ON SOME ONE PARAMETER FAMILIES OF GENUS 2 ALGEBRAIC CURVES AND HALF TWISTS

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0. Introduction.

In recent years one parameter families of algebraic curves in genus 2 have attracted great interest (see for example [Mc] and the bibliography quoted there). Although the families we consider here also define Teichmüller disks the point of view we will take in this paper is somewhat different. We will be interested in families for which one can both explicitly describe the hyperbolic structure and give the form of the equations. The hyperbolic structure will be described by Fenchel-Nielsen coordinates depending in each case on two real parameters, the length of a specific geodesic and a twist parameter. The equations will depend on one complex parameter. Two such families are well known, surfaces with an order 3 automorphism and surfaces with an order 4 automorphism, we will consider these but we will also consider others, two in detail, chosen for the simplicity of both equation and Fenchel-Nielsen coordinates, and give indications on some more.

The important point and one of the main aspect of this paper is that all these families share the fact that there is a natural action of $\mathrm{PSL}_2(\mathbb{Z})$ on the Teichmüller subspace defined by the set of Fenchel-Nielsen coordinates, that yields an explicitly describable action of the permutation group $\mathfrak{S}_3$ on the parameter space for the equations.

To be more specific let

$$T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{0.1}$$

be representatives in $\mathrm{SL}_2(\mathbb{Z})$ of standard generators of $\mathrm{PSL}_2(\mathbb{Z})$. Then in all cases T will act by half-Dehn twists along certain geodesics. The action of R is more difficult to describe in full generality but is always based on variants of the following. In [Bu-Si2] it was shown how to associate to Fenchel-Nielsen coordinates a well defined hyperbolic octagon such that the surface is obtained by identifications of opposite edges of the octagon (see section **1** for details). Conversely from such an octagon we can recover Fenchel-Nielsen coordinates. In the simplest cases R corresponds to a rotation of the octagon performing a circular permutation on 4 of the Weierstrass points.

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In all cases the induced action of T on the corresponding subspace of moduli space is generically non-trivial. In the simple cases, as above, the induced action of R will be trivial on moduli space but not on a double cover that will serve as parameter space for the equations. On the other hand $T\,R$ will again act by half-Dehn twists along another set of geodesics and the induced action will be generically non-trivial on moduli space and distinct from that of T. In the not so simple cases where the action of R is more intricate it will turn out that the induced action of R itself is sometimes non-trivial on moduli space (see sections **3** and **6**).

Returning to the point of view of [Mc] we note that there are also families of translation surfaces with a natural action of $\mathrm{SL}_2(\mathbb{Z})$, precisely those described by a theorem of Gutkin and Judge [Gu-Ju] and obtained as $\mathrm{SL}_2(\mathbb{R})$ orbits of square-tiled surfaces also called origami (see for example [Hu-Le]). In section **5** we will show that one of the families we have constructed coincides precisely with the family of translation surfaces obtained from three squares giving for this family both equations and a description of the hyperbolic geometry. This is probably the most striking result of this paper.

The methods used are largely inspired by the generalization found by Aline Aigon [Ai] for the D_5 action on the two parameter space of genus 2 curves with a non-hyperelliptic involution described in [Bu-Si1].

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1. Octagons, equations and the basic half-twist transformations.

There are many ways to define hyperbolic surfaces of genus 2, but in the sequel we will describe then in terms of fundamental polygons, an octagon in sections **2** and **3** a dodecagon in section **4**. On the other hand a more synthetic and practical way to describe a surface is Fenchel-Nielsen coordinates. None of these descriptions are unique but we will need to associate to a given set of Fenchel-Nielsen an explicit and uniquely defined octagon or dodecagon. For octagons we will do this in two ways (in fact those given in [Bu-Si2]) one (the most used) that we proceed to describe here leaving the second to section **3** and the representations by dodecagons to section **4**.

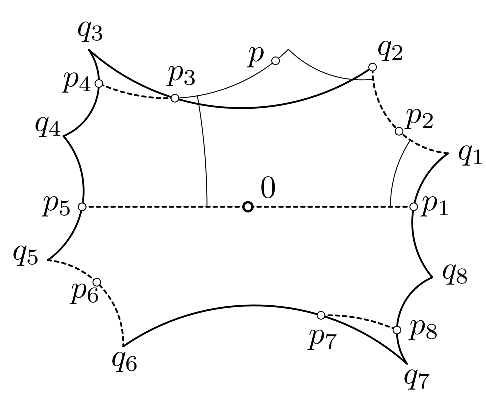

Figure 1

For this we first note that to a hyperbolic surface of genus 2 we can always associate a hyperbolic octagon in the unit disk, symmetric with respect to the origin and such that the sum of the interior angles is 2π. The corresponding fuchsian group is the group generated by identification of opposite sides. The preimages of the Weierstrass points are in this case the origin, the vertices (labeled q_i in figure 1 and that are identified in the surface), and the hyperbolic midpoints of the sides (see figure 1 where the midpoints are the p_i).

We want to associate such an octagon to a given set of Fenchel-Nielsen coordinates and vice-versa. We briefly recall the method of [Bu-Si2] to do this. Let $\mathscr{H}$ be the rectangular hyperbolic hexagon with side lengths $\ell_1, \widehat{\ell_3}, \ell_2, \widehat{\ell_1}, \ell_3, \widehat{\ell_2}$ (recall that the $\widehat{\ell_i}$ depend on the ℓ_i — see [Bu], p.454) and let t_1, t_2 and t_3 be 3 real numbers. Embed $\mathscr{H}$ isometrically in the unit disk with the edge of length ℓ_3 on the real line the first vertex at distance t_3 of the origin (to the left if t_3 is positive or to the right if it is negative). Shift the remaining vertices by t_1 and t_2, with a similar sign convention, to obtain points p_2, q_2, p and p_3 as illustrated in figure 1. Let h_n be the elliptic transformation of order 2 centered at point n. Then the remaining vertices of the octagon can be constructed by means of the h_{p_i} and h_0 (see figure 1 and [Bu-Si2] for more details). This yields an octagon for the surface with Fenchel-Nielsen coordinates $(2\ell_1, tw_1, 2\ell_2, tw_2, 2\ell_3, tw_3)$ where $tw_i = t_i/\ell_i$. By construction the pants decomposition we are considering is defined by the geodesic arcs

$$[q_1, q_2],\ [p_3, p_4] \cup [p_8, p_7] \text{ and } [p_5, p_1] \tag{1.1}$$

and clearly the octagon constructed in this way is uniquely defined by the Fenchel-Nielsen coordinates. Conversely starting from an octagon as above one can compute the corresponding Fenchel-Nielsen coordinates by means of formulae given in [Bu], p.454 and p.38–39 (see [Bu-Si2]).

The Fuchsian group defining the surface depends only on the octagon (since by definition it is the group generated by identifications of opposite sides). For practical reasons we will use the following description of this group. Let h_n be the order 2 elliptic transformation centered at the point n. Then the fuchsian group G for the surface is generated by

$$g_1 = h_{p_1} \cdot h_0,\ g_2 = h_{p_2} \cdot h_0,\ g_3 = h_{p_3} \cdot h_0 \text{ and } g_4 = h_{p_4} \cdot h_0\ . \tag{1.2}$$

We are now ready to introduce the elementary moves that we will elaborate upon in different contexts. These moves are half-twists, and their importance has been pointed out by Aline Aigon in [Ai]. We will do this in terms of octagons and in terms of the Fuchsian group, associated as above to the octagon.

Let S be the surface $\mathbb{D}/G$, G as above, and let γ be the closed geodesic image of the union of the geodesic arcs $[p_3, p_4]$ and $[p_7, p_8]$. Let p_3' be the hyperbolic midpoint of $[p_3, p_4]$ (see figure 2) and p_4' be the symmetric of p_3' with respect to p_4 (or in other words $p_4' = h_{p_4}(p_3')$). The surface S' obtained by performing a half twist along γ has fuchsian group generated by g_1, g_2, $g_3' = h_{p_3'} \cdot h_0$ and $g_4' = h_{p_4'} \cdot h_0$. The fundamental octagon for S' is shown in figure 2. It has vertices $q_1, q_2, q_3', q_4, q_5, q_6, q_7', q_8$ where $q_3' = h_{p'3}(q_2)$, which is the same as $h_{p_4'}(q_4)$, and $q_7' = h_{p_7'}(q_6)$. An informal formulation of this is that we have replaced the Weierstrass points defined by p_3 and p_4 by the midpoints of the arcs $[p_3, p_4]$ and $[p_7, p_8]$.

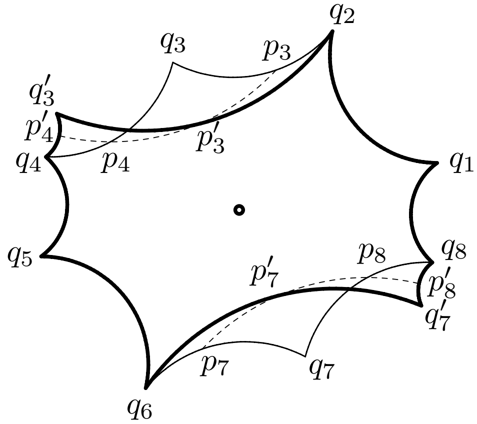

Figure 2

There are many other possibilities, but the description is basically the same (see [Ai] for a more complete treatment and [Si2] for other indications). This operation can also be formulated in terms of Fenchel-Nielsen coordinates. For the example we are considering it is,

$$(\ell_1, tw_1, \ell_2, tw_2, \ell_3, tw_3) \mapsto (\ell_1, tw_1, \ell_2, tw_2 + \tfrac{1}{2}, \ell_3, tw_3) \tag{1.3}$$

In general there is no obvious algebraic relation between the equations for S and S' but this will be the case if for example the point p_3' is a center of symmetry for the surface S. To make things more precise we need to introduce specific uniformizing functions.

For the first let F be the G-equivariant meromorphic even function on the unit disk, two to one on the interior of the octagon $\setminus\{0\}$ and such that

$$F(0) = 0,\ F(p_1) = 1,\ F(p_3) = \infty\ . \tag{1.4}$$

Then an equation for S is

$$y^2 = x(x-1)(x - F(q_1))(x - F(p_2))(x - F(p_4)) \tag{1.5}$$

and F is of course the uniformizing function yielding the x-coordinate (see [Bu-Si2]).

If p_3' is a center of symmetry for S we will need a second form. Let tr be the Möbius transformation sending 0 to -1, $F(q_1)$ to 1 and $F(p_3')$ to ∞. Write F_s for the composed map $tr \circ F$. Then

$$y^2 = (x^2-1)(x^2-a)(x^2-b) \quad \text{where} \quad a = F_s(p_2)^2,\ b = F_s(p_4)^2 \tag{1.6}$$

and again F_s is a uniformizing function (see [Si2]). Note that in this context we have

$$F_s(0) = -1,\ F_s(q_i) = 1,\ F_s(p_1) = -F_s(p_2),\ F_s(p_3) = -F_s(p_4) \tag{1.7}$$

(see again [Si2]).

In the case (1.6), that is when p_3' is a center of symmetry and more precisely $F_s(p_3') = \infty$, Aline Aigon [Ai] has shown that an equation for S' (obtained by a half-twist) is

$$y^2 = (x^2-1)(x^2-a')(x^2-b') \quad \text{where} \quad a' = \frac{a(1-b)}{a-b},\ b' = 1-b \tag{1.8}$$

(see also for this form [Si2]).

2. Genus 2 curves with an order 3 automorphism.

In this section we consider the one (complex) parameter family of genus 2 curves with an order 3 automorphism. There are various ways to describe this family, some probably simpler than others but for technical reasons we will use a less known way. But first we need an easy lemma.

2.1 Lemma. *Let f be a Möbius transformation of order 3. If f satisfies*

$$f(-z) = -f(f(z)) \tag{2.1.1}$$

then f is of the form

$$f : z \mapsto \frac{3\alpha + z}{1 - z/\alpha}, \quad \alpha \in \mathbb{C},\ \alpha \neq 0 \tag{2.1.2}$$

Proof. We first note that f is conjugate to a rotation. Let $j = \exp(i\,\pi/3)$. Writing f as a matrix in $\mathrm{SL}_2(\mathbb{C})$ it is conjugate to $\begin{pmatrix} j & 0 \\ 0 & j^{-1} \end{pmatrix}$. Conjugating by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ it is readily shown that the condition (2.1.1) imposes $b = -\frac{ad}{c}$. The result easily follows from this.

