Complete Kneser Transversals

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joint work with
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Let us consider 8 points in $\mathbb{R}^3$ general position.
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**Question:** Is there a transversal line to all tetrahedra?
NEVER

- There are at most 3 points inside H
- There are at least 5 points outside H
- There are 3 points in the same side of H
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So, the line passing through $x$ and $y$ gives the desired transversal.
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So, the line passing through $x$ and $y$ gives the desired transversal.

Question : Let $A$ be a set of 7 points in $\mathbb{R}^3$ in general position. Is there a transversal line to all tetrahedra of $A$?
Introduction
Kneser hypergraphs
Rado’s central point theorem
Complete Kneser transversals
Radon partitions
Stability and instability
Some computational results

Sometimes YES

Sometimes NO
Kneser Transversal

Let $k, d, \lambda \geq 1$ be integers with $d \geq \lambda$.

$m(k, d, \lambda) \overset{\text{def}}{=} \text{the maximum positive integer } n \text{ such that every set } X \text{ of } n \text{ points (not necessarily in general position) in } \mathbb{R}^d \text{ has the property that the convex hull of all } k\text{-set of } X \text{ have a transversal } (d - \lambda)\text{-plane (called Kneser Transversal).}$
Kneser Transversal

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$M(k, d, \lambda) \overset{\text{def}}{=} \text{the minimum positive integer } n \text{ such that for every set of } n \text{ points in general position in } \mathbb{R}^d \text{ the convex hull of the } k\text{-sets does not have a transversal } (d - \lambda)\text{-plane.}$
• $m(k, d, \lambda) < M(k, d, \lambda)$. 
\begin{itemize}
  \item $m(k, d, \lambda) < M(k, d, \lambda)$.
  \item $m(4, 3, 2) = 6$ and $M(4, 3, 2) = 8$.
\end{itemize}
• $m(k,d,\lambda) < M(k,d,\lambda)$.
• $m(4,3,2) = 6$ and $M(4,3,2) = 8$.

Theorem (Arocha, Bracho, Montejano, R.A., 2011)

$$M(k,d,\lambda) = \begin{cases} d + 2(k - \lambda) + 1 & \text{if } k \geq \lambda, \\ k + (d - \lambda) + 1 & \text{if } k \leq \lambda. \end{cases}$$
Kneser hypergraphs

A hypergraph $H$ is a pair $(V, \mathcal{H})$ where $V$ (vertices) is a finite set and $\mathcal{H}$ (hyperedges) is a collection of subsets of $V$. 
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The Kneser hypergraph $K^{\lambda+1}(n, k)$ is the hypergraph $(V, \mathcal{H})$ where $V$ is the collection of all $k$-elements subsets of a $n$-set and $\mathcal{H} = \{(S_1, \ldots, S_\rho)| 2 \leq \rho \leq \lambda + 1, \ S_1 \cap \cdots \cap S_\rho = \emptyset\}$. 
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Remark Kneser graphs are obtained when $\lambda = 1$. 

Knese hypergraph when $n = 5$, $k = 2$ and $\lambda = 1$ (Petersen graph)
A coloring of a hypergraph $H$ is a function that assigns colors to the vertices such that no hyperedge of $H$ is monochromatic.
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A collection of vertices $\{S_1, \ldots, S_\rho\}$ of $K^{\lambda+1}(n, k)$ are in the same color class if and only if either

a) $\rho \leq \lambda + 1$ and $S_1 \cap \cdots \cap S_\rho \neq \emptyset$ or

b) $\rho > \lambda + 1$ and any $(\lambda + 1)$-subfamily $\{S_{i_1}, \ldots, S_{i_{\lambda+1}}\}$ of $\{S_1, \ldots, S_\rho\}$ is such that $S_{i_1} \cap \cdots \cap S_{i_{\lambda+1}} \neq \emptyset$. 
Proposition (Arocha, Bracho, Montejano, R.A., 2011) If \( \chi(K_{\lambda+1}(n, k)) \leq d - \lambda + 1 \) then \( n \leq m(k, d, \lambda) \).
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Theorem (Arocha, Bracho, Montejano, R.A., 2011)

