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Combustion and Flame 142 (2005) 153–159
Combustion and Flame
www.elsevier.com/locate/combustflame

Thermoacoustic instabilities: Should the Rayleigh criterion be extended to include entropy changes?

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Received 1 September 2004; received in revised form 22 February 2005; accepted 28 February 2005

Available online 11 April 2005

Abstract

The Rayleigh criterion (which measures the correlation between pressure and heat release) is the standard tool used to investigate and predict combustion instabilities in both experimental and numerical studies. However, the Rayleigh term is just one of the terms appearing in the acoustic energy equation. The recent development of large eddy simulations for combustion chambers allows complete closure of the budget and analysis of all terms in this equation. This task leads to unexpected difficulties and requires some basic work, as multiple definitions of the energy of fluctuations in a reacting compressible flow can be derived. The objective of this article is to revisit the theoretical derivations of the fluctuation energy equations. Two forms of energy are defined: The first is the classic acoustic energy (AE) introduced by various authors. The second is the fluctuation energy (FE) presented by B.T. Chu [Acta Mech. (1965) 215–234]. Both equations are rederived in a compact manner starting from full nonlinear forms. It is shown that the classic Rayleigh criterion naturally appears as the source term of the AE equation, while the FE form leads to a different criterion stating that temperature and heat release must be in phase for the instability to be fed by the flame/acoustics coupling. The FE form also integrates the fluctuations of three variables (pressure, velocity, entropy), while the AE form uses only pressure and velocity perturbations. It is shown that only the FE form should be used in flames, in contradiction to many current studies performed for combustion instabilities.

Keywords: Fluctuation energy; Rayleigh criterion; Large eddy simulations

1. Introduction

It has long been known that the coupling between acoustic waves and flames in industrial systems can lead to high-amplitude instabilities [1–3]. In addition to inducing oscillations of all physical quantities (pressure, velocities, temperature, etc.), these instabilities can increase the amplitude of the flame motion and, in extreme cases, destroy part of the burner. A commonly used criterion for assessing the stability of a combustor is the Rayleigh criterion [4], which states that if pressure and heat release fluctuations are in phase, the instability is fed by the flame/acoustics coupling. Formally, this criterion may take the form

$$\iiint_{\Omega} p_1 q_1 \,\mathrm{d}\Omega > 0, \tag{1}$$

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0010-2180/$ – see front matter
doi:10.1016/j.combustflame.2005.02.013

where p_1 and q_1 stand for pressure and heat release fluctuations, respectively, and Ω is the flow domain. The sign of the above integral may change with the phase of the oscillation, and Eq. (1) is often integrated over a period to characterize the stability of the system at a given frequency.

The validity of the Rayleigh criterion can be discussed by introducing a more general analysis tool: the budget of fluctuation energy. Such budgets require the analysis of acoustic quantities such as acoustic fluxes through boundaries and are impossible to construct experimentally. However, recent large eddy simulation tools [5–8] have opened new possibilities in this field by giving access to all unsteady variables at any grid point during combustion oscillations. A better understanding of the underlying physics can be obtained by using the fluctuation energy equation, rather than the Rayleigh criterion, to analyze the large eddy simulation (LES) results.

The objective of this article is to present fundamental derivations of the fluctuation energy equation and to discuss them in the framework of thermoacoustic instabilities. A long-term motivation of this work is to provide the theoretical basis for LES analysis and an understanding of combustion instabilities: by closing the fluctuation energy equation in a given combustor, it is expected to achieve progress similar to the landmark work performed for turbulence modeling, where the budgets of kinetic energy near walls have allowed an understanding of turbulent phenomena that was out of reach before [9,10]. The first step toward this ambitious goal is to derive the correct energy equation and, more importantly, to determine which energy should be used to measure fluctuations in a reacting compressible flow: this apparently rarely studied topic is the objective of the present work.

Section 2 first shows how the classic acoustic energy (AE) equation is obtained by linearizing a global energy equation directly deduced from the Navier–Stokes equations and how the Rayleigh criterion enters this budget as one of the source terms. The validity of the Rayleigh criterion is then discussed in terms of energy budget. While the AE equation of Section 2 has been discussed and used by various authors, Section 3 actually shows that this is not the adequate form to be used in reacting flows and proposes an extended formulation. This formulation, called here fluctuation energy (FE) equation, was first introduced by Chu [11], but is obviously not very well known in the combustion community: it accounts for entropy fluctuations and leads to a new criterion for thermoacoustic instabilities. It is rederived in a more compact way here and corrected for terms missing in Chu's derivation in Section 3.

