A FUNCTIONAL RELATION FOR ACCESSORY PARAMETERS FOR GENUS 2 ALGEBRAIC CURVES WITH AN ORDER 4 AUTOMORPHISM

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0. Introduction.

Let C be a hyperelliptic algebraic curve of genus $g \geqslant 2$ and let

$$y^2 = \prod_{i=1}^{n}(x - a_i), \quad n = 2g+2 \text{ or } 2g+1 \tag{0.1}$$

be an equation for C. Then the conformal universal cover of C is the unit disk $\mathbb{D}$. Let $F : \mathbb{D} \to \mathbb{P}^1(\mathbb{C})$ be the uniformizing map yielding the x coordinate in C and let φ be a local inverse. Then it is well known that φ satisfies a differential equation of the form,

$$\left(\frac{\varphi''}{\varphi'}\right)' - \frac{1}{2}\left(\frac{\varphi''}{\varphi'}\right)^2 = \sum\left(\frac{3}{8}\frac{1}{(x-a_i)^2} + \frac{\mathfrak{b}_i}{x-a_i}\right) , \tag{0.2}$$

where the $\mathfrak{b}_i$'s are the so called accessory parameters. In the general case the $\mathfrak{b}_i$ satisfy three (resp. two) relations if $n = 2g+2$ (resp. $n = 2g+1$) (see [Ne] chap. V for more details). If the curve defined by (0.1) has extra automorphisms the quadratic-differential behaviour of the Schwarzian derivative on the left hand side of (0.2) induces extra relations. Unfortunately except in special cases, these extra relations are still not enough to be able to compute the $\mathfrak{b}_i$'s (see [Gi-Go] for a lucid treatment of this question). In fact, although they have been studied for over a century, these accessory parameters are largely mysterious and to quote Nehari "the determination of the $n-3$ independent constants [...] is an exceedingly difficult task".

In [Ai] Aline Aigon-Dupuy, generalizing results of [Bu-Si1], has shown the existence of natural hyperbolic transformations that can also be described on the level of equations. In this paper we will show that the transformations induced on the $\mathfrak{b}_i$'s can also be described. For curves with an order 4 automorphism the result we obtain is the following.

Let C be a genus 2 hyperelliptic algebraic curve defined by

$$y^2 = x(x^2-1)(x-\alpha)(x-1/\alpha) . \tag{0.3}$$

Let $\mathfrak{b}_1$, $\mathfrak{b}_2$, $\mathfrak{b}_3$, $\mathfrak{b}_4$ and $\mathfrak{b}_5$ be the accessory parameters associated respectively to -1, 0, 1, α and $1/\alpha$. The standard relations on the $\mathfrak{b}_i$'s (see (4.1) and (4.2)) imply in this case that

$$\mathfrak{b}_1 = -\mathfrak{b}_3 = \tfrac{3}{8},\ \mathfrak{b}_2 = -\mathfrak{b}_4 - \mathfrak{b}_5 \ \text{ and } \ \mathfrak{b}_5 = -\tfrac{1}{4}\,\alpha\,(3 + 4\alpha\mathfrak{b}_4)\ . \tag{0.4}$$

From this it follows that for curves with equations of the form (0.3) the accessory parameters only depend on α and $\mathfrak{b}_4$.

For such curves our result is (see section **4**),

Theorem. *With notations as above let* $\mathfrak{B}(\alpha) = \mathfrak{b}_4\,\alpha\,(\alpha^2 - 1)$. *Then,*

$$\mathfrak{B}(\sqrt{1-\alpha^2}) = -\tfrac{3}{8} - \mathfrak{B}(\alpha)\ .$$

Similar relations exist for genus 2 curves with smaller automorphisms groups but although the computations can fairly easily be achieved using a symbolic computation software, their size prohibits from presenting them here. For this reason we will only briefly present some results in the more general setting.

It has been known for some time, that for theoretical reasons, Whittaker's conjecture [Wh] concerning accessory parameters cannot be true in general. On the other hand explicit counter examples are hard to come by. Numerical counter examples have been given in [Ch-Ch], and in [Gi-Go] Girondo and González present an explicit counter example in genus 3. One of the consequences of the above theorem is that it yields an infinite family of counter examples in genus 2.

1. Fundamental octagons and equations for genus 2 surfaces with automorphisms.

Let S be a genus 2 hyperbolic surface with, in addition to the hyperelliptic involution τ, an orientation preserving involution φ. From a geometric point of view such surfaces can easily be constructed. Start with a symmetric pair of pants, that is one with two boundary components of equal length. Then paste this pair of pants to a second copy using the same twist parameter for the components of equal lengths. In this way we obtain a surface with Fenchel-Nielsen coordinates $(l_1, tw_1, l_1, tw_1, l_2, tw_2)$. The symmetry of the pairs of pants extends to a symmetry of the surface. The Weierstrass points being on the boundary components this symmetry is necessarily distinct from the hyperelliptic involution.

The surface S has a second non-trivial involution $\psi = \tau \circ \varphi$. Both φ and ψ have two fixed points respectively n_1, n_2 and m_1, m_2. Since the fixed points n_i of φ are not on the boundary of the pants it is easily checked that the fixed points m_i of ψ are on the boundary component of length l_2 of the pants decomposition.

