We denote by $t(q_i)\in H_{\alpha_i}$ (resp. $b(q_i)\in H_{\alpha_i}$) the result of these concentrations, see Figure 8.

To show that the scalar $\mathrm{tr}_{\alpha_i}(G_{\alpha_i}^{d_i}v_i)$ is independent of the manner of concentrating the algebraic decoration on the point p_i amounts then to verify that $\mathrm{tr}_{\alpha_i}(G_{\alpha_i}^{d_i}v(q_i))$ does not depend on the choice of the point q_i, where $v(q_i)=t(q_i)b(q_i)$.

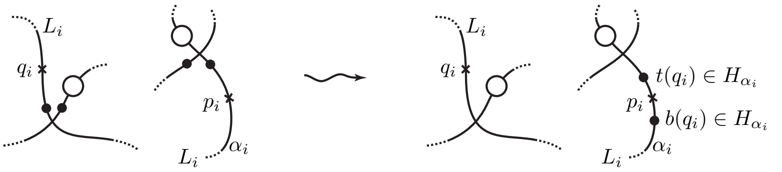


Figure 8

If q_i moves through an arc of the flat diagram for L_i which does not contain any algebraic decoration, then $t(q_i)$ and $b(q_i)$ clearly remain unchanged and thus $\mathrm{tr}_{\alpha_i}(G_{\alpha_i}^{d_i} v(q_i))$ also.

Suppose that q_i goes through a point decorated by some element $a \in H_\delta$ (for some $\delta \in \pi$). Consider two points q_i and q_i' located respectively above and below the point decorated by a (see Figure 9). Let $\mathcal{A}$ (resp. $\mathcal{A}'$) be the arc of the flat diagram of L_i delimited by q_i and p_i (resp. q_i' and p_i) which does not contain the point q_i' (resp. q_i). As above there is an unique manner to concentrate the algebraic decoration of the arcs $\mathcal{A}$ and $\mathcal{A}'$ on two points located just above and below p_i (see Figure 9). Moreover, using the rules of Figure 7, there is an unique way to collapse the labelled discs of the arc $\mathcal{A}$ (resp. $\mathcal{A}'$) into a unique labelled disc located above q_i (resp. below q_i'). Denote by $\alpha \in \pi$ (resp. $\alpha' \in \pi$) the label of this disc (see Figure 9).

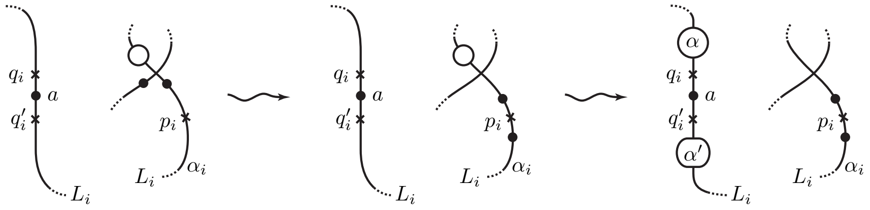


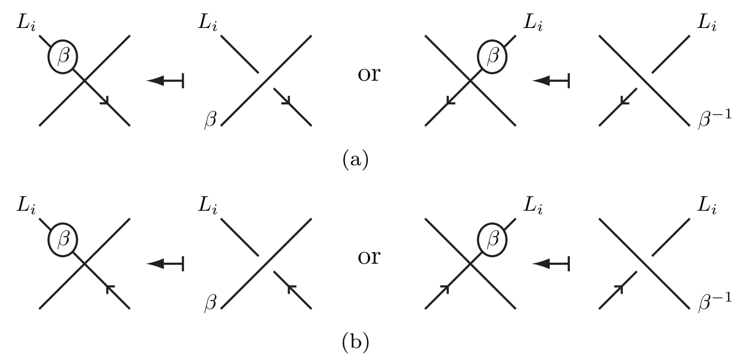
Figure 9

Lemma 2.4. $\varphi_{\alpha'^{-1}} = \varphi_\alpha$.

Proof. Consider the initial flat diagram of L_i (i.e., the one obtained just after applying Step (B)) and traverse it downwards from q_i. Starting with $\gamma = 1 \in \pi$, each time a disc labelled by some $\beta \in \pi$ is encountered, replace γ by $\gamma\beta^\nu$, where $\nu = 1$ (resp. $\nu = -1$) if the labelled disc is traversed downwards (resp. upwards). By this procedure, after a complete turn around the flat diagram of L_i, we obtain an element $\gamma_{\text{end}} \in \pi$. Now each labelled disc of the flat diagram for L_i comes from a crossing of the diagram of L, see Step (B). Thus $\gamma \leftarrow \gamma\beta$ results from the situation depicted in Figure 10(a) and $\gamma \leftarrow \gamma\beta^{-1}$ results from the situation depicted in Figure 10(b). Therefore (recall that L is arranged with blackboard framing) the result γ_{end} is the image under g of the (homotopy) longitude $\widetilde{L}_i$ (which is here oriented downwards from q_i). Since the π-link (L, z, g) is H-compatible, we have that $\gamma_{\text{end}} \in Z(\pi)$ and $\varphi_{\gamma_{\text{end}}} = \mathrm{id}$. Moreover the steps $\gamma \leftarrow \gamma\beta$ and $\gamma \leftarrow \gamma\beta^{-1}$ are clearly compatible with the rules of Figure 7 and so $\gamma_{\text{end}} = \alpha'\alpha$. Therefore $\varphi_{\alpha'^{-1}}^{-1}\varphi_\alpha = \varphi_{\alpha'}\varphi_\alpha = \varphi_{\alpha'\alpha} = \varphi_{\gamma_{\text{end}}} = \mathrm{id}$. Hence $\varphi_{\alpha'^{-1}} = \varphi_\alpha$. □

Finally there is two cases to consider: the algebraic decoration concentrated just above p_i can arise from either the arc $\mathcal{A}$ or the arc $\mathcal{A}'$, see Figure 11.

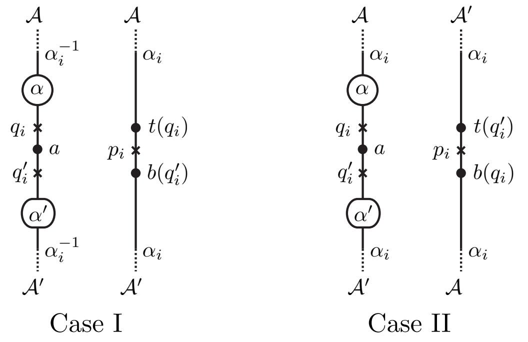
In Case I, there exists $k \in \mathbb{Z}$ (resp. $l \in \mathbb{Z}$) such that $k + \frac{1}{2}$ (resp. $l + \frac{1}{2}$) is the Whitney degree of the arc $\mathcal{A}$ oriented upwards from q_i (resp. the arc $\mathcal{A}'$ oriented



(a)

(b)

Figure 10



Case I

Case II

Figure 11

downwards from q'_i), that is, half of the number of half-turns of the tangent vector to the curve as one traverses it in the given direction (with the sign convention $\curvearrowright = +\frac{1}{2}$ and $\curvearrowleft = -\frac{1}{2}$). In this setting we have that $d_i = -(k+\frac{1}{2}) + (l+\frac{1}{2}) = -k+l$. Then, using (1.31) and Lemma 2.4, we obtain

$$t(q'_i) = (S_{\alpha_i^{-1}} S_{\alpha_i})^k S_{\alpha_i^{-1}}(\varphi_\alpha(a)) t(q_i) = G^k_{\alpha_i} S_{\alpha_i^{-1}}(\varphi_\alpha(a)) G^{-k}_{\alpha_i} \cdot t(q_i) \tag{2.4}$$

and

$$b(q_i) = b(q'_i)(S_{\alpha_i^{-1}} S_{\alpha_i})^l S_{\alpha_i^{-1}}(\varphi_{\alpha'^{-1}}(a)) = b(q'_i) G^l_{\alpha_i} S_{\alpha_i^{-1}}(\varphi_\alpha(a)) G^{-l}_{\alpha_i}. \tag{2.5}$$

Therefore

$$\begin{aligned}
\mathrm{tr}_{\alpha_i}(G^{d_i}_{\alpha_i} v(q'_i)) &= \mathrm{tr}_{\alpha_i}(G^{d_i}_{\alpha_i} t(q'_i) b(q'_i)) \\
&= \mathrm{tr}_{\alpha_i}(G^{d_i}_{\alpha_i} G^k_{\alpha_i} S_{\alpha_i^{-1}}(\varphi_\alpha(a)) G^{-k}_{\alpha_i} t(q_i) b(q'_i)) \quad \text{by (2.4)} \\
&= \mathrm{tr}_{\alpha_i}(G^l_{\alpha_i} S_{\alpha_i^{-1}}(\varphi_\alpha(a)) G^{-l}_{\alpha_i} G^{d_i}_{\alpha_i} t(q_i) b(q'_i)) \quad \text{since } d_i = -k+l \\
&= \mathrm{tr}_{\alpha_i}(G^{d_i}_{\alpha_i} t(q_i) b(q'_i) G^l_{\alpha_i} S_{\alpha_i^{-1}}(\varphi_\alpha(a)) G^{-l}_{\alpha_i}) \quad \text{by (1.32)} \\
&= \mathrm{tr}_{\alpha_i}(G^{d_i}_{\alpha_i} t(q_i) b(q_i)) \quad \text{by (2.5)} \\
&= \mathrm{tr}_{\alpha_i}(G^{d_i}_{\alpha_i} v(q_i)).
\end{aligned}$$

In Case II, there exists $k \in \mathbb{Z}$ (resp. $l \in \mathbb{Z}$) such that k (resp. l) is the Whitney degree of the arc $\mathcal{A}$ oriented upwards from q_i (resp. $\mathcal{A}'$ oriented downwards from q'_i). Then $d_i = k - l$ and, using (1.31) and Lemma 2.4, we obtain that

$$t(q_i) = (S_{\alpha_i^{-1}} S_{\alpha_i})^l (\varphi_{\alpha'^{-1}}(a)) t(q'_i) = G^l_{\alpha_i} \varphi_\alpha(a) G^{-l}_{\alpha_i} t(q'_i) \tag{2.6}$$

and

$$b(q_i') = b(q_i)(S_{\alpha_i^{-1}} S_{\alpha_i})^k(\varphi_\alpha(a)) = b(q_i)G_{\alpha_i}^k \varphi_\alpha(a) G_{\alpha_i}^{-k}. \tag{2.7}$$

Therefore

$$\begin{aligned}
\operatorname{tr}_{\alpha_i}(G_{\alpha_i}^{d_i} v(q_i')) &= \operatorname{tr}_{\alpha_i}(G_{\alpha_i}^{d_i} t(q_i')b(q_i')) \\
&= \operatorname{tr}_{\alpha_i}(G_{\alpha_i}^{d_i} t(q_i')b(q_i)G_{\alpha_i}^k \varphi_\alpha(a) G_{\alpha_i}^{-k}) \quad \text{by (2.7)} \\
&= \operatorname{tr}_{\alpha_i}(G_{\alpha_i}^k \varphi_\alpha(a) G_{\alpha_i}^{-k} G_{\alpha_i}^{d_i} t(q_i')b(q_i)) \quad \text{by (1.32)} \\
&= \operatorname{tr}_{\alpha_i}(G_{\alpha_i}^{d_i} G_{\alpha_i}^l \varphi_\alpha(a) G_{\alpha_i}^{-l} t(q_i')b(q_i)) \quad \text{since } d_i = k - l \\
&= \operatorname{tr}_{\alpha_i}(G_{\alpha_i}^{d_i} t(q_i)b(q_i)) \quad \text{by (2.4)} \\
&= \operatorname{tr}_{\alpha_i}(G_{\alpha_i}^{d_i} v(q_i)).
\end{aligned}$$

In every case, we get that $\operatorname{tr}_{\alpha_i}(G_{\alpha_i}^{d_i} v(q_i')) = \operatorname{tr}_{\alpha_i}(G_{\alpha_i}^{d_i} v(q_i))$. The scalar $\operatorname{tr}_{\alpha_i}(G_{\alpha_i}^{d_i} v_i)$ is hence independent of the manner of concentrating the algebraic decoration on p_i.

Let us show that $\operatorname{tr}_{\alpha_i}(G_{\alpha_i}^{d_i} v_i)$ does not depend on the choice of the point p_i. Firstly, if we move p_i across an extremum, then the color α_i is replaced by α_i^{-1}, the element v_i is replaced by $S_{\alpha_i^\nu}^\nu(v_i)$, where $\nu = +1$ if we move the point p_i across a maximum from left to right or across a minimum from right to left and $\nu = -1$ otherwise, and the Whitney degree d_i is replaced by $-d_i$. Now

$$\begin{aligned}
\operatorname{tr}_{\alpha_i^{-1}}(G_{\alpha_i^{-1}}^{-d_i} S_{\alpha_i^\nu}^\nu(v_i)) &= \operatorname{tr}_{\alpha_i^{-1}}(S_{\alpha_i^\nu}^\nu(G_{\alpha_i}^{d_i}) S_{\alpha_i^\nu}^\nu(v_i)) \quad \text{by (1.29)} \\
&= \operatorname{tr}_{\alpha_i^{-1}}(S_{\alpha_i^\nu}^\nu(v_i G_{\alpha_i}^{d_i})) \\
&= \operatorname{tr}_{\alpha_i}(v_i G_{\alpha_i}^{d_i}) \quad \text{by (1.33)} \\
&= \operatorname{tr}_{\alpha_i}(G_{\alpha_i}^{d_i} v_i) \quad \text{by (1.32)}.
\end{aligned}$$

Thus $\operatorname{tr}_{\alpha_i}(G_{\alpha_i}^{d_i} v_i)$ remains unchanged by moving p_i across an extremum.

