

A theoretical study of the eigenfrequencies of the arterial tree

Nicoud F ¹

¹ CNRS UMR 5149, Institut de Mathématiques et Modélisation de Montpellier, Université Montpellier II, France

Introduction

The pulsatile flow in the arterial tree can be described by a set of two linear partial differential equations for the pressure and flow rate (Reuderink et al., 1889). Modeling the arterial tree as a network of interconnected 1D segments and accounting for appropriate boundary impedances, the pulsatile flow is then described by an homogeneous linear system whose unknown are the amplitude of the forward and backward waves in each 1D segment. The problem can then be interpreted as a non linear eigenvalue problem that is being solved in this theoretical study.

Materials and methods

In a 1D segment whose cross section area A and compliance C do not depend on the axial position, the pressure fluctuations at pulsation ω read ($j^2 = -1$):

$$p^i(x, t) = \text{Re}[\hat{p}(x) \exp(-j\omega t)]$$

$$\hat{p}(x) = p^+ \exp(jkx) + p^- \exp(-jkx)$$

where p^+ and p^- stand for the complex amplitudes of the forward and backward waves respectively and are determined to fulfil the boundary conditions at each edge of the segment. The complex wave number $k = \sqrt{\omega(\rho\omega + jf_v)C/A}$ involves the blood density ρ and the viscous drag function f_v derived from the Womersley solution. In the case of a 1,5 D network of N inter-connected homogeneous segments, the appropriate jump relations (expressing the pressure and flow rate conservation) and boundary conditions (input impedance) must be fulfilled by the $2N$ (2 wave amplitudes for each segment) unknowns $[P]^T = [p_1^+, \dots, p_N^+, p_1^-, \dots, p_N^-]$. The wave amplitudes are then solution of a linear homogeneous system of the type $[\Omega(\omega)][P] = 0$ where $[\Omega(\omega)]$ is a square matrix of size $2N$. Non trivial solutions exist only for those values of the pulsation making $[\Omega(\omega)]$ singular, viz. $\det([\Omega(\omega)]) = 0$. Periodic (neither damped nor amplified) fluctuations then correspond to pulsations with zero imaginary part ($\Im(\omega) = 0$) while $\Im(\omega) < 0$ denotes damped modes.

Results and discussion

The above equation was solved for a 22-segments model of the human arterial tree (based from Avolio, 1980)

with appropriate input impedances at the ascending aorta (AA) and, among others, sub-clavian, carotid, renal, iliac arteries (Nichols et al., 1998). Four roots were found between 0 and 12 Hz (Fig. 1, left). Any velocity/pressure fluctuation in the arterial tree tends to be damped by viscous/viscoelastic effects and by the capillary resistance which is modeled by the distal impedances. Periodic fluctuations are obtained when the energy input from the heart, which is modeled by the input impedance at the AA, compensates the energy losses. When using the experimental values of the aortic input impedance, only damped modes were found (white symbols in Fig 1, left). However, all these modes can be made periodic (black symbols, Fig1, left) by slightly modifying the inlet impedance of the AA (Fig. 1, right). The difference between the experimental and modified input impedance is within the experimental error, meaning that **a)** the results are consistent with *in vivo* periodic pressure fluctuations; **b)** the present formalism may serve as a way to generate self-consistent data for low order models. As a side effect of the net energy flux at the AA, the pressure eigenmodes are not stationary, excluding the existence of pressure nodes at fixed locations in the arterial tree. This reconciles the present results with the classical argument that eigenmodes cannot exist in an arterial system (Nichols et al., 1998) because of large dissipative effects.

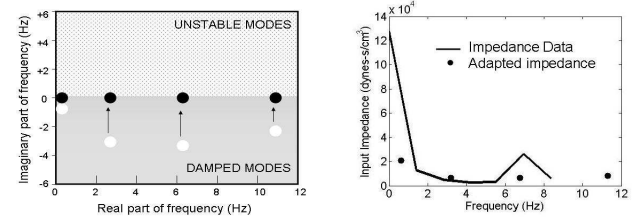


Fig. 1. **Left:** Eigenmodes in the frequency plane. **Right:** Original and modified input impedance (modulus)

Conclusion

Eigenfrequencies have been theoretically determined for a 1,5 D model of the arterial tree. Even if the corresponding eigenmodes are not stationary, these frequencies realize the exact balance between the damping and amplifying effects. One question of interest for further studies is the possibility for one of these frequencies to become unstable.

References

- Reuderink, P.J. et al., Linear and Nonlinear one-dimensional models of pulse wave transmission at high Womersley numbers, **J. Biomechanics.**, 1989, 22: 819-827
- Avolio, A.P., A multibranched model of the human arterial system, **Medical and Biological Eng. and Computing**, 1980, 18:709-718
- Nichols, W. et al., McDonald's Blood Flow in Arteries, Fourth Edition, Oxford University Press, 1998