By the method developed in [Bu-Si2] section 2 we can associate to the Fenchel-Nielsen coordinates $(l_1, tw_1, l_1, tw_1, l_2, tw_2)$ a fundamental octagon for the surface. The one we will consider is described in figure 1 where γ_1, union of the geodesic arcs $[p_1, p_8] \cup [p_4, p_5]$, and γ_2, union of the geodesic arcs $[p_2, p_3] \cup [p_7, p_6]$, are of length l_1 while γ_3, the geodesic arc $[q_5, q_1]$, is of length l_2 and the side identification is the identification of opposite sides. The Weierstrass points are 0, the p_i's and the q_i's. Since the fixed points m_1 and m_2 are on γ_3 they must exchange the two Weierstrass points on γ_3, hence in the octagon they are respectively the hyperbolic midpoints of 0 and q_1 and 0 and q_5 (see figure 2).

From this it is also easily seen that φ exchanges p_1 and p_2. Hence m_1 is also the midpoint of p_1 and p_2. The construction of the other points m_2, n_1 and n_2 is similar and described in figure 2.

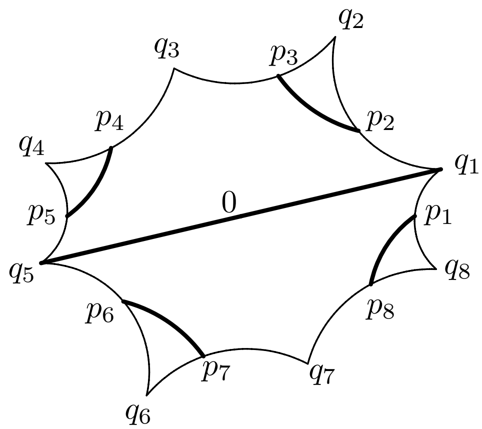

Figure 1

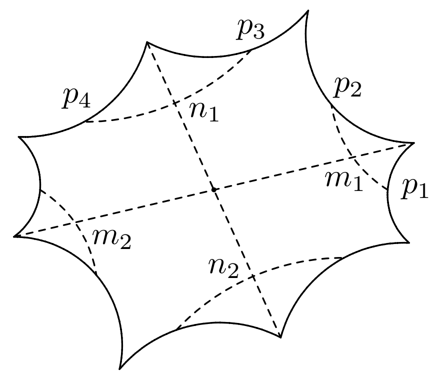

Figure 2

We now consider the uniformizing function F described in [Bu-Si2] proposition (1.4). The algebraic curve C_1 defined by S admits for equation $y^2 = x(x-1)(x-a_1)(x-a_2)(x-a_3)$ where,

$$F(0) = 0,\ F(p_1) = F(p_5) = 1,\ F(p_3) = F(p_7) = \infty, \tag{1.1}$$
$$F(q_i) = a_1,\ F(p_2) = F(p_7) = a_2 \text{ and } F(p_4) = F(p_8) = a_3\ .$$

Now φ exchanges 0 and q_1 and p_3 with a point in the G-orbit of p_8. This means that on the algebraic curve C_1 the involution φ is induced by $x \mapsto \dfrac{x-a_1}{x/a_3 - 1}$. But since φ also exchanges p_1 and p_2 we have the relation

$$a_2 = a_3 \frac{a_1 - 1}{a_3 - 1}\ . \tag{1.2}$$

On the other hand we can always find for S an algebraic equation of the form

$$y^2 = P(x) = (x^2 - a^2)(x^2 - 1)(x^2 - b^2)\ , \tag{1.3}$$

where a and b are complex numbers and a^2 and b^2 are distinct from each other and from 0 and 1. Passing from (1.3) to (1.1) is done via the transformation

$$tr : z \mapsto \frac{a-b}{a-1} \cdot \frac{z+1}{z+b} \tag{1.4}$$

which in turn gives,

$$a_1 = \frac{a-b}{a-1} \cdot \frac{2}{1+b},\ a_2 = \frac{a-b}{a-1} \cdot \frac{a+1}{a+b} \text{ and } a_3 = \frac{a-b}{a-1} \cdot \frac{1+b}{2b}\ . \tag{1.5}$$

Conversely this system can always be solved because of (1.2). In fact there will be two sets of solutions if (a, b) is one $(1/a, 1/b)$ is another.

In terms of equation (1.3) φ is simply induced by $z \mapsto -z$ and m_1 will be mapped by $F_s = tr^{-1} \circ F$ to 0 or ∞. Because of the remark made on the solutions of (1.5) there is an arbitrary choice involved here. We make the convention that (a, b) have been chosen such that m_1 is mapped to 0. In which case n_1 is mapped to ∞. This fixed we find that,

$$F_s(q_i) = -F_s(0) = 1,\ F_s(p_2) = -F_s(p_1) = a \text{ and } F_s(p_4) = -F_s(p_3) = b\ . \tag{1.6}$$

2. Action of half-twists on equations.

We summarize here some of the results of [Ai] generalizing results of [Bu-Si1] that will be needed in the sequel. Let S be a genus 2 surface with a non-trivial involution φ, as in section **1**, and let γ be a simple closed geodesic passing through two Weierstrass points and the two fixed points of φ or ψ. Then one can produce a new surface S' by cutting S along γ and repasting with a rotation of angle π of one of the sides of γ. We will call such a transformation a half-twist along γ.

We are going to give a more precise description in terms of octagons for a special case that we will use in the sequel.

