

A velocity transformation for heat and mass transfer

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The Van Driest transformation for compressible boundary layers with heat transfer is studied in the low-Mach-number limit. The limiting case leads to a bi-logarithmic type of profile for the mean velocity. A (purely mathematical) analogy with incompressible boundary layers having mass transfer at the wall is identified. © 2000 American Institute of Physics. [S1070-6631(99)00712-6]

The supersonic compressible turbulent boundary layer with or without heat transfer is fairly well documented.¹⁻³ Dimensional analysis of the inner layer shows that the law of the wall can be described in terms of two nondimensional wall parameters, the friction Mach number $M_\tau = u_\tau/c_w$ and the heat flux parameter $B_q = q_w/(\rho_w C_p u_\tau T_w)$, where u_τ is the friction velocity $\sqrt{\tau_w/\rho_w}$, q_w the heat flux, C_p the constant-pressure specific heat, and T_w and c_w the temperature and speed of sound at the wall. Many experimental and numerical data support the validity of the Van Driest⁴ transformation of the velocity into the form of the incompressible logarithmic law

$$U_{\text{VD}}^+ = \int_0^{u^+} \left(\frac{\rho}{\rho_w} \right)^{1/2} du^+ = \frac{1}{\kappa} \ln y^+ + C, \quad (1)$$

where the additive constant C is in principle a function of both M_τ and B_q . The superscript ‘+’ denotes the usual wall scaling $U^+ = U/u_\tau$ and $y^+ = \rho_w u_\tau y/\mu_w$. The case of significant heat transfer but small Mach number has received little attention by experimenters. Therefore the usefulness of the Van Driest transformation to retrieve the classical logarithmic law of the wall cannot yet be satisfactorily demonstrated in this case. However, since the transformation is based on the assumption that turbulence structure is unaltered by large temperature variations it should be as valid at low Mach numbers as at high ones.

In this Brief Communication, the behavior of the transformation in the limiting case where B_q is finite but $M_\tau \rightarrow 0$ is studied, and a mathematical analogy between heat transfer and transpiration is identified. Note that $M_\tau = M_\infty \sqrt{c_f/2}$ so that for typical ‘low speed’ values of $M_\tau \approx 0.05$ and $c_f \approx 0.003$ we have $M_\tau \approx 0.002$, large enough for round-off errors in numerical work to be negligible.

The ‘mixing length’ formula (obtainable by simple local-scaling arguments without the erroneous mixing length concept) reads, in the fully turbulent region:

$$\frac{\partial U}{\partial y} = \frac{(\tau/\rho)^{1/2}}{\kappa y}, \quad (2)$$

where τ and ρ are the local shear stress and density. In the case of nonzero M_τ , U_{VD}^+ can be deduced¹ as a function of U^+ by integrating the ‘mixing length’ formula in the fully turbulent region, assuming a constant turbulent Prandtl number to derive the temperature and hence the density. We obtain

$$U_{\text{VD}}^+ \approx \frac{1}{R} \left[\sin^{-1} \left(\frac{R(U^+H)}{D} \right) - \sin^{-1} \left(\frac{RH}{D} \right) \right], \quad (3)$$

where $R = M_\tau \sqrt{\text{Pr}_t(\gamma-1)/2}$, $H = B_q/((\gamma-1)M_\tau^2)$, and $D = \sqrt{C_1 + R^2 H^2}$.

If Eq. (3) is expanded as a power series in M_τ , we obtain

$$U_{\text{VD}}^+ \sim \frac{2\sqrt{C_1}}{\text{Pr}_t B_q} \left[1 - \sqrt{1 - U^+ \frac{\text{Pr}_t B_q}{C_1}} \right] + O(M_\tau^2). \quad (4)$$

Although data for C_1 are scarce and unreliable, it is reasonable to assume $C_1 \approx 1$.¹ Then, writing $\mathcal{W} \equiv -\text{Pr}_t B_q$ as the strength of the (not necessarily small) perturbation associated with wall heating or cooling, Eq. (4) can be rewritten in the form:

$$U_{\text{VD}}^+ \approx \frac{2}{\mathcal{W}} [\sqrt{1 + \mathcal{W}U^+} - 1] + O(M_\tau^2). \quad (5)$$

The first-order term in Eq. (5), i.e., the limit of the Van Driest transformation for low-Mach-number flow with small M_τ but finite \mathcal{W} , is formally equivalent to the well-known (e.g., Simpson⁵) transformation of the velocity profile for incompressible turbulent wall flows with uniform injection, where again $U_{\text{VD}}^+ = (1/\kappa) \ln y^+ + C$. In this case, the perturbation parameter is $\mathcal{W} = V_{\text{inj}}^+$, the characteristic transpiration velocity being in wall units. Thus, the ‘mixing length’ formulas suggest that an incompressible wall flow subject to injection or suction has the same mean velocity profile as a low-Mach-number flow (with variable density) subject to cooling or heating. It must be stressed at once that this curious mathematical analogy is most unlikely to have any simple physical basis.

Consider the x -component momentum equation in the two cases, neglecting streamwise gradients. In the flow with

heat transfer, we have the familiar constant-stress layer, $\tau = \tau_w$, but combining the mixing length formula with constant Pr_t gives $\rho_w/\rho \approx T/T_w \approx 1 - Pr_t B_q U^+$. In the transpiration case, the density is constant, i.e., $\rho = \rho_w$, but the x -component momentum equation, integrated in the y direction, gives $\tau = \tau_w + \rho_w V_{inj} U$. In both cases π/ρ is of the form $(\tau_w/\rho_w)(1 + \mathcal{W}U^+)$, and, inserting this into Eq. (2), Eq. (5) follows. If U_{VD}^+ is given by Eq. (1) then U^+ obeys the so-called *bi-logarithmic law*.

We present this *mathematical* analogy as a curiosity rather than something with specific physical content, and we make no comment on behavior in the viscous wall region below the validity of Eq. (2); but perhaps investigators of

strong heat transfer and of transpiration may find an exchange of views worth while. This note is dedicated to the memory of Professor Hieu Ha Minh.

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