MULTIPLICITY ONE AT FULL CONGRUENCE LEVEL

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Abstract. Let F be a totally real field in which p is unramified. Let $\overline{r} : G_F \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ be a modular Galois representation which satisfies the Taylor–Wiles hypotheses and is tamely ramified and generic at a place v above p. Let $\mathfrak{m}$ be the corresponding Hecke eigensystem. We describe the $\mathfrak{m}$-torsion in the mod p cohomology of Shimura curves with full congruence level at v as a $\mathrm{GL}_2(k_v)$-representation. In particular, it only depends on $\overline{r}|_{I_{F_v}}$ and its Jordan–Hölder factors appear with multiplicity one. The main ingredients are a description of the submodule structure for generic $\mathrm{GL}_2(\mathbf{F}_q)$-projective envelopes and the multiplicity one results of [EGS15].

1. Introduction

Fix a prime p and a totally real field $F/\mathbf{Q}$. Fix a modular Galois representation $\overline{r} : G_F \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ with corresponding Hecke eigensystem $\mathfrak{m}$. Fix a place $v|p$ of F. The mod p local-global compatibility predicts that the $\mathfrak{m}$-torsion subspace, which we denote by π, in the mod p cohomology of a Shimura curve with infinite level at v realizes the mod p Langlands correspondence for $\mathrm{GL}_2(F_v)$ (see [Bre10]), generalizing the case of modular curves ([Col10, Eme11, Paš13]). The goal of the mod p local Langlands program is then to describe π in terms of the restriction to the decomposition group at v, $\overline{r}|_{G_v}$, though it is not even known whether π depends only on $\overline{r}|_{G_v}$. One of the major difficulties is that little is known about supersingular representations outside of the case of $\mathrm{GL}_2(\mathbf{Q}_p)$ (see [AHHV14]).

We now assume that p is unramified in F and that $\overline{r}|_{G_v}$ is 1-generic (see Definition 4.1). Let $K = \mathrm{GL}_2(\mathcal{O}_v)$ and $I_1 \subset K$ be the usual pro-p Iwahori subgroup. [BDJ10] and [Bre14] conjecturally describe the K-socle and I_1-invariants of π—in particular they should satisfy mod p multiplicity one. [Gee11b] and [EGS15] later confirmed these conjectures. [Bre14] shows that such a π (also satisfying other properties known for $\mathfrak{m}$-torsion in completed cohomology) must contain a member of a family of representations constructed in [BP12]. If $f = 1$, this family has one element, and produces the mod p Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$. For $f > 1$, each family is infinite (see [Hu10]), and so, in contrast to the $\mathrm{GL}_2(\mathbf{Q}_p)$-case, a naïve one-to-one correspondence cannot exist. Moreover, the K-socle and the I_1-invariants are not sufficient to specify a single mod p $\mathrm{GL}_2(\mathbf{Q}_{p^f})$-representation when $f > 1$.

However, [EGS15] proves a stronger multiplicity one result than what is used in the construction of [BP12], namely a result for any lattice in a tame type that has irreducible cosocle. In this paper, we strengthen this result in tame situations as follows (cf. Corollary 5.4). Let $K(1) \subset K$ be the first congruence subgroup.

Theorem 1.1. *Suppose that $\overline{r}$ is 1-generic and tamely ramified at v and satisfies the Taylor–Wiles hypotheses. Then the $\mathrm{GL}_2(k_v)$-representation $\pi^{K(1)}$ is isomorphic to the representation $D_0(\overline{r}|_{G_v})$ (which depends only on $\overline{r}|_{I_v}$) constructed in* [BP12]. *In particular, its Jordan–Hölder constituents appear with multiplicity one.*

If the Jordan–Hölder constituents of a $\mathrm{GL}_2(k_v)$-representation appear with multiplicity one, we say that the representation is *multiplicity free*.

Corollary 1.2. *There exists a supersingular* $\mathrm{GL}_2(F_v)$*-representation* π *such that* $\pi^{K_v(1)}$ *is multiplicity free.*

Remark 1.3. We know of no purely local proof of this result.