2.2 Corollary. *Let C be a genus 2 curve with an order 3 automorphism. Then an equation for C can always be written in the form*

$$y^2 = (x^2 - {t_0}^2)(x^2 - f_3(t_0)^2)(x^2 - f_3(f_3(t_0))^2)$$

for some $t_0 \neq 0,\ \pm 1,\ \pm 3,\ \pm i\sqrt{3}$ and where f_3 is

$$f_3 : t \mapsto \frac{3+t}{1-t}\ .$$

Proof. If C has an order 3 automorphism then its automorphism group contains the dihedral group D_3. On the other hand the equation of a hyperelliptic curve with an involution, distinct from the hyperelliptic one, can always be normalized so that the involution is induced by $x \mapsto -x$. We are thus led to condition (2.1.1) and the order 3 automorphism must be induced by a function of the form (2.1.2). Dividing the x-coordinate of the Weierstrass points by α if necessary we obtain the result.

An important fact motivating our choice is that

(2.3) the orbit of 0 under f_3 is $0, 3, -3$ and the orbit of infinity is $\infty, -1, 1$.

Note that these points are the fixed points of the transformations inducing non hyperelliptic involutions in C.

In practice we will need a slightly different version of **2.2**

2.4 Corollary. *Let C be a genus 2 curve with an order 3 automorphism. Then an equation for C can always be written in the form*

$$y^2 = x^6 - (a+18)\,x^4 + (2\,a+81)\,x^2 - a$$

with $a \neq 0$ or -27.

Moreover two curves C and C' defined by a and a' are isomorphic if and only if $a' = a$ or $a' = 729/a$.

Proof. For the first part just take $a = (t_0 + f_3(t_0) + f_3(f_3(t_0)))^2$.

For the second assume first that the reduced automorphism group (the quotient of $\mathrm{Aut}(C)$ by the hyperelliptic involution) is D_3. In this case the only involutions of C or C' are induced by the transformations with fixed points as in (2.3). Now an isomorphism is induced by a Möbius transformation f which must preserve globally the two orbits of (2.3). It is easily checked that this is only possible if f is in the group generated by $x \mapsto -x$, f_3 and $x \mapsto 3/x$. This yields $a' = a$ or $a' = 729/a$. If the reduced automorphism group is larger than D_3 then it is of order 12, and it is easily checked that this is only possible if $a = 27$.

From the hyperbolic point of view the corresponding surfaces are also easy to describe. Start with a pair of pants with all three geodesic boundary components of equal hyperbolic lengths. Paste two copies of such a pair of pants in such a way that none of the geodesic boundaries are separating and with the same twist parameter for each. In this case the order 3 symmetry of the pair of pants extends to a conformal order 3 automorphism of the resulting genus 2 surface. In terms of Fenchel-Nielsen coordinates these surfaces have coordinates of the form $(\ell, tw, \ell, tw, \ell, tw)$, where ℓ is the hyperbolic length and tw is the twist parameter.

As in section **1** we will use the representation by means of an octagon. We use the construction indicated in **1** figure 1 starting with a rectangular hexagon admitting an order 3 symmetry as indicated in figure 3.

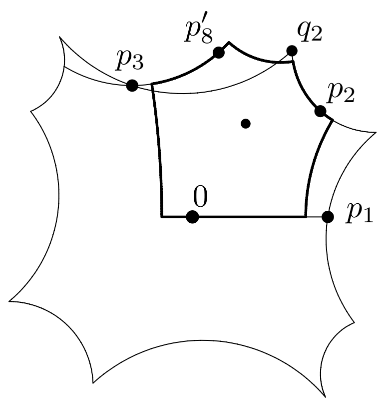

Figure 3

The order 3 automorphism being induced by the order 3 transformation on the hexagon one can easily deduce its action on the Weierstrass points (which are the images of 0, the vertices and the midpoints of the sides). In particular if we denote by $\widetilde{f}_3$ this transformation we have $\widetilde{f}_3(p_1) = q_2$, $\widetilde{f}_3(q_2) = p_3$ and $\widetilde{f}_3(p_3) = p_1$.

2.5 Remark. By construction the Fenchel-Nielsen coordinates $(\ell, tw, \ell, tw, \ell, tw)$ correspond to the pants decomposition given in (1.1). On the other hand if we rotate the octagon in such a way that p_3 becomes a positive real we get new Fenchel-Nielsen coordinates. These correspond to the pants decomposition indicated in figure 5. Noting this, it can easily be checked by following the action of $\widetilde{f}_3$ (note that p_8' is identified in the surface with p_8 and p_4) that the new Fenchel-Nielsen coordinates are again of the form $(\ell', tw', \ell', tw', \ell', tw')$.

2.6 Lemma. *Let $(\ell, tw, \ell, tw, \ell, tw)$ correspond to the octagon of figure 3 and let $(\ell', tw', \ell', tw', \ell', tw')$ correspond to the octagon obtained by the rotation indicated in* **2.5***.*

Let $L = \cosh(\ell/2)$, $L' = \cosh(\ell'/2)$, $Tw = \cosh(tw\,\ell/2)$ and $Tw' = \cosh(tw'\,\ell'/2)$. We then have

$$L' = Tw^2 \frac{2L-1}{L-1} - 1$$

$$Tw' = \sqrt{\frac{Tw^2(L+1)(2L-1) - 2(L^2-1)}{2Tw^2(2L-1) - 3(L-1)}}$$

and

$$tw' = -\operatorname{sign}(tw)\operatorname{arccosh}(Tw')/\operatorname{arccosh}(L')$$

Proof. To find ℓ' we only need, by remark **2.5**, to compute the hyperbolic distance between 0 and p_3. For this note that in the construction we started with a rectangular hexagon with 3 sides of lengths $\ell/2$. The hyperbolic cosine of the remaining sides is then $L/(L-1)$ (see [Bu] p.454). We can now easily compute the length of $[0, p_3]$ using formula (2.3.2) in [Bu] p.38. This yields L'. The transformation being symmetric we can apply it twice and solve in Tw' to recover L. Finally the reason for the sign change is the same as the one given in [Ok].

Let, as in section **1**, the points p_i be the midpoints of the sides of the octagon (see figure 5). The fuchsian group G is, as in (1.2), the group generated by $g_i = h_{p_i} h_0$, $1 \leqslant i \leqslant 4$, where as before h_p is the order 2 elliptic transformation centered at the point p.

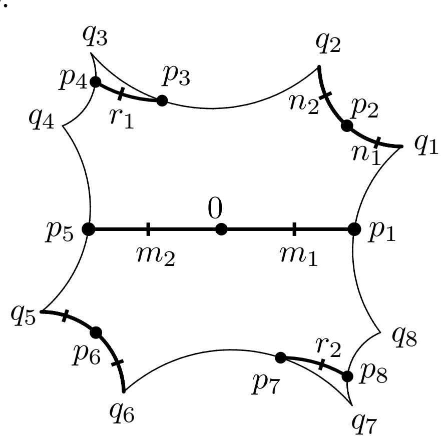

Figure 4

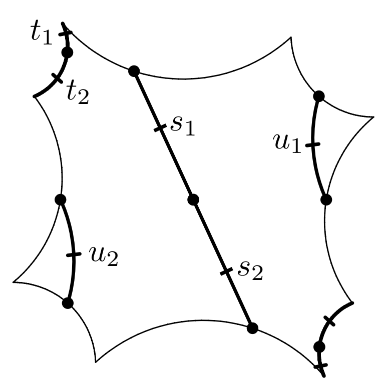

Figure 5

Let u_1 be the midpoint of $[p_1, p_2]$ (see figures 4 and 5) and r_1 be the midpoint of $[p_3, p_4]$ (see figure 4). Obviously we are in the situation described in (1.6) and we can use the uniformizing function F_s for which we have in addition to the relations (1.7)

$$F_s(u_1) = 0, \quad \text{and} \quad F_s(r_1) = \infty$$

By **2.2** we can easily find t_0 such that $t_0 F_s(p_3) = f_3(t_0)$ and $t_0 F_s(p_1) = f_3^2(t_0)$. Hence if we replace F_s by $F_1 = t_0\, F_s$ we have

$$F_1(q_i) = -F_1(0) = t_0,\ F_1(p_3) = -F_1(p_4) = f_3(t_0),\ F_1(p_1) = -F_1(p_2) = f_3(f_3(t_0))$$

(note that $p_4 = -p_8$ and that $p_8' = g_3(p_8)$). In particular the algebraic curve has an equation of the form given in **2.2**.

Let m_1 be the midpoint of $[0, p_1]$, n_1 the midpoint of $[q_1, p_2]$, r_1 the midpoint of $[p_3, p_4]$ and so on as indicated in figures 5 and 6.

We still have $F_1(u_1) = 0$, but $\widetilde{f_3}(u_1) = g_3(-t_2)$ (g_3 as in (1.2) and $\widetilde{f_3}$ as above) and $\widetilde{f_3}(g_3(-t_2)) = m_1$, hence by 2.3, $F_1(t_1) = F_1(t_2) = 3$ and $F_1(s_1) = F_1(s_2) = -3$. For the same reasons $F_1(m_1) = -1$ and $F_1(n_1) = 1$.

2.7 Lemma. *Let G be, as above, the fuchsian group generated by $g_1, \ldots, g_4$. Let $\widetilde{G}$ be the extended group generated by G, $\widetilde{f_3}$, h_0 and h_{m_1} (where, as before, h_p is the order 2 elliptic transformation centered at p). Then the quotient $\mathbb{D}/\widetilde{G}$ of the unit disk by this extended group is the Riemann sphere with four elliptic points, one of order 3 and three of order 2.*

Proof. By construction the quotient of $\mathbb{D}$ by G is a genus 2 surface S with an automorphism group containing the dihedral group D_3. The quotient of S by the hyperelliptic involution, which is induced by h_0, is the sphere with 6 marked points of order 2. By the above we may assume that these points are $\pm t_0$, $\pm f_3(t_0)$ and $\pm f_3(f_3(t_0))$. On this sphere $\widetilde{f_3}$ induces f_3. Taking the quotient yields the sphere with 4 marked points, $b = (t_0 + f_3(t_0) + f_3(f_3(t_0)))$, $(-t_0 + f_3(-t_0) + f_3(f_3(-t_0))) = -b$ and the images of the fixed points of f_3, $\pm 3\,i\sqrt{3}$. On this quotient h_{m_1} induces the same action as h_{n_1}, h_{r_1} and so forth, in other words h_{m_1} induces the transformation $x \mapsto -x$. Hence the final quotient is the sphere with the marked points $a = b^2$, -27 of order 3, and the images of the fixed points 0 and ∞. Note that the a mentioned here is the same as the a in **2.4**. Note also that 0 is also the image in the last quotient of s_i, u_i and t_i while ∞ is also the image of m_i, n_i and r_i. Summarizing we have,

2.8 Corollary. *Let G and $\widetilde{G}$ be as in **2.7**. Let*

$$y^2 = x^6 - (a + 18)\, x^4 + (2\, a + 81)\, x^2 - a$$

be an equation for the algebraic curve defined by $\mathbb{D}/G$. Then the quotient $\mathbb{D}/\widetilde{G}$ is the Riemann sphere with the four marked points

$$-27 \quad \textit{of order } 3 \quad \textit{and} \quad 0,\ a \textit{ and } \infty \quad \textit{of order } 2.$$

2.9 Remark. To do the same construction as the one we have just made for the rotated octagon introduced in **2.6** we note that we have replaced p_1 by p_3, p_2 by p_4 and so on. The role played by u_1 is now played by r_1 and it also exchanges the orbits of these two points. By (2.3) and the remarks made on F_1 the corresponding uniformizing function F_1' will be $3/F_1$. In particular t_0 will be replaced by $3/t_0$ (note that $3/f_3(t) = f_3(f_3(3/t))$) and a by $729/a$ (cf. **2.4**).

We now proceed to the main result of this section,

2.10 Proposition. *Let S be genus 2 surface with an order 3 automorphism. Let $(\ell, tw, \ell, tw, \ell, tw)$ be Fenchel-Nielsen coordinates for S and let*

$$y^2 = x^6 - (a+18)\, x^4 + (2a+81)\, x^2 - a$$

be an equation for the corresponding algebraic curve.

Then the surface S' with Fenchel-Nielsen coordinates $(\ell, tw+\frac{1}{2}, \ell, tw+\frac{1}{2}, \ell, tw+\frac{1}{2})$ has for algebraic equation

$$y^2 = x^6 - (a'+18)\, x^4 + (2a'+81)\, x^2 - a' \quad \textit{with} \quad a' = -(27+a)\ .$$

Proof. For notational reasons we prove this for $tw - \frac{1}{2}$ instead of $tw + \frac{1}{2}$. Since the transformation $a \mapsto a'$ is an involution this is innocuous.

We are applying here simultaneously three half-twist along the non intersecting closed geodesics yielding the pants decomposition. The obvious generalization of the construction made in section **1** shows that the group G' generated by

$$g_1' = h_{m_1} \cdot h_{m_2},\ g_2' = h_{n_1} \cdot h_{m_2},\ g_3' = h_{r'2} \cdot h_{m_2},\ g_4' = h_{r_1} \cdot h_{m_2}$$

(where $r_2' = g_3(r_2)$) is a fuchsian group for S' (see figure 6).