2. The acoustic energy equation

The derivation leading to the classic form of the AE equation usually starts from the linearized equations for density and velocity and combines them to form an energy [3,11–13]. Here a different path is followed: a nonlinear equation for "energy" is derived from the Navier–Stokes equations first and is linearized afterward.

2.1. A nonlinear "energy" equation in reacting flows

To simplify the derivation, all species are supposed to share the same molar weight and constant heat capacities. This assumption is not necessary to derive the generalized acoustic energy equation, although it makes the algebra simpler. It is valid for air flames but must be revisited for the case of H_2–O_2 mixtures, for example. In what follows, D/Dt stands for the particular (total) derivative while ρ, $\mathbf{u}$, T, P, and s are the density, velocity, temperature, pressure, and entropy per unit mass of mixture, respectively. C_p and C_v are the usual heat capacity (per unit mass) at fixed pressure and volume, respectively. Moreover, $\vec{\boldsymbol{\tau}}$ is the stress tensor whose components are $\tau_{ij} = \mu(\partial u_i/\partial x_j + \partial u_j/\partial x_i) - 2/3\delta_{ij}\,\mathrm{div}(\mathbf{u})$, and μ is the dynamic viscosity. λ is the heat diffusivity, and $\gamma = C_\mathrm{p}/C_\mathrm{v}$ and $r = C_\mathrm{p} - C_\mathrm{v}$ are the mass heat capacity ratio and difference. Note that the viscous terms (molecular diffusion of momentum and heat) are kept in the present analysis, although they are usually neglected for the analysis of acoustic perturbations. The reason is that their inclusion leads to a simple justification for the extended fluctuation energy as shown in Section 3. They are neglected in the different stability criteria discussed throughout this article.

Under the above assumptions, the momentum equation reads

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla P + \nabla\cdot\vec{\boldsymbol{\tau}}. \qquad (2)$$

Taking the dot product of Eq. (2) with $\mathbf{u}$, one obtains directly

$$\rho\frac{D\mathbf{u}^2/2}{Dt} + \nabla\cdot(P\mathbf{u}) = P\nabla\cdot\mathbf{u} + \mathbf{u}\cdot(\nabla\cdot\vec{\boldsymbol{\tau}}). \qquad (3)$$

To obtain an energy-like term for pressure on the left-hand side (l.h.s), it is convenient to use the transport equation for pressure to express the pressure dilation term in Eq. (3). This equation can be obtained using the sensible energy equation [3]

$$\rho\frac{De_\mathrm{s}}{Dt} = -P\nabla\cdot\mathbf{u} + q + \nabla\cdot(\lambda\nabla T) + \vec{\boldsymbol{\tau}} : \vec{\nabla}\mathbf{u} \qquad (4)$$

together with the continuity equation $D\rho/Dt = -\rho\nabla\cdot\mathbf{u}$ and the classic relation $\rho e_\mathrm{s} = P/(\gamma - 1)$

valid for perfect gas. It reads

$$\frac{DP}{Dt} = -\gamma P \nabla \cdot \mathbf{u} + (\gamma - 1) \times \left(q + \nabla \cdot (\lambda \boldsymbol{\nabla} T) + \vec{\boldsymbol{\tau}} : \vec{\boldsymbol{\nabla}} \mathbf{u}\right), \tag{5}$$

where q is the rate of heat release by unit of volume. Injecting Eq. (5) into Eq. (3) and introducing the speed of sound $c = \sqrt{\gamma P/\rho}$ yield

$$\rho \frac{D\mathbf{u}^2/2}{Dt} + \frac{1}{\rho c^2} \frac{DP^2/2}{Dt} + \nabla \cdot (P\mathbf{u}) = \frac{\gamma - 1}{\gamma}\left(q + \nabla \cdot (\lambda \boldsymbol{\nabla} T) + \vec{\boldsymbol{\tau}} : \vec{\boldsymbol{\nabla}} \mathbf{u}\right) + \mathbf{u} \cdot (\boldsymbol{\nabla} \cdot \vec{\boldsymbol{\tau}}). \tag{6}$$

The l.h.s. of this equation is formally close to the classic equation for the acoustic energy. One should, however, stress that it is exact and nonlinear and remains valid for any fluctuations of the pressure and velocity fields.