$\chi(K^{\lambda+1}(n, k)) \leq n - k - \left\lceil \frac{k}{\lambda} \right\rceil + 2.$
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Corollary (Arocha, Bracho, Montejano, R.A., 2011)

\[
d - \lambda + k + \left\lceil \frac{k}{\lambda} \right\rceil - 1 \leq m(k, d, \lambda).
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$$\chi(K^{\lambda+1}(n, k)) > \begin{cases} n - 2k + \lambda & \text{if } k \geq \lambda, \\ n - 2k & \text{if } k \leq \lambda. \end{cases}$$
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Corollary (Arocha, Bracho, Montejano, R.A., 2011) \( d - \lambda + k + \lceil \frac{k}{\lambda} \rceil - 1 \leq m(k, d, \lambda) \).

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\end{cases}
\]

Theorem (Lovász) \( \chi(K^2(n, k)) = n - 2k + 2 \).
Conjecture $m(k, d, \lambda) = d - \lambda + k + \left\lceil \frac{k}{\lambda} \right\rceil - 1$. 

Theorem (Arocha, Bracho, Montejano, R.A., 2011) The conjecture is true if either

a) $d = \lambda$

b) $\lambda = 1$

c) $k \leq \lambda$

d) $\lambda = k - 1$

e) $k = 2, 3$. 

J. L. Ramírez Alfonsín

Complete Kneser Transversals
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The conjecture is true if either
a) $d = \lambda$ or
b) $\lambda = 1$ or
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e) $k = 2, 3$. 
Rado’s central point theorem

Rado’s theorem If $X$ is a bounded measurable set in $\mathbb{R}^d$ then there exists a point $x \in \mathbb{R}^d$ such that

$$\text{measure}(P \cap X) \geq \frac{\text{measure}(X)}{d + 1}$$

for each half-space $P$ that contains $x$. 
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A generalization of the discrete version of Rado’s result.

Theorem (Arocha, Bracho, Montejano, R.A. 2011)
Let $X$ be a finite set of $n$ points in $\mathbb{R}^d$. Then, there is a $(d - \lambda)$-plane $L$ such that any closed half-space $H$ through $L$ contains at least $\left\lfloor \frac{n-d+2\lambda}{\lambda+1} \right\rfloor + (d - \lambda)$ points of $X$. 
Let $k, d, \lambda \geq 1$ be integers with $d \geq \lambda$. $m^\ast(k, d, \lambda)$ def = the maximum positive integer $n$ such that every set $X$ of $n$ points (not necessarily in general position) in $\mathbb{R}^d$ has the property that the convex hull of all $k$-set of $X$ have a transversal $(d - \lambda)$-plane containing $(d - \lambda) + 1$ points of $X$ (called Complete Kneser Transversal).

We clearly have that $m^\ast(k, d, \lambda) \leq m(k, d, \lambda)$.
Complete Kneser transversal

Let $k, d, \lambda \geq 1$ be integers with $d \geq \lambda$. 
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Let $k, d, \lambda \geq 1$ be integers with $d \geq \lambda$.

$m^*(k, d, \lambda) \overset{\text{def}}{=} \text{the maximum positive integer } n \text{ such that every set } X \text{ of } n \text{ points (not necessarily in general position) in } \mathbb{R}^d \text{ has the property that the convex hull of all } k\text{-set of } X \text{ have a transversal } (d - \lambda)\text{-plane containing } (d - \lambda) + 1 \text{ points of } X \text{ (called Complete Kneser Transversal).}$
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Let $k, d, \lambda \geq 1$ be integers with $d \geq \lambda$.

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We clearly have that

$$m^* (k, d, \lambda) \leq m (k, d, \lambda)$$
Proposition $m^*(k, d, k) = d$. 
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Proof (easy): For any set of $d$ or less points in $\mathbb{R}^d$ choose any set $T$ with $d - k + 1$ points. Then, $\text{aff}(T)$ is a complete Kneser transversal since $T$ have non-empty intersection with any $k$-set.
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On the other hand, if we choose $d + 1$ affinely independent points in $\mathbb{R}^d$ then any $(d - k + 1)$-set $T$ will leave $k$ points in its complement, and thus $\text{aff}(T)$ cannot be a complete Kneser transversal.
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• We assume $k \geq \lambda + 1$. 
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- We assume $k \geq \lambda + 1$.
- It turns out that the function $m^*$ has two different behaviours:
  