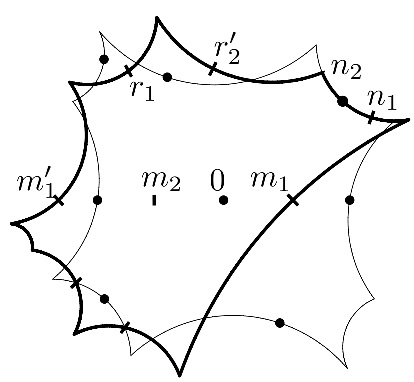

Figure 6

Writing $S' = \mathbb{D}/G'$ we note that, since $\widetilde{f}_3(n_2) = m_1$, $\widetilde{f}_3(n_1) = m_2$ and so forth, $\widetilde{f}_3$ also induces an order 3 automorphism on $\mathbb{D}/G' = S'$. Also h_0 induces an involution, distinct from the hyperelliptic involution which is induced by h_{m_2}. Hence if $\widetilde{G'}$ is the group generated by G', h_{m_2}, h_0 and $\widetilde{f}_3$ then $\mathbb{D}/\widetilde{G'}$ is the sphere with the four marked points -27 (of order 3), 0, a' and ∞.

On the other hand we obviously have $\widetilde{G} = \widetilde{G'}$. Hence the two quotients are isomorphic the difference being that now m_i, n_i and r_i are sent to a', in place of ∞ while 0, p_i and q_i are sent to ∞ in place of a. In other words there is a Möbius transformation taking $-27, 0, a, \infty$ to $-27, 0, a', \infty$, keeping -27 and 0 fixed and sending a to ∞. It is $z \mapsto -(27+a)\frac{z}{z-a}$. The point a' being the image of ∞ is then equal to $-(27+a)$.

Combining **2.9** and **2.10** we have,

2.11 Corollary. *Let $\mathscr{T}_{2,3}$ be the subspace of the genus 2 Teichmüller space formed by surfaces with Fenchel-Nielsen coordinates of the form $(\ell, tw, \ell, tw, \ell, tw)$. Let $\mathscr{S}_a = \widehat{\mathbb{C}} \smallsetminus \{-27, 0, \infty\}$ be the parameter space for curves with equations of the form* **2.4***. Let Γ_3 be the group generated by $\gamma_1 : (\ell, tw) \mapsto (\ell, tw + \frac{1}{2})$ and $\gamma_2 : (\ell, tw) \mapsto (\ell', tw')$ with ℓ' and tw' as in* **2.6**.

Then Γ_3 induces on $\mathscr{S}_a$ the action of the permutation group $\mathfrak{S}_3$ generated by $\sigma_1 : a \mapsto -(27 + a)$ and $\sigma_2 : a \mapsto 729/a$.

2.12 Remark. By **2.4** $\mathscr{S}_a$ is a ramified double cover of the moduli space of genus 2 curves with an order 3 automorphism. The ramification is at $a = 27$ which defines the curve with automorphism group of order 24.

The action described in **2.11** does not go down to moduli as a group action. This is because σ_2 induces a trivial action whereas the action induced by $\sigma_1\sigma_2$ differs from the action induced by σ_1.

To be complete we need,

2.13 Lemma. *Let γ_1 and γ_2 be as in* **2.11***. Then ${\gamma_2}^2 = 1$ and $(\gamma_1\gamma_2)^3 = 1$. In particular the group Γ_3 induces an action of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathscr{T}_3$.*

Proof. It would be nice to have a direct geometric proof but unfortunately we have not as yet found one. Hence we proceed with an ugly computational proof. For this we first note that ${\gamma_2}^2 = 1$ is obvious, and that $(\gamma_1\gamma_2)^3 = 1$ is equivalent to $\gamma_1\gamma_2\gamma_1 = \gamma_2{\gamma_1}^{-1}\gamma_2$. Expressing γ_2, γ_1 and ${\gamma_1}^{-1}$ in terms of L and Tw (notations as in **2.6**) we have,

$$T_2 = (L, Tw) \mapsto \left(Tw^2 \frac{2L-1}{L-1} - 1,\ \sqrt{\frac{Tw^2(L+1)(2L-1) - 2(L^2-1)}{2Tw^2(2L-1) - 3(L-1)}} \right)$$

$$T_1 = (L, Tw) \mapsto \left(L,\ Tw\sqrt{\frac{L+1}{2}} + \sqrt{Tw^2 - 1}\sqrt{\frac{L-1}{2}} \right) \tag{2.14}$$

$$T_1^{-1} = (L, Tw) \mapsto \left(L,\ Tw\sqrt{\frac{L+1}{2}} - \sqrt{Tw^2 - 1}\sqrt{\frac{L-1}{2}} \right)$$

Because of the sign change indicated in **2.6**, $\gamma_1\gamma_2\gamma_1$ will be represented by ${T_1}^{-1}\, T_2\, T_1$ and $\gamma_2{\gamma_1}^{-1}\gamma_2$ by $T_2\, T_1\, T_2$. It can be proved by brute force that these are indeed equal but the computations involved, although reasonable enough for the length, turn out to be extremely heavy for the twist. On the other hand the computations simplify considerably for surfaces with zero twist i.e, with $Tw = 1$. We find fairly directly that $T_2\, T_1\, T_2$ maps $(L, 1)$ to

$$\left(\frac{(2\,L^2 - L + 1)}{2(L-1)},\ \frac{1}{2}\sqrt{\frac{(2\,L^2 - 3\,L + 3)\,(2\,L - 1)}{(L^2 - L + 1)\,(L-1)}} \right). \tag{2.15}$$

On the other hand ${T_1}^{-1}\, T_2\, T_1$ maps $(L, 1)$ to

$$\left(\frac{(2\,L^2 - L + 1)}{2(L-1)},\right.$$
$$\left. \frac{\sqrt{3 + 2\,L^3 - L^2}\sqrt{2\,L^2 + L - 1} - \sqrt{(2\,L-1)\,(L-1)^2}\sqrt{2\,L^2 - 3\,L + 3}}{4\sqrt{(L^2 - L + 1)\,(L-1)}} \right).$$

Squaring the numerator of the second term one easily finds that both transformations coincide if $Tw = 1$.

Remains to show that the sign condition is satisfied. For this we note that $\gamma_2(\ell, 0) = (\ell', 0)$ and $\gamma_1{}^{-1}(\ell', 0) = (\ell'', -\frac{1}{2})$. Hence $\gamma_2\gamma_1{}^{-1}\gamma_2(\ell, 0)$ will be of the form (ℓ''', tw) with $tw \geqslant 0$. A value of $tw = \frac{1}{2}$ corresponds to $Tw = \sqrt{\frac{L+1}{2}}$. Applying T_2 to this we have

$$T_2\left(L, \sqrt{\frac{L+1}{2}}\right) = \left(\frac{(L+1)(2\,L-1)}{2\,(L-1)}, \sqrt{\frac{3+2\,L^3-L^2}{4(L^2-L+1)}}\right)$$

from which it is easy to conclude that for the twist parameter tw of $\gamma_2\gamma_1(\ell, 0)$ we will have $tw \geqslant -\frac{1}{2}$ and hence the twist parameter of $\gamma_1\gamma_2\gamma_1(\ell, 0)$ will be positive.

We conclude that $\gamma_1\gamma_2\gamma_1$ and $\gamma_2\gamma_1{}^{-1}\gamma_2$ coincide on the full set of surfaces with zero twist. On the other hand, by **2.11**, these transformations induce holomorphic maps on $\mathscr{S}_a$. Since they are obviously continuous we hence conclude that they coincide everywhere.

2.16 Examples. We limit here to examples that can immediately be deduced from the results of **2.11**.

— The fixed point of σ_2 is $a = 27$. As noted earlier this is the curve with a larger automorphism group. It is easily checked that it corresponds to $\ell = 2\,\text{arccosh}(2)$, $tw = 0$, which is the fixed point of γ_2.

— The fixed point of σ_1 is $a = -\frac{27}{2}$. But $-\frac{27}{2} = \sigma_2(\sigma_1(27))$, hence we can recover the Fenchel-Nielsen coordinates by applying $\gamma_2\gamma_1$ to the preceding case. We find $\ell = 2\,\text{arccosh}(\frac{7}{2})$ and $tw = \frac{1}{4}$.

— $\sigma_1\sigma_2$ has two fixed points, $-27\frac{1\pm i\sqrt{3}}{2}$, and these are exchanged by σ_2. To find the Fenchel-Nielsen coordinates we look for fixed points of $\gamma_1\gamma_2$. We find that $\ell = 2\,\text{arccosh}(L)$, where L is the solution 2.2057... of equation $8\,x^3 - 12\,x^2 - 12\,x - 1 = 0$ and $tw = \frac{1}{4}$. Other considerations show that this in fact corresponds to $a = -27\frac{1-i\sqrt{3}}{2}$. The other value for a corresponds to the same value of ℓ but $tw = -\frac{1}{4}$. This is a fixed point of $\gamma_1^{-1}\gamma_2$. Both curves, with $tw = \pm\frac{1}{4}$, are of course isomorphic.

2.17 Remark. In this section we have only considered the pants decompositions indicated in figure 4 and 5 but it is sometimes useful to consider other rotations of these.

An example of this is the following. Consider the surface defined by the coordinates $(\ell, \frac{1}{2}, \ell, \frac{1}{2}, \ell, \frac{1}{2})$ and assume that we are using the pants decomposition (1.1). We can also use the pants decomposition defined by the arcs $[p_6, p_2]$, $[p_3, p_4] \cup [p_8, p_7]$ and $[q_8, q_1]$. Elementary computations of the same type as the ones used in the proof of 2.6 show that the Fenchel-Nielsen coordinates are now $(\ell', \frac{1}{2}, \ell, 0, \ell', \frac{1}{2})$ where

$$L' = \cosh(\ell'/2) = \frac{3L-1}{2(L-1)} \quad \text{if} \quad L = \cosh(\ell/2)\ . \tag{2.18}$$

In particular we have $\ell' = \ell$ if $L = (5+\sqrt{17})/4$. But in this case the associated algebraic curve is isomorphic to one of its transforms under the transformations indicated in (1.7). Looking how this isomorphism acts on the Weiesrstrass points one can show that the curve corresponds to $t_0 = i\sqrt{10-2\sqrt{17}}/2$ and hence has an equation of type **2.4** with $a = -23 + \sqrt{17}$.

3. Genus 2 curves with an order 4 automorphism and cousins.

We start this section by reviewing and reformulating some of the results of [Si2].

To this end let $\mathscr{H}$ be a rectangular hyperbolic hexagon with side lengths $(\ell_1, \ell_2, \ell_1, \ell_2/2, \ell_3, \ell_2/2)$, in that order (see figure 7). Such hexagons exist, but the values of the ℓ_i are far from independent. In fact writing $L_i = \cosh(\ell_i)$ we have the relations

$$L_2 = \frac{L_1+1}{L_1-1} \quad \text{and} \quad L_3 = 2\,L_1 + 1 \tag{3.1}$$

(see [Bu], p. 454).

Embed isometrically this hexagon in the unit disk with the side of length ℓ_3 on the real axis with midpoint at the origin. Following the construction indicated in figure 7 it is easy to construct a fundamental octagon for the surface with Fenchel-Nielsen coordinates $(2\ell_1, tw, 2\ell_1, tw, 2\ell_3, \frac{1}{2})$ (where again, the surface is obtained by identifying opposite sides). The pants decomposition we are using is not the one indicated in (1.1) but the one given by $[p_1, p_2] \cup [p_6, p_5]$, $[p_3, p_4] \cup [p_8, p_7]$ and $[q_5, q_1]$.

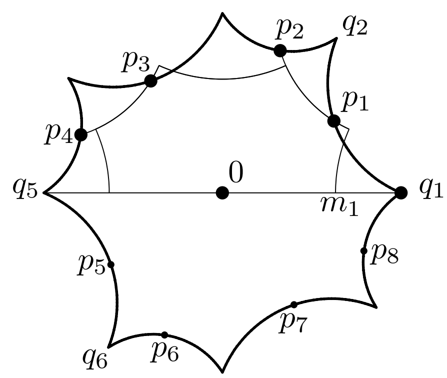

Figure 7

Since the edge of the hexagon opposite to ℓ_3 is twice the length of those opposite to ℓ_1 and ℓ_2 it is readily checked that the octagon we have obtained is stable under rotation by $\pi/2$. In other words the surface has an order 4 automorphism.

By [Si2] (1.6) and (3.2) we have that the uniformizing map F_s of (1.7) satisfies

$$F_s(q_1) = -F_s(0) = 1,\ F_s(p_2) = -F_s(p_1) = a,\ F_s(p_4) = -F_s(p_3) = 1/a\ . \tag{3.2}$$

Hence an equation for the algebraic curve is,

$$y^2 = (x^2 - a^2)(x^2 - 1)(x^2 - 1/a^2)\ . \tag{3.3}$$

We also note that by [Si2] section **3**, since m_1 is the midpoint of 0 and q_1, we have $F_s(m_1) = -i$.