Secondly, if we move p_i through a disc labelled by β, then the color α_i is replaced by $\beta^\nu \alpha_i \beta^{-\nu}$, where $\nu = +1$ (resp. $\nu = -1$) if we move the point p_i upwards (resp. downwards) through the labelled disc, the element v_i is replaced by $\varphi_{\beta^\nu}(v_i)$, and the Whitney degree d_i remains unchanged. Now

$$\begin{aligned}
\operatorname{tr}_{\beta^\nu \alpha_i \beta^{-\nu}}(G_{\beta^\nu \alpha_i \beta^{-\nu}}^{d_i} \varphi_{\beta^\nu}(v_i)) &= \operatorname{tr}_{\beta^\nu \alpha_i \beta^{-\nu}}(\varphi_{\beta^\nu}(G_{\alpha_i}^{d_i}) \varphi_{\beta^\nu}(v_i)) \\
&= \operatorname{tr}_{\beta^\nu \alpha_i \beta^{-\nu}}(\varphi_{\beta^\nu}(G_{\alpha_i}^{d_i} v_i)) \\
&= \operatorname{tr}_{\alpha_i}(G_{\alpha_i}^{d_i} v_i) \quad \text{by (1.34)}.
\end{aligned}$$

Therefore $\operatorname{tr}_{\alpha_i}(G_{\alpha_i}^{d_i} v_i)$ remains unchanged by moving p_i through a labelled disc. The scalar $\operatorname{tr}_{\alpha_i}(G_{\alpha_i}^{d_i} v_i)$ is hence independent of the choice of the point p_i on the flat diagram of L_i.

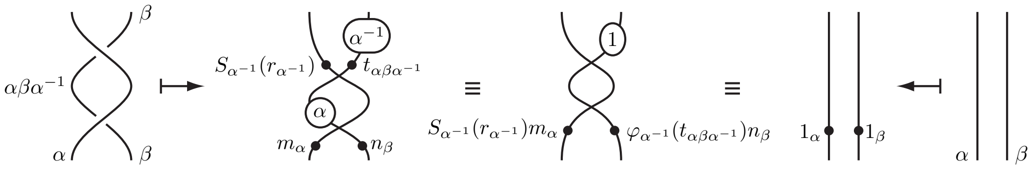
Proof of (ii) and (iii). By Lemma 2.1, it suffices to verify that if we apply Steps (B), (C), and (D) to two equivalent π-colored link diagrams (which represent H-compatible π-links), then we get the same scalar. Recall that two π-colored link diagrams are equivalent if one can be obtained from the other by a finite sequence of isotopies which preserve the colors of the vertical segments and of moves I_α-V_α of Figure 2.

It is straightforward that $\mathrm{Inv}_{\{H,\mathrm{tr}\}}$ remains unchanged under isotopies (in the class of generic link diagrams) which preserve the colors of the vertical segments and under the move I_α.

To show the invariance under the move $\mathrm{II}_{\alpha,\beta}$, write $R_{\alpha,\beta} = m_\alpha \otimes n_\beta$ and $R_{\alpha^{-1},\alpha\beta\alpha^{-1}} = r_{\alpha^{-1}} \otimes t_{\alpha\beta\alpha^{-1}}$. We have that

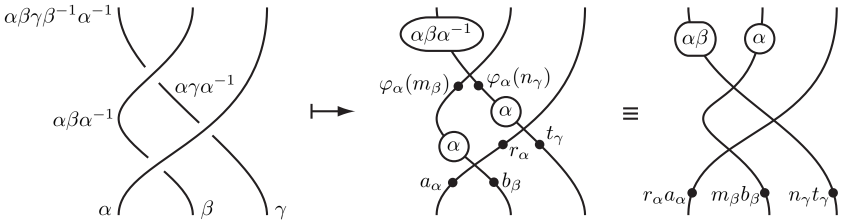
$$\begin{aligned}
&S_{\alpha^{-1}}(r_{\alpha^{-1}})m_\alpha \otimes \varphi_{\alpha^{-1}}(t_{\alpha\beta\alpha^{-1}})n_\beta \\
&\quad = \quad (S_{\alpha^{-1}} \otimes \mathrm{id}_{H_\beta})(\mathrm{id}_{H_{\alpha^{-1}}} \otimes \varphi_{\alpha^{-1}})(R_{\alpha^{-1},\alpha\beta\alpha^{-1}}) \cdot R_{\alpha,\beta} \\
&\quad = \quad (S_{\alpha^{-1}} \otimes \mathrm{id}_{H_\beta})(\varphi_\alpha \otimes \mathrm{id}_{H_\beta})(R_{\alpha^{-1},\beta}) \cdot R_{\alpha,\beta} \quad \text{by (1.11)} \\
&\quad = \quad R^{-1}_{\alpha,\beta} \cdot R_{\alpha,\beta} \quad \text{by (1.13)} \\
&\quad = \quad 1_\alpha \otimes 1_\beta.
\end{aligned}$$

Therefore:

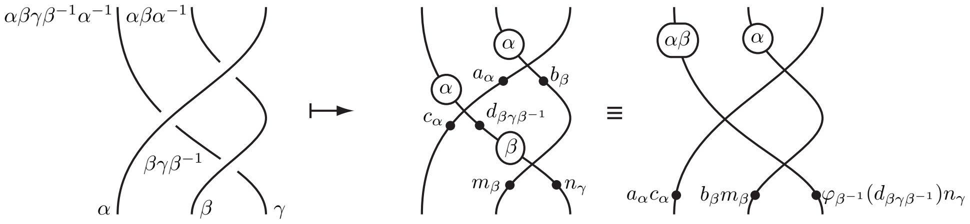


Here the symbol "$\equiv$" means that the flat diagrams are related by a finite sequence of isotopies (in the class of generic flat diagrams) and of moves of Figures 5, 6, and 7. The invariance under the first equivalence of $\mathrm{II}_{\alpha,\beta}$ is then verified. For the second one, this can be done similarly.

To show the invariance under the move $\mathrm{III}_{\alpha,\beta,\gamma}$, write $R_{\alpha,\beta} = a_\alpha \otimes b_\beta$, $R_{\beta,\gamma} = m_\beta \otimes n_\gamma$, and $R_{\alpha,\gamma} = r_\alpha \otimes t_\gamma$. By (1.11), we have that $R_{\alpha\beta\alpha^{-1},\alpha\gamma\alpha^{-1}} = (\varphi_\alpha \otimes \varphi_\alpha)(R_{\beta,\gamma}) = \varphi_\alpha(m_\beta) \otimes \varphi_\alpha(n_\gamma)$. Then:



Moreover, writing $R_{\alpha,\beta\gamma\beta^{-1}} = c_\alpha \otimes d_{\beta\gamma\beta^{-1}}$, we have that:

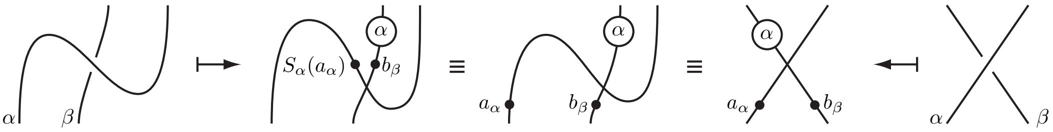


Now

$$\begin{aligned}
&r_\alpha a_\alpha \otimes m_\beta b_\beta \otimes n_\gamma t_\gamma \\
&\quad = \quad (R_{\beta,\gamma})_{\alpha 23}(R_{\alpha,\gamma})_{1\beta 3}(R_{\alpha,\beta})_{12\gamma} \\
&\quad = \quad (R_{\alpha,\beta})_{12\gamma}[(\mathrm{id}_{H_\alpha} \otimes \varphi_{\beta^{-1}})(R_{\alpha,\beta\gamma\beta^{-1}})]_{1\beta 3}(R_{\beta,\gamma})_{\alpha 23} \quad \text{by (1.15)} \\
&\quad = \quad a_\alpha c_\alpha \otimes b_\beta m_\beta \otimes \varphi_{\beta^{-1}}(d_{\beta\gamma\beta^{-1}})n_\gamma.
\end{aligned}$$

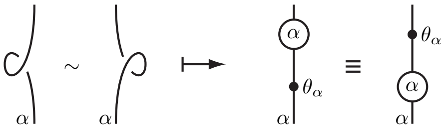
Hence the invariance under the move $\mathrm{III}_{\alpha,\beta,\gamma}$ is verified.

The invariance under the first equivalence of the move $\mathrm{IV}_{\alpha,\beta}$ follows from:

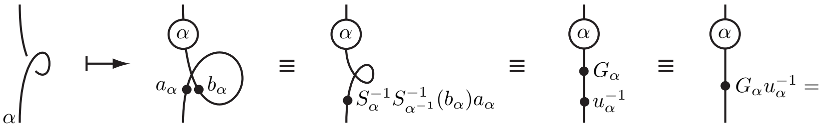


where $R_{\alpha,\beta} = a_\alpha \otimes b_\beta$. For the second one, this can be done similarly.

To show the invariance under the move V_α, we first remark that:



Indeed, write $R_{\alpha,\alpha} = a_\alpha \otimes b_\alpha$. Since $u_\alpha^{-1} = S_\alpha^{-1} S_{\alpha^{-1}}^{-1}(b_\alpha)a_\alpha$ by (1.16), we have that:



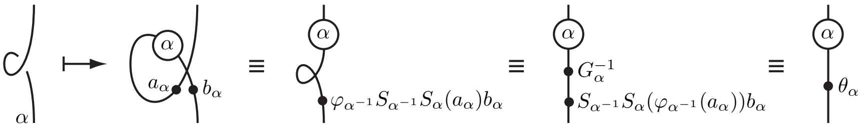
Moreover, since

$$\begin{aligned}
u_{\alpha^{-1}}^{-1} &= m_{\alpha^{-1}}(\mathrm{id}_{H_{\alpha^{-1}}} \otimes S_\alpha S_{\alpha^{-1}})\sigma_{\alpha^{-1},\alpha^{-1}}(R_{\alpha^{-1},\alpha^{-1}}) \quad \text{by (1.16)}\\
&= m_{\alpha^{-1}}(\mathrm{id}_{H_{\alpha^{-1}}} \otimes S_\alpha S_{\alpha^{-1}})\sigma_{\alpha^{-1},\alpha^{-1}}(S_{\alpha^{-1}}^{-1}\varphi_{\alpha^{-1}} \otimes S_{\alpha^{-1}}^{-1})(R_{\alpha,\alpha}) \quad \text{by (1.14)}\\
&= S_{\alpha^{-1}}^{-1}(b_\alpha)S_\alpha(\varphi_\alpha(a_\alpha))
\end{aligned}$$

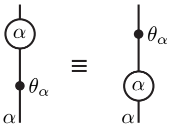
and so

$$\begin{aligned}
&G_\alpha^{-1} S_{\alpha^{-1}} S_\alpha(\varphi_\alpha(a_\alpha))b_\alpha\\
&\quad= S_{\alpha^{-1}}(S_{\alpha^{-1}}^{-1}(b_\alpha)S_\alpha(\varphi_\alpha(a_\alpha))G_{\alpha^{-1}}) \quad \text{by (1.29)}\\
&\quad= S_{\alpha^{-1}}(u_{\alpha^{-1}}^{-1}G_{\alpha^{-1}})\\
&\quad= S_{\alpha^{-1}}(\theta_{\alpha^{-1}})\\
&\quad= \theta_\alpha \quad \text{by (1.25)},
\end{aligned}$$

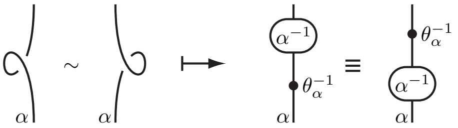
we have that:



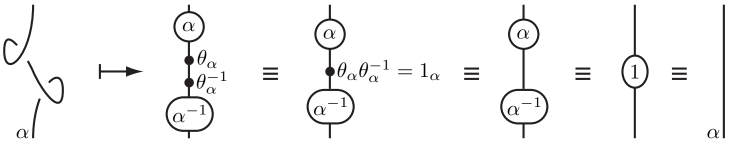
We can conclude by remarking that $\varphi_\alpha(\theta_\alpha) = \theta_\alpha = \varphi_{\alpha^{-1}}(\theta_\alpha)$ by (1.26) and so that:



We can show similarly that:



It's then easy to verify the invariance under the last move of V_α:



This completes the proof of Theorem 2.3.