Let the surface S (with the additional symmetry) be represented by an octagon $\mathscr{P}$ as in figure 1 and let G be the fuchsian group identifying opposite sides of the octagon (and where of course we ask that $S = \mathbb{D}/G$). Let γ be the simple closed geodesic obtained by taking the union of the geodesic arcs $[p_3, p_4]$ and $[p_8, p_7]$. Transforming by a half-twist along γ consists in replacing q_3 by q_3' so that the midpoint of $[q_2, q_3']$ is the midpoint n_1 of $[p_3, p_4]$ and replacing q_7 by q_7' so that the midpoint of $[q_6, q_7']$ is the midpoint n_2 of $[p_7, p_8]$ (see figure 3).

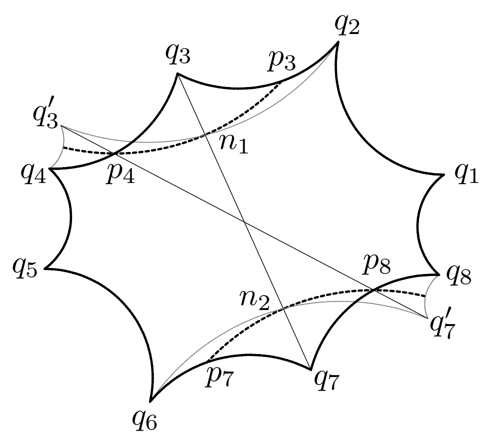

Figure 3

Call h_p be the elliptic transformation of order 2 with center p. We can reformulate the above by saying that

$$q_3' = h_{n_1}(q_2) \quad \text{and} \quad q_7' = h_{n_2}(q_6) \ .$$

Now since n_1 is the midpoint of $[p_3, p_4]$ we have the relation $h_{p_4} h_{n_1} h_{p_3} = h_{n_1}$ and by the remarks made in section **1** we have

$$h_{p_3}(q_2) = q_3, \ h_{n_1}(q_3) = 0 \quad \text{and hence} \quad q_3' = h_{p_4}(0) \ .$$

Hence p_4 is the midpoint of $[0, q_3']$ and the midpoint of $[q_3', q_4]$ is $h_{p_4}(n_1)$ which is in the G-orbit of n_2. In the same way p_8 is the midpoint of $[0, q_7']$ and the midpoint of $[q_7', q_8]$ is $h_{p_8}(n_2)$ (which is in the G-orbit of n_1).

Let G be as before the group identifying opposite sides of the octagon with vertices $q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8$ and let G' be the group identifying opposite sides of the new octagon with vertices $q_1, q_2, q_3', q_4, q_5, q_6, q_7', q_8$, we then have

2.1 Lemma. *Let $\widetilde{G}$ be the extended group generated by G and the elliptic transformations h_0 and h_{n_1}, similarly let $\widetilde{G'}$ be the group generated by G', h_0 and h_{p_4}. Then $\widetilde{G} = \widetilde{G'}$.*

PROOF: Call g_1 the transformation identifying the side with vertices q_4, q_5 with the side with vertices q_8, q_1. We have $g_1 \circ h_0 = h_{p_1}$ and similarly $h_0 \circ g_1 = h_{p_5}$ (where as in figure 1 p_1 is the midpoint of $[q_8, q_1]$ and p_5 is the midpoint $[q_4, q_5]$). From this it easily follows that $\widetilde{G}$ is generated by the order 2 elliptic transformations h_{p_i} centered at the p_i's, h_0 and h_{n_1}. Since $h_{n_2} = h_0 h_{n_1} h_0$ and, as noted above n_1 (resp. n_2) is the midpoint of $[q_2, q_3']$ (resp. $[q_6, q_7']$). The result easily follows.

2.2 Proposition (Aline Aigon-Dupuy [Ai]). *Let G and G' be as above. Let $\alpha = a^2$ and $\beta = b^2$ (with a and b as in (1.6)) and let*

$$y^2 = (x^2 - \alpha)(x^2 - 1)(x^2 - \beta) \tag{2.2.1}$$

be the equation described in (1.3) for the algebraic curve defined by $S = \mathbb{D}/G$, then,

$$y^2 = (x^2 - \alpha')(x^2 - 1)(x^2 - \beta') \tag{2.2.2}$$

where,

$$\alpha' = \frac{\alpha(1-\beta)}{\alpha - \beta} \quad \text{and} \quad \beta' = 1 - \beta \ ,$$

is an equation for the algebraic curve defined by $S' = \mathbb{D}/G'$.

PROOF: This is a special case of a more general result of Aline Aigon-Dupuy [Ai]. The proof we give here is slightly different but more adapted for what we want to do next.

The quotient of S by the hyperelliptic involution, which is induced by h_0, is just the Riemann sphere with 6 marked points $\pm a$, ± 1 and $\pm b$. By the remarks made in section **1** the involution induced by h_{n_1} is now $z \mapsto -z$. From this it follows that the full quotient $\mathbb{D}/\widetilde{G}$ is the Riemann sphere with 5 marked points 0, 1, $a^2 = \alpha$, $b^2 = \beta$ and ∞. We also note by (1.6) that β is the image of p_3 and p_4 while ∞ is the image of n_1 and n_2 and also of p_4' (the midpoint of $[q_3', q_4]$) and p_8' (the midpoint of $[q_7', q_8]$) which are in the same orbits.