We again have an action of $\mathrm{PSL}_2(\mathbb{Z})$. For this let γ_1 be the transformation keeping the lengths fixed but replacing tw by $tw + \frac{1}{2}$. The transformation induced on the equation is taken care of by

3.4 Proposition. *Let S be a genus 2 surface with an order 4 automorphism and with F-N coordinates $(2\ell_1, tw, 2\ell_1, tw, 2\ell_3, \frac{1}{2})$ where $\cosh(\ell_3) = 2\cosh(\ell_1) + 1$. Let*

$$y^2 = (x^2 - a^2)(x^2 - 1)(x^2 - 1/a^2)$$

be the associated equation (as in (3.3)).

Then the surface S' with F-N coordinates $(2\ell_1, tw + \frac{1}{2}, 2\ell_1, tw + \frac{1}{2}, 2\ell_3, \frac{1}{2})$ has equation

$$y^2 = (x^2 + a^2)(x^2 - 1)(x^2 + 1/a^2)\ .$$

This is (3.2) of [Si2].

To define γ_2 we again rotate the octagon so that q_2 becomes a positive real. This can equivalently be viewed as changing the pants decomposition.

3.5 Lemma. *Let S be a genus 2 surface with an order 4 automorphism and with Fenchel-Nielsen coordinates $(2\ell_1, tw, 2\ell_1, tw, 2\ell_3, \frac{1}{2})$ where $\cosh(\ell_3) = 2\cosh(\ell_1)+1$. Let this set of coordinates correspond to the octagon of figure 7 and pants decomposition defined by $[p_1, p_2] \cup [p_6, p_5]$, $[p_3, p_4] \cup [p_8, p_7$ and $[q_5, q_1]$. Then the Fenchel-Nielsen coordinates for the pants decomposition $[p_2, p_3] \cup [p_7, p_6]$, $[p_4, p_5] \cup [p_1, p_8]$ and $[q_6, q_2]$ (see figure 7) are of the form $(2\ell_1', tw', 2\ell_1', tw', 2\ell_3', \frac{1}{2})$ with again $\cosh(\ell_3') = 2\cosh(\ell_1') + 1$. Moreover writing $L_1 = \cosh(\ell_1)$, $L_1' = \cosh(\ell_1')$, $Tw = \cosh(tw\,\ell_1)$ and $Tw' = \cosh(tw'\,\ell_1')$ we have*

$$\begin{aligned} L_1' &= Tw^2\,\frac{2\,L_1}{L_1 - 1} - 1, \\ Tw' &= \sqrt{\frac{L_1^2 Tw^2 + L_1 Tw^2 - L_1^2 + 1}{2\,L_1 Tw^2 - L_1 + 1}}, \\ tw' &= -\operatorname{sign}(tw)\operatorname{arccosh}(Tw')/\operatorname{arccosh}(L_1') \end{aligned}$$

Proof. We first note that the fact that the octagon is stable under rotation by $\pi/2$ ensures that the new Fenchel-Nielsen coordinates are of the announced form. To compute these we note two facts. The first is that the midpoint of $[0, q_2]$ is also the midpoint $[p_1, p_2]$. The second is that the hexagon we started with can be decomposed into four isometric copies of a trirectangular quadrangle and remaining angle $\pi/4$ (at 0). With the formulae [Bu] p.454 it is now easy to compute the length of the geodesic arc $[0, q_2]$. The hyperbolic cosine of this length is

$$2\,\frac{2\,Tw^2 L_1}{L_1 - 1} - 1\ .$$

But this is $2\,L_1' + 1$ hence the value of L_1' found. Since the transformation is an involution we can solve it backwards to find Tw'. The sign condition on tw' is again the same as the one explained in [Ok].

Starting with ℓ_1' and tw' and associating an equation of the form (3.3), with the same method as before, will not lead to the same value for a. In fact as explained in [Si2] this yields the equation,

(3.6) $$y^2 = (x^2 - a''^2)(x^2 - 1)(x^2 - 1/a''^2) \quad\text{with}\quad a'' = i\,\frac{i - a}{i + a}\ .$$

Summarizing we have obtained

3.7 Corollary. *Let $\mathscr{T}_{2,4}$ be the subspace of the genus 2 Teichmüller space formed by surfaces with Fenchel-Nielsen coordinates of the form $(2\ell_1, tw, 2\ell_1, tw, 2\ell_3, \frac{1}{2})$ with* $\cosh(\ell_3) = 2\cosh(\ell_1) + 1$. *Let*

$$y^2 = (x^2 - 1)(x^2 - a^2)(x^2 - 1/a^2)$$

be the associated equations of the form **3.3**. *Let Γ_4 be the group generated by $\gamma_1 : (\ell_1, tw) \mapsto (\ell_1, tw + \frac{1}{2})$ and $\gamma_2 : (\ell_1, tw) \mapsto (\ell_1', tw')$ with ℓ_1' and tw' as in* **3.5**.

Then Γ_4 induces on the a-space the transformations $a \mapsto -i\,a$ and $a \mapsto i\,\frac{i-a}{i+a}$.

3.8 Remarks. 1. We again have ${\gamma_2}^2 = 1$ and $(\gamma_1\gamma_2)^3 = 1$. The proof is of the same nature as the proof of **2.13**, and quite similar. Expressed in terms of L and Tw both $\gamma_1\gamma_2\gamma_1$ and $\gamma_2{\gamma_1}^{-1}\gamma_2$ transform $(L, 1)$ into

$$\left(\frac{L^2+1}{L-1}, \sqrt{\frac{L\,(L^2-L+2)}{(L^2+1)(L-1)}}\right) .$$

The rest of the argument is the same.

2. The way we have defined the transformations on the equations in **3.7** is needed for the next part of this section. On the other hand the action of the permutation group $\mathfrak{S}_3$ is not clearly visible in this formulation. To recover this, simply write the equation in the form

$$y^2 = (x^2-1)(x^4 + \alpha\,x^2 + 1) \quad \text{with} \quad \alpha = -(a^2 + 1/a^2) .$$

Then the induced actions of γ_1 and γ_2 are given by

$$\alpha \mapsto -\alpha \qquad \text{and} \qquad \alpha \mapsto 2\,\frac{\alpha+6}{\alpha-2}$$

from which we recover the desired $\mathfrak{S}_3$ action.

We want to apply (1.8) to the present situation. For this we need to change octagons and consider the octagon associated by (1.1) to the Fenchel-Nielsen coordinates $(2\ell_1, tw, 2\ell_3, \frac{1}{2}, 2\ell_1, tw)$. In this case with the associated F_s function we have equation

$$y^2 = (x^2-1)(x^2 - {a'}^2)(x^2 - {b'}^2) . \tag{3.9}$$

To obtain a' and b' in terms of a and b we use the properties of F_s, indicated in section **1**, to compare where points are mapped in the curves defined by equations (3.3) and (3.9). Doing this we find that to pass from (a, b) to (a', b') we need a Möbius transformation that maps $-i$ to ∞, $-a$ to 1 and $1/a$ to -1. This is

$$\varphi : z \mapsto \frac{a-i}{a+i} \cdot \frac{z-i}{z+i} .$$

In this context we have,

$$a' = \varphi(a) = \frac{(i-a)^2}{(i+a)^2} \quad \text{and} \quad b' = \varphi(1) = i\,\frac{i-a}{i+a} . \tag{3.10}$$

We are now ready to introduce the family we want to consider.

3.11 Proposition. *Let S be a surface with Fenchel-Nielsen coordinates of the form* $(2\ell_1, tw, 2\ell_3, 0, 2\ell_1, tw)$ *with* $\cosh(\ell_3) = 2\cosh(\ell_1) + 1$. *Then the algebraic curve corresponding to S has an equation of the form*

$$y^2 = (x^2 - \alpha)(x^2 - 1)(x^2 - \alpha - 1) \quad \alpha \neq 0,\ \pm 1\ . \tag{3.11.1}$$

Proof. We first note that S has an equation of the form $y^2 = (x^2-\alpha)(x^2-1)(x^2-\beta)$ (see section **1**). Since S is obtained by applying a half-twist to the surface we just considered we have by (1.8)

$$\alpha = \frac{a'^2(1-b'^2)}{a'^2 - b'^2} \quad \text{and} \quad \beta = 1 - b'^2\ .$$

By 3.10 we find that

$$\alpha = a' \quad \text{and} \quad \beta = 1 + a'\ . \tag{3.12}$$

To define γ_1 and γ_2 on this family we simply transport the γ_i operating on $\mathscr{T}_{2,4}$.

3.13 Proposition. *Let S be a surface with Fenchel-Nielsen coordinates of the form* $(2\ell_1, tw, 2\ell_3, 0, 2\ell_1, tw)$ *with* $\cosh(\ell_3) = 2\cosh(\ell_1) + 1$ *and let*

$$y^2 = (x^2 - b^2)(x^2 - 1)(x^2 - b^2 - 1)$$

be its equation defined in **3.11**.

Let S' be defined by $(2\ell_1, tw + \frac{1}{2}, 2\ell_3, 0, 2\ell_1, tw + \frac{1}{2})$, *then an equation for S' is*

$$y^2 = (x^2 - b'^2)(x^2 - 1)(x^2 - b'^2 - 1) \quad \textit{with} \quad b' = i\frac{i+b}{i-b}\ . \tag{3.13.1}$$

Let S'' be defined by $(2\ell_1', tw, 2\ell_3', 0, 2\ell_1', tw)$, ℓ_1' *and* ℓ_3' *as in* **3.5**. *Then an equation for S'' is*

$$y^2 = (x^2 - b''^2)(x^2 - 1)(x^2 - b''^2 - 1) \quad \textit{with} \quad b'' = \frac{1-b}{1+b}\ . \tag{3.13.2}$$

Proof. We first note that the curve defined by $y^2 = (x^2-1)(x^2-\alpha)(x^2-\beta)$ is isomorphic to the one defined by $y^2 = (x^2 - 1/\alpha)(x^2-1)(x^2-\alpha/\beta)$. Now if $\beta = 1+\alpha$ then $\beta/\alpha = 1 + 1/\alpha$ hence replacing b by $-b$ in (3.13.1) and (3.13.2) also leads to equations for S' and S''. This is also the case if we replace b by $1/b$.

We have seen in **3.7** that replacing tw by $tw + \frac{1}{2}$, in the surface with an order 4 automorphism, consists in replacing a by $-i\,a$. Hence the a' of (3.10) is replaced by $\dfrac{(a-1)^2}{(a+1)^2}$. The result for b' easily follows.

For S'' the proof is the same with γ_2 of **3.7** in place of γ_1.

3.14 Remark. The situation here although quite similar to the one in **3.7** is subtlety different because in general S and S'' are not isomorphic (see **3.15**). In fact the we will see in **3.18** that the induced $\mathfrak{S}_3$ action will be on the moduli space of the curves defined in **3.11**.

3.15 Lemma. *Let C and C' be the curves with respective equations $y^2 = (x^2 - \alpha)(x^2-1)(x^2-\alpha-1)$ and $y^2 = (x^2-\alpha')(x^2-1)(x^2-\alpha'-1)$. Then C is isomorphic to C' if and only if $\alpha' = \alpha$ or $\alpha' = 1/\alpha$.*

Proof. We generalize the argument given at the beginning of the proof of **3.13**. Consider the two curves with equations $y^2 = (x^2-1)(x^2-\alpha)(x^2-\beta)$ and $y^2 = (x^2-1)(x^2-\alpha')(x^2-\beta')$ and assume their automorphism groups do not contain elements of order 4. Then the two curves are isomorphic if and only if $\{\alpha',\beta'\}$ is, up to order, one of the six pairs

$$\{\alpha,\beta\},\ \{1/\alpha,1/\beta\},\ \{1/\alpha,\beta/\alpha\},\ \{\alpha/\beta,1/\beta\},\ \{\alpha,\alpha/\beta\},\ \{\beta/\alpha,\beta\}\ . \tag{3.16}$$

Replacing β by $\alpha+1$ we obtain the result for curves without automorphisms of order 4.

If C does have an automorphism of order 4, replacing if necessary $\{\alpha,\beta\}$ by one of the pairs in (3.16), we may assume that $\beta = 1/\alpha$. In this case we must also consider the pairs obtained by applying (3.16) to

$$\left\{-\left(\frac{\sqrt{\alpha}-i}{\sqrt{\alpha}+i}\right)^2, -\left(\frac{\sqrt{\alpha}+i}{\sqrt{\alpha}-i}\right)^2\right\}\ . \tag{3.17}$$

A case by case study shows that, up to isomorphism, the only pairs to consider are $\{\alpha,\alpha+1\}$ for α equal to $\frac{\sqrt{5}-1}{2}$, $-\frac{\sqrt{5}+1}{2}$ or $\frac{i\sqrt{3}-1}{2}$ in which cases reexamining the lists one can directly show the result to be true.