Proof. We can and do choose $\overline{r}$ such that $\overline{r}|_{G_v}$ is generic and irreducible by [GK14, Corollary A.3]. Then the $\mathrm{GL}_2(F_v)$-socle π' of π is supersingular (and irreducible) by [EGS15, Corollary 10.2.3] and [BP12, Theorem 1.5(i)], and $\pi'^{K_v(1)} \subset \pi^{K_v(1)}$ is multiplicity free by Theorem 1.1. □

The theorem is obtained by combining results of [EGS15] with a description of the submodule structure of generic $\mathrm{GL}_2(k_v)$-projective envelopes (see Theorem 3.14). Note that this theorem rules out infinitely many representations constructed in the proof of [Hu10, Theorem 4.17] from appearing in completed cohomology. It is not clear to the authors whether the results of [Bre14, EGS15] uniquely characterize π when $\overline{r}$ is tamely ramified.

We now make a brief remark on the genesis of this paper. The second and third authors arrived independently at a proof of Theorem 1.1 (in an unreleased preprint) following a different argument, but related to the strategy presented here which was outlined in an unreleased preprint by the first author. Relating the two approaches led to this collaboration. After our paper had been written, we were notified that Hu and Wang also obtained a similar result independently [HW].

We now give a brief overview of the paper. In Section 2, we describe the extension graph, which simplifies the combinatorics of Serre weights. Section 3 is the technical heart of the paper, where we describe the submodule structure of generic $\mathrm{GL}_2(\mathbf{F}_q)$-projective envelopes. In Section 4, we use the results of Section 3 to give two different characterizations of a construction of [BP12]. Finally, in Section 5, we derive our main result.

1.1. **Acknowledgments.** Many of the ideas in this article, especially the combinatorics of Section 2 and proof of Proposition 3.8, came out of the joint work [LLHLM16] of the first two authors with Bao V. Le Hung and Brandon Levin. We thank them heartily for their collaboration. We thank Florian Herzig for answering questions and providing references on modular representation theory. We thank Matthew Emerton for numerous long discussions about p-adic Langlands. We thank Yongquan Hu for answering a question about $K(1)$-invariants. The debt this article owes to the work of Christophe Breuil, Matthew Emerton, Toby Gee, Vytautas Paškūnas, and David Savitt will be obvious to the reader.

1.2. **Notation.** We introduce some notation that will be in force throughout. If F is any field, we write $\overline{F}$ for a separable closure of F and $G_F := \mathrm{Gal}(\overline{F}/F)$ for the absolute Galois group of F. If F is a global field and v is a place of F, we fix an embedding $\overline{F} \hookrightarrow \overline{F}_v$, and we write $I_v \subset G_v$ to denote the inertia and decomposition subgroups at v of G_F. We further write $\varpi_v \in F_v$ to denote an uniformizer.

Throughout the paper, the place v will divide p, and $F_v/\mathbf{Q}_p$ will be an unramified extension of degree f. Let $q = p^f$. We fix a coefficient field $\mathbf{F}$ which is a finite extension of $\mathbf{F}_q$. Without further mention, all representations will be over $\mathbf{F}$. We fix an embedding $\iota_0 : \mathbf{F}_q \hookrightarrow \mathbf{F}$. The letters i and j will denote elements of $\mathbf{Z}/f$. Let $\iota_i = \iota_0 \circ \varphi^i$ be the i-th Frobenius twist of ι_0.

Let G be the algebraic group $\mathrm{Res}_{\mathbf{F}_q/\mathbf{F}_p}\mathrm{GL}_2$. Let T (resp. T_{GL_2}) be the diagonal torus in $\mathrm{Res}_{\mathbf{F}_q/\mathbf{F}_p}\mathrm{GL}_2$ (resp. in GL_2). Let W be the Weyl group of (G,T) (or sometimes the analogous version for SL_2). Let $X^*(T)$ be the character group which is identified with $\mathbf{Z}^2$ in the standard way. In particular if $\mu_i \in X^*(T_{\mathrm{GL}_2})$ and $\mu_i^{(i)} = \mu_i \circ \iota_i$ then $\sum \mu_i^{(i)} \mapsto (\mu_i)_i$ under this identification. Let $X^0(T) \subset X^*(T)$ be the subgroup generated by $\det \circ \iota_i$. We write $C_0 \subseteq X^*(T)$ for the base alcove for $X^*(T)$. Concretely we have $\mu \in C_0$ if and only if $0 < \langle \mu + \eta, \alpha^\vee \rangle < p$ for all positive coroots $\alpha^\vee$. We say that a weight μ is *p-restriced* if it belongs to the base alcove C_0. It is customary to write $X_1(T)$ for the set of p-restricted weights.