  $$\alpha(d, \lambda) = \frac{\lambda - 1}{\left\lfloor \frac{d}{2} \right\rfloor} \geq 1$$
  
  $$\alpha(d, \lambda) = \frac{\lambda - 1}{\left\lfloor \frac{d}{2} \right\rfloor} < 1$$
Radon’s theorem Let $X$ be a set of $d + 2$ points in $\mathbb{R}^d$ in general position. Then, there exists a unique partition $X = X_1 \cup X_2$ such that $\text{conv}(X_1) \cap \text{conv}(X_2) \neq \emptyset$. 
Radon’s theorem Let $X$ be a set of $d + 2$ points in $\mathbb{R}^d$ in general position. Then, there exists a unique partition $X = X_1 \cup X_2$ such that $\text{conv}(X_1) \cap \text{conv}(X_2) \neq \emptyset$.

Lemma Let $X$ be any set of $d + 2$ distinct points in $\mathbb{R}^d$ and let $\lfloor \frac{d+2}{2} \rfloor \leq t \leq d + 1$. Then, $X$ can be partitioned into disjoint sets $S$ and $T$ such that $|T| = t$ and $\text{conv}(S) \cap \text{aff}(T) \neq \emptyset$. 
Theorem If $\alpha(d, \lambda) < 1$ then $d - \lambda + 1 + k \leq m^*(k, d, \lambda)$. 
Theorem If $\alpha(d, \lambda) < 1$ then $d - \lambda + 1 + k \leq m^*(k, d, \lambda)$.

Proof: Let $X$ be a collection of $d - \lambda + 1 + k$ points in $\mathbb{R}^d$. 

J. L. Ramírez Alfonsín
Theorem  If $\alpha(d, \lambda) < 1$ then $d - \lambda + 1 + k \leq m^*(k, d, \lambda)$.

Proof : Let $X$ be a collection of $d - \lambda + 1 + k$ points in $\mathbb{R}^d$.

• Since $k \geq \lambda + 1$ then $|X| \geq d + 2$. Let $Y$ be a $(d + 2)$-subset of $X$. 
Theorem If \( \alpha(d, \lambda) < 1 \) then \( d - \lambda + 1 + k \leq m^*(k,d,\lambda) \).

Proof: Let \( X \) be a collection of \( d - \lambda + 1 + k \) points in \( \mathbb{R}^d \).

- Since \( k \geq \lambda + 1 \) then \( |X| \geq d + 2 \). Let \( Y \) be a \( (d + 2) \)-subset of \( X \).
- Since \( \alpha(d, \lambda) < 1 \) then \( \left\lfloor \frac{d+2}{2} \right\rfloor \leq d - \lambda + 1 \leq d + 1 \).
Theorem If $\alpha(d, \lambda) < 1$ then $d - \lambda + 1 + k \leq m^*(k, d, \lambda)$.

Proof: Let $X$ be a collection of $d - \lambda + 1 + k$ points in $\mathbb{R}^d$.

- Since $k \geq \lambda + 1$ then $|X| \geq d + 2$. Let $Y$ be a $(d + 2)$-subset of $X$.
- Since $\alpha(d, \lambda) < 1$ then $\lfloor \frac{d+2}{2} \rfloor \leq d - \lambda + 1 \leq d + 1$.
- By Lemma, the set $Y$ can be partitioned into disjoint sets $S$ and $T$ such that $|T| = d - \lambda + 1$ and $\text{conv}(S) \cap \text{aff}(T) \neq \emptyset$. 

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Theorem If $\alpha(d, \lambda) < 1$ then $d - \lambda + 1 + k \leq m^*(k, d, \lambda)$.