3.18 Proposition. *Let $\mathscr{T}_{2,a}$ be the subspace of Teichmüler space of genus 2 surfaces with F-N coordinates of the form $(2\ell_1, tw, 2\ell_3, 0, 2\ell_1, tw)$ with $\cosh(\ell_3) = 2\cosh(\ell_1)+1$. Then the transformations $\gamma_1 : (\ell_1, tw) \mapsto (\ell_1, tw+\frac{1}{2})$ and $\gamma_2 : (\ell_1, tw) \mapsto (\ell_1', tw')$, ℓ_1' and tw' as in* **3.5***, induce an action of the symmetric group $\mathfrak{S}_3$ on the moduli space of such surfaces.*

Proof. By **3.11** the curves associated to the surfaces have equations of the form

$$y^2 = (x^2-\alpha)(x^2-1)(x^2-\alpha-1) \quad \alpha \neq 0,\ \pm 1\ .$$

By **3.15** the moduli space of such curves is the quotient of $\mathbb{C}\setminus\{-1,0,1\}$ by $\alpha \mapsto 1/\alpha$. Hence the map $\alpha \mapsto \alpha + 1/\alpha$ defines a map from $\mathbb{C}\setminus\{-1,0,1\}$ to the moduli space. Let $\beta = \alpha + 1/\alpha$, then by **3.13**, γ_1 and γ_2 induce the transformations

$$\tau_1 : \beta \mapsto 2\,\frac{6-\beta}{2+\beta} \quad \text{and} \quad \tau_2 : \beta \mapsto 2\,\frac{\beta+6}{\beta-2}\ . \tag{3.19}$$

Since these two transformations generate an $\mathfrak{S}_3$ we are done.

3.20 Remark. There is another way to prove the results on the family defined in **3.11**. To see this we need a second description which is in fact the one given in [Bu-Si1] where a real component of the family was introduced (see [Bu-Si1], (6.2)). For this we consider the genus 3 curve with equation,

$$y^2 = (x^2-x_1^2)(x^2-\frac{1}{x_1^2})\left(x^2 - \frac{(x_1-1)^2}{(x_1+1)^2}\right)\left(x^2 - \frac{(x_1+1)^2}{(x_1-1)^2}\right) \quad \text{where} \quad x_1 = \sqrt{\frac{b-1}{b+1}}$$

This is a double cover of the genus 2 curve with equation (3.11.1) — see [Bu-Si1] (5.10). Such a curve has an obvious order 4 automorphism induced by $x \mapsto \frac{x-1}{x+1}$ and hence the automorphism group contains the Dihedral group D_4. Taking the quotient under this group we find the sphere with four elliptic points one of order 4 and 3 of order 2. With this we obtain a description similar to the one used in the proofs of **2.11** and **3.7** (see [Si2] for the second).

To have a hyperbolic description corresponding to this second point of view we note that we have been working with Fenchel-Nielsen coordinates for the pants decomposition (1.1). But if we consider the coordinates for the pants decomposition $[q_3, q_4]$, $[p_5, p_6] \cup [p_2, p_1]$ and $[p_7, p_5]$ then the Fenchel-Nielsen coordinates are of the form $(2\ell_2, tw_2, 4\ell_2, tw_2/2, 2\ell_2, tw_2)$ (this follows from the remarks made on the hexagon made at the beginning of this section). With the same methods as before one can compute $L_2 = \cosh(\ell_2)$ and $Tw_2 = \cosh(tw_2 \operatorname{arccosh}(\ell_2))$. We have

$$\begin{aligned} L_2 =& \frac{TwL_1}{\sqrt{L_1(L_1-1)}} \\ Tw_2 =& \sqrt{\frac{L_1^2Tw^2 + L_1Tw^2 - L_1^2 + 1}{2L_1Tw^2 - L_1 + 1}}\ . \end{aligned}$$

From this it is fairly easy to recover the hyperbolic structure of the genus 3 double cover.

3.21 Examples. We first look for fixed points for the transformations τ_1 and τ_2 of (3.19) but in most cases with the notations of **3.13** or **3.11**.

— Up to isomorphism the fixed point in moduli for the action of τ_2 corresponds to $b = \sqrt{2} - 1$. The Fenchel-Nielsen coordinates for this curve were computed in [Bu-Si1], (8.3). We have $L_1 = 1 + \sqrt{2}$, $tw = 0$.

— Up to isomorphism the fixed point in moduli for the action of τ_1 corresponds to $b = i(1 - \sqrt{2})$. But this corresponds to the image under $\tau_2\tau_1$ of the preceding example. Hence to recover the Fenchel-Nielsen coordinates we only need to compute $\gamma_2(\gamma_1(L_1, tw)) = (2 + 2\sqrt{2}, -1/4)$.

— Applying γ_1 to the first example we find the fixed point of $\tau_2\tau_1\tau_2$ which corresponds to $b = (1+i)/\sqrt{2}$ and of course to $L_1 = 1 + \sqrt{2}$, $tw = \frac{1}{2}$.

— The fixed point of $\tau_1\tau_2$ corresponds, again up to isomorphism, to $b = (1-i)(1-\sqrt{3})/2$. This is the curve associated to the last example of [Si2] hence $L_1 = 1+\sqrt{3}$, $tw = 1/4$.

Also of interest are the curves in the family with larger automorphism groups. We have already encountered in the proof of **3.15** those with an order 4 automorphism.

— The case $\alpha = b^2 = (\sqrt{5} - 1)/2$ was computed in [Bu-Si1] (8.1) it corresponds to $L_1 = (1 + \sqrt{5})/2$, $tw = 0$.

— For $\alpha = -(1+\sqrt{5})/2$ we note that this can be obtained by applying $\tau_2\tau_1\tau_2$ to $\alpha' + 1/\alpha'$ with $\alpha' = \sqrt{(\sqrt{5} - 1)/2}$. Hence to obtain the Fenchel-Nielsen coordinates we apply $\gamma_2\gamma_1\gamma_2$ to $((1 + \sqrt{5})/2, 0)$. This yields $L_1 = (5 + 3\sqrt{5})/2$ and $Tw = \sqrt{150 + 30\sqrt{5}}/10$. This curve was also considered in a different form in [Bu-Si1] (8.11).

— For $\alpha = b^2 = (i\sqrt{3} - 1)/2$ we note that this corresponds to applying τ_1 to $\alpha' + 1/\alpha'$ with $\alpha' = {b'}^2 = (2 - \sqrt{3})^2$. But $b = 2 - \sqrt{3}$ is the second example of [Bu-Si1] (8.2), hence $L_1 = 3$, $tw = 1/2$.

For those with an order 3 automorphism it is far more difficult to be systematic. We know nevertheless of two. They have been obtained by a fairly different method than the one we have used here, so we will only indicate the results.

— The first corresponds to $L_1 = 4 + \sqrt{17}$, $tw = \frac{1}{2}$. Assuming this corresponds to the pants decomposition (1.1), it has a second pants decomposition given by the arcs $[q_3, q_4]$, $[p_6, p_2]$ and $[p_3, p_5] \cup [p_1, p_7]$ all of length $2\operatorname{arccosh}((5 + \sqrt{17})/4)$. The twist parameters are $\frac{1}{2}$, 0, $\frac{1}{2}$. In other words this surface is isometric to the one considered in **2.17**. As noted there it has an equation of the form **2.4** with $a = -23 + \sqrt{17}$.

— The second corresponds to $L_1 = (3 + \sqrt{17})/4$, $tw = \frac{1}{2}$. But this is of the form (2.18) with $L = (5 + \sqrt{17})/2$. Hence the order 3 automorphism by **2.17**. Again an equation is easiest expressed in the form **2.4** with $a = \dfrac{-16767 + 729\sqrt{17}}{512}$. Note that σ_2 of this value is $-23 - \sqrt{17}$ (compare with the above).

4. Quotients of genus 5 surfaces with an order 6 automorphism.

The third family we want to study was indirectly introduced in [Si1] section **6** and the construction we are going to give here is a very mild variation of the one given there. We will consider dodecagons as fundamental domains in this section but revert to octagons in the next.

We start with a rectangular dodecagon $\mathscr{D}$ with edges alternatively of hyperbolic lengths ℓ and ℓ' (see figure 8). These lengths are of course related, in fact we have $\sinh(\ell/2)\sinh(\ell'/2) = \sqrt{3}/2$ (see [Bu], p.454). Moreover if we call ℓ'' (resp. ℓ''') the length of the separating horizontal (resp. vertical) geodesic we have $\cosh(\ell''/2) = 2\cosh(\ell/2)$ (resp. $\cosh(\ell'''/2) = 2\cosh(\ell'/2)$).

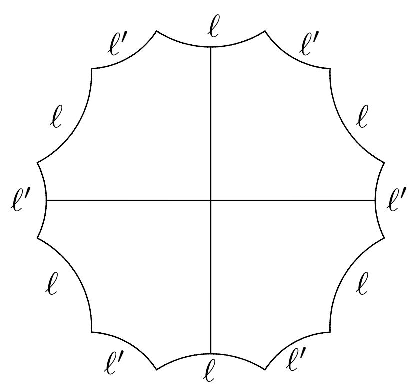

Figure 8

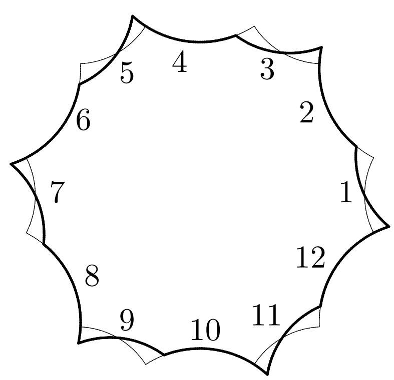

Figure 9

Paste two copies of the dodecagon $\mathscr{D}$ along the edges of length ℓ'. This yields a sphere with 6 disks removed and geodesic boundary components of length 2ℓ. Now we can paste two copies of such a sphere to obtain a genus 5 surface S_1, which is hyperelliptic by its construction from two isometric spheres. We can do this last pasting with twist parameters, and if we do so with the same twist parameter tw

on the six boundary components we will then obtain a genus 5 surface with an order 6 automorphism (induced by rotation of angle $\pi/3$ in the dodecagon). It also has involutions distinct from the hyperelliptic involution. In particular it is easy to construct a fixed point free involution φ that exchanges the closed geodesics of length 2ℓ, fixes globally two of length $2\ell'$ and exchanges two pairs of length $2\ell'$.

Since φ is fixed point free the quotient $S_2 = S_1/\varphi$ is of genus 3. Moreover from the construction of φ we can give a geometric description of S_2 as follows. Consider a sphere with 4 disks removed and three geodesic boundaries of length 2ℓ and one of length $2\ell''$. Then S_2 is obtained by pasting two copies of such a sphere using the twist parameter tw (the same as above) on the geodesics of length 2ℓ and twist 0 on the one of length $2\ell''$. From this it follows that S_2 is hyperellptic. Moreover by construction it also admits non-hyperlliptic involutions.

One of these involutions is again fixed point free. The quotient of S_2 by this involution is of genus 2 with Fenchel-Nielsen coordinates $(2\ell, tw, \ell, 2tw, \ell'', 0)$. This defines the family of genus 2 surfaces we want to consider.

4.1 Proposition. *Let S be a genus 2 surface with Fenchel-Nielsen coordinates $(2\ell, tw, \ell, 2tw, \ell'', 0)$ and additional relation $\cosh(\ell''/2) = 2\cosh(\ell/2)$. Then the algebraic curve defined by S has an equation of the form*

$$y^2 = x(x-1)\left(x^3 + a\,x^2 - \frac{8}{3}a\,x + \frac{16}{9}a\right) \quad a \neq 0,\ -9\ .$$

Proof. The proof is essentially the same as the proof of [Si1] (6.1) and we briefly outline this proof. Since the genus 5 surface S_1 is hyperelliptic with an additional non-hyperelliptic involution φ it has an equation of the form

$$y^2 = \prod_{i=1}^{6}(x^2 - x_i^2)\ , \tag{4.2}$$

where the action of φ is induced by $x \mapsto -x$. Moreover since it has an order 6 automorphism, we may assume that the x_i are globally stable under the action of

$$f : x \mapsto \frac{3x - \sqrt{3}}{\sqrt{3}x + 3} \tag{4.3}$$

(note for further use that $f^3(t) = -1/t$ and that $f(-t) = -f^5(t)$).

In this context the genus 3 quotient S_2 of S_1 by φ has equation $y^2 = x\prod_{i=1}^{6}(x - x_i^2)$. Let $y_i = \frac{x_i^2+1}{x_i^2-1}$. Relabeling the y_i if necessary we obtain that S_2 has an equation of the form $y^2 = (x^2-1)\prod_{i=1}^{3}(x^2 - y_i^2)$. This change of coordinates for the Weierstrass points is exactly what is needed to recover S simply or more precisely the genus 2 quotient S of S_2 has for equation

$$y^2 = x(x-1)\prod_{i=1}^{3}(x - y_i^2)\ . \tag{4.4}$$

Expanding (4.4) we obtain **4.1** after a tedious but elementary computation.