2.2. Linearization and acoustic energy

Equation (6) can be linearized as follows: Consider the simple case of small-amplitude fluctuations (index 1) superimposed to a zero Mach number mean flow (index 0). The instantaneous pressure, density, and velocity fields can then be written as $P = p_0 + p_1$, $\rho = \rho_0 + \rho_1$, and $\mathbf{u} = \mathbf{u}_1$, where $p_1 \ll p_0$, $\rho_1 \ll \rho_0$, and $\mathbf{u}_1^2 \ll c_0^2$, where $c_0 = \sqrt{\gamma P_0/\rho_0}$ is the mean speed of sound. For simplicity, the temporal fluctuations of the diffusivity, viscosity, and heat capacities are neglected. Note that the zero Mach number assumption for the mean flow implies $\boldsymbol{\nabla} p_0 = 0$ (from Eq. (2)) and $q_0 + \nabla \cdot (\lambda \boldsymbol{\nabla} T_0) = 0$ (from Eq. (4)). It also implies that the approximation $D/Dt \approx \partial/\partial t$ holds for any fluctuating quantity because, with $u_0 = 0$, the nonlinear convective term is always of second order. Thus, Eq. (5) becomes an equation for the pressure fluctuations:

$$\frac{Dp_1}{Dt} = -\gamma P \nabla \cdot \mathbf{u}_1 + (\gamma - 1) \times \left(q_1 + \nabla \cdot (\lambda \boldsymbol{\nabla} T_1) + \vec{\boldsymbol{\tau}}_1 : \vec{\boldsymbol{\nabla}} \mathbf{u}_1\right). \tag{7}$$

With $P = p_0 + p_1$, it is clear that $P^2 = p_0^2 + 2p_0 p_1 + p_1^2$, and an estimation of $D(P^2/2)/Dt = p_0 Dp_1/Dt + D(p_1^2/2)/Dt$ is

$$\frac{DP^2/2}{Dt} \simeq \frac{\partial p_1^2/2}{\partial t} - \gamma P p_0 \nabla \cdot \mathbf{u}_1 + p_0(\gamma - 1)\left(q_1 + \nabla \cdot (\lambda \boldsymbol{\nabla} T_1) + \vec{\boldsymbol{\tau}}_1 : \vec{\boldsymbol{\nabla}} \mathbf{u}_1\right). \tag{8}$$

This equation is valid to third order because $\mathbf{u}_1 \cdot \boldsymbol{\nabla} p_1^2$ was neglected in front of $\partial p_1^2/\partial t$. It further implies that

$$\frac{1}{\rho c^2} \frac{DP^2/2}{Dt} \simeq \frac{1}{\rho_0 c_0^2} \frac{\partial p_1^2/2}{Dt} - p_0 \nabla \cdot \mathbf{u}_1 + \frac{\gamma - 1}{\gamma}\left(1 - \frac{p_1}{p_0}\right) \times \left(q_1 + \nabla \cdot (\lambda \boldsymbol{\nabla} T_1) + \vec{\boldsymbol{\tau}}_1 : \vec{\boldsymbol{\nabla}} \mathbf{u}_1\right). \tag{9}$$

Moreover, approximating the total derivative of $\mathbf{u}^2$ by its partial derivative and neglecting higher-order terms, the following third-order approximation of the first term of Eq. (6) can be obtained easily:

$$\rho \frac{D(\mathbf{u}^2/2)}{Dt} \approx \rho_0 \frac{\partial(\mathbf{u}_1^2/2)}{\partial t}. \tag{10}$$

The mean pressure P_0 being constant over space and the mean velocity being $u_0 = 0$, the following expression holds for the flux term of Eq. (6):

$$\nabla \cdot (P\mathbf{u}) = p_0 \nabla \cdot \mathbf{u}_1 + \nabla \cdot (p_1 \mathbf{u}_1). \tag{11}$$

Finally, by injection of Eqs. (9), (10), and (11) into Eq. (6), the following equation is derived:

$$\frac{\partial e_1}{\partial t} + \nabla \cdot (p_1 \mathbf{u}_1) = \frac{\gamma - 1}{\gamma p_0}\left(q_1 + \nabla \cdot (\lambda \boldsymbol{\nabla} T_1)\right) p_1 + \mathbf{u}_1 \cdot (\boldsymbol{\nabla} \cdot \vec{\boldsymbol{\tau}}_1). \tag{12}$$