2.6. **Basic properties of** $\mathrm{Inv}_{\{H,\mathrm{tr}\}}$. Throughout this subsection $H = \{H_\alpha\}_{\alpha\in\pi}$ will denote a ribbon Hopf π-coalgebra endowed with a π-trace $\mathrm{tr} = (\mathrm{tr}_\alpha)_{\alpha\in\pi}$.

Let (L, z, g) be a H-compatible π-link. Fix $\alpha \in \pi$. Then $(L, z, \alpha g \alpha^{-1})$ is clearly a H-compatible π-link.

Lemma 2.5. $\mathrm{Inv}_{\{H,\mathrm{tr}\}}(L, z, \alpha g\alpha^{-1}) = \mathrm{Inv}_{\{H,\mathrm{tr}\}}(L, z, g)$.

When $\alpha \in \mathrm{Im}(g)$, this follows from the invariance of $\mathrm{Inv}_{\{H,\mathrm{tr}\}}$ under the moves of Figure 2.

Proof. The lemma follows directly from the facts that, for all $\alpha, \beta, \gamma \in \pi$, the φ_α are algebra isomorphisms, $R_{\alpha\beta\alpha^{-1},\alpha\gamma\alpha^{-1}} = (\varphi_\alpha \otimes \varphi_\alpha)(R_{\beta,\gamma})$, $S_{\alpha\beta\alpha^{-1}}\varphi_\alpha = \varphi_\alpha S_\beta$, $\varphi_\alpha(G_\beta) = G_{\alpha\beta\alpha^{-1}}$, and $\mathrm{tr}_{\alpha\beta\alpha^{-1}}\varphi_\alpha = \mathrm{tr}_\beta$. □

Let (L, z, g) be a H-compatible π-link. Suppose that L is the disjoint union of two framed links L_1 and L_2. Since L_1 and L_2 are contained in two disjoint 3-balls in S^3, by the Van Kampen theorem, there exist two morphisms $g_1 : \pi_1(S^3 \setminus L_1, z) \to \pi$ and $g_2 : \pi_1(S^3 \setminus L_2, z) \to \pi$ such that the diagram of Figure 12 is commutative, where the horizontal arrows are induced by the embeddings $(S^3\setminus L, z) \hookrightarrow (S^3\setminus L_1, z)$ and $(S^3 \setminus L, z) \hookrightarrow (S^3 \setminus L_2, z)$. It is straightforward that (L_1, z, g_1) and (L_2, z, g_2) are H-compatible.

$$\begin{tikzcd}
\pi_1(S^3 \setminus L_1, z) \arrow[dr, "g_1"'] & \pi_1(S^3 \setminus L, z) \arrow[l] \arrow[r] \arrow[d, "g"] & \pi_1(S^3 \setminus L_2, z) \arrow[dl, "g_2"] \\
& \pi &
\end{tikzcd}$$

FIGURE 12

Lemma 2.6. $\mathrm{Inv}_{\{H,\mathrm{tr}\}}(L, z, g) = \mathrm{Inv}_{\{H,\mathrm{tr}\}}(L_1, z, g_1)\,\mathrm{Inv}_{\{H,\mathrm{tr}\}}(L_2, z, g_2)$.

Proof. Choose a π-colored diagram for (L, z, g) such that the diagrams for L_1 and L_2 are disjoint. It suffices then to remark that the π-colored sub-diagram consisting in L_i $(i = 1, 2)$ is a π-colored diagram for (L_i, z, g_i). □

2.7. **Examples.** In this subsection, we give some examples of computations of the invariant $\mathrm{Inv}_{\{H,\mathrm{tr}\}}$ of Theorem 2.3 which show that $\mathrm{Inv}_{\{H,\mathrm{tr}\}}$ is not trivial.

Example 2.7. Fix an integer $n \geq 2$, set $\pi = \mathbb{Z}/n\mathbb{Z}$, and define a bicharacter $c : \pi \times \pi \to \mathbb{C}^*$ of π by setting $c(a \,(\mathrm{mod}\; n\mathbb{Z}), b \,(\mathrm{mod}\; n\mathbb{Z})) = e^{\frac{2i\pi}{n}ab}$. Let us consider the ribbon Hopf π-coalgebra $H = \mathbb{C}^c$ (see Example 1.3) endowed with the π-trace $\mathrm{tr} = (\mathrm{id}_\mathbb{C})_{\alpha\in\pi}$. Let $O_k \subset S^3$ be the framed trivial knot with framing $k \in \mathbb{Z}$ and let $z_k \in S^3 \setminus O_k$. For any $l \in \mathbb{Z}$, define $g_l : \pi_1(S^3 \setminus O_k, z_k) \cong \mathbb{Z} \to \pi$ by

$g_l(1)=l\,(\mathrm{mod}\; n\mathbb{Z})$. The π-link (O_k,z_k,g_l) is clearly H-compatible (since $\pi=\mathbb{Z}/n\mathbb{Z}$ is commutative and the crossing of $H=\mathbb{C}^c$ is trivial). One easily gets that

$$\begin{aligned}\mathrm{Inv}_{\{H,\mathrm{tr}\}}(O_k,z_k,g_l) &= tr_{g_l(1)}(G_{g_l(1)}^{-1}\theta_{g_l(1)}^{k})\\ &= c(l\,(\mathrm{mod}\; n\mathbb{Z}),l\,(\mathrm{mod}\; n\mathbb{Z}))^k\\ &= e^{\frac{2i\pi kl^2}{n}}.\end{aligned}$$

In particular $\mathrm{Inv}_{\{H,\mathrm{tr}\}}(O_1,z_1,g_0)=1\neq e^{\frac{2i\pi}{n}}=\mathrm{Inv}_{\{H,\mathrm{tr}\}}(O_1,z_1,g_1)$.

Example 2.8. Consider the trefoil T as in Figure 13(a). The Wirtinger presentation of the group of T is $\pi_1(T)=\langle x,y,z\,|\,xy=yz=zx\rangle$. Let $g:\pi_1(T)\to\pi$. Denote $\alpha=g(x)$, $\beta=g(y)$, and $\gamma=g(z)$. The coloration by g of the diagram of T is depicted in Figure 13(b). Let $H=\{H_\alpha\}_{\alpha\in\pi}$ be a ribbon Hopf π-coalgebra endowed with a trace tr. Suppose that the π-trefoil represented by Figure 13(b) is H-compatible. Write $R_{\beta^{-1},\alpha^{-1}}=\sum_i a_i\otimes b_i$, $R_{\gamma,\alpha}=\sum_j c_j\otimes d_j$, and $R_{\alpha,\beta}=\sum_k e_k\otimes f_k$. The detailed application of Steps (B) and (C) of Section 2.4 is given in Figure 13(c). Therefore we get that

$$\mathrm{Inv}_{\{H,\mathrm{tr}\}}(T,g)=\sum_{i,j,k}\mathrm{tr}_\alpha\Big(G_\alpha^2\varphi_{\gamma^{-1}}S_\beta^{-1}(a_iS_{\beta^{-1}}^{-1}(f_k))d_je_kS_{\alpha^{-1}}(b_i)S_{\alpha^{-1}}\varphi_\beta S_\gamma(c_j)\Big).$$

Fix an integer $n\geq 2$, set $\pi=\mathbb{Z}/n\mathbb{Z}$, and consider the ribbon Hopf π-coalgebra $H=\mathbb{C}^c$ (see Example 1.3), where $c:\pi\times\pi\to\mathbb{C}^*$ is the bicharacter of π given by $c(a\,(\mathrm{mod}\; n\mathbb{Z}),b\,(\mathrm{mod}\; n\mathbb{Z}))=e^{\frac{2i\pi}{n}ab}$. The family $\mathrm{tr}=(\mathrm{id}_\mathbb{C})_{\alpha\in\pi}$ is a π-trace for H. Note that all π-links are H-compatible (since $\pi=\mathbb{Z}/n\mathbb{Z}$ is commutative and the crossing of $H=\mathbb{C}^c$ is trivial). For $l\in\mathbb{Z}/n\mathbb{Z}$, we define

$$g_l:\left\{\begin{array}{ccc}\pi_1(T)=\langle x,y,z\,|\,xy=yz=zx\rangle & \to & \mathbb{Z}/n\mathbb{Z}\\ x,\quad y,\quad z & \mapsto & l\end{array}\right..$$

Then

$$\mathrm{Inv}_{\{\mathbb{C}^c,\mathrm{tr}\}}(T,g_l)=c(-l,-l)\,c(l,l)\,c(l,l)=\exp\Big(\frac{6i\pi l^2}{n}\Big).$$

For example, for $n=6$, we get that $\mathrm{Inv}_{\{H,\mathrm{tr}\}}(T,g_0)=1\neq -1=\mathrm{Inv}_{\{H,\mathrm{tr}\}}(T,g_1)$.

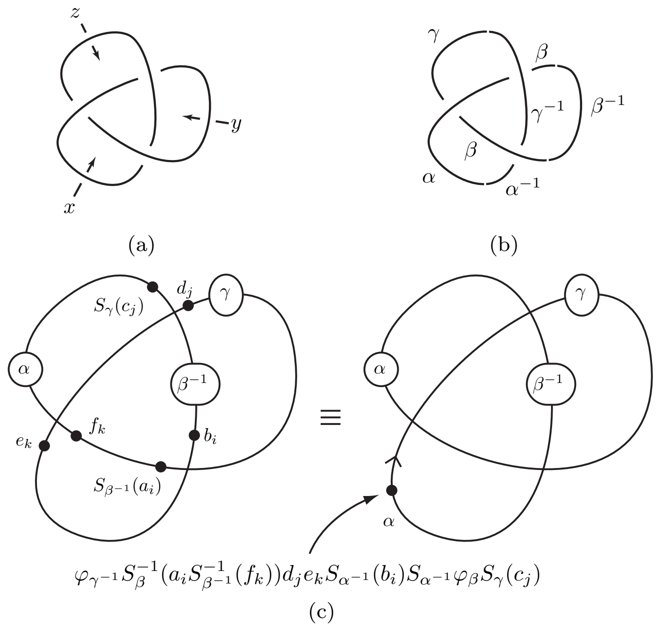
3. Invariants of π-manifolds

Our goal in this section is to normalize the invariant of π-links constructed in the previous section to an invariant of principal π-bundles over 3-manifolds.

3.1. **π-manifolds.** Recall that π is a discrete group. Following [18], a *π-manifold* is a couple (M,ξ) where M is a closed, connected, and oriented 3-manifold and ξ is a principal π-bundle over M, that is, since π is discrete, a regular covering $\tilde M\to M$ with group of automorphisms π. The space $\tilde M$ (resp. M) is called the *total space* (resp. *base space*) of ξ. Two π-manifolds (M,ξ) and (M',ξ') are said to be *equivalent* if there exists an homeomorphism $\tilde h:\tilde M\to\tilde M'$ which preserves the action of π and induces an orientation-preserving homeomorphism $h:M\to M'$.

A π-manifold (M,ξ) is said to be *pointed* when the total space $\tilde M$ of ξ is endowed with a base point $\tilde x\in\tilde M$. Two pointed π-manifolds $(M,\xi,\tilde x)$ and $(M',\xi',\tilde x')$ are said to be *equivalent* if there exists an equivalence $\tilde h:\tilde M\to\tilde M'$ between them such that $h(\tilde x)=h(\tilde x')$.

Let $(M,\xi,\tilde x)$ be a pointed π-manifold. Denote by $x\in M$ the image of $\tilde x\in\tilde M$ under the covering $\tilde M\to M$. We can associate to the pointed π-manifold $(M,\xi,\tilde x)$ a morphism $f:\pi_1(M,x)\to\pi$, called *monodromy of ξ at $\tilde x$*, by the following procedure: any loop γ in (M,x) uniquely lifts to a path $\tilde\gamma$ in $\tilde M$ beginning at $\tilde x$. The path $\tilde\gamma$ ends at $\alpha\cdot\tilde x$ for a unique $\alpha\in\pi$. The monodromy is defined by



(a) (b)

(c)

Figure 13

$f([\gamma]) = \alpha$, where $[\gamma]$ denotes the homotopy class in $\pi_1(M,x)$ of the loop γ. This leads to the triple (M,x,f).

Conversely, a triple (M,x,f) where M is a closed, connected, and oriented 3-manifold, $x \in M$, and $f : \pi_1(M,x) \to \pi$ is a group homomorphism leads to a pointed π-manifold uniquely determined up to equivalence (see [6, Proposition 14.1]). When convenient, we will adopt this second point of view. In particular, under this point of view, two pointed π-manifolds (M,x,f) and (M',x',f') are equivalent if there exists an orientation-preserving homeomorphism $h : M \to M'$ such that $h(x) = x'$ and $f' \circ h_* = f$, where $h_* : \pi_1(M,x) \to \pi_1(M',x')$ is the induced group isomorphism.

3.2. **Surgery along π-links.** For any framed link L in S^3, we will denote by S^3_L the 3-manifold obtained from S^3 by surgery along L (see [12]) and by $i_L : S^3 \setminus L \hookrightarrow S^3_L$ the (canonical) embedding. A pointed π-manifold (M,x,f) is said to be *obtained from S^3 by surgery along a π-link* (L,z,g) if there exists an orientation-preserving homeomorphism $h : S^3_L \to M$ such that $i_L(z) = h^{-1}(x)$ and $g = f \circ h_* \circ (i_L)_*$, where $h_* : \pi_1(S^3_L, h^{-1}(x)) \to \pi_1(M,x)$ and $(i_L)_* : \pi_1(S^3 \setminus L, z) \to \pi_1(S^3_L, i_L(z))$ are the induced group homomorphisms.