The same can be done for S' and $\widetilde{G'}$ for which we find the sphere with 5 marked points 0, 1, α', β' and ∞. The only difference is that β' is now the image of n_1 while ∞ is the image of p_4. Since by (2.1) the two marked spheres are isomorphic there is a Möbius transformation, mapping one set of points on to the other. It is moreover easy to deduce from the previous remarks that it keeps 0 and 1 fixed and sends β to ∞. Hence it is,

$$z \mapsto \frac{z(1-\beta)}{z-\beta} \ . \tag{2.3}$$

Taking the image of α and ∞ under this map we obtain α' and β'.

2.4 Remark. We have limited (2.2) to one possible half-twist along the simple closed geodesic γ passing through the images of p_3 and p_4 and n_1 and n_2 but we could have treated in the same fashion other half-twists. We summarize below some of these other possibilities, and refer to [Ai] for a more complete treatment.

half-twist along geodesic through	α'	β'
$p_3, n_1, p_4, p_8, n_2, p_7$	$\frac{\alpha(1-\beta)}{\alpha-\beta}$	$1-\beta$
$p_1, m_1, p_2, p_6, m_2, p_5$	$\frac{\alpha}{\alpha-1}$	$\frac{\alpha-\beta}{\alpha-1}$
$0, n_1, q_3, q_7, n_2$	$\frac{\alpha}{\alpha-1}$	$\frac{\beta}{\beta-1}$
$0, m_1, q_1, q_5, m_2$	$1-\alpha$	$1-\beta$
$n_1, p_5, p_1, n_2, p_2, p_6$	$1-\alpha$	$\frac{\beta(1-\alpha)}{\beta-\alpha}$
$m_1, p_3, p_7, m_2, p_8, p_4$	$\frac{\beta-\alpha}{\beta-1}$	$\frac{\beta}{\beta-1}$

For the geodesic through $n_1, p_5, p_1, n_2, p_2, p_6$ see figure 4 (the geodesic through $m_1, p_3, p_7, m_2, p_8, p_4$, is similar). These transformations can of course be combined in various ways. We give an example that will be used in the next section: if we do simultaneous half-twists along the geodesic $p_3, n_1, p_4, p_8, n_2, p_7$ and along the geodesic $p_1, m_1, p_2, p_6, m_2, p_5$ then α and β are transformed into

$$\alpha' = \frac{\alpha(\beta-1)}{\beta(\alpha-1)} \quad \text{and} \quad \beta' = \frac{\beta-1}{\alpha-1}\ . \tag{2.5}$$

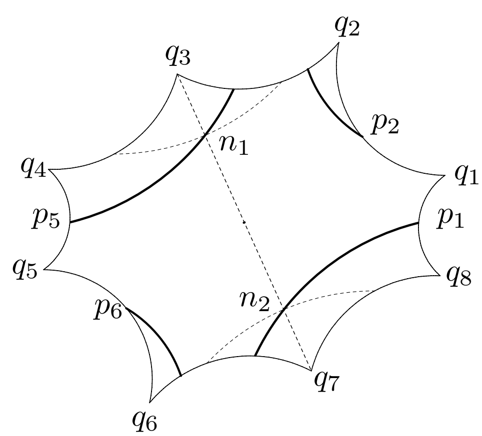

Figure 4

3. Action of half-twists on curves with an automorphism of order 4.

Let S be a Riemann surface with an order 4 automorphism. Then, as is well known, the fixed points of such an automorphism are Weierstrass points. In particular we can choose the octagon representing a fundamental domain for S to be of the form described in **1** and moreover stable under rotation by $\pi/2$ (see figure 5). Let the equation of the corresponding curve be of the form (1.3). The automorphism of order 4 leaves 0 fixed, leaves the $q_i' s$ globally fixed and exchanges m_1 and n_1. Hence in terms of (1.3) it leaves 1 and -1 fixed while it exchanges points above 0 and ∞, in other words it is induced by $x \mapsto 1/x$ and $b = 1/a$. We recover in this way the well known fact that such curves admit an equation of the form,

$$y^2 = (x^2 - a^2)(x^2 - 1)(x^2 - 1/a^2) \tag{3.1}$$

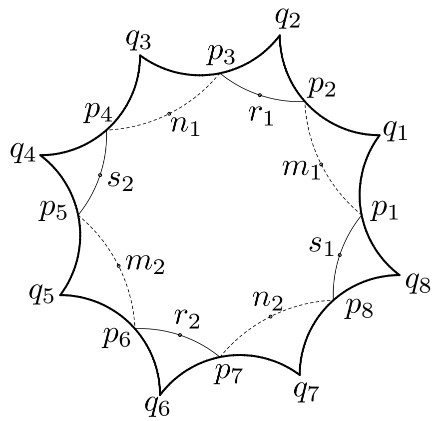

Figure 5

We also have additional involutions induced by $x \mapsto -1/x$, with fixed points points over $\pm i$. Obviously these points correspond to the midpoints of the other arcs of equal lengths, that is the points $\pm i$ are the images of r_1 and r_2 (the midpoints of p_2, p_3 and p_6, p_7 respectively) and of s_1 and s_2 ((the midpoints of p_4, p_5 and p_8, p_1 respectively). There is also a choice involved here which corresponds to choosing a versus $-a$. This is of no consequence for what we want to do but we assume that a is such that r_1 and r_2 are sent to the points over i.