Equation (12) is a conservation equation for the classic acoustic energy:

$$e_1 = \rho_0 \mathbf{u}_1^2/2 + \frac{1}{\rho_0 c_0^2} p_1^2/2. \tag{13}$$

Neglecting the viscous terms gives the classic equation for the acoustic energy [3]:

$$\frac{\partial e_1}{\partial t} + \nabla \cdot (p_1 \mathbf{u}_1) = \frac{\gamma - 1}{\gamma p_0} p_1 q_1. \tag{14}$$

The expected Rayleigh term $(p_1 q_1)$ appears as a source term on the r.h.s. of this equation. The first obvious observation is that the acoustic energy growth rate depends on the Rayleigh term but also on the acoustic fluxes $\nabla \cdot (p_1 \mathbf{u}_1)$ so that, according to Eq. (14), the Rayleigh criterion is only a necessary condition for instability to occur. If Eq. (14) is integrated over the whole combustor Ω, a more proper instability criterion is obtained stating that the combustor will oscillate (the total acoustic energy will grow) if

$$\iiint_{\Omega} \frac{\gamma - 1}{\gamma p_0} p_1 q_1 \, \mathrm{d}\Omega > \iint_{\Sigma} p_1 \mathbf{u}_1 \cdot \mathbf{n} \, \mathrm{d}\Sigma \tag{15}$$

or, in other words, if the source term due to combustion is larger than the acoustic losses on the combustor inlet and outlet surface Σ. The practical implication of Eq. (15) is that the classic Rayleigh criterion should not be used alone but should also include acoustic losses. This task is difficult in experiments because it requires the evaluation of acoustic fluxes $p_1\mathbf{u}_1$ on the boundaries of the burner, but it can be done in LES. Closure of Eq. (14) was performed for the first time in a self-excited combustor in 2004 by Martin et al. [14] and showed similar orders of magnitude for the Rayleigh term and the acoustic losses.

As no assumption has been made for defining the small-amplitude fluctuations (except that they are indeed small compared with the characteristic scales), Eq. (12) describes a priori the evolution of the energy associated with the three classic types of disturbances put together (acoustic, entropy, vorticity). However, as a direct consequence of the zero Mach number assumption, the pulsation ω[1] associated with the entropy and vortical modes is zero, corresponding to spatial disturbances that do not oscillate over time. Thus the present analysis essentially deals with acoustic perturbations. However, we see in the next section that the fluctuating flow is not isentropic and that an extended energy form is required.

3. The fluctuation energy equation

3.1. Why the acoustic energy form should not be used in flames

Equation (12) is a valid equation for e_1 but there is no evidence that this is the relevant quantity to describe the level of fluctuations in a turbulent reacting flow. Actually, the following simple example suggests that e_1 is not an appropriate measure of the fluctuation activity in a non-isentropic flow. Let us consider a hypothetical flow that initially contains only entropic linear fluctuations but no acoustic waves. The amount of AE associated with this "Flow 1" would be exactly zero, as no velocity/pressure fluctuations are associated with the entropy mode (see Eq. (13)). Hence "Flow 1" would be given the same amount of energy as the corresponding steady flow. A more disturbing observation is that, taking Flow 1 as an initial condition, it is easy to show that e_1 would increase as soon as the diffusivity λ is not zero, even in the absence of combustion: indeed, in this simple nonreacting but initially non-isentropic flow, Eq. (7) shows that p_1 cannot remain zero because $\nabla\cdot(\lambda\boldsymbol{\nabla}T_1)\neq 0$ if an entropy fluctuation exists initially. Hence, e_1 increases at time $t=0$ while the amount of fluctuations present in the flow necessarily decreases due to diffusivity. This result is clearly not satisfactory and shows that the classic acoustic energy defined by Eq. (13) is inadequate in a non-isentropic flow. In his article, Chu [11] gave a tentative definition of what should be a fluctuation energy in such flows and insisted that it must decrease when viscosity/diffusivity effects cannot be neglected. This definition obviously does not apply to the AE form of Eq. (13). The fact that the quantity e_1, which comes from classic acoustic theory, is not relevant to purely entropic fluctuations is not very surprising. This means, however, that another quantity should be used in the case of reacting flows to characterize the global amount of fluctuations properly and that this energy should include entropy fluctuations.