Lemma 3.1. *Every pointed π-manifold can be obtained from S^3 by surgery along a π-link.*

Proof. Let (M,x,f) be a pointed π-manifold. Since M is a closed, connected, and oriented 3-manifold, it can be obtained from S^3 by (integer) surgery, i.e., there exist a framed link $L \subset S^3$ and an orientation-preserving homeomorphism $h : S^3_L \to M$. Moreover L can always be chosen such that $h^{-1}(x) \in i_L(S^3 \setminus L)$. Let $z \in S^3 \setminus L$ such that $i_L(z) = h^{-1}(x)$. Set $g = f \circ h_* \circ (i_L)_*$. Hence (M,x,f) is obtained from S^3 by surgery along (L,z,g). $\square$

3.3. **Invariants of π-manifolds.** Let $H = \{H_\alpha\}_{\alpha\in\pi}$ be a finite type unimodular ribbon Hopf π-coalgebra and $\lambda = (\lambda_\alpha)_{\alpha\in\pi}$ be a (non-zero) right π-integral for H

such that $\lambda_1(\theta_1) \neq 0$ and $\lambda_1(\theta_1^{-1}) \neq 0$, where $\theta = \{\theta_\alpha\}_{\alpha\in\pi}$ denotes the twist of H. By Theorem 1.2, $\mathrm{tr}^\lambda = (x \in H_\alpha \mapsto \mathrm{tr}^\lambda_\alpha(x) = \lambda_\alpha(G_\alpha x) \in \Bbbk)_{\alpha\in\pi}$ is a π-trace for H, where $G = (G_\alpha)_{\alpha\in\pi}$ is the spherical π-grouplike element of H.

Lemma 3.2. *If (M, x, f) is a pointed π-manifold obtained from S^3 by surgery along a π-link (L, z, g), then (L, z, g) is H-compatible.*

Proof. Let C be a component of L, γ be a path in $S^3 \setminus L$ connecting z to $\widetilde{C}$ and ν be an orientation of $\widetilde{C}$. By definition of the surgery, $i_L(\widetilde{C})$ bounds a disk in S^3_L. Therefore $[i_L(\gamma)^{-1} i_L(\widetilde{C}) i_L(\gamma)] = 1$ in $\pi_1(S^3_L, i_L(z))$, that is, $(i_L)_*(\lambda_{(\gamma,\nu)}) = 1$, where $\lambda_{(\gamma,\nu)} = [\gamma^{-1}\widetilde{C}\gamma] \in \pi_1(S^3 \setminus L, z)$ (here the oriented circle $\widetilde{C}$ is viewed as a loop based on the point $\gamma(1)$). Since (M, x, f) is obtained from S^3 by surgery along (L, z, g), there exists an orientation-preserving homeomorphism $h : S^3_L \to M$ such that $i_L(z) = h^{-1}(x)$ and $g = f \circ h_* \circ (i_L)_*$. Then $g(\lambda_{(\gamma,\nu)}) = f \circ h_* \circ (i_L)_*(\lambda_{(\gamma,\nu)}) = f \circ h_*(1) = 1$ and hence $g(\lambda_{(\gamma,\nu)}) \in Z(\pi)$ and $\varphi_{g(\lambda_{(\gamma,\nu)})} = \varphi_1 = \mathrm{id}$. □

Let (M, ξ) be a π-manifold. Choose a point $\tilde{x}$ in the total space $\widetilde{M}$ of ξ. Denote by x the projection of $\tilde{x}$ under the covering $\widetilde{M} \to M$ and by $f : \pi_1(M, x) \to \pi$ the monodromy of ξ at $\tilde{x}$. By Lemma 3.1, we can present the pointed π-manifold (M, x, f) by a surgery along a π-link (L, z, g). Set

$$\tau_H(M, \xi) = \lambda_1(\theta_1)^{b_-(L)-n_L}\, \lambda_1(\theta_1^{-1})^{-b_-(L)}\, \mathrm{Inv}_{\{H,\mathrm{tr}^\lambda\}}(L, z, g),$$

where $b_-(L)$ is the number of strictly negative eigenvalues of the linking matrix of the framed link L (with framing numbers on the diagonal) and n_L is the number of components of L. Note that this scalar is well-defined since $\lambda_1(\theta_1)$ and $\lambda_1(\theta_1^{-1})$ are supposed to be non-zero and (L, z, g) is H-compatible (by Lemma 3.2).

Theorem 3.3. *Let $H = \{H_\alpha\}_{\alpha\in\pi}$ be a finite type unimodular ribbon Hopf π-coalgebra and $\lambda = (\lambda_\alpha)_{\alpha\in\pi}$ be a right π-integral for H such that $\lambda_1(\theta_1) \neq 0$ and $\lambda_1(\theta_1^{-1}) \neq 0$, where $\theta = \{\theta_\alpha\}_{\alpha\in\pi}$ denotes the twist of H. Then τ_H is an invariant of π-manifolds.*

The theorem is proven in Section 3.4.

Recall that the space of right π-integrals for H is one-dimensional and remark that the invariant τ_H remains unchanged if we replace λ by a scalar multiple $k\lambda$, with $k \in \Bbbk^*$. Therefore τ_H does not depend of the choice of the (non-zero) right π-integral for H used to compute it.

When $\pi = 1$, for any closed, connected, and oriented 3-manifold M, $\tau_H(M, M)$ is equal to $(\lambda_1(\theta_1^{-1})/\lambda_1(\theta_1))^{\frac{1}{2}\dim H_1(M)}$ times the Hennings' invariant of M (in the Kauffman-Radford formulation of [10]) calculated from the ribbon Hopf algebra H_1^{op} (endowed with the R-matrix $R_{1,1}^{-1}$ and the twist θ_1^{-1}) and the right integral λ_1. Note that here a square root of $\lambda_1(\theta_1^{-1})/\lambda_1(\theta_1)$ is assumed to exist.

Recall that, given a topological group G, a principal G-bundle is called *flat* when its transition functions are locally constant. Therefore equivalence class of flat principal G-bundle are in one-to-one correspondence with equivalence class of principal G_d-bundle, where G_d denotes the group G endowed with the discrete topology. Hence, when the group π is not discrete, the invariant τ_H may be viewed as an invariant of flat principal π-bundles over 3-manifolds.

The next example shows that the invariant τ_H is not trivial.

Example 3.4. Consider the ribbon Hopf $(\frac{1}{N}\mathbb{Z})/\mathbb{Z}$-coalgebra $A = \{A_\alpha\}_{\alpha\in(\frac{1}{N}\mathbb{Z})/\mathbb{Z}}$ of Example 1.4, where $N \geq 1$, which is studied in Appendix A. We restrict to the case $r = 2$. Let us denote by $(\lambda_\alpha)_{\alpha\in(\frac{1}{N}\mathbb{Z})/\mathbb{Z}}$ the right $(\frac{1}{N}\mathbb{Z})/\mathbb{Z}$-integral of Lemma A.1. Fix $p \geq 1$ and let ξ be a principal π-bundle over the lens space $L(p, 1)$. Denote by $f : \pi_1(L(p, 1)) \cong \mathbb{Z}/p\mathbb{Z} \to (\frac{1}{N}\mathbb{Z})/\mathbb{Z}$ the monodromy of ξ and set

$\alpha = f(1) \in (\frac{1}{N}\mathbb{Z})/\mathbb{Z}$. Note that $p\alpha = 0$. Since the lens space $L(p,1)$ is obtained by surgery of S^3 along the trivial knot with framing p, we have that

$$\tau_A(L(p,1),\xi) = \lambda_0(\theta_0)^{-1}\,\lambda_\alpha(\theta_\alpha^p).$$

By Lemma A.4, $\lambda_0(\theta_0) = -\frac{i}{2}$ and $\lambda_\alpha(\theta_\alpha^p) = -\frac{i}{2}p$ if $\alpha = 0$ and $\lambda_\alpha(\theta_\alpha^p) = 0$ otherwise. Therefore

$$\tau_A(L(p,1),\xi) = \begin{cases} p & \text{if } \xi \text{ is the trivial bundle,} \\ 0 & \text{otherwise.} \end{cases}$$

To obtain more interesting examples (from the topological point of view), one may start from ribbon Hopf π-coalgebras with non-trivial crossing. To produce examples of such Hopf π-coalgebras (in particular for π non abelian), it would be useful to define and study crossed Lie (co)algebras, their enveloping (co)algebras, and their quantum deformations in a similar way as the machinery of quantum groups (see, e.g., [9, 16]).

3.4. **Proof of Theorem 3.3.** Let us first show that $\tau_H(M,\xi)$ does not depend on the choice of the base point $\tilde{x}$ in the total space $\tilde{M}$ of the π-manifold (M,ξ). Let $\tilde{x}'$ be another point in $\tilde{M}$. Denote by x (resp. x') the projection of $\tilde{x}$ (resp. $\tilde{x}'$) under the covering $\tilde{M} \to M$ and by $f : \pi_1(M,x) \to \pi$ (resp. $f' : \pi_1(M,x') \to \pi$) the monodromy of ξ at $\tilde{x}$ (resp. $\tilde{x}'$). Let (L,z,g) a π-link along which the pointed π-manifold (M,x,f) is obtained by a surgery. Recall that there exists an orientation-preserving homeomorphism $h : S^3_L \to M$ such that $i_L(z) = h^{-1}(x)$ and $g = f \circ h_* \circ (i_L)_*$, where $i_L : S^3 \backslash L \hookrightarrow S^3_L$ is the (canonical) embedding and $(i_L)_*$ and h_* are the homomorphisms induced in homotopy by i_L and h respectively. Without loss of generality, we can assume that $x' \in h \circ i_L(S^3 \setminus L)$. Let $z' \in S^3 \setminus L$ such that $i_L(z') = h^{-1}(x')$. Since $S^3 \setminus L$ is connected, there exists a path $\gamma : [0,1] \to S^3 \setminus L$ connecting $z = \gamma(0)$ to $z' = \gamma(1)$. Define $\phi_\gamma : \pi_1(S^3 \backslash L, z') \to \pi_1(S^3 \backslash L, z)$ by setting $\phi_\gamma([\ell]) = [\gamma^{-1}\ell\gamma]$ for any loop ℓ in $(S^3 \setminus L, z')$. Set $g' = g \circ \phi_\gamma : \pi_1(S^3 \setminus L, z') \to \pi$. Note that the π-links (L,z,g) and (L,z',g') are equivalent: they are ambiently isotopic via an isotopy of the identity map id_{S^3} which pushes z along γ and is constant in a neighborhood of L. The path $\rho = h \circ i_L \circ \gamma : [0,1] \to M$ connects the point $\rho(0) = h(i_L(z)) = x$ to the point $\rho(1) = h(i_L(z')) = x'$. Define $\phi_\rho : \pi_1(M,x') \to \pi_1(M,x)$ by setting $\phi_\rho([\ell]) = [\rho^{-1}\ell\rho]$ for any loop ℓ in (M,x'). Note that, by construction,

$$\phi_\rho \circ h_* \circ (i_L)_* = h_* \circ (i_L)_* \circ \phi_\gamma : \pi_1(S^3 \setminus L, z') \to \pi_1(M,x).$$

Then $g' = g \circ \phi_\gamma = f \circ h_* \circ (i_L)_* \circ \phi_\gamma = (f \circ \phi_\rho) \circ h_* \circ (i_L)_*$ and so the pointed π-manifold $(M, x', f \circ \phi_\gamma)$ is obtained by surgery along the π-link (L,z',g'). Since π is a discrete group, the path $\rho : [0,1] \to M$ uniquely lifts to a path $\tilde{\rho} : [0,1] \to \tilde{M}$ such that $\tilde{\rho}(0) = \tilde{x}$. Since $\tilde{x}'$ and $\tilde{\rho}(1)$ belong to the same fiber (over x'), there exists $\alpha \in \pi$ such that $\tilde{\rho}(1) = \alpha \cdot \tilde{x}'$. Using the definition of the monodromy, we obtain that $f' = \alpha^{-1}(f \circ \phi_\gamma)\,\alpha$. Therefore

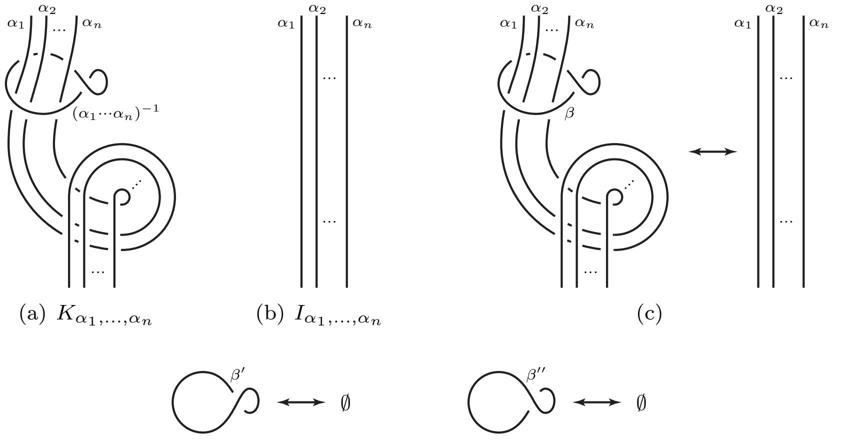
$$\alpha^{-1}g'\alpha = (\alpha^{-1}(f \circ \phi_\rho)\,\alpha) \circ h_* \circ (i_L)_* = f' \circ h_* \circ (i_L)_*$$

and so the pointed π-manifold (M,x',f') is obtained by surgery along the π-link $(L,z',\alpha^{-1}g'\alpha)$. Finally, recalling that (L,z,g) and (L,z',g') are equivalent (H-compatible) π-links, we have

$$\begin{aligned} \mathrm{Inv}_{\{H,\mathrm{tr}^\lambda\}}(L,z',\alpha^{-1}g'\alpha) &= \mathrm{Inv}_{\{H,\mathrm{tr}^\lambda\}}(L,z',g') && \text{Lemma 2.5} \\ &= \mathrm{Inv}_{\{H,\mathrm{tr}^\lambda\}}(L,z,g) && \text{by Theorem 2.3.} \end{aligned}$$

Hence $\tau_H(M,\xi)$ does not depend on the choice of the base point $\tilde{x}$ in $\tilde{M}$.