Applying half-twists along $p_3, n_1, p_4, p_8, n_2, p_7$ and $p_1, m_1, p_2, p_6, m_2, p_8$ (see figure 6) we find, by (2.5), that this replaces a^2 and $1/a^2$ by $-a^2$ and $-1/a^2$ hence the equation

$$y^2 = (x^2 + a^2)(x^2 - 1)(x^2 + 1/a^2) \tag{3.2}$$

for this transform.

But in this situation we can also apply simultaneous half-twists along p_2, r_1, p_3, p_7, r_2, p_6 and $p_8, s_1, p_1, p_5, s_2, p_4$ (see figure 7). To be able describe the equation we obtain we need to send i to 0 and $-i$ to ∞. For this we transform the equation (3.1) by means of $z \mapsto i\,\frac{z-i}{z+i}$. This yields the equation

$$y^2 = \left(x^2 + \left(\frac{a-i}{a+i}\right)^2\right)(x^2-1)\left(x^2 + \left(\frac{a+i}{a-i}\right)^2\right) \tag{3.3}$$

and applying (2.5) we find

$$y^2 = \left(x^2 - \left(\frac{a-i}{a+i}\right)^2\right)(x^2-1)\left(x^2 - \left(\frac{a+i}{a-i}\right)^2\right) \tag{3.4}$$

for the transform.

To simplify later computations we need to describe another equation for S. Replacing b by $1/a$ in (1.4) and using the transformation $z \mapsto tr(z) - 1$ we find that the algebraic curve corresponding to S admits an equation of the form

$$y^2 = x(x^2-1)\left(x - \frac{2a}{a^2+1}\right)\left(x - \frac{a^2+1}{2a}\right) \tag{3.5}$$

for which the automorphism of order 4 is again induced by $x \mapsto 1/x$. More importantly, replacing F by $F-1$ we have the existence of an uniformizing function, that we will denote F_1, such that

$$
\begin{aligned}
&F_1(0) = -1,\ F_1(p_1) = F(p_5) = 0,\ F_1(p_3) = F_1(p_7) = \infty,\ F_1(q_i) = 1,\\
&F_1(p_2) = F_1(p_7) = \frac{2a}{a^2+1},\ F_1(p_4) = F_1(p_8) = \frac{a^2+1}{2a}\\
&F_1(m_1) = F_1(m_2) = a,\ F_1(n_1) = F_1(n_2) = \frac{1}{a}\\
&F_1(r_1) = F_1(r_2) = \frac{i+a}{1+ia},\ F_1(s_1) = F_1(s_2) = \frac{1+ia}{i+a}.
\end{aligned}
\tag{3.6}
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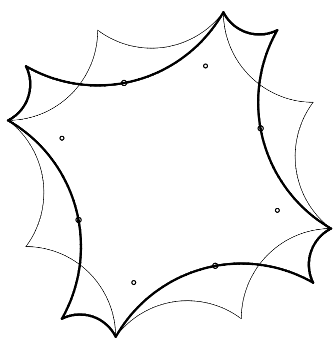

Figure 6

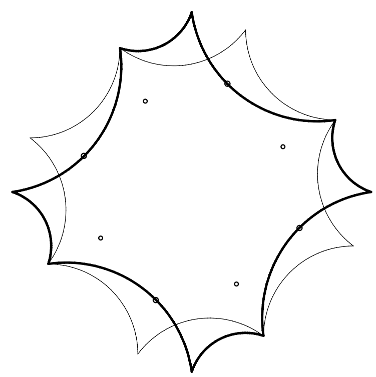

Figure 7

Finally we also find that the involution induced by the order 2 elliptic transformation with center m_1 (or n_1 for that matter) is induced on the algebraic curve defined by (3.1) by the transformation,

$$x \mapsto \frac{x-\alpha}{\alpha\, x - 1} \quad \text{where} \quad \alpha = \frac{2a}{a^2+1}\ . \tag{3.7}$$

From now on we will use this α as parameter for the curves and write the equation (3.5) in the form $y^2 = x(x^2-1)(x-\alpha)(x-1/\alpha)$.

Let G_1, G_2 and G_3 be the 3 fuchsian groups for the curves defined by (3.1), (3.2) and (3.4) and let $\widetilde{G}_1$, $\widetilde{G}_2$ and $\widetilde{G}_3$ be the groups extended by the elliptic elements inducing the order 4 and order 2 automorphisms indicated above. Then for the same reasons as those given in (2.1) the three groups $\widetilde{G}_i$ are equal. We describe the total quotient in the next lemma.

3.8 Lemma. *Let $\mathscr{P}$ be a geodesic octagon in the unit disk $\mathbb{D}$ stable under rotation of angle $\pi/2$. Let G be the group generated by identification of opposite sides of $\mathscr{P}$. let $y^2 = x(x^2-1)(x-\alpha)(x-1/\alpha)$ be an equation of the algebraic curve corresponding to $\mathbb{D}/G$.*

Let $\widetilde{G}$ be the group generated by G, rotation of angle $\pi/2$, centered at 0, and h_{m_1}. Then $\mathbb{D}/\widetilde{G}$ is the sphere with the 4 marked points 0, α^2, 1 and ∞. Moreover 0 is an elliptic point of order 4 while the three others are of order 2.