3.2. An extended fluctuation energy

A natural way to proceed is then to start from Eq. (6) and combine it with an equation for entropy. Starting from the Gibbs equation,

$$T\,\mathrm{d}s = C_\mathrm{v}\,\mathrm{d}T - \frac{P}{\rho^2}\,\mathrm{d}\rho = \mathrm{d}e_\mathrm{s} - \frac{P}{\rho^2}\,\mathrm{d}\rho, \tag{16}$$

and using the continuity equation, the state equation ($p=\rho rT$), as well as Eq. (4), lead to

$$\frac{Ds}{Dt} = \frac{r}{P}\big(q + \nabla\cdot(\lambda\boldsymbol{\nabla}T) + \vec{\boldsymbol{\tau}} : \vec{\boldsymbol{\nabla}}\mathbf{u}\big). \tag{17}$$

Multiplying Eq. (17) by Ps/rC_p and adding it to Eq. (6) directly give

$$\rho\frac{D\mathbf{u}^2/2}{Dt} + \frac{1}{\rho c^2}\frac{DP^2/2}{Dt} + \frac{P}{rC_\mathrm{p}}\frac{Ds^2/2}{Dt} + \nabla\cdot(P\mathbf{u}) = \frac{s+r}{C_\mathrm{p}}\big(q + \nabla\cdot(\lambda\boldsymbol{\nabla}T) + \vec{\boldsymbol{\tau}} : \vec{\boldsymbol{\nabla}}\mathbf{u}\big) + \mathbf{u}\cdot(\boldsymbol{\nabla}\cdot\vec{\boldsymbol{\tau}}). \tag{18}$$

This equation is exact and can be linearized like Eq. (6) in Section 2.2. Suppose that an entropy perturbation of small amplitude s_1 is superimposed on the mean flow s_0. Then, since $Ds_0/Dt = \mathbf{u}_1\cdot\boldsymbol{\nabla}s_0$, Eq. (17) gives

$$\frac{Ds_1}{Dt} = \frac{r}{P}\big(q_1 + \nabla\cdot(\lambda\boldsymbol{\nabla}T_1) + \vec{\boldsymbol{\tau}}_1 : \vec{\boldsymbol{\nabla}}\mathbf{u}_1\big) - \mathbf{u}_1\cdot\boldsymbol{\nabla}s_0, \tag{19}$$

so that, with

$$\frac{Ds^2/2}{Dt} = \frac{Ds_1^2/2}{Dt} + \frac{Ds_1s_0}{Dt} + \frac{Ds_0^2/2}{Dt}, \tag{20}$$

the following equation holds to third order:

$$\frac{P}{rC_\mathrm{p}}\frac{Ds^2/2}{Dt} \simeq \frac{P_0}{rC_\mathrm{p}}\frac{\partial s_1^2/2}{\partial t} + \frac{1}{C_\mathrm{p}}\big(q_1 + \nabla\cdot(\lambda\boldsymbol{\nabla}T_1) + \vec{\boldsymbol{\tau}}_1 : \vec{\boldsymbol{\nabla}}\mathbf{u}_1\big)$$

[1] Indeed, $\omega = ku_0$, where k is the wavenumber of the perturbation and $u_0 = 0$ under the zero Mach number assumption.

$$+\frac{P_0}{rC_p}s_1\mathbf{u}_1\cdot\nabla s_0. \tag{21}$$

The other terms on the l.h.s. of Eq. (18) have already been linearized in Section 2.2, and, keeping only the second-order terms, the linearized form of Eq. (18) becomes an equation for the fluctuation energy (FE equation) e_{tot}:

$$\frac{\partial e_{tot}}{\partial t}+\nabla\cdot(p_1\mathbf{u}_1)=\frac{T_1}{T_0}\big(q_1+\nabla\cdot(\lambda\nabla T_1)\big)$$
$$-\frac{P_0}{rC_p}s_1\mathbf{u}_1\cdot\nabla s_0+\mathbf{u}_1\cdot(\nabla\cdot\vec{\boldsymbol{\tau}}_1). \tag{22}$$