It remains to show that τ_H is an invariant of pointed π-manifolds. Let us describe the Kirby moves (in the form of Fenn and Rourke) in terms of π-colored link diagrams. Two π-links are said to be related by a *Kirby* 1*-move* (resp. a *special Kirby* (± 1)*-move*) if they may be presented by π-colored diagrams which can be obtained one from the other by interchanging the π-colored tangle diagram $K_{\alpha_1,\dots,\alpha_n}$ of Figure 14(a) with the π-colored tangle diagram $I_{\alpha_1,\dots,\alpha_n}$ of Figure 14(b), where $n \geq 1$ and $\alpha_1,\dots,\alpha_n \in \pi$ (resp. by adding or deleting a disjoint diagram of a circle with framing ± 1 whose vertical segments are colored by the neutral element 1 of π).



(a) $K_{\alpha_1,\dots,\alpha_n}$ (b) $I_{\alpha_1,\dots,\alpha_n}$ (c)

(d) (e)

FIGURE 14. π-colored Kirby moves

Lemma 3.5. *Let (M,x,f) and (M',x',f') be two equivalent pointed π-manifolds. Suppose that (L,z,g) and (L',z',g') are two π-links along which (M,x,f) and (M',x',f') are respectively obtained from S^3 by surgery. Then there exists a finite sequence $(L_0,z_0,g_0),\dots,(L_n,z_n,g_n)$ of π-links such that $(L_0,z_0,g_0)=(L,z,g)$, $(L_n,z_n,g_n)=(L',z',g')$ and, for any $1 \leq i \leq n$, $(L_{i-1},z_{i-1},g_{i-1})$ and (L_i,z_i,g_i) are equivalent π-links or are related by a Kirby 1-move or a special Kirby (± 1)-move.*

Proof. Since (M,x,f) and (M',x',f') are obtained from S^3 by surgery along (L,z,g) or (L',z',g'), there exist two orientation-preserving homeomorphisms $h : S^3_L \to M$ and $h' : S^3_{L'} \to M'$ such that $i_L(z) = h^{-1}(x)$, $i_{L'}(z') = {h'}^{-1}(x')$, $g = f \circ h_* \circ (i_L)_*$, and $g' = f' \circ h'_* \circ (i_{L'})_*$. Since (M,x,f) and (M',x',f') are equivalent, there exists an orientation-preserving homeomorphism $\phi : M \to M'$ such that $\phi(x) = x'$ and $f' \circ \phi_* = f$. It is implicit in the proof given in [11] of the Kirby theorem, refined in [5] and [15], that the (orientation-preserving) homeomorphism ${h'}^{-1} \circ \phi \circ h : S^3_L \to S^3_{L'}$ can be decomposed into isotopies, Kirby 1-moves, and special Kirby (± 1)-moves, i.e., that there exist a finite sequence $L_0 = L, L_1, \dots, L_n = L'$ of framed links in S^3 and a finite sequence $h_1 : S^3_{L_0} \to S^3_{L_1}, \dots, h_n : S^3_{L_{n-1}} \to S^3_{L_n}$ of orientation-preserving homeomorphisms such that ${h'}^{-1} \circ \phi \circ h = h_n \circ \cdots \circ h_1$ and h_i comes from an isotopy, a Kirby 1-move or a special Kirby (± 1)-move between L_{i-1} and L_i.

Without loss of generality, we can assume that $h_i \circ \cdots \circ h_1 \circ h^{-1}(x) \in i_{L_i}(S^3 \setminus L_i)$ for any $1 \leq i \leq n$. Let $z_i \in S^3 \setminus L_i$ such that $i_{L_i}(z_i) = h_i \circ \cdots \circ h_1 \circ h^{-1}(x)$. Note that $z' = z_n$. Set $(L_0,z_0,g_0) = (L,z,g)$ and define $g_i = f \circ h_* \circ (h_1^{-1})_* \circ \cdots \circ (h_i^{-1})_* \circ (i_{L_i})_* : \pi_1(S^3 \setminus L_i, z_i) \to \pi$ for any $1 \leq i \leq n$. Since

$$g_n = f \circ h_* \circ (h_1^{-1})_* \circ \cdots \circ (h_n^{-1})_* \circ (i_{L_n})_* = f \circ \phi_*^{-1} \circ h'_* \circ (i_{L'})_* = f' \circ h'_* \circ (i_{L'})_* = g$$

we have that $(L_n, z_n, g_n) = (L', z', g')$.

Fix $1 \le i \le n$. If h_i comes from an isotopy of S^3 between L_{i-1} and L_i, then it is straightforward that $(L_{i-1}, z_{i-1}, g_{i-1})$ and (L_i, z_i, g_i) are equivalent π-links. Suppose that h_i comes from a Kirby move between L_{i-1} and L_i. Then there exists a open 3-ball U in S^3 (inside which the Kirby move is performed) such that $S^3 \setminus (L_i \cup U) = S^3 \setminus (L_{i-1} \cup U)$ and $i_{L_i | S^3 \setminus (L_i \cup U)} = h_i \circ i_{L_{i-1} | S^3 \setminus (L_{i-1} \cup U)}$. Moreover U can be chosen so that $z_i \in S^3 \setminus (L_i \cup U)$. Then $z_{i-1} = z_i$ since $i_{L_i}(z_i) = h_i \circ \cdots \circ h_1 \circ h^{-1}(x) = h_i(i_{L_{i-1}}(z_{i-1})) = i_{L_i}(z_{i-1})$. Therefore the following diagram is commutative:

$$\begin{tikzcd}
\pi_1(S^3 \setminus (L_{i-1} \cup U), z_{i-1}) \arrow[rr, equal] \arrow[d] & & \pi_1(S^3 \setminus (L_i \cup U), z_i) \arrow[d] \\
\pi_1(S^3 \setminus L_{i-1}, z_{i-1}) \arrow[r, "g_{i-1}"] & \pi & \pi_1(S^3 \setminus L_i, z_i) \arrow[l, "g_i"']
\end{tikzcd}$$

Hence $(L_{i-1}, z_{i-1}, g_{i-1})$ and (L_i, z_i, g_i) can be presented by π-colored link diagrams which are identical except for pieces shown in Figure 14(c), 14(d), or 14(e), where $n \ge 1$ and $\alpha_1, \dots, \alpha_n, \beta \in \pi$, $\beta' \in \pi$, or $\beta'' \in \pi$. Now, since g_{i-1} and g_i vanish on the (homotopy) longitudes (see the proof of Lemma 3.2), we have that $\alpha_1 \cdots \alpha_n \beta = 1$ and so $\beta = (\alpha_1 \cdots \alpha_n)^{-1}$, $\beta' = 1$, or $\beta'' = 1$. Therefore $(L_{i-1}, z_{i-1}, g_{i-1})$ and (L_i, z_i, g_i) are related by a Kirby 1-move or a special Kirby (±1)-move. □

By Lemma 3.5, it remains to show that if (L, z, g) and (L', z', g') are two H-compatible π-links which are equivalent, related by a special Kirby (±1)-move, or related by a Kirby 1-move, then we have that

$$\begin{aligned} &\lambda_1(\theta_1)^{b_-(L)-n_L}\, \lambda_1(\theta_1^{-1})^{-b_-(L)}\, \mathrm{Inv}_{\{H,\mathrm{tr}^\lambda\}}(L,z,g) \\ &= \lambda_1(\theta_1)^{b_-(L')-n_{L'}}\, \lambda_1(\theta_1^{-1})^{-b_-(L')}\, \mathrm{Inv}_{\{H,\mathrm{tr}^\lambda\}}(L',z',g'). \end{aligned} \tag{3.1}$$

When (L, z, g) and (L', z', g') are equivalent H-compatible π-links, (3.1) follows directly from Theorem 2.3 and from the facts that $b_-(L) = b_-(L')$ and $n_L = n_{L'}$ (since L and L' are in particular isotopic framed links).

Suppose that a π-colored diagram of (L', z', g') is obtained from one of (L, z, g) by adding an unknotted circle C^ν with framing $\nu = \pm 1$, unlinked with the other components of L, whose vertical segments are colored by the neutral element 1 of π. Using the computations of Figure 15, we obtain that $\mathrm{Inv}_{\{H,\mathrm{tr}^\lambda\}}(L', z', g') = \lambda_1(\theta_1^\nu)\, \mathrm{Inv}_{\{H,\mathrm{tr}^\lambda\}}(L, z, g)$. Since $n_{L'} = n_{L \amalg C^\nu} = n_L + 1$ and $b_-(L') = b_-(L \amalg C^\nu)$ equals $b_-(L)$ if $\nu = 1$ or $b_-(L) + 1$ if $\nu = -1$, we get the equality (3.1).

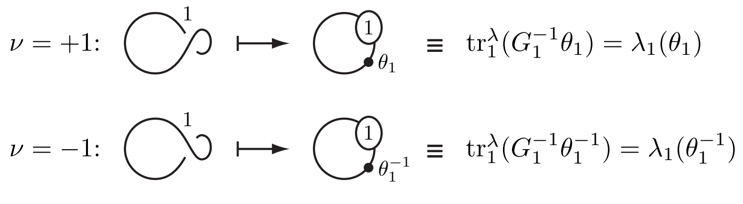
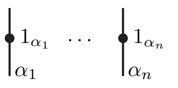


FIGURE 15

Suppose that a π-colored diagram of (L, z, g) is obtained from one of (L', z', g') by replacing the π-colored tangle diagram $K_{\alpha_1,\dots,\alpha_n}$ of Figure 14(a) with the π-colored tangle diagram $I_{\alpha_1,\dots,\alpha_n}$ of Figure 14(b) for some $n \ge 1$ and $\alpha_1, \dots, \alpha_n \in \pi$. In this case $b_-(L') = b_-(L)$ and $n_{L'} = n_L + 1$. Therefore we have to show that $\mathrm{Inv}_{\{H,\mathrm{tr}^\lambda\}}(L', z', g') = \lambda_1(\theta_1)\, \mathrm{Inv}_{\{H,\mathrm{tr}^\lambda\}}(L, z, g)$. Hence it suffices to verify that:

(3.2) $K_{\alpha_1,\dots,\alpha_n} \longmapsto \lambda_1(\theta_1)$



Let us show (3.2) by induction on $n \geq 1$. For $n = 1$, let $\alpha \in \pi$. Write $R_{\alpha,\alpha^{-1}} = a_\alpha \otimes b_{\alpha^{-1}}$ and $R_{\alpha^{-1},\alpha} = c_{\alpha^{-1}} \otimes d_\alpha$. Since

$$\begin{aligned}
&\mathrm{tr}^{\lambda}_{\alpha^{-1}}(G^{-1}_{\alpha^{-1}}\varphi_\alpha(b_{\alpha^{-1}})\theta_{\alpha^{-1}}c_{\alpha^{-1}})\, a_\alpha\varphi_{\alpha^{-1}}(d_\alpha)\theta_\alpha \\
&\qquad = \lambda_{\alpha^{-1}}(\theta_{\alpha^{-1}}b_{\alpha^{-1}}c_{\alpha^{-1}})\,\theta_\alpha\varphi_\alpha(a_\alpha)d_\alpha \quad \text{by (1.24)}\\
&\qquad = (\lambda_{\alpha^{-1}} \otimes \mathrm{id}_{H_\alpha})((\theta_{\alpha^{-1}} \otimes \theta_\alpha)\cdot(\sigma_{\alpha,\alpha^{-1}}(\varphi_\alpha \otimes \mathrm{id}_{H_{\alpha^{-1}}})(R_{\alpha,\alpha^{-1}}))\cdot R_{\alpha^{-1},\alpha})\\
&\qquad = (\lambda_{\alpha^{-1}} \otimes \mathrm{id}_{H_\alpha})\Delta_{\alpha^{-1},\alpha}(\theta_1)\\
&\qquad = \lambda_1(\theta_1)\,1_\alpha \quad \text{by (1.4)},
\end{aligned}$$

we have the equalities depicted in Figure 16. Hence (3.2) is true for $n = 1$.