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- Since $k \geq \lambda + 1$ then $|X| \geq d + 2$. Let $Y$ be a $(d + 2)$-subset of $X$.
- Since $\alpha(d, \lambda) < 1$ then $\left\lfloor \frac{d+2}{2} \right\rfloor \leq d - \lambda + 1 \leq d + 1$.
- By Lemma, the set $Y$ can be partitioned into disjoint sets $S$ and $T$ such that $|T| = d - \lambda + 1$ and $\text{conv}(S) \cap \text{aff}(T) \neq \emptyset$.
- We claim that $\text{aff}(T)$ is a complete Kneser transversal for $X$. 
Theorem If $\alpha(d, \lambda) < 1$ then $d - \lambda + 1 + k \leq m^*(k, d, \lambda)$.

Proof: Let $X$ be a collection of $d - \lambda + 1 + k$ points in $\mathbb{R}^d$.

- Since $k \geq \lambda + 1$ then $|X| \geq d + 2$. Let $Y$ be a $(d + 2)$-subset of $X$.
- Since $\alpha(d, \lambda) < 1$ then $\left\lfloor \frac{d+2}{2} \right\rfloor \leq d - \lambda + 1 \leq d + 1$.
- By Lemma, the set $Y$ can be partitioned into disjoint sets $S$ and $T$ such that $|T| = d - \lambda + 1$ and $\text{conv}(S) \cap \text{aff}(T) \neq \emptyset$.
- We claim that $\text{aff}(T)$ is a complete Kneser transversal for $X$. Since $|X| = d - \lambda + 1 + k$ then there is exactly one $k$-set not intersected by $T$. But this $k$-set contains $S$ for which $\text{conv}(S) \cap \text{aff}(T) \neq \emptyset$. 
Cyclic polytope

The cyclic polytope is the convex hull of a finite set of points in the moment curve in $\mathbb{R}^d$ (defined as the map $\gamma : \mathbb{R} \to \mathbb{R}^d, t \mapsto (t, t^2, \ldots, t^d)$).
**Cyclic polytope**

The **cyclic polytope** is the convex hull of a finite set of points in the **moment curve** in \( \mathbb{R}^d \) (defined as the map \( \gamma : \mathbb{R} \to \mathbb{R}^d, t \mapsto (t, t^2, \ldots, t^d) \)).

Let \( k, d, \lambda \geq 1 \) be integers with \( d \geq \lambda \).

\[ \eta(k, d, \lambda) \overset{\text{def}}{=} \text{the maximum number of vertices that the cyclic polytope in } \mathbb{R}^d \text{ can have, so that it has a complete Kneser } (d - \lambda)\text{-transversal to the convex hull of its } k\text{-sets of vertices.} \]
Cyclic polytope

The cyclic polytope is the convex hull of a finite set of points in the moment curve in \( \mathbb{R}^d \) (defined as the map \( \gamma : \mathbb{R} \to \mathbb{R}^d, t \mapsto (t, t^2, \ldots, t^d) \)).

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\[ m^*(k, d, \lambda) \leq \eta(k, d, \lambda) \]
Theorem If $\alpha(d, \lambda) \geq 1$ then $m^*(k, d, \lambda) = d - \lambda + 1 = \eta(k, d, \lambda)$. 
Theorem If $\alpha(d, \lambda) \geq 1$ then $m^*(k, d, \lambda) = d - \lambda + 1 = \eta(k, d, \lambda)$.
Let $\beta(\lambda, j) = \frac{j + \lambda - 1}{2}$ for each $j$ with $j + \lambda$ odd.
Theorem If \( \alpha(d, \lambda) \geq 1 \) then \( m^*(k, d, \lambda) = d - \lambda + 1 = \eta(k, d, \lambda) \).

Let \( \beta(\lambda, j) = \frac{j + \lambda - 1}{2} \) for each \( j \) with \( j + \lambda \) odd.

\[
z(k, d, \lambda) \overset{\text{def}}{=} d - \lambda + 1 + \max_{\substack{j \in \{\lambda + 1, \ldots, d - \lambda + 2\} \atop j + \lambda \text{ is odd}}} \left( \left\lfloor \frac{k - 1}{\beta(\lambda, j)} \right\rfloor \right) \cdot j + (k - 1) \mod_{\beta(\lambda, j)}
\]

\[
Z(k, d, \lambda) \overset{\text{def}}{=} d - \lambda + 1 + \left\lfloor (2 - \alpha(d, \lambda))(k - 1) \right\rfloor
\]
Theorem If $\alpha(d, \lambda) \geq 1$ then $m^*(k, d, \lambda) = d - \lambda + 1 = \eta(k, d, \lambda)$.