Proof. The quotient by G and the hyperelliptic involution is of course the Riemann sphere with the six marked points $-1, 0, 1, \alpha, 1/\alpha, \infty$. On this quotient the order 4 automorphism induces $x \mapsto 1/x$. Taking the quotient by this we obtain the sphere with the four points $-2, 2, \alpha + 1/\alpha, \infty$. The automorphisms of order 2 induces on this quotient an involution that exchanges -2 and 2, $\alpha + 1/\alpha$ and ∞ (see (3.7)) hence we can write it as

$$\mu : x \mapsto \frac{(\alpha + 1/\alpha)\, x - 4}{x - \alpha - 1/\alpha} \, . \tag{3.9}$$

The fixed points of this transformation are 2α and $2/\alpha$ which are the images under the map $x \mapsto x + 1/x$ of r_1, r_2, s_1 and s_2 for the first and of m_1, m_2, n_1 and n_2 for the second (see (3.6)). Using the map $x \mapsto x + \mu(x)$ we find that the total quotient is the sphere with the 4 marked points 0, 4α, $4/\alpha$ and ∞ which is of course isomorphic to the sphere with the points 0, α^2, 1 and ∞. Tracing down which points gets mapped where we obtain the assertion on the orders.

Call α_1 (resp. α_2) the equivalent of α for (3.2) (resp. (3.4)). Passing from (3.1) to (3.2) replaces p_1 and p_3 by m_1 and n_1 and hence in the total quotient, described in (3.8), exchanges 1 and ∞ and leaves 0 fixed. Hence α_1 is equal to

$$\alpha_1 = \frac{\alpha}{\sqrt{\alpha^2 - 1}} = \frac{1}{\sqrt{1 - 1/\alpha^2}} \tag{3.10}$$

(note that we do not have to worry about the determination of the square root because replacing α_1 by $-\alpha_1$ yields isomorphic curves).

In the same way we find that $1/\alpha_2$ is equal to

$$\frac{1}{\alpha_2} = \frac{1}{\sqrt{1 - \alpha^2}} \, . \tag{3.11}$$

4. Uniformizing function and accessory parameters.

In general the accessory parameters are needed to compute the uniformizing function and if one knows the uniformizing function one does not really need the accessory parameters. In this section we will nevertheless undertake the seemingly useless task of recovering the accessory parameters from the uniformizing function. Fortunately in the case we are looking at this is relatively easy and simple.

Consider a curve with equation $y^2 = \prod_{i=1}^{5} (x - a_i)$ and let, as in the introduction, $\mathfrak{b}_1, \ldots, \mathfrak{b}_5$ be the corresponding accessory parameters. Then as is well known we have the following two relations,

$$\sum_{i=1}^{5} \mathfrak{b}_i = 0 \quad \text{and} \quad \sum_{i=1}^{5} (a_i \mathfrak{b}_i + \tfrac{3}{8}) = \tfrac{3}{8} \tag{4.1}$$

(see for example [Ne] Chap. V).

The Schwarzian derivative behaving as a quadratic differential under Möbius transformations on the right we have the following transformation rule,

$$\text{If } \; a_i' = \frac{a\, a_i + b}{c\, a_i + d} \quad \text{then} \quad \mathfrak{b}_i' = \frac{(c\, a_i + d)(4c\, a_i \mathfrak{b}_i + 3c + 4d\, \mathfrak{b}_i)}{4(ad - bc)} \tag{4.2}$$

provided a_i and a'_i are different from ∞. If a'_i is the image of ∞ then one can recover the accessory parameter with (4.1).

If we let $a_1 = -1$, $a_2 = 0$, $a_3 = 1$, $a_4 = \alpha$ and $a_5 = 1/\alpha$ we find

$$\mathfrak{b}_1 = -\mathfrak{b}_3 = \tfrac{3}{8},\ \mathfrak{b}_4 = \mathfrak{b},\ \mathfrak{b}_5 = -\alpha(\alpha\mathfrak{b} + \tfrac{3}{4}),\ \mathfrak{b}_2 = \alpha(\alpha\mathfrak{b} + \tfrac{3}{4}) - \mathfrak{b}\ . \tag{4.3}$$

Let F be as in section **1** (note that this is $F_1 + 1$, but that replacing the a_i by $a_i + 1$ does not change the $\mathfrak{b}_i$). Let S_C be the right hand side of (0.2) for 0, 1, 2, $1 + \alpha$, $1 + 1/\alpha$ and the $\mathfrak{b}_i$ as in (4.3) then,

$$\begin{aligned} y_1 &= z^{1/4}\Big(1 - \tfrac{3}{8}\, z + \frac{64\,\alpha^3\mathfrak{b} - 9\,\alpha^2 - 18\,\alpha - 64\,\alpha\,\mathfrak{b} - 57}{384\,(\alpha+1)^2}\, z^2 + \cdots \\ y_2 &= z^{3/4}\Big(1 - \tfrac{1}{8}\, z + \frac{64\,\alpha^3\mathfrak{b} - 15\,\alpha^2 - 30\,\alpha - 64\,\alpha\,\mathfrak{b} - 63}{640\,(\alpha+1)^2}\, z^2 + \cdots \end{aligned} \tag{4.4}$$

are two independent solutions of $y''(z) + \frac{1}{2} S_C\, y(z) = 0$. The general solution to (0.2) is then of the form

$$\varphi = \frac{a\, y_1 + b\, y_2}{c\, y_1 + d\, y_2} \quad \text{with} \ \ ad - bc \neq 0 \tag{4.5}$$