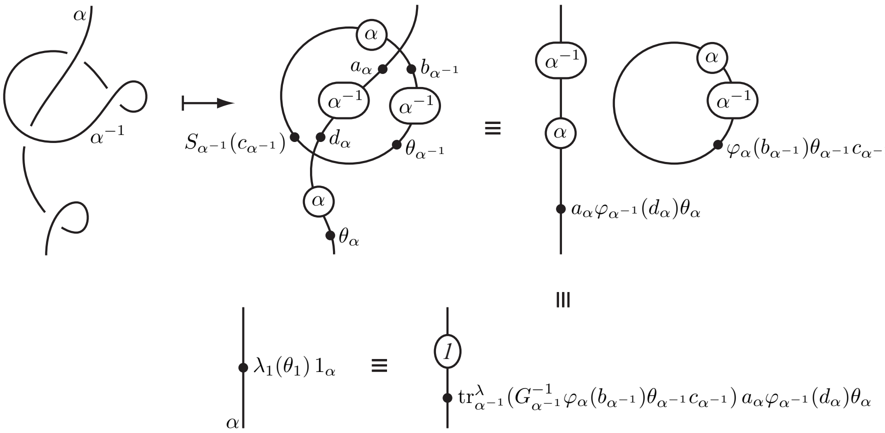


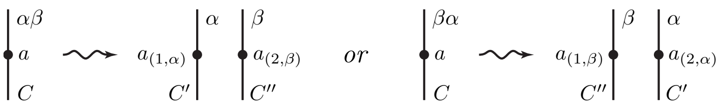
FIGURE 16

Suppose that (3.2) is true for $n \geq 1$ and let $\alpha_1, \dots, \alpha_{n+1} \in \pi$. Denote by C the component of the π-colored (n,n)-tangle diagram $K_{\alpha_1,\dots,\alpha_n\alpha_{n+1}}$ colored by $\alpha_n\alpha_{n+1}$ and by C' (resp. C'') the component of the π-colored $(n+1,n+1)$-tangle diagram $K_{\alpha_1,\dots,\alpha_n,\alpha_{n+1}}$ colored by α_n (resp. α_{n+1}). Note that if α and β are the colors of two parallel vertical segments of C' and C'', then the color of the corresponding vertical segment of C is either $\alpha\beta$ or $\beta\alpha$ depending if the segment of C' is on the left or on the right of the segment of C''. Using the hypothesis of induction and since $\Delta_{\alpha_n,\alpha_{n+1}}(1_{\alpha_n\alpha_{n+1}}) = 1_{\alpha_n} \otimes 1_{\alpha_{n+1}}$, we have that (3.2) for $n+1$ follows from the next lemma.

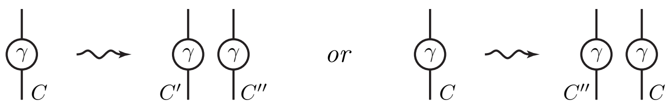
Lemma 3.6. *The flat diagram obtained from $K_{\alpha_1,\dots,\alpha_n,\alpha_{n+1}}$ can be deduced from the one obtained from $K_{\alpha_1,\dots,\alpha_n\alpha_{n+1}}$ by the following splitting procedure:*

(a) *the algebraic decoration and the labelled discs of the components other than C remain unchanged;*

(b) *a segment of C containing some algebraic element is split as follows:*



(c) *a segment of C containing a labelled disc is split as follows:*



Moreover, these splitting rules are compatible with the rules of Figure 5.

Proof. Fix a crossing c of the π-colored tangle diagram $K_{\alpha_1,\dots,\alpha_n\alpha_{n+1}}$. We have to consider three cases: any, one, or two strands of the crossing c is part of the component C. Firstly, if any of the two strands of c belongs to C, then c remains unchanged in $K_{\alpha_1,\dots,\alpha_n,\alpha_{n+1}}$. Suppose secondly that only one strand of c is part of C. There is height cases to consider (depending of the type of the crossing, the position of C in c, and the relative position of C' and C'' in $K_{\alpha_1,\dots,\alpha_n,\alpha_{n+1}}$). For example, if the position of C in c is from bottom-left to upper-right, the four cases are depicted in Figure 17.

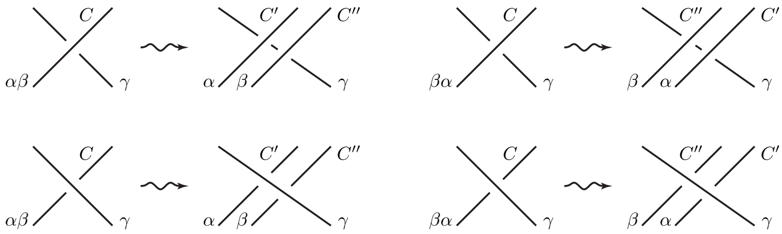
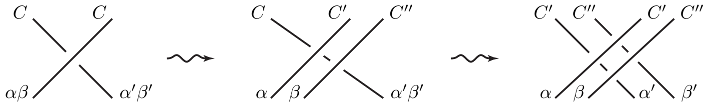


FIGURE 17

The compatibility of the splitting rules with Step (B) of Section 2.4 follows from the quasitriangularity of Hopf π-coalgebra H. For example, for the first case of Figure 17, if we write $R_{\alpha\beta,\gamma} = r_{\alpha\beta}\otimes s_\gamma$, $R_{\alpha,\beta\gamma\beta^{-1}} = a_\alpha\otimes b_{\beta\gamma\beta^{-1}}$, and $R_{\beta,\gamma} = c_\beta\otimes d_\gamma$, then

$$r_{\alpha\beta(1,\alpha)}\otimes r_{\alpha\beta(2,\beta)}\otimes s_\gamma = a_\alpha\otimes c_\beta\otimes\varphi_{\beta^{-1}}(b_{\beta\gamma\beta^{-1}})d_\gamma \qquad \text{by (1.9)}$$

and so the diagram of Figure 18 is commutative. The others cases of Figure 17 can be done similarly. Suppose thirdly that the two strands of c are part of C. There is also height cases to consider (depending of the type of the crossing and the relative positions of C' and C''). Here the compatibility with the splitting can be formally done by decomposing through the previous case. For example:



Finally, the compatibility of the splitting rules with the ones of Figure 5 comes from the anti-(co)multiplicativity of the antipode S and the (co)multiplicativity of the crossing φ. For example, let $\alpha,\beta,\gamma\in\pi$ and $a,b\in H_{\alpha\beta}$. Since $S_{\alpha\beta}(a)_{(1,\beta^{-1})}\otimes S_{\alpha\beta}(a)_{(2,\alpha^{-1})} = S_\beta(a_{(2,\beta)})\otimes S_\alpha(a_{(1,\alpha)})$, $(ab)_{(1,\alpha)}\otimes(ab)_{(2,\beta)} = a_{(1,\alpha)}b_{(1,\alpha)}\otimes a_{(2,\beta)}b_{(2,\beta)}$, and $\varphi_\gamma(a_{(1,\alpha)})\otimes\varphi_\gamma(a_{(2,\beta)}) = \varphi_\gamma(a)_{(1,\alpha)}\otimes\varphi_\gamma(a)_{(2,\beta)}$, the diagrams of Figure 19 are commutative. □

3.5. **Basic properties of τ_H.** Throughout this subsection H will denote a finite type unimodular ribbon Hopf π-coalgebra and $\lambda = (\lambda_\alpha)_{\alpha\in\pi}$ a right π-integral for H such that $\lambda_1(\theta_1)\neq 0$ and $\lambda_1(\theta_1^{-1})\neq 0$, where $\theta = \{\theta_\alpha\}_{\alpha\in\pi}$ denotes the twist of H.

Let (M_1,ξ_1) and (M_2,ξ_2) be two π-manifolds. Choosing base points of their total spaces leads to two pointed π-manifolds (M_1,x_1,f_1) and (M_2,x_2,f_2). Take closed 3-balls $B_1\subset M_1$ and $B_2\subset M_2$ such that $x_1\in\partial B_1$ and $x_2\in\partial B_2$. Glue $M_1\setminus\mathrm{Int}B_1$ and $M_2\setminus\mathrm{Int}B_2$ along a homeomorphism $h:\partial B_1\to\partial B_2$ chosen so that $h(x_1)=x_2$ and that the orientations in $M_1\setminus\mathrm{Int}B_1$ and $M_2\setminus\mathrm{Int}B_2$ induced by those in M_1, M_2 are compatible. This gluing yields a closed, connected, and oriented 3-manifold $M_1\#M_2$ endowed with a base point $x=h(x_1)=x_2$. By

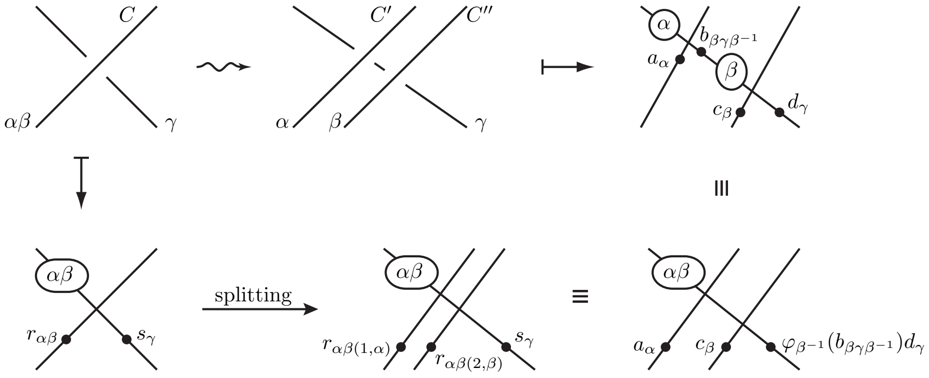


FIGURE 18

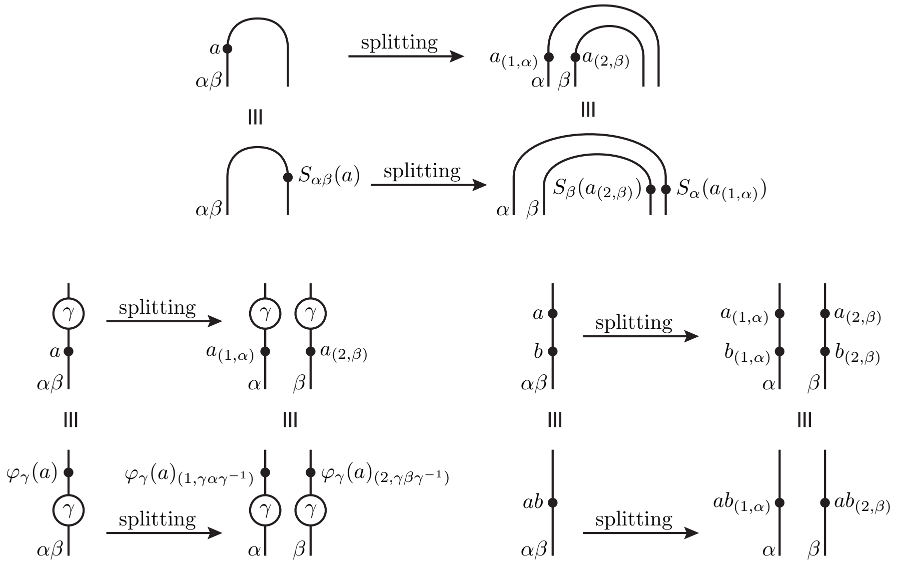


FIGURE 19

the Van Kampen theorem, since $\partial B_2 \cong h(\partial B_1)$ is simply-connected, there exists an unique group homomorphism $f : \pi_1(M_1\#M_2, x) \to \pi$ such that the diagram of Figure 20(a) is commutative, where the horizontal arrows are induced by the embeddings $(M_1, x_1) \hookrightarrow (M_1\#M_2, x)$ and $(M_2, x_2) \hookrightarrow (M, x)$. We de-

$$\begin{tikzcd}
\pi_1(M_1,x_1) \arrow[r] \arrow[dr, "f_1"'] & \pi_1(M_1\#M_2,x) \arrow[d, "f"] & \pi_1(M_2,x_2) \arrow[l] \arrow[dl, "f_2"] \\
 & \pi &
\end{tikzcd}$$

(a)

$$\begin{tikzcd}
\pi_1(S^3 \setminus L_1,z) & \pi_1(S^3 \setminus (L_1 \amalg L_2),z) \arrow[l] \arrow[r] \arrow[d, "g"] & \pi_1(S^3 \setminus L_2,z) \arrow[dl, "g_2"] \\
 & \pi &
\arrow[from=1-1, to=2-2, "g_1"']
\end{tikzcd}$$

(b)

FIGURE 20

note by $(M_1\#M_2, \xi_1\#\xi_2)$ the underlying π-manifold of the pointed π-manifold $(M_1\#M_2, x, f)$.

Lemma 3.7. $\tau_H(M_1\#M_2, \xi_1\#\xi_2) = \tau_H(M_1, \xi_1)\, \tau_H(M_2, \xi_2)$.