4.5 Remark. The quotient map from the genus 5 surface S_1 to the genus 2 surface S is induced by the map

$$\psi : x \mapsto \frac{(x^2+1)^2}{(x^2-1)^2} \ .$$

Under this map the fixed points of f, which corresponds to $x = \pm i$ are mapped to the point with $x = 0$. The fixed points of the non-hyperelliptic involutions of S_1 corresponds to the midpoints of the edges of $\mathscr{D}$ in figure 8. These correspond to x either in the orbit of 0 under f or in the orbit of 1. From our construction we may assume that $x = 0$ corresponds to the point at the right end of the horizontal axis of figure 8. Since by construction f is induced by rotation of angle $\pi/3$ in the dodecagon, this choice imposes that the upper end point of the vertical axis corresponds to $x = -1$. Note for further use that $\psi(0) = 1$ and $\psi(-1) = \infty$.

There is a second description of the surfaces defined in **4.1**. Start again with the doecagon $\mathscr{D}$ but shift the end points of the edges of lengths ℓ by $tw\,\ell$ as shown in figure 9. Then S is obtain by the identifications

$$1\text{—}7,\ 2\text{—}12,\ 3\text{—}5,\ 4\text{—}10,\ 6\text{—}8,\ 9\text{—}11 \ . \tag{4.6}$$

Let G be the Fuchsian group generated by these identifications and let F_D be the even G-equivariant uniformizing function from the unit disk to the sphere such that F_D sends 0 to 0, sends the midpoint of the arc labeled 1 to 1 and the midpoint of the arc labeled 4 to infinity. By **4.5** this means that this function F_D is the uniformizing function giving the x-coordinate in the equation defined in **4.1**.

4.7 Lemma. *Let m be the midpoint of the arc labeled 2 in figure 9 and let n be the midpoint of the arc labeled 3. Then $F_D(m) = 4/3$ and $F_D(n) = 4$.*

Proof. We consider again the genus 5 surface S_1 and its equation of the form (4.2). The midpoints of the arcs of length ℓ and ℓ' have x coordinate in the orbits of 1 and 0 under the function f of (4.3). In particular we have the points with x coordinate $\sqrt{3} - 2$ and $-1/\sqrt{3}$. Applying the map ψ of **4.5** to these points we obtain $4/3$ and 4. The result follows from the geometric construction of the genus 2 quotient given at the beginning of this section.

The Fenchel-Nielsen coordinates of **4.1** correspond to the pants decomposition given by the arcs (2—12)∪(6—8), (4—10) and the horizontal axis in figure 9.

We also have a second pants decomposition with the same properties, namely we can consider the pants decomposition defined by the arcs (3—5)∪(9—11), (1—7) and the line joining the midpoints of arc 4 to the midpoint of arc 10. Again this can be viewed as obtained by rotating the dodecagon. Applying the formulae in [Bu], p. 38–39 and 454 we can easily compute the Fenchel-Nielsen coordinates for this second decomposition. If we let $L = \cosh(\ell)$, $Tw = \cosh(tw\,\ell)$ (note that here we use ℓ and not $\ell/2$),

$$\begin{aligned} L_2 &= Tw^2\,\frac{2L+1}{L-1} - 1 \\ Tw_2 &= \sqrt{\frac{2Tw^2L^2 + 3Tw^2L + Tw^2 - 2L^2 + 2}{4Tw^2L + 2Tw^2 - L + 1}} \\ tw_2 &= -\operatorname{sign}(tw)\operatorname{arccosh}(Tw_2)/\operatorname{arccosh}(L_2) \end{aligned} \tag{4.8}$$

then the Fenchel-Nielsen coordinates are $(2\ell_2, tw_2, \ell_2, 2tw_2, \ell_2'', 0)$ where $\cosh(\ell_2) = L_2$ and $\cosh(\ell_2''/2) = 2\cosh(\ell_2/2)$.

The function playing the role of the uniformizing function F_D for this second decomposition is clearly seen to be $z \mapsto F_D(z)/(F_D(z)-1)$. Applying the map $x \mapsto x/(x-1)$ the equation of **4.1** becomes

$$y^2 = x(x-1)\left(x^3 + a_2\, x^2 - \frac{8}{3}a_2\, x + \frac{16}{9}a_2\right) \quad \text{with} \quad a_2 = \frac{-9\,a}{9+a} \tag{4.9}$$

(cf. [Si1] (6.8)).

We are now ready to describe the action of $\mathrm{PSL}_2(\mathbb{Z})$ on the family of surfaces defined in **4.1**. We will use,

$$\begin{aligned} &\gamma_1 : (2\ell, tw, \ell, 2tw, \ell'', 0) \mapsto (2\ell, tw + \tfrac{1}{2}, \ell, 2tw+1, \ell'', 0)\\ &\gamma_2 : (2\ell, tw, \ell, 2tw, \ell'', 0) \mapsto (2\ell_2, tw_2, \ell_2, 2tw_2, \ell_2'', 0) \end{aligned} \tag{4.10}$$

where ℓ_2 and tw_2 are defined by (4.8).

The action induced by γ_2 on the a-parameter space is of course the one defined in (4.9). For γ_1 we have

4.11 Proposition. *Let S be a surface with coordinates $(2\ell, tw, \ell, 2tw, \ell'', 0)$, with $\cosh(\ell''/2) = 2\cosh(\ell/2)$. And let as in* 4.1

$$y^2 = x(x-1)\left(x^3 + a\, x^2 - \frac{8}{3}a\, x + \frac{16}{9}a\right)$$

be the associated equation for the surface. Then the surface with Fenchel-Nielsen coordinates $(2\ell, tw + \frac{1}{2}, \ell, 2tw+1, \ell'', 0)$ has for equation

$$y^2 = x(x-1)\left(x^3 + a_1\, x^2 - \frac{8}{3}a_1\, x + \frac{16}{9}a_1\right) \quad \textit{with} \quad a_1 = -9 - a\ .$$

Proof. Let as before S_1 be the genus 5 surface we have used in the construction of S. We also assume that S_1 has the equation considered in the proof of **4.1**. Consider the group of automorphism of S_1 generated by the hyperelliptic involution, the automorphism induced by f and the one induced by $x \mapsto -x$. The quotient of S_1 under this group is a sphere with 4 elliptic points one of order 6, the image of the fixed points of f and 3 of order 2 which are respectively the images of the Weierstrass points, the image of the points in the orbit under f of the fixed points of $x \mapsto -x$ and finally the image of the points in the orbit under f of the fixed points of $x \mapsto 1/x$.

Now the map

$$\varphi : x \mapsto -(\psi(x) + \psi(f(x)) + \psi(f(f(x)))) \tag{4.12}$$

(with ψ as in **4.5**) is a 12 to 1 map satisfying $\varphi(f(x)) = \varphi(-x) = \varphi(x)$. In other words it induces the quotient map from S_1 to the sphere. The image of the Weierstrass points is of course a, the image of i is 0, while $\varphi(0) = -9$ and $\varphi(1) = \infty$.

Let G be a Fuchsian group uniformizing S_1, and let $\widetilde{G}$ be the group generated by G and elliptic transformations inducing f and $x \mapsto -x$. The quotient of $\mathbb{D}$ by $\widetilde{G}$ is again the same sphere with the 4 marked points.

The same argument as the one used in the proof of **2.10** shows that replacing tw by $tw + \frac{1}{2}$ in the construction of S_1 is equivalent to replacing a by ∞ while keeping 0 and -9 fixed. This is achieved by $z \mapsto -(9+a)z/(z-a)$. Since a_1 is the image of ∞ under this map we obtain the result.

4.13 Remarks. 1) We again have ${\gamma_2}^2 = 1$ and $(\gamma_1\gamma_2)^3 = 1$. Unfortunately we do not have in this case either, a direct geometric proof. The only proof we know of follows the same lines as the proof of **2.13** and is just as ugly. In terms of the L and Tw introduced in (4.8) $\gamma_1\gamma_2\gamma_1$ and $\gamma_2{\gamma_1}^{-1}\gamma_1$ transform $(L, 1)$ into

$$\left(\frac{2\,L^2 + L + 3}{2(L-1)},\ \sqrt{\frac{4\,L^3 + 9\,L + 5}{(L^2 + L + 1)(L-1)}}\right)$$

2) As can easily be seen the fundamental dodecagon of figure 9 is globally invariant under rotation by $\pi/3$. This rotation is of course incompatible with the identification of edges used to define the genus 2 surface S. On the other hand we can construct a genus 3 double cover with an order 3 automorphism (this surface is distinct from the surface S_2 used in the construction of S). To do this consider the curve with equation

$$y^2 = P(x) = x\,(x+3)\,(x - t^2)\,\left(x - (f_3(t))^2\right)\,\left(x - (f_3(f_3(t)))^2\right) \tag{4.14}$$

f_3 as in **2.2**. If we apply the transformation $x \mapsto \frac{x}{3} + 1$ to the roots of P an elementary computation shows that the curves defined by (4.14) also have equations of the form given in **4.1**. Hence (4.14) is just another description of the family.

But now the curve defined by

$$y^2 = (x^2+3)\,(x^2 - t^2)\,\left(x^2 - (f_3(t))^2\right)\,\left(x^2 - (f_3(f_3(t)))^2\right) \tag{4.15}$$

is a genus 3 double cover with an order 3 automorphism induced by f_3. We will use this in the next section.

4.16 Examples. The group generated by $a \mapsto a_1$ and $a \mapsto a_2$ is again isomorphic to the permutation group $\mathfrak{S}_3$. Up to isomorphy we have three fixed points. The first is given by $a = -18$ and was described in [Si1] 6.8 it corresponds to $\cosh(\ell) = 1 + \sqrt{3}$, $tw = 0$.

The second is defined by $a = -\frac{9}{2}$. This corresponds to $\gamma_2\gamma_1$ of the preceding. Hence to $\cosh(\ell) = 1 + \frac{\sqrt{3}}{2}$ and $tw = -\frac{1}{2}$. Note that this curve is isomorphic to the one defined by $a = 9$, which corresponds to the transform by γ_1 of the first example.

The third is given by $a = -\frac{9}{2}(1 + i\sqrt{3})$ it corresponds to a fixed point of $\gamma_1\gamma_2$. This yields for $\cosh(\ell)$ the solution $3.1454\ldots$ of equation $8x^3 - 12x^2 - 36x - 17$ and $tw = \frac{1}{4}$.

4.17 Remark. The construction we have made at the beginning of section **3** can be generalized. Let $\mathscr{H}$ be a hexagon with side lengths $(\ell_1, \widehat{\ell_3}, \ell_2, \widehat{\ell_1}, \ell_3, \widehat{\ell_2})$ and assume that $\cosh(\ell_3) = \cosh(\ell_1) + \cosh(\ell_2) + 1$. Then $\widehat{\ell_3} = \widehat{\ell_1} + \widehat{\ell_2}$. Following a construction similar to the one made at the beginning of section **3** we find that the surfaces with Fenchel-Nielsen coordinates $(2\ell_1, tw_1, 2\ell_2, tw_2, 2\ell_3, \frac{1}{2})$, with $\cosh(\ell_3) = \cosh(\ell_1) + \cosh(\ell_2) + 1$, have a non-hyperelliptic automorphism of order 2 (computation of this type are made in the second part of section 8 of [Bu-Si1]).

Our last example is obtained by taking $\cosh(\ell) = 7$ and $tw = \frac{1}{2}$. Since we have $7 = 2+4+1$ we are in the situation described in the above remark and the surface has a non-hyperelliptic automorphism of order 2. This implies certain relations between the x-coordinates of the Weierstrass points. In this concrete situation we can be very explicit although the arguments involved are rather lengthy. To summarize them denote by q_1 the upper end point of the arc labeled 1 in figure 9, denote by q_2 the other end point of the arc labeled 2 and finally denote by q_3 the remaining end point of the arc labeled 3. Consider the uniformizing map F_D introduced earlier and write $\alpha_i = F_D(q_i)$. The condition is then $\alpha_1(\alpha_2 - \alpha_3) = \alpha_3(\alpha_2 - 1)$ (this is quite similar to the conditions introduced in [Si1] 4.1). Call s_1, s_2, s_3 the 3 solutions of $\left(x^3 + a\,x^2 - \frac{8}{3}a\,x + \frac{16}{9}a\right) = 0$. Then depending on the order chosen for the s_i the equation

$$\frac{s_1(s_2 - s_3)}{s_3(s_2 - 1)} = 1$$

has the 3 possible solutions $a = 0$, $a = -8$ or $a = 9/80$. The solution $a = 0$ is of course excluded. The solution $a = -8$ does not lead to a curve with a non-trivial automorphism. The way to prove this is rather cumbersome and the simplest is the rather brutal method that consists in computing the 120 possible equations of the form $y^2 = x(x-1)P(x)$, with $\deg(P) = 3$ and checking that in the list obtained there are no repetitions. Hence we are left with the sole possibility $a = 9/80$. This has some consequences, one worth mentioning is that the list of approximate values given in (8.6) of [Bu-Si1] are in fact exact.