(see [Ne] p. 204). The first terms of a series expansion of (4.5) are

$$\frac{a}{c} + \frac{bc - ad}{c^2}\sqrt{z} - \frac{(bc - ad)\, d}{c^3}\, z + \cdots \tag{4.6}$$

On the other hand F is an even function with a zero of order 2 at 0, hence its inverse φ is of the form $\sqrt{z}\,\varphi_1(z)$ with φ_1 holomorphic and non zero near 0. This implies that we must have $a = 0$ and $d = 0$ in (4.6) and in (4.5). From this we conclude that the first order terms of φ are

$$c_1\sqrt{z}\,\Big(1 + \tfrac{1}{4}\, z + \frac{32\,\alpha\mathfrak{b} - 32\,\alpha^3\mathfrak{b} + 45\,\alpha^2 + 90\,\alpha + 69}{480(\alpha+1)}\, z^2 + \cdots \tag{4.7}$$

where c_1 is a constant. Inverting we obtain,

$$F(z) = \frac{z^2}{{c_1}^2} - \frac{z^4}{2\,{c_1}^4} + \frac{1}{60}\,\frac{\big(9 + 30\,\alpha + 15\,\alpha^2 + 8\,\mathfrak{b}\alpha^3 - 8\,\mathfrak{b}\alpha\big)\, z^6}{(\alpha+1)^2\, {c_1}^6} + \cdots\ . \tag{4.8}$$

To find the map to the total quotient we compose $F - 1$ with,

$$z \mapsto z + 1/z,\ z \mapsto z + \mu(z) \ \text{ and } \ z \mapsto \alpha\, z/4$$

where μ is as in (3.9). The first terms of the power expansion of this function is,

$$-\frac{\alpha^2 z^4}{{c_1}^4\,(\alpha+1)^2} - \frac{1}{30}\,\frac{\big(8\,\alpha^3\mathfrak{b} + 15\,\alpha^2 - 8\,\alpha\,\mathfrak{b} + 9\big)\,\alpha^2\, z^8}{(\alpha+1)^4\, {c_1}^8} + \cdots\ . \tag{4.9}$$

For the two transforms by,

$$z \mapsto z/(z-1) \;\text{ and }\; z \mapsto z/(z-\alpha^2)$$

respectively, this yield the expansions,

$$\frac{\alpha^2 z^4}{{c_1}^4\,(\alpha+1)^2} + \frac{1}{30}\,\frac{\alpha^2\left(8\,\alpha^3\mathfrak{b} - 15\,\alpha^2 - 8\,\alpha\,\mathfrak{b} + 9\right)z^8}{{c_1}^8\,(\alpha+1)^4} + \cdots \tag{4.10}$$

$$\frac{z^4}{{c_1}^4\,(\alpha+1)^2} + \frac{1}{30}\,\frac{\left(8\,\alpha^3\mathfrak{b} + 15\,\alpha^2 - 8\,\alpha\,\mathfrak{b} - 21\right)z^8}{{c_1}^8\,(\alpha+1)^4} + \cdots\,. \tag{4.11}$$

Substituting c_3 for c_1, β for α and $\mathfrak{b}'$ for $\mathfrak{b}$ in (4.9) and equating the first term with the first term of (4.11) yields,

$$c_3^2 = \frac{\beta(\alpha+1)c_1^2 i}{\beta+1}\,. \tag{4.12}$$

Replacing c_3 by this value in the second terms we find,

$$\mathfrak{b}' = \frac{6\,\beta^2 - 9 - 8\,\beta^2\alpha^3\mathfrak{b} - 15\,\alpha^2\beta^2 + 8\,\beta^2\alpha\,\mathfrak{b}}{8\,\beta\,(\beta^2-1)}\,. \tag{4.13}$$

By (3.11) this is the accessory parameter for $\beta = \frac{1}{\sqrt{1-\alpha^2}}$. Using (4.2) we find that the parameter for $\sqrt{1-\alpha^2}$ is $-\beta(\beta\mathfrak{b}' + \frac{3}{4})$. Multiplying by $\frac{1}{\beta}\frac{1-\beta^2}{\beta^2}$ to obtain $\mathfrak{B}(\sqrt{1-\alpha^2})$ we have,

$$\mathfrak{B}(\sqrt{1-\alpha^2}) = \frac{12\,\beta^2 - 15 - 8\,\beta^2\alpha^3\mathfrak{b} - 15\,\alpha^2\beta^2 + 8\,\beta^2\alpha\,\mathfrak{b}}{8\,\beta^2} \tag{4.14}$$

and finally substituting $\frac{1}{1-\alpha^2}$ for β^2 in (4.14) we have

$$\mathfrak{B}(\sqrt{1-\alpha^2}) = -\tfrac{3}{8} - \alpha(\alpha^2-1)\mathfrak{b} = \; -\tfrac{3}{8} - \mathfrak{B}(\alpha) \tag{4.15}$$

as announced and this proves the theorem of the introduction.