Here, e_{tot} is defined by

$$e_{tot}=\rho_0\mathbf{u}_1^2/2+\frac{1}{\rho_0c_0^2}p_1^2/2+\frac{P_0}{rC_p}s_1^2/2. \tag{23}$$

Equation (22) was first derived by Chu [11] except for the term in ∇s_0 which vanishes in the case where the mean flow entropy s_0 is uniform over space. It generalizes the classic AE form (Eq. (6)) to the case of entropy/acoustic fluctuations and degenerates naturally to it in isentropic flows. Note that, in the general case, this FE form must integrate the fluctuations of three variables p_1, u_1, and s_1, while the acoustic energy e_1 included only two.[2] The fact that entropy is present in this extended energy does not mean that the entropy mode of fluctuation has been added to the analysis. Indeed, the zero Mach number was still used for the linearization process leading to the extended form Eq. (23). This means that only acoustic fluctuations are oscillating over time because the entropy and vortical modes correspond to the null frequency. However, Eq. (19) shows that the acoustic modes are not isentropic when unsteady heat release and/or a mean entropy gradient are present. The extended form of the fluctuation energy is a natural way of accounting for this property.

Integrated over space, this equation suggests that the classic Rayleigh criterion $\iiint_V p_1q_1\,\mathrm{d}V$ should be replaced by $\iiint_V T_1q_1\,\mathrm{d}V$ to characterize the stability of a combustor. Specifically, the global fluctuation energy in the system grows when (neglecting viscous effects)

$$\iiint_{\Omega}\left(\frac{T_1q_1}{T_0}-\frac{P_0}{rC_p}s_1\mathbf{u}_1\cdot\nabla s_0\right)\mathrm{d}\Omega$$
$$>\iint_{\Sigma}p_1\mathbf{u}_1\cdot\mathbf{n}\,\mathrm{d}\Sigma. \tag{24}$$

This criterion extends the classic Rayleigh criterion to the case where the net energy flux at the boundaries cannot be neglected and the entropy fluctuations are significant. It also extends the initial equation of Chu to the case where the entropy field is not constant over space, which is always the case when combustion occurs. The second term on the l.h.s. was not present in Chu's article [11] and does not seem to have been discussed earlier. It describes how the overall fluctuation energy decreases when a positive fluctuation of entropy ($s_1>0$) is convected toward region with larger mean entropy ($\mathbf{u}_1\cdot\nabla s_0>0$), which is an expected result. A rough estimate of the order of magnitude of this term (without accounting for the phase shifts between the different terms) can be obtained as follows. Under the zero Mach number assumption, the mean pressure field P_0 is constant so that $\nabla s_0=C_p/T_0\nabla T_0$. Moreover, $|u_1|\propto|p_1|/\rho_0c_0$ for acoustic perturbations and, assuming that the main source of entropy fluctuations is the unsteady heat release, one obtains $\omega|s_1|\propto r|q_1|/P_0$, where ω is the pulsation. Introducing $L_f=(|\nabla T_0|/T_0)^{-1}$, the characteristic thickness of the flame brush, and $\lambda=2\pi c_0/\omega$, the characteristic acoustic wavelength, the maximum (assuming optimal phase shifts) order of magnitude of the ∇s_0 term in Eq. (24) is given by $|p_1||q_1|\times\lambda/(2\pi(\gamma-1)L_f)\times(\gamma-1)/\gamma P_0$. The equivalent estimate of the classic Rayleigh term in Eq. (15) leads to $(\gamma-1)|p_1||q_1|/\gamma P_0$. As L_f can be much smaller than λ, this means that the additional term related to the nonuniform mean entropy field is *potentially* larger than the classic Rayleigh term. Specific postprocessing of unsteady LES data is necessary to address this issue more precisely.

More generally, the derivation of Eq. (24) shows how arbitrary the stability criteria can be, because their form depends on the choice of the energy used to characterize the fluctuations. This issue is discussed in the next section.