Proof. Let (L_1, z_1, g_1) and (L_2, z_2, g_2) be two π-links along which (M_1, x_1, f_1) and (M_2, x_2, f_2) are respectively obtained from S^3 by surgery. Without loss of generality, we can suppose that L_1 and L_2 are disjoint (in S^3) and that $z_1 = z_2$. Set $z = z_1 = z_2$ and let $\omega_i : \pi_1(S^3 \setminus L_i, z) \to \pi_1(S^3 \setminus L_1 \amalg L_2, z)$. As in Lemma 2.6, there exists an unique group homomorphism $g : \pi_1(S^3 \setminus L_1 \amalg L_2, z) \to \pi$ such that the diagram of Figure 20(b) is commutative, where the horizontal arrows are induced by the embeddings $(S^3 \setminus L_1 \amalg L_2, z) \hookrightarrow (S^3 \setminus L_1, z)$ and $(S^3 \setminus L_1 \amalg L_2, z) \hookrightarrow (S^3 \setminus L_2, z)$. Then $(L_1 \amalg L_2, z, g)$ is a π-link along which $(M_1 \# M_2, x, f)$ is obtained from S^3 by surgery. One easily concludes using the facts that $b_-(L_1 \amalg L_2) = b_-(L_1) + b_-(L_2)$, $n_{L_1 \amalg L_2} = n_{L_1} + n_{L_2}$, and $\mathrm{Inv}_{\{H,\mathrm{tr}\}}(L_1 \amalg L_2, z, g) = \mathrm{Inv}_{\{H,\mathrm{tr}\}}(L_1, z, g_1)\,\mathrm{Inv}_{\{H,\mathrm{tr}\}}(L_2, z, g_2)$ (by Lemma 2.6). □

Appendix A

In this appendix, we give some results concerning the Hopf $(\frac{1}{N}\mathbb{Z})/\mathbb{Z}$-coalgebra $A = \{A_\alpha\}_{\alpha \in (\frac{1}{N}\mathbb{Z})/\mathbb{Z}}$ of Example 1.4. They are used for topological purpose in Section 3.4.

Fix $N \geq 1$ and $r \geq 2$ and set $t = \exp(\frac{i\pi}{2r})$ and $q = t^2 = \exp(\frac{i\pi}{r})$. Recall (see Example 1.4) that, for any $\alpha \in (\frac{1}{N}\mathbb{Z})/\mathbb{Z}$, A_α is the associative algebra over $\mathbb{C}$ with generators $a^{\frac{1}{N}}$, e, and f, subject to the following relations:

$$\begin{array}{lll} a^{\frac{1}{N}} e = q^{\frac{1}{N}}\, e a^{\frac{1}{N}} & a^{\frac{1}{N}} f = q^{-\frac{1}{N}}\, f a^{\frac{1}{N}} & ef - fe = \frac{a^2 - a^{-2}}{q - q^{-1}} \\ e^r = 0 & f^r = 0 & a^{4r} = t^{-4r\alpha}. \end{array}$$

The family $A = \{A_\alpha\}_{\alpha \in \pi}$ is a Hopf π-coalgebra by setting:

$$\begin{array}{lll} \Delta_{\alpha,\beta}(a^{\frac{1}{N}}) = a^{\frac{1}{N}} \otimes a^{\frac{1}{N}} & \Delta_{\alpha,\beta}(e) = e \otimes a^{-1} + a \otimes e & \Delta_{\alpha,\beta}(f) = f \otimes a^{-1} + a \otimes f \\ \epsilon(a) = 1 & \epsilon(e) = 0 & \epsilon(f) = 0 \\ S_\alpha(a^{\frac{1}{N}}) = a^{-\frac{1}{N}} & S_\alpha(e) = -q^{-1}\, e & S_\alpha(f) = -q\, f. \end{array}$$

When $A = \{A_\alpha\}_{\alpha \in (\frac{1}{N}\mathbb{Z})/\mathbb{Z}}$ is endowed with the trivial crossing (that is, $\varphi_\beta|_{A_\alpha} = \mathrm{id}_{A_\alpha}$), it is a ribbon Hopf $(\frac{1}{N}\mathbb{Z})/\mathbb{Z}$-coalgebra with R-matrix

$$R_{\alpha,\beta} = \frac{1}{4r} \sum_{n=0}^{r-1} \sum_{k,l \in \mathbb{Z}/4r\mathbb{Z}} \frac{(q - q^{-1})^n}{[n]!} t^{-(l+\alpha)n + (k-\beta)(l+\alpha-n) - n} f^n a^{k-\beta} \otimes e^n a^{-(l+\alpha)}$$

and twist $\theta_\alpha = a^{2(r-1)} u_\alpha^{-1}$, where the u_α are the Drinfeld elements of A.

Note that $\{a^m e^k f^l \mid 0 \leq k, l < r,\ m \in \frac{1}{N}\mathbb{Z},\ 0 \leq m < 4r\}$ is a basis for A_α.

Lemma A.1. *For any $\alpha \in (\frac{1}{N}\mathbb{Z})/\mathbb{Z}$, set $\lambda_\alpha = \overline{a^{2(r-1)} e^{r-1} f^{r-1}}$, where the bar over the expression denotes the characteristic function of this element of the algebra A_α. Then $(\lambda_\alpha)_{\alpha \in (\frac{1}{N}\mathbb{Z})/\mathbb{Z}}$ is a right $(\frac{1}{N}\mathbb{Z})/\mathbb{Z}$-integral for A.*

Proof. We first recall that, if x, y are elements of an associative $\mathbb{C}$-algebra such that $yx = w\, xy$ for some $w \in \mathbb{C} \setminus \{1\}$, then, for any $n \geq 1$,

(A.1)
$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_w x^{n-k} y^k, \text{ where } [n]_w = \frac{w^n - 1}{w - 1} \text{ and } \begin{bmatrix} n \\ k \end{bmatrix}_w = \frac{[n]_w!}{[k]_w! [n-k]_w!}.$$

Fix $0 \leq k, l < r$ and $m \in \frac{1}{N}\mathbb{Z}$ with $0 \leq m < 4r$. For any $\alpha, \beta \in \pi$, using (A.1), we have

$$\Delta_{\alpha,\beta}(e^k) = (e \otimes a^{-1} + a \otimes e)^k = \sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix}_{q^2} (e \otimes a^{-1})^{k-i} (a \otimes e)^i = \sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix}_{q^2} e^{k-i} a^i \otimes a^{i-k} e^i$$

and

$$\Delta_{\alpha,\beta}(f^l) = (a\otimes f+f\otimes a^{-1})^l = \sum_{j=0}^{l} \begin{bmatrix} l \\ j \end{bmatrix}_{q^2} (a\otimes f)^{l-j}(f\otimes a^{-1})^j = \sum_{j=0}^{l} \begin{bmatrix} l \\ j \end{bmatrix}_{q^2} a^{l-j}f^j\otimes f^{l-j}a^{-j}.$$

Therefore

$$\begin{aligned}\Delta_{\alpha,\beta}(a^m e^k f^l) &= \sum_{i=0}^{k}\sum_{j=0}^{l} \begin{bmatrix} k \\ i \end{bmatrix}_{q^2} \begin{bmatrix} l \\ j \end{bmatrix}_{q^2} a^m e^{k-i} a^{i+l-j} f^j \otimes a^{m+i-k} e^i f^{l-j} a^{-j} \\ &= \sum_{i=0}^{k}\sum_{j=0}^{l} \begin{bmatrix} k \\ i \end{bmatrix}_{q^2} \begin{bmatrix} l \\ j \end{bmatrix}_{q^2} q^{-(k-i)(i+l-j)-j(l-j)+ij}\, a^{m+i+l-j}e^{k-i}f^j \otimes a^{m+i-k-j}e^i f^{l-j}.\end{aligned}$$

Since $0 \leq j \leq l \leq r-1$ and $0 \leq k-i \leq k \leq r-1$, a necessary condition for $\lambda_\alpha(a^{m+i+l-j}e^{k-i}f^j) = \overline{(a^{2(r-1)}e^{r-1}f^{r-1})}(a^{m+i+l-j}e^{k-i}f^j)$ to be non-zero is that $j = l = r-1$, $k = r-1$ and $i = 0$, and so $m = 2(r-1)$. Thus $(\lambda_\alpha \otimes \mathrm{id}_{A_\beta})\Delta_{\alpha,\beta}(a^m e^k f^l)$ equals $a^{2(r-1)+0-(r-1)-(r-1)}e^0 f^{(r-1)-(r-1)} = 1_\beta$ if $m = 2(r-1)$, $k = r-1$, and $l = r-1$ and equals 0 otherwise. Hence $(\lambda_\alpha \otimes \mathrm{id}_{A_\beta})\Delta_{\alpha,\beta}(a^m e^k f^l) = \lambda_{\alpha\beta}(a^m e^k f^l)1_\beta$ and so $(\lambda_\alpha)_{\alpha\in(\frac{1}{N}\mathbb{Z})/\mathbb{Z}}$ is a right $(\frac{1}{N}\mathbb{Z})/\mathbb{Z}$-integral for A. □

We fix $\alpha \in (\frac{1}{N}\mathbb{Z})/\mathbb{Z}$ and denote by c the unique element of $\frac{1}{N}\mathbb{Z} \cap [0,1[$ such that $\alpha \equiv c \bmod 1$. For any $i \in \mathbb{Z}/4r\mathbb{Z}$, we set

$$\Lambda_i^\alpha = \frac{1}{4r} \sum_{j\in\mathbb{Z}/4r\mathbb{Z}} t^{(i+c)j} a^j \in A_\alpha.$$

Lemma A.2. *In A_α, we have that $a^n = \sum_{i\in\mathbb{Z}/4r\mathbb{Z}} t^{-n(c+i)}\Lambda_i^\alpha$ for any $n \in \mathbb{Z}$.*

Proof. Let $n \in \mathbb{Z}$. Write $n = 4rq + p$ where $q, p \in \mathbb{Z}$ and $0 \leq p < 4r$. Then

$$\begin{aligned}\sum_{i\in\mathbb{Z}/4r\mathbb{Z}} t^{-n(c+i)}\Lambda_i^\alpha &= \frac{1}{4r} \sum_{i,j\in\mathbb{Z}/4r\mathbb{Z}} t^{-n(c+i)} t^{(i+c)j} a^j \\ &= \sum_{j=0}^{4r-1} \Big(\frac{1}{4r} \sum_{i=0}^{4r-1} t^{(j-p)i}\Big)\, t^{-4rqc} t^{(j-p)c} a^j \qquad \text{since } t^{-4rqi} = 1 \\ &= t^{-4rqc} \sum_{j=0}^{4r-1} \delta_{j,p}\, t^{(j-p)c} a^j \\ &= t^{-4rqc}\, a^p \;=\; a^n \qquad \text{since } a^{4r} = t^{-4rc}.\end{aligned}$$

□

By Lemma A.2 and the fact that $\Lambda_i^\alpha \Lambda_j^\alpha = \delta_{i,j}\,\Lambda_i^\alpha$, where $\delta_{i,j}$ is the Kronecker symbol, we obtain that the set $\{\Lambda_i^\alpha \mid i \in \mathbb{Z}/4r\mathbb{Z}\}$ forms a basis of orthogonal idempotents for the algebra $\mathbb{C}\langle a\rangle \subset A_\alpha$.