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$$Z(k, d, \lambda) \overset{\text{def}}{=} d - \lambda + 1 + \left\lfloor (2 - \alpha(d, \lambda))(k - 1) \right\rfloor$$

Theorem If $\alpha(d, \lambda) < 1$ then $z(k, d, \lambda) \leq \eta(k, d, \lambda) \leq Z(k, d, \lambda)$. 
**Asymptotics**

**Theorem** If \( \alpha(d, \lambda) < 1 \) then \( \lim_{k \to \infty} \frac{\eta(k, d, \lambda)}{k} = 2 - \alpha(d, \lambda) \).
Asymptotics

Theorem If $\alpha(d, \lambda) < 1$ then $\lim_{k \to \infty} \frac{\eta(k, d, \lambda)}{k} = 2 - \alpha(d, \lambda)$.

Corollary If $\alpha(d, \lambda) < 1$ then $m^*(k, d, 2) < m(k, d, 2)$ for $k$ large enough and $d \geq 3$. 
Question: Is the existence of a Kneser Transversal invariant of the order type?
Question: Is the existence of a Kneser Transversal invariant of the order type? NO
Stability and instability

A Kneser transversal is said to be stable (resp. instable) if the given set of points can be slightly perturbated (move each point to, not more than $\epsilon > 0$ distance of their original position) such that the new configuration of points admits (if there is any) only complete Kneser transversals (resp. the new configuration of points does not admit a Kneser transversal).
Codimension 2 and 3

Theorem Let $X = \{x_1, x_2, \ldots, x_n\}$ be a collection of $n = d + 2(k - \lambda)$ points in general position in $\mathbb{R}^d$. Suppose that $L$ is a $(d - \lambda)$-plane transversal to the convex hulls of all $k$-sets of $X$ with $\lambda = 2, 3$ and $k \geq \lambda + 2$ and $d \geq 2(\lambda - 1)$. Then, either

1. $L$ is a complete Kneser transversal (i.e., it contains $d - \lambda + 1$ points of $X$) or
2. $|L \cap X| = d - 2(\lambda - 1)$ and the other $2(k - 1)$ points of $X$ are matched in $k - 1$ pairs in such a way that $L$ intersects the corresponding closed segments determined by them.
Theorem Let $\epsilon > 0$ and let $X = \{x_1, \ldots, x_n\}$ be a finite collection of points in $\mathbb{R}^d$. Suppose that $n = d + 2(k - \lambda)$, $k - \lambda \geq 2$ and $\lambda = 2, 3$. Then, there exists $X' = \{x'_1, \ldots, x'_n\}$, a collection of points in $\mathbb{R}^d$ in general position such that $|x_i - x'_i| < \epsilon$, for every $i = 1, \ldots, n$, and with the property that every transversal $(d - \lambda)$-plane to the convex hull of the $k$-sets of $X'$ is complete (i.e., it contains $d - \lambda + 1$ points of $X'$).
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Theorem Let $\lambda = 2, 3$, $k - \lambda \geq 2$ and $d \geq 2(\lambda - 1)$. Then,

$$m(k, d, \lambda) < d + 2(k - \lambda).$$
Some computational results

We know that $m(4, 3, 2) = 6$ and $M(4, 3, 2) = 8$. 
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Question What about transversal lines to all tetrahedra in configurations of 7 points in $\mathbb{R}^3$?
Complete Kneser lines: determined by oriented matroids
Complete Kneser lines: determined by oriented matroids
Kneser lines: a bit more complicated

Representation in $\mathbb{R}^3$

Projection in $\mathbb{R}^2$
Theorem Among the 246 different order types of 7 points in general position in $\mathbb{R}^3$ there are:

$A = 124$ admitting a complete Kneser line to the tetrahedra.

$B = 124$ admitting a representation for which there is a non-complete Kneser line to the tetrahedra.

We have $|A \cap B| = 46$, $|A \setminus B| = |B \setminus A| = 78$ and $|A \cup B| = 44$. Moreover, for each of the 78 order types of $B \setminus A$ there exists a representation for which there is no Kneser transversal line.