3.3. About the choice of an energy form

At this point, two energies (Table 1) have been defined, leading to two different equations (Table 2) but also to different instability criteria (Eqs. (15) and (24)). Table 3 compares these criteria with the classic Rayleigh criterion in the simplest case where heat diffusivity and viscosity are both zero and mean entropy s_0 is constant. The criteria are also integrated with time over a period of the instability τ. The first form (AE) leads to a stability criterion (Eq. (15)) ex-

Table 1
Definitions of acoustic energy (AE) and fluctuation energy (FE)

AE	$e_1=\rho_0\mathbf{u}_1^2/2+\frac{1}{\rho_0c_0^2}p_1^2/2$
FE	$e_{tot}=\rho_0\mathbf{u}_1^2/2+\frac{1}{\rho_0c_0^2}p_1^2/2+\frac{P_0}{rC_p}s_1^2/2$

[2] Another expression for e_{tot} is $e_{tot}=\rho_0\mathbf{u}_1^2/2+(c_0^2/\gamma\rho_0)\rho_1^2/2+(\rho_0C_v/T_0)T_1^2/2$.

Table 2
Conservation equations for acoustic energy (AE) and fluctuation energy (FE)

AE	$\frac{\partial e_1}{\partial t} + \nabla \cdot (p_1 \mathbf{u}_1) = \frac{\gamma - 1}{\gamma p_0}(q_1 + \nabla \cdot (\lambda \nabla T_1))p_1 + \mathbf{u}_1 \cdot (\nabla \cdot \vec{\boldsymbol{\tau}}_1)$
FE	$\frac{\partial e_{\rm tot}}{\partial t} + \nabla \cdot (p_1 \mathbf{u}_1) = \frac{T_1}{T_0}(q_1 + \nabla \cdot (\lambda \nabla T_1)) - \frac{P_0}{r C_{\rm p}} s_1 \mathbf{u}_1 \cdot \nabla s_0 + \mathbf{u}_1 \cdot (\nabla \cdot \vec{\boldsymbol{\tau}}_1)$

Table 3
Summary of criteria for combustion instability for zero thermal diffusivity, zero viscosity, and constant mean entropy[a]

Classic Rayleigh	$\iiint_\Omega p_1 q_1 \, \mathrm{d}\Omega > 0$
Extended Rayleigh	$\frac{(\gamma - 1)}{\gamma p_0} \iiint_\Omega p_1 q_1 \, \mathrm{d}\Omega > \iint_\Sigma p_1 \mathbf{u}_1 \cdot \mathbf{n} \, \mathrm{d}\Sigma$
Chu	$\frac{1}{T_0} \iiint_\Omega T_1 q_1 \, \mathrm{d}\Omega > \iint_\Sigma p_1 \mathbf{u}_1 \cdot \mathbf{n} \, \mathrm{d}\Sigma$

[a] These criteria should also be integrated over time but this integration is not indicated here for clarity.

tending the Rayleigh criterion, while the second one (FE) leads to a criterion (Eq. (24)) that is very different. Interestingly, the AE Rayleigh criterion predicted instability when pressure and heat release fluctuations were in phase, while the FE criterion requires temperature and heat release to be in phase.

A relevant question is then to determine which of these two forms is the most adequate. This can be done by pursuing the simple test case mentioned in Section 3.1: Consider a domain with zero fluxes on boundaries and no combustion source term. A "good" energy, according to Chu's definition [11], should only decrease in this situation and this decrease should be caused by dissipation. For simplification, the thermal diffusivity and molecular viscosity are assumed to be constant for this exercise, and the gradients of the mean entropy s_0 and heat capacity ratio γ are neglected. Starting from the equations of Table 2, integrating over the whole domain, and setting q_1 and all boundary fluxes to zero lead to the equations

$$\frac{\partial}{\partial t} \iiint_\Omega e_1 \, \mathrm{d}\Omega = -\lambda \frac{\gamma - 1}{\gamma p_0} \iiint_\Omega \nabla p_1 \cdot \nabla T_1 \, \mathrm{d}\Omega - \iiint_\Omega \vec{\boldsymbol{\tau}}_1 : \vec{\nabla} \mathbf{u}_1 \, \mathrm{d}\Omega \tag{25}$$

and

$$\frac{\partial}{\partial t} \iiint_\Omega e_{\rm tot} \, \mathrm{d}\Omega = -\frac{\lambda}{T_0} \iiint_\Omega (\nabla T_1)^2 \, \mathrm{d}\Omega - \iiint_\Omega \vec{\boldsymbol{\tau}}_1 : \vec{\nabla} \mathbf{u}_1 \, \mathrm{d}\Omega. \tag{26}$$