5. Translation surfaces obtained from 3 squares.

In this section we want to link the family introduced in **4.1** with surfaces in the $\mathrm{SL}_2(\mathbb{R})$ orbit of the translation surface tiled by 3 squares.

But before that we need to recall a few facts. Let C be a real genus 2 curve with three real components. Then it always admits an equation of the form,

$$(5.1) \quad y^2 = P(x) = x(x-0)(x-a_1)(x-a_2)(x-a_3) \quad \text{with} \;\; 1 < a_1 < a_2 < a_3 \,.$$

Let γ_1 be the pull back in C of $[0, 1]$, γ_2 the pull back of $[1, a_1]$ and so on up to γ_5 the pull back of $[a_3, \infty]$. Finally let γ_6 be the pull back of $[-\infty, 0]$. The γ_i are simple closed curves in C but to obtain cycles we need to orient them. For this we do the following.

Since P is non zero in the upper half plane $\mathbb{H}$, and the latter is simply connected, we can choose on $\mathbb{H}$ a determination of the square root $\sqrt{P(x)}$. Obviously we can extend this determination to $\mathbb{R}$. We take the one which is positive on $[0, 1]$. It will then be negative on $[a_1, a_2]$ and positive on $[a_3, \infty]$. It will also be pure imaginary with positive imaginary part on $[0,1]$, pure imaginary with negative imaginary part on $[a_1,a_2]$ and be pure imaginary with negative imaginary part on $[-\infty,0]$.

With this determination of the square root we choose on the γ_i the orientation defined by the map $x\mapsto(x,\sqrt{P(x)})$ and x increasing. The intersection numbers at the roots of P are now easy to compute they are $(\gamma_k,\gamma_{k+1})=1$ (k mod 6). As a consequence we have

$$\gamma_3=-\gamma_1-\gamma_5 \quad \text{and} \quad \gamma_6=-\gamma_2-\gamma_4\ . \tag{5.2}$$

We generalize this convention, let a_1, a_2 and a_3 be distinct complex numbers different from 0 and 1. Let α_1 be a simple arc in the complex plane joining 0 and 1 and not passing through any of the a_i. Let α_2 join 1 and a_1 and not passing through the the other a_i and intersecting α_1 only in 1. Construct in the same way α_3 from a_1 to a_2, α_4 from a_2 to a_3, α_5 from a_3 to ∞ and α_6 from 0 to ∞ so the α_i only intersect in one point. Let γ_i be the pull back in C of α_i. We can choose on each α_i a determination of the square root $\sqrt{P(x)}$ so that the induced orientation on the γ_i is such that the intersection numbers are $(\gamma_k,\gamma_{k+1})=1$ (k mod 6). This is the convention we will use when dealing with both

$$\int_{\gamma_i}\frac{a\,dx+b\,x\,dx}{y} \quad \text{or} \quad \int_{\alpha_i}\frac{a\,dx+b\,x\,dx}{\sqrt{P(x)}}\ . \tag{5.3}$$

Note that with this convention we again have (5.2).

5.4 Proposition. *Let C be the curve defined by*

$$y^2=P_a(x)=x(x-1)\left(x^3+a\,x^2-\frac{8}{3}a\,x+\frac{16}{9}a\right)\ .$$

Then for some positive real λ, $\left(C,\lambda\,\dfrac{x\,dx}{y}\right)$ is a translation surface in the $\mathrm{SL}_2(\mathbb{R})$ orbit of the translation surface tiled by 3 squares and holomorphic differential dz.

In particular such surfaces are in the family **4.1**.

Proof. Let C be the curve defined by $y^2=P_a(x)$. Let $\{0,1,a_1,a_2,a_3\}$ be the roots of P_a. By **4.13** we know that applying $x\mapsto 3\,(x-1)$ to this set yields a set of the form $\{-3,0,{t_0}^2,f_3(t_0)^2,f_3(f_3(t_0))^2\}$ (f_3 as in **2.2**).

We need to be a little more precise. Let a_1 be the image under F_D of the upper end point of arc 1 in figure 9, a_2 the image of the upper endpoint of arc 2 and a_3 the upper end point of arc 3. Then choose t_0 so that a_1 is mapped to ${t_0}^2$, a_2 to $f_3(f_3(t_0))^2$ and a_3 to $f_3(t_0)^2$.

On the curve C_1 defined by $w^2=z\,(z+3)\,(z-{t_0}^2)\,(z-f_3(t_0)^2)\,(z-f_3(f_3(t_0))^2)$ the differential $\dfrac{x\,dx}{y}$ becomes, up to multiplication by $\sqrt{3}$,

$$\omega_1=3\,\frac{dz}{w}+\frac{z\,dz}{w}\ .$$

Let C_2 be the genus 3 double cover of C_1 defined by $w^2=(z^2+3)\,(z^2-{t_0}^2)\,(z^2-f_3(t_0)^2)\,(z^2-f_3(f_3(t_0))^2)$. The quotient map from C_2 to C_1 is defined by $(z,w)\mapsto(z^2,z\,w)$. From this it follows that the differential ω_1 lifts to

$$\omega_2=2\left(3\,\frac{dz}{w}+\frac{z^2\,dz}{w}\right)\ . \tag{5.5}$$

Now C_2 has the order 3 automorphism induced by f_3 and defined by

$$\varphi_3 : (z, w) \mapsto \left(\frac{3+z}{1-z}, \frac{16\, w}{(1-z)^4} \right) . \tag{5.6}$$

It is readily checked that ω_2 is an invariant differential under this automorphism.

Let α be simple arc in the z-plane joining t_0 and $-t_0$ passing through 0 but not through $\pm i\sqrt{3}$, $\pm f_3(f_3(t_0))$ or $\pm f_3(t_0)$. Call α_1 the pull back in C_2 of α and let α_2 the image of α_1 under f_3 and α_3 the image of α_2. Similarly let d be a half line from $f_3(t_0)$ to infinity and not passing through any of the points $\pm t_0$,$\pm f_3(f_3(t_0))$ or $-f_3(t_0)$. Call β_1 the pull back of $d \cup -d$ and define β_2 and β_3 in the same way as α_2 and α_3. Since the differential ω_2 is φ_3 invariant, we have

$$\int_{\alpha_1} \omega_2 = \int_{\alpha_2} \omega_2 = \int_{\alpha_3} \omega_2 \quad \text{and} \quad \int_{\beta_1} \omega_2 = \int_{\beta_2} \omega_2 = \int_{\beta_3} \omega_2 \, . \tag{5.7}$$

Call γ_1 the image in C of α_1, γ_2 the image of β_3, γ_3 the image of α_2 and finally γ_4 the image of β_1. Choose, as in the beginning of this section, an orientation on the γ_i such that the intersection number of γ_k and γ_{k+1} is 1.

Now the map from C_2 to C_1 is two to one on α_1 and β_1 but one to one on α_2 and α_3. From this, (5.2) and (5.7) we conclude that

$$\begin{aligned} &\int_{a_1}^{a_2} \frac{x\,dx}{y} = -2 \int_0^1 \frac{x\,dx}{y} = -2 \int_{a_3}^{\infty} \frac{x\,dx}{y} \quad \text{and} \\ &\int_{a_2}^{a_3} \frac{x\,dx}{y} = -2 \int_1^{a_1} \frac{x\,dx}{y} = -2 \int_{\infty}^{0} \frac{x\,dx}{y} \, . \end{aligned} \tag{5.8}$$

In other words the situation is the one described on the right of figure 10 and this ends the proof.

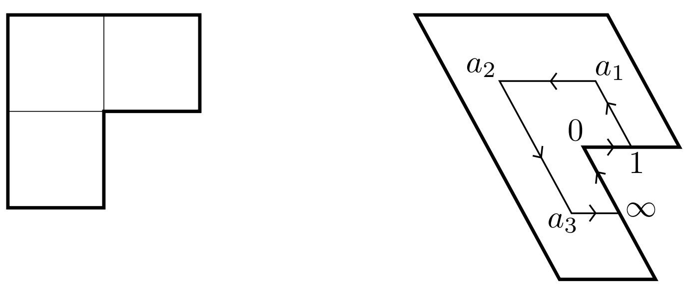

Figure 10

5.9 Remark. Our next objective is to identify the transformations considered in section **4** in the context of translation surfaces. To formulate our results we will need to use a slightly different action of $\mathrm{SL}_2(\mathbb{R})$ than the one normally considered. Let the translation surface S be obtained by pasting three copies of a parallelogram with coordinates 0, z_1, z_2 and z_1+z_2 in $\mathbb{C}$ (see figure 11). Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$, then the image of S under A for the action we want to consider is the surface S' obtained from the parallelogram 0, w_1, w_2 and $w_1 + w_2$, where $\left(\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right) = A \left(\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right)$. When $z_1 = 1$ and $z_2 = i$ this action is just the transpose of the standard action.

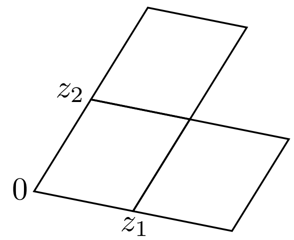

Figure 11

For the second transformation described in (4.10) we have a relatively easy description. Consider the fundamental dodecagon for the surface. The circuit we have used to obtain the integrals of (5.8) and hence the circuit shown in figure 10 is illustrated on the left of figure 12.

If we let

$$\alpha = \int_0^1 \frac{x\,dx}{y} \quad \text{and} \quad \beta = \int_1^{a_1} \frac{x\,dx}{y}$$

then by (5.8) the sequence of integrals along the arcs of the circuit will be α, β, $-2\,\alpha$, $-2\,\beta$, α and β.

Now consider the circuit indicated on the right of figure 11. Taking into account the identifications (4.6) we find for the values of the integrals along this circuit (starting again in 0) are,

$$-\beta,\ \alpha,\ 2\,\beta,\ -2\,\alpha,\ -\beta,\ \alpha\ . \tag{5.10}$$

But as explained in section **4** the transformation $(\ell, tw) \mapsto (\ell_2, tw_2)$ of (4.10) corresponds precisely to the rotation bringing the point p_6 to the horizontal axis. Hence the transformation corresponds to $(z_1, z_2) \mapsto (w_1, w_2)$, where

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\ . \tag{5.11}$$

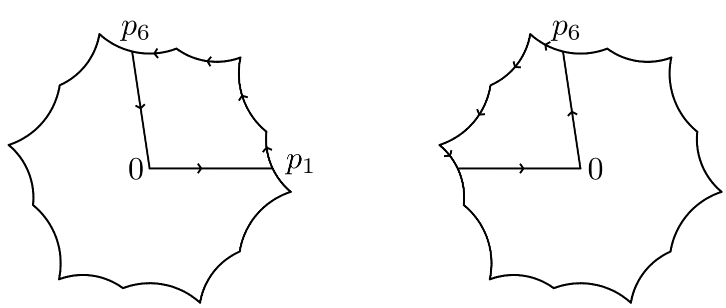

Figure 12

The description of the transformation $(\ell, tw) \mapsto (\ell, tw + \frac{1}{2})$ is a little more involved. To make the arguments easier to follow we are going to change models on both sides. For the translation surface this is easy we cut two of the parallelograms forming the L shaped polygon and re-paste using the identifications. We obtain in this way a star shaped polygon as illustrated on the right of figure 11. The identifications are now by opposite sides.

For the hyperbolic model, we use the octagon with Fenchel-Nielsen coordinates $(\ell'', 0, \ell, 2\,tw, 2\,\ell, tw)$ and pants decomposition (1.1).