4.16 Remarks. 1) For the curves we are considering, Whittaker's conjecture for the accessory parameters [Wh] would imply that

$$\mathfrak{b} = \frac{3 - 12\alpha^2}{20\,\alpha(\alpha^2-1)}\,.$$

Using (4.15) one can easily check that if Whittaker's conjecture is satisfied by a curve then it is not satisfied by its two transforms. This give an infinite family of counter examples to Whittaker's conjecture. Explicit ones are given below (4.18).

2) The formula (4.15) actually extends to the degenerate cases when $\alpha = 0$ or $\alpha = \pm 1$. We first note that when α tends to 0 or ± 1 the total quotient tends to a sphere with three marked points. Hence at the limit we can easily compute the uniformizing function. We then can use Kra's theorem on the degeneration

of accessory parameters [Kr] to find the limit for $\mathfrak{B}$. We find $\mathfrak{B}(0) = \frac{1}{4}$ and $\mathfrak{B}(1) = -\frac{5}{8}$.

3) We can do computations similar to the ones we have made in this section for the transformations described in section **2**. Unfortunately their size prevents form presenting them here so we will only indicate one of the results we have found. Let, as in (2.2), C be the algebraic curve with equation $y^2 = (x^2 - a^2)(x^2 - 1)(x^2 - b^2)$. Let $\mathfrak{b}_1$, $\mathfrak{b}_2$ and $\mathfrak{b}_3$ be the accessory parameters corresponding to a, 1 and b respectively (of course $\mathfrak{b}_2$ can be expressed in terms of $\mathfrak{b}_1$ and $\mathfrak{b}_3$ but we will need it for symmetry). Let $\mathfrak{B}_1 = a\mathfrak{b}_1$, $\mathfrak{B}_2 = \mathfrak{b}_2$ and $\mathfrak{B}_3 = b\mathfrak{b}_3$ and let $\mathfrak{B}'_1$, $\mathfrak{B}'_2$ and $\mathfrak{B}'_3$ be the corresponding terms for the curve C' with equation (2.2.2). Then

$$\begin{aligned} \mathfrak{B}'_1 &= \mathfrak{B}_1 + \frac{a^2}{b^2}\left(\mathfrak{B}_2 + \mathfrak{B}_3\right) \\ \mathfrak{B}'_2 &= \mathfrak{B}_2 + \frac{1}{b^2}\left(\mathfrak{B}_1 + \mathfrak{B}_3\right) \\ \mathfrak{B}'_3 &= \mathfrak{B}_3 - \frac{a^2}{b^2}\left(\mathfrak{B}_2 + \mathfrak{B}_3\right) - \frac{1}{b^2}\left(\mathfrak{B}_1 + \mathfrak{B}_3\right) \end{aligned} \tag{4.17}$$

Changing the octagon for C and hence a and b, we can in fact recover from this the $\mathfrak{B}_i$ for the other transforms of (2.4).

4.18 Applications. The curve with equation,

$$y^2 = x(x^2 - 1)(x - i)(x + i)$$

has an automorphism group of order 48, from this it is easy to compute the accessory parameters (which are in fact well known). The $\mathfrak{b}$ corresponding to the root i is $\frac{3}{8}\,i$. If we apply the transformation we obtain the curve with equation

$$y^2 = x(x^2 - 1)(x - \tfrac{\sqrt{2}}{2})(x - \sqrt{2})$$

and the accessory parameters for $\frac{\sqrt{2}}{2}$ and $\sqrt{2}$ are respectively $\frac{3}{8}\sqrt{2}$ and $-\frac{9}{16}\sqrt{2}$.

The curve with equation

$$y^2 = x(x^2 - 1)(x - \tfrac{1}{2})(x - 2)$$

has an automorphism group of order 24. The accessory parameter is 0 for $\frac{1}{2}$ and $-\frac{3}{8}$ for 2. In this case we have two distinct transforms. The first has equation

$$y^2 = x(x^2 - 1)(x - \sqrt{3}\,i)(x + i/\sqrt{3})$$

and the parameter is $\frac{5}{32}\sqrt{3}\,i$ for $\sqrt{3}\,i$ and is $-\frac{9}{32}\sqrt{3}\,i$ for $-i/\sqrt{3}$.

The second has equation

$$y^2 = x(x^2 - 1)(x - \sqrt{3}/2)(x - 2/\sqrt{3})$$

and the accessory parameter is $\sqrt{3}$ for $\sqrt{3}/2$ and $-\frac{9}{8}\sqrt{3}$ for $2/\sqrt{3}$.

Fenchel-Nielsen coordinates for the curves we have just described can be found in [Bu-Si1].

There is also a curve that is transformed into an isomorphic curve for both transformations. This is,

$$y^2 = x(x^2-1)\left(x - \frac{\sqrt{3}+i}{2}\right)\left(x - \frac{\sqrt{3}-i}{2}\right) .$$

Combining (4.2) and (4.15) we can easily compute the accessory parameters, in particular for $\frac{\sqrt{3}+i}{2}$ we have $\frac{-\sqrt{3}+9\,i}{16}$. In fact for this example we also know the Fenchel-Nielsen coordinates. These can be computed using the methods of [Bu-Si2] section **2** and the construction given here. The Fenchel-Nielsen coordinates are $(L, 1/4, L, 1/4, L', 1/2)$ where $L = 2\operatorname{arccosh}(1+\sqrt{3})$ and $L' = 2\operatorname{arccosh}(3+2\sqrt{3})$ as indicated in [Bu-Si2].

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