The last term on the r.h.s. is shared by both energy forms and is always negative, because it is the classic dissipation function related to the velocity fluctuations. This means that both e_1 and $e_{\rm tot}$ decrease when only viscosity is present. This is, however, not the case when only diffusivity is present and Eqs. (25) and (26) provide a better understanding of Chu's criterion to choose an energy definition: if the flow is isentropic, pressure (p_1) and temperature (T_1) fluctuations are in phase and the r.h.s. term of Eq. (25) is always negative so that the AE form e_1 is a proper estimate of energy. In all other cases, however, the r.h.s. term of Eq. (25) can take any sign, increasing or decreasing the energy and making the AE form e_1 a quantity of limited interest. On the other hand, the r.h.s. term of Eq. (26) is a truly dissipative term in all flows, even if they are not isentropic. This suggests that only the FE form of the energy should be used in flames. Whether this linearized form is valid in flames that exhibit very large entropy fluctuations remains to be checked. Postprocessing of LES fields of compressible reacting turbulent flows is also needed to estimate the magnitude of the different terms of the criterion derived for the FE form, Eq. (24). Specifically, such data would be necessary to study the behavior of the $T_1 q_1$ term with respect to its classic $p_1 q_1$ counterpart, as well as the magnitude of the term related to the mean entropy gradient.

4. Conclusion

This article describes the construction of conservation equations for fluctuation energies in reacting flows. The AE form is first constructed starting from a nonlinear energy equation. It is shown that the usual Rayleigh term is the source term of this equation, but that another term (acoustic losses) plays a significant role in the budget of this equation. Second, it is shown that the AE form is insufficient to describe fluctuations in flames where entropy waves play a role. A new energy (fluctuation energy) is defined and its conservation equation is derived. This equation shows that a different stability criterion is obtained in which temperature and heat release must be in phase to trigger the instability, while the Rayleigh criterion predicts instability when pressure and heat release are in phase. Another source term due to entropy gradients is also exhibited. These fundamental equations are believed to be building blocks to analyze results produced by recent compressible large eddy simulations of turbulent flames in which all terms of the energy equations can be examined. In the long term, closing budgets of fluctuation energies in LES of unstable combustors could impact the understanding of combustion instabilities like budgets of turbulent kinetic energy did for turbulence near wall-bounded flows [9,10] 15 years ago.

Acknowledgment

A significant part of this work was achieved during the 2004 Summer Program of the Center for Turbulence Research. Fruitful discussions with Professor S. Lele and J. Freund are gratefully acknowledged, as is the support of CTR and CERFACS for both authors.

References

[1] A.A. Putnam, Combustion Driven Oscillations in Industry, American Elsevier, 1971.
[2] F.A. Williams, Combustion Theory, Benjamin Cummings, Menlo Park, CA, 1985.
[3] T. Poinsot, D. Veynante, Theoretical and Numerical Combustion, R.T. Edwards, 2001.
[4] L. Rayleigh, Nature 18 (1878) 319–321.
[5] Y. Huang, H.G. Sung, S.Y. Hsieh, V. Yang, J. Prop. Power 19 (2003) 782–794.
[6] K. Mahesh, G. Constantinescu, P. Moin, J. Comput. Phys. 197 (2004) 215–240.
[7] C.D. Pierce, P. Moin, J. Fluid Mech. 504 (2004) 73–97.
[8] L. Selle, G. Lartigue, T. Poinsot, R. Koch, K.-U. Schildmacher, W. Krebs, B. Prade, P. Kaufmann, D. Veynante, Combust. Flame 137 (2004) 489–505.
[9] N. Mansour, J.W. Kim, P. Moin, J. Fluid Mech. 194 (1988) 15–44.
[10] P. Moin, K. Mahesh, Ann. Rev. Fluid Mech. 30 (1998) 539–578.
[11] B.T. Chu, Acta Mech. (1965) 215–234.
[12] L. Landau, E. Lifchitz, Physique Theorique, Tome 6, Mecanique des Fluides, Librairie du Globe–MIR, 1984.
[13] D.G. Crighton, A. Dowling, J. Ffowcs Williams, M. Heckl, F. Leppington, Modern Methods in Analytical Acoustics, Springer-Verlag, New York, 1992.
[14] C. Martin, L. Benoit, F. Nicoud, T. Poinsot, in: Proc. of the Summer Program, Center for Turbulence Research, NASA Ames/Stanford University, 2004, pp. 377–394.