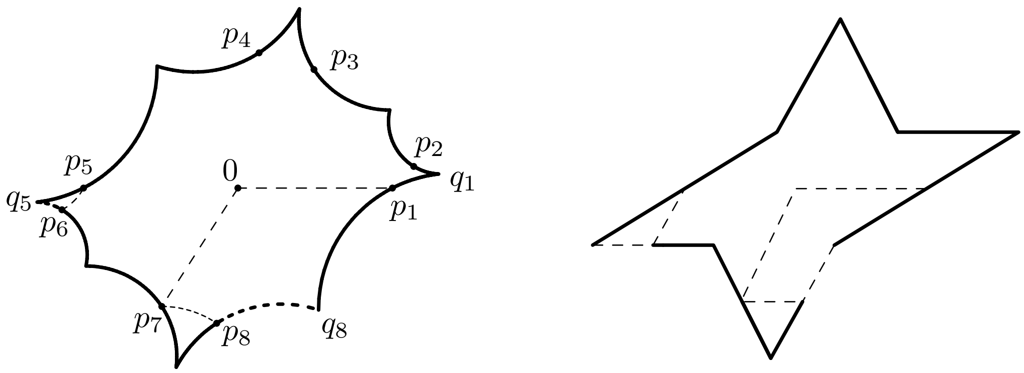

Figure 13

Now consider the uniformizing function F of (1.4). Let $b_1 = F(p_2)$, $b_2 = F(q_1)$ and $b_3 = F(p_4)$ (see figure 13). In general the b_i are distinct from the a_i introduced in the proof of **5.4**. Let

$$tr : z \mapsto \frac{b_1 - b_3}{b_1 - b_2} \cdot \frac{z - b_2}{z - b_3}$$

and let $F_1 = tr \cdot F$. Then

$$(5.12) \qquad F_1(q_i) = 0,\ F_1(p_2) = F_1(p_6) = 1 \text{ and } F_1(p_4) = F_1(p_8) = \infty\ .$$

Moreover comparing lengths one can check that

$$(5.13) \qquad F_1(0) = a_2,\ F_1(p_1) = F_1(p_5) = a_1 \text{ and } F_1(p_3) = F_1(p_7) = a_3\ .$$

In particular the integration circuit we have been using is the one illustrated by doted arcs on the left of figure 13 and starts at q_5.

We want now to describe the transformation

$$(5.14) \qquad (\ell'', 0, \ell, 2\,tw, 2\,\ell, tw) \mapsto (\ell'', 0, \ell, 2\,tw + 1, 2\,\ell, tw + \frac{1}{2})$$

in terms of octagons and Fuchsian groups.

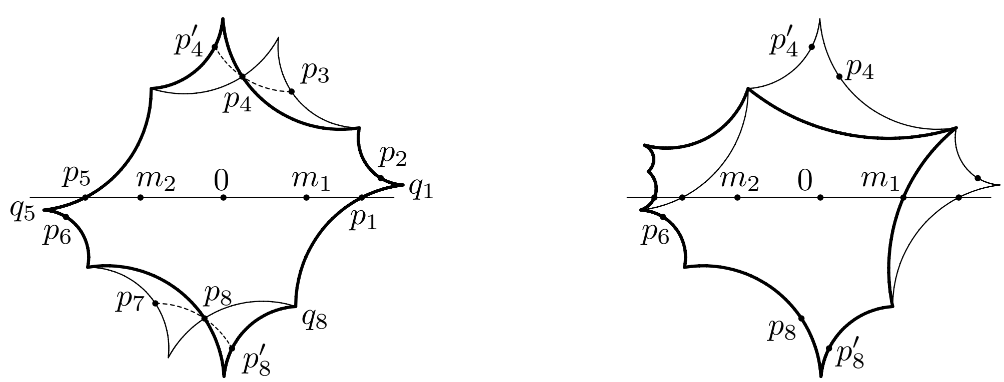

Figure 14

We split the transformation (5.14) in two. In the first step we transform the initial octagon into the one for the coordinates $(\ell'', 0, \ell, 2\,tw + 1, 2\,\ell, tw)$. This is shown on the left of figure 14. We complete by the action of a half twist along the horizontal geodesic $[p_5, p_1]$ to obtain an octagon representing $(\ell'', 0, \ell, 2\,tw + 1, 2\,\ell, tw + \frac{1}{2})$ (see the right of figure 14).

Call G the group generated by,

$$g_1 = h_0 \cdot h_{p_5}\,,\ g_2 = h_0 \cdot h_{p_6}\,,\ g_3 = h_0 \cdot h_{p_7}\ , g_4 = h_0 \cdot h_{p_8}$$

Where as before h_p is the elliptic transformation of order 2 centered at p. Let $p'_8 = h_{p_8}(p_7)$ and let

$$g'_4 = h_0 \cdot h_{p'_8}\,,\ g''_2 = h_{m_2} \cdot h_{p_6}\,,\ g''_3 = h_{m_2} \cdot h_{p_8}\,,\ g''_4 = h_{m_2} \cdot h_{p'_8}$$

Call G' the group generated by g_1, g_2, g_4 and g'_4 and call G'' the group generated by g_1, g''_2, g''_3 and g''_4. The groups G, G' and G'' are the groups identifying the opposite edges of the three octagons of figure 14.

Let C be the algebraic curve of equation $y^2 = P_a(x)$, P_a as in **5.4**, and associated to the hyperbolic surface by **4.1**. The function F_1 we have introduced in (5.13) and (5.14) is the uniformizing function for the group G, giving the x-coordinate. Let π_1 be the G-uniformizing map from the unit disk to C such that the x-coordinate of $\pi_1(z)$ is $F_1(z)$ and such that the induced orientation on the arcs along which we integrate coincides with the one we have used above.

5.15 Lemma. *Let*

$$\omega = \pi_1^* \left(\frac{x\,dx}{y} \right)\ .$$

Then ω is an invariant differential for the three groups G, G' and G''.

PROOF. For G this is by definition. For G' all we need to note is that p'_4 being in the orbit of p_8 under G, $h_{p'_8}$ induces the hyperelliptic involution. Hence ${h_{p'_8}}^*(\omega) = -\omega$. Since h_0 acts in the same way we have the invariance under g'_4.

We have $F_1(0) = a_2$ and $F_1(p_1) = F_1(p_5) = a_1$. On the other hand m_1 and m_2 are the midpoints of $[0, p_1]$ and $[p_5, 0]$. By **4.7** we conclude that $F_1(m_1) = F_1(m_2) = 4/3$.

Now consider the curve C_2 of genus 3 defined in the proof of **5.4** and let $\varphi : C_2 \to C$ be the map defined by

$$\varphi : (x, y) \mapsto \left(\frac{x^2}{3} + 1, \frac{x\,y}{9\,\sqrt{3}} \right)\ .$$

Note for further use that the image of the points with x-coordinate ± 1 in C_2 are the points with x-coordinate $4/3$ in C.

Let π_2 be the map from the unit disk to C_2 such that $\varphi \cdot \pi_2 = \pi_1$. From the above we obtain that $\pi_2(m_2)$ is a point with x-coordinate ± 1 in C_2.

Let $f : (x, y) \mapsto (-x, y)$ in C_2 and let $\omega_2 = \varphi^* \left(\frac{x\,dx}{y} \right)$. Up to multiplication by a scalar this is the same as the ω_2 of (5.5) and from this we obviously have $f^*(\omega_2) = -\omega_2$. Also since ω_2 is φ_3 invariant (φ_3 as in (5.6)) we have $\left(\varphi_3^{-1}\, f\, \varphi_3\right)^* (\omega_2) = -\omega_2$. But since $\pi_2(m_2)$ is a fixed point of $\varphi_3^{-1}\, f\, \varphi_3$, this involution is induced by h_{m_2}.

Hence $h^*_{m_2}(\pi^*_2(\omega_2) = -\pi^*_2(\omega_2)$ or since by definition of ω_2 and π_2, $\pi^*_2(\omega_2) = \omega$, we have

$$h^*_{m_2}(\omega) = -\omega \ .$$

With the same argument as for G' we conclude that ω is G'' invariant.

If

$$\alpha = \int_{q_5}^{p_6} \omega \ \text{ and } \ \beta = \int_{p_6}^{p_5} \omega$$

then by (5.8), (5.12) and (5.13) we also have

$$\int_{p_1}^{0} \omega = -2\,\alpha, \ \int_{0}^{p_7} \omega = -2\,\beta, \ \int_{p_7}^{p_8} = \alpha, \ \int_{p_8}^{q_8} \omega = \beta \ .$$

5.16 Lemma. *We have*

$$\int_{p_6}^{p'_5} \omega = \beta - \alpha, \ \int_{m_1}^{m_2} \omega = -2\,\alpha, \ \int_{m_2}^{p_8} \omega = 2\,\alpha - 2\,\beta, \ \int_{p_8}^{p'_8} \omega = \alpha, \ \int_{p'_8}^{q_8} \omega = \beta - \alpha$$

(see figure 15).

Proof. The arguments used in the proofs of **5.4** and **5.15** show that

$$\int_{m_1}^{0} \omega = \int_{0}^{m_2} \omega = \int_{m_2}^{p_5} \omega = \int_{p_5}^{p'_5} \omega = \frac{1}{2}\int_{p_1}^{0} \omega = -\alpha \ .$$

This proves the assertion about the integral between m_1 and m_2. Moreover since we clearly have

$$\int_{p_6}^{p'_5} \omega = \int_{p_6}^{p_5} \omega + \int_{p_5}^{p'_5} \omega = -\alpha + \beta$$

the assertion for the integral between p_6 and p'_5 is also proved.

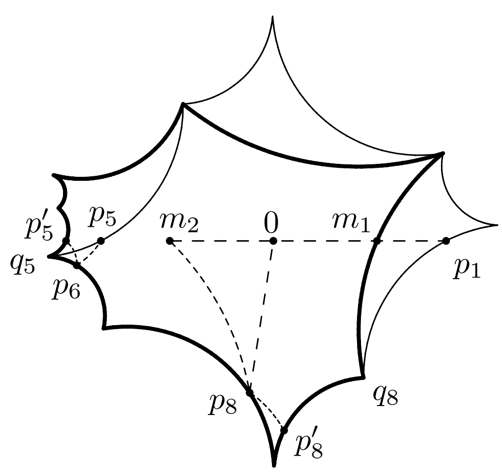

Figure 15

For obvious reasons (see figures 14 and 15) we have

$$\int_{p_8}^{p'_8} \omega = \int_{p_7}^{p_8} \omega, \ \int_{m_2}^{p_8} \omega = \int_{0}^{p_8} \omega - \int_{0}^{m_2} \omega \ \text{ and } \ \int_{p'_8}^{q_8} \omega = \int_{p_8}^{q_8} \omega - \int_{p_7}^{p_8} \omega \ .$$

This proves the remaining three assertions of the lemma.

Calling S the surface $\mathbb{D}/G$ an S'' the surface $\mathbb{D}/G''$ we have shown that the values of the integrals of ω along the circuits are respectively

$$\alpha,\ \beta,\ -2\,\alpha,\ -2\,\beta,\ \alpha,\ \beta$$
$$\alpha,\ \beta-\alpha,\ -2\,\alpha,\ 2\,\alpha-2\,\beta,\ \alpha,\ \beta-\alpha\ .$$

In other words (S'',ω) is the image of (S,ω) under $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ where the action is the one indicated in remark **5.9**.

Summarizing we have proved

5.17 Theorem. *Let (S,ω) be the translation surface associated to the curve $y^2 = P_a(x)$ as in* **5.4**. *Then the surface S has Fenchel-Nielsen coordinates of the form $(2\,\ell, tw, \ell, 2\,tw, \ell'', 0)$ with $\cosh(\ell''/2) = 2\,\cosh(\ell/2)$.*

Moreover if (S',ω') is the transform of (S,ω) under $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ for the action described in **5.9***, then S' has Fenchel-Nielsen coordinates $(2\ell, tw-\frac{1}{2}, \ell, 2\,tw-1, \ell'', 0)$. If (S'',ω'') is the image of (S,ω) under $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ then the Fenchel-Nielsen coordinates are $(2\ell_2, tw_2, \ell_2, 2\,tw_2, \ell''_2, 0)$ where ℓ_2 and tw_2 are defined by* (4.8).

5.18 Examples. We can identify in terms of translation surfaces the first three examples of **4.16**. The first, which has an automorphism of order 4 is obviously the surface defined by three squares itself. The second is the transform of the first under $\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$ (action as defined in **5.9**) or equivalently under $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$.

For the third example we note that $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ must act as a rotation on the L-type polygon. From this it is not hard to see that the corresponding surface is the one tiled by 6 equilateral triangles.

6. Conclusion and questions.

There are other families to which we can apply the methods developed in sections **2**, **3** and **4**. For instance one can apply the transformations of Aline Aigon to the family of surfaces with an order 3 automorphism and proceed as in section **3**. We can also do this for transforms of the family with an order 4 automorphism we have not considered in this paper. Again for all these we will have an action of $\mathrm{PSL}_2(\mathbb{Z})$ generated by half twists. The first obvious question is to describe these also as translation surfaces. For real curves with 3 real components and an order 4 automorphism there is an easy interpretation in terms of "Swiss crosses" (see [Mc]), this will be developed in a forthcoming paper.

At the other end we have $\mathrm{SL}_2(\mathbb{R})$ orbits of surfaces tiled by squares and one of the natural questions is: is the action of $\mathrm{SL}_2(\mathbb{Z})$ also generated by half twists?

In the other direction one can also ask the following. If (S,ω) is a translation surface and (S',ω') is the transform of (S,ω) under $U \in \mathrm{SL}_2(\mathbb{R})$ can one express the Fenchel-Nielsen coordinates of S' in terms of those of S?

In fact we can generalize further and consider primitive Teichmüller disks and ask about the hyperbolic counterpart of the actions of $\mathrm{SL}(S,\omega)$ and $\mathrm{SL}_2(\mathbb{R})$.

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