Lemma A.3. *$\theta_\alpha = t^{-c^2}\,\Gamma_\alpha\, a^{2(r-1)-2c} \sum_{n=0}^{r-1} \frac{(q-q^{-1})^n}{[n]!} t^{n^2+3n}\, a^{-2n}e^n f^n$, where $\Gamma_\alpha = \sum_{j\in\mathbb{Z}/4r\mathbb{Z}} t^{j^2}\Lambda_j^\alpha$.*

Proof. Recall that $\alpha \equiv c \bmod 1$. By (1.16), we have

$$\begin{aligned}
u_\alpha^{-1} &= m_\alpha(\mathrm{id}_{H_\alpha} \otimes S_{-\alpha}S_\alpha)\sigma_{\alpha,\alpha}(R_{\alpha,\alpha}) \\
&= \frac{1}{4r}\sum_{n=0}^{r-1}\sum_{k,l\in\mathbb{Z}/4r\mathbb{Z}} \frac{(q-q^{-1})^n}{[n]!} t^{-(l+\alpha)n+(k-\alpha)(l+\alpha-n)-n}\, e^n a^{-(l+\alpha)} S_{-\alpha}S_\alpha(f^n a^{k-\alpha}) \\
&= \frac{1}{4r}\sum_{n=0}^{r-1}\sum_{k,l\in\mathbb{Z}/4r\mathbb{Z}} \frac{(q-q^{-1})^n}{[n]!} t^{-(l+\alpha)n+(k-\alpha)(l+\alpha-n)+3n}\, e^n a^{-(l+\alpha)} f^n a^{k-\alpha} \\
&= \frac{1}{4r}\sum_{n=0}^{r-1}\sum_{k,l\in\mathbb{Z}/4r\mathbb{Z}} \frac{(q-q^{-1})^n}{[n]!} t^{(l+\alpha)n+(k-\alpha)(l+\alpha-n)+3n}\, a^{k-l-2\alpha} e^n f^n \\
&\quad (j = l-n,\ i = k-l+2n) \\
&= \frac{1}{4r}\sum_{n=0}^{r-1}\sum_{i,j\in\mathbb{Z}/4r\mathbb{Z}} \frac{(q-q^{-1})^n}{[n]!} t^{j^2+n^2+3n-\alpha^2+i(j+\alpha)}\, a^{i-2n-2\alpha} e^n f^n \\
&= \sum_{n=0}^{r-1} \frac{(q-q^{-1})^n}{[n]!} t^{n^2+3n-c^2} \sum_{j\in\mathbb{Z}/4r\mathbb{Z}} t^{j^2}\Big(\frac{1}{4r}\sum_{i\in\mathbb{Z}/4r\mathbb{Z}} t^{(j+c)i}a^i\Big)\, a^{-2n-2c} e^n f^n \\
&= \sum_{n=0}^{r-1} \frac{(q-q^{-1})^n}{[n]!} t^{n^2+3n-c^2} \Big(\sum_{j\in\mathbb{Z}/4r\mathbb{Z}} t^{j^2}\Lambda_j^\alpha\Big)\, a^{-2n-2c} e^n f^n \\
&= t^{-c^2}\,\Gamma_\alpha\, a^{-2c} \sum_{n=0}^{r-1} \frac{(q-q^{-1})^n}{[n]!} t^{n^2+3n}\, a^{-2n} e^n f^n .
\end{aligned}$$

We conclude by using the fact that $\theta_\alpha = a^{2(r-1)}u_\alpha^{-1}$. □

Lemma A.4. *Suppose that $r = 2$. Let $\alpha \in (\frac{1}{N}\mathbb{Z})/\mathbb{Z}$ and $p \geq 1$ with $p\alpha = 0$. Then*

$$\lambda_\alpha(\theta_\alpha^p) = \begin{cases} -\frac{ip}{2} & \textit{if } \alpha = 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Note that $q = \exp(\frac{i\pi}{2}) = i$. Recall that $\alpha = c + \mathbb{Z}$. Since $p\alpha = 0$, we have that $pc \in \mathbb{Z}$. By Lemma A.3, we have

$$\theta_\alpha = t^{-c^2}\,\Gamma_\alpha\, a^{2-2c}(1 + (i - i^{-1})t^4 a^{-2} ef) = t^{-c^2}\,\Gamma_\alpha\, a^{2-2c}(1 - 2ia^{-4}X),$$

where $X = a^2 ef$. Note that $aX = Xa$. Since

$$X^2 = a^2efa^2ef = a^4efef = a^4e(ef - \frac{a^2 - a^{-2}}{i - i^{-1}})f = \frac{1}{-2i}(-a^6ef + a^2ef) = \frac{1}{-2i}(1-a^4)X,$$

and so $X^n = \frac{(1-a^4)^{n-1}}{(-2i)^{n-1}}X$ for any $n \geq 1$, we obtain that

$$\begin{aligned}
\theta_\alpha^p &= t^{-pc^2}\,\Gamma_\alpha^p\, a^{2p-2pc}(1 - 2ia^{-4}X)^p \\
&= t^{-pc^2}\,\Gamma_\alpha^p\, a^{2p-2pc}\Big(1 + \sum_{n=1}^{p} \tbinom{p}{n}(-2i)^n a^{-4n} X^n\Big) \\
&= t^{-pc^2}\,\Gamma_\alpha^p\, a^{2p-2pc}\Big(1 + \sum_{n=1}^{p} \tbinom{p}{n}(-2i)^n a^{-4n} \frac{(1-a^4)^{n-1}}{(-2i)^{n-1}} X\Big) \\
&= U_\alpha + V_\alpha X,
\end{aligned}$$

where $U_\alpha = t^{-pc^2}\,\Gamma_\alpha^p\, a^{2p-2pc} \in \mathbb{C}\langle a\rangle$ and

$$V_\alpha = -2i\, t^{-pc^2}\,\Gamma_\alpha^p\, a^{2p-2pc} \sum_{n=1}^{p} \tbinom{p}{n} a^{-4n}(1-a^4)^{n-1} \in \mathbb{C}\langle a\rangle.$$

Since $\{\Lambda_j^\alpha \mid j \in \mathbb{Z}/8\mathbb{Z}\}$ is a set of orthogonal idempotents and by using Lemma A.2, we have

$$\begin{aligned}
\Gamma_\alpha^p &= \Big(\sum_{j\in\mathbb{Z}/8\mathbb{Z}} t^{j^2}\Lambda_j^\alpha\Big)^p = \sum_{j\in\mathbb{Z}/8\mathbb{Z}} t^{pj^2}\Lambda_j^\alpha,\\
a^{2p-2pc} &= \sum_{j\in\mathbb{Z}/8\mathbb{Z}} t^{-(2p-2pc)(c+j)}\Lambda_j^\alpha,\\
a^{-4n} &= \sum_{j\in\mathbb{Z}/8\mathbb{Z}} t^{4n(c+j)}\Lambda_j^\alpha,
\end{aligned}$$

and $(1-a^4)^{n-1} = \Big(\sum_{j\in\mathbb{Z}/8\mathbb{Z}} (1-t^{-4(c+j)})\,\Lambda_j^\alpha\Big)^{n-1} = \sum_{j\in\mathbb{Z}/8\mathbb{Z}} (1-t^{-4(c+j)})^{n-1}\Lambda_j^\alpha$. Therefore

$$\begin{array}{rcl}
V_\alpha & = & -2i\,t^{-pc^2}\displaystyle\sum_{n=1}^{p}\sum_{j\in\mathbb{Z}/8\mathbb{Z}} \binom{p}{n}\, t^{pj^2-(2p-2pc)(c+j)+4n(c+j)}(1-t^{-4(c+j)})^{n-1}\,\Lambda_j^\alpha\\
\text{(A.2)} & = & -2i\,t^{pc^2-2pc}\displaystyle\sum_{n=1}^{p}\sum_{j\in\mathbb{Z}/8\mathbb{Z}} \binom{p}{n}\, t^{pj^2-(2p-2pc)j+4n(c+j)}(1-t^{-4(c+j)})^{n-1}\,\Lambda_j^\alpha
\end{array}$$

Remark that if we write $V_\alpha = \sum_{j\in\mathbb{Z}/8\mathbb{Z}} v_j\,\Lambda_j^\alpha$ with $v_j \in \mathbb{C}$, then

$$\begin{aligned}
\lambda_\alpha(\theta_\alpha^p) &= \lambda_\alpha(U_\alpha) + \lambda_\alpha(V_\alpha X)\\
&= 0 + \sum_{j\in\mathbb{Z}/8\mathbb{Z}} v_j\,\lambda_\alpha\Big(\Lambda_j^\alpha X\Big) \qquad \text{since } U_\alpha \in \mathbb{C}\langle a\rangle\\
&= \sum_{j\in\mathbb{Z}/8\mathbb{Z}} v_j\,\lambda_\alpha\Big(\frac{1}{8}\sum_{k\in\mathbb{Z}/8\mathbb{Z}} t^{(j+c)k}\,a^k X\Big)\\
&= \frac{1}{8}\sum_{j,k\in\mathbb{Z}/8\mathbb{Z}} v_j\,t^{(j+c)k}\,\overline{(a^2ef)}(a^{k+2}ef)\\
&= \frac{1}{8}\sum_{j\in\mathbb{Z}/8\mathbb{Z}} v_j.
\end{aligned}$$

Hence, using (A.2),

$$\begin{aligned}
\lambda_\alpha(\theta_\alpha^p) &= -\frac{i}{4}\,t^{pc^2-2pc}\sum_{j\in\mathbb{Z}/8\mathbb{Z}}\sum_{n=1}^{p}\binom{p}{n}\,t^{pj^2-(2p-2pc)j+4n(c+j)}(1-t^{-4(c+j)})^{n-1}\\
&= -\frac{i}{4}\,t^{pc^2-2pc}\sum_{j\in\mathbb{Z}/8\mathbb{Z}} t^{pj^2-(2p-2pc)j}\sum_{n=1}^{p}\binom{p}{n}\,(t^{4(c+j)})^n(1-t^{-4(c+j)})^{n-1}.
\end{aligned}$$

If $\alpha = 0$ (that is, $c = 0$), then

$$\begin{aligned}
\lambda_0(\theta_0^p) &= -\frac{i}{4} \sum_{j\in\mathbb{Z}/8\mathbb{Z}} t^{pj^2-2pj} \sum_{n=1}^{p} \tbinom{p}{n} (-1)^{jn}(1-(-1)^j)^{n-1} \\
&= -\frac{i}{4}\Big(\sum_{j\in\mathbb{Z}/8\mathbb{Z},j \text{ even}} t^{pj^2-2pj} \sum_{n=1}^{p} \tbinom{p}{n}\, 0^{n-1} + \sum_{j\in\mathbb{Z}/8\mathbb{Z},j \text{ odd}} t^{pj^2-2pj} \sum_{n=1}^{p} \tbinom{p}{n} (-1)^n 2^{n-1}\Big) \\
&= -\frac{i}{8}\Big(\sum_{j\in\mathbb{Z}/8\mathbb{Z},j \text{ even}} t^{pj^2-2pj} p + \sum_{j\in\mathbb{Z}/8\mathbb{Z},j \text{ odd}} t^{pj^2-2pj} \sum_{n=1}^{p} \tbinom{p}{n} (-2)^n\Big) \\
&= -\frac{i}{8}\Big(p \sum_{j\in\mathbb{Z}/8\mathbb{Z},j \text{ even}} t^{pj^2-2pj} + ((-1)^p-1) \sum_{j\in\mathbb{Z}/8\mathbb{Z},j \text{ odd}} t^{pj^2-2pj}\Big) \\
&= -\frac{i}{8}\Big(p(1+1+t^{8p}+t^{24p}) + ((-1)^p-1)\,(t^{-p}+t^{3p}+t^{15p}+t^{35p})\Big) \\
&= -\frac{i}{8}\Big(4p + 2t^{-p}((-1)^p-1)\,(1+(-1)^p)\Big) \\
&= -\frac{i\,p}{2}.
\end{aligned}$$

Suppose that $\alpha \neq 0$ (that is, $c \neq 0$). For any $j \in \mathbb{Z}/8\mathbb{Z}$, set $x_j = t^{-4(c+j)} = (-1)^j \exp(-i\pi c) \neq \pm 1$. Then

$$\begin{aligned}
\sum_{n=1}^{p} \tbinom{p}{n} (t^{4(c+j)})^n (1-t^{-4(c+j)})^{n-1} &= \sum_{n=1}^{p} \tbinom{p}{n}\, x_j^{-n} (1-x_j)^{n-1} \\
&= x_j^{-p}(1-x_j)^{-1} \sum_{n=1}^{p} \tbinom{p}{n} (x_j)^{p-n}(1-x_j)^n \\
&= x_j^{-p}(1-x_j)^{-1}(1^p - x_j^p) \;=\; -\frac{1-x_j^{-p}}{1-x_j}.
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_\alpha(\theta_\alpha^p) &= \frac{i}{4}\, t^{pc^2-2pc} \sum_{j\in\mathbb{Z}/8\mathbb{Z}} t^{pj^2-(2p-2pc)j} \frac{1-x_j^{-p}}{1-x_j} \\
&= \frac{i}{4}\, t^{pc^2-2pc} \Big(\frac{1-x_0^{-p}}{1-x_0} \sum_{j\in\mathbb{Z}/8\mathbb{Z},j \text{ even}} t^{pj^2-(2p-2pc)j} \\
&\qquad + \frac{1-(-x_0)^{-p}}{1+x_0} \sum_{j\in\mathbb{Z}/8\mathbb{Z},j \text{ odd}} t^{pj^2-(2p-2pc)j} \Big) \\
&= \frac{i}{4}\, t^{pc^2-2pc} \Big(\frac{1-x_0^{-p}}{1-x_0} (1+t^{4pc}+t^{8p+8pc}+t^{24p+12pc}) \\
&\qquad + \frac{1-(-x_0)^{-p}}{1+x_0} (t^{-p+2pc}+t^{3p+6pc}+t^{15p+10pc}+t^{35p+14pc}) \Big) \\
&= \frac{i}{2}\, t^{pc^2-2pc} \Big(\frac{1-x_0^{-p}}{1-x_0} (1+x_0^{-p}) + \frac{1-(-x_0)^{-p}}{1+x_0} t^{-p+2pc} (1+(-x_0)^{-p}) \Big) \\
&= \frac{i}{2}\, t^{pc^2-2pc} \Big(\frac{1-x_0^{-2p}}{1-x_0} + t^{-p+2pc} \frac{1-x_0^{-2p}}{1+x_0} \Big) \\
&= 0 \quad \text{since } pc \in \mathbb{Z} \text{ and so } x_0^{-2p} = \exp(2i\pi pc) = 1.
\end{aligned}$$

This completes the proof of the lemma. □

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Département des Sciences Mathématiques, Université Montpellier II, Case Courrier 051, Place Eugène Bataillon, 34095 Montpellier Cedex 5, France
E-mail address: `virelizi@math.univ-montp2.fr`