Let $G^{\mathrm{der}} = \mathrm{Res}_{\mathbf{F}_q/\mathbf{F}_p}\mathrm{SL}_2$ and T^{der} be the standard torus. We write $\Lambda_W = X^*(T^{\mathrm{der}})$ for the weight lattice for G^{der} and $\Lambda_R \subset \Lambda_W$ for the root lattice. We consider the usual section $\Lambda_W \to X^*(T)$ obtained by identifying the fundamental dominant weights $\omega_i \in \Lambda_W$ with the character $((1,0)\delta_{ij})_j \in (\mathbf{Z}^2)^f$, δ_{ij} being the usual Kroneker delta. We embed $\Lambda_R \subset X^*(T)$ so that its image in $X^*(Z)$ is trivial. Let $\eta = \sum_i \omega_i^{(i)}$. Let π be the action of Frobenius on $X^*(T)$ so that, for instance, $\pi\omega_i = \omega_{i+1}$.

For a dominant character $\mu \in X^*(T)$ we write $V(\mu)$ for the Weyl module defined in [Jan03], II.2.13(1). It has a unique simple G-quotient $L(\mu)$. If $\mu = \sum_i \mu_i^{(i)}$ is p-restricted then $L(\mu) = \otimes_i L(\mu_i)^{(i)}$ by the Steinberg tensor product theorem, see [Her09, Theorem 3.9] (as usual $L(\mu_i)^{(i)}$ denotes the i-th Frobenius twist of $L(\mu_i)$). Let Γ be the group $G(\mathbf{F}_p) \cong \mathrm{GL}_2(\mathbf{F}_q)$. Let $F(\mu)$ be the Γ-representation $L(\mu)|_\Gamma$, which remains irreducible by [Her09, A.1.3]. Note that $F(\mu) \cong F(\lambda)$ if and only if $\mu \cong \lambda \mod (p-\pi)X^0(T)$.

Let $\widetilde{W}_a$ denote the affine Weyl group for G^{der}. It is the semidirect product $\Lambda_R \rtimes W$ acting in the usual way on Λ_W. Similar comments apply to the extended affine Weyl group $\widetilde{W}$ of G, defined as the semidirect product $X^*(T) \rtimes W$. If $\lambda \in \Lambda_R$ (resp. $\lambda \in X^*(T)$) we write t_λ for the image of $\lambda \in \Lambda_R$ (resp. $\lambda \in X^*(T)$) under the usual embedding $\Lambda_R \hookrightarrow \widetilde{W}_a$ (resp. $X^*(T) \hookrightarrow \widetilde{W}$), i.e. t_λ is the translation by λ in the dominant direction. Note that we can extend the Frobenius action on the affine Weyl groups by declaring $(\pi s)_j = s_{j+1}$ for $s \in W$.

For $\widetilde{w} \in \widetilde{W}$ we will use $\cdot$ to denote the dot action $\widetilde{w} \cdot \mu = \widetilde{w}(\mu + \eta) - \eta$. Let C_0 be the p-resticted dominant alcove for G and A the p-resticted dominant alcove for G^{der}. Let $W_+ \subset \widetilde{W}$ be the stabilizer of A under the dot action. For instance, when $f = 1$, the set W_+ is formed by the elements id and $(12)t_{-p\omega}$.

2. The extension graph

In this section, we describe what is called the extension graph in [LLHLM16, §2] for GL_2. The modifications from GL_3 are simple.

Definition 2.1. Let $S_e = \{\omega_i\}_i$. For $J \subset S_e$, let

$$\omega_J = \sum_{\omega \in J} \omega. \tag{2.1}$$

The following lemma is easily checked.

Lemma 2.2. *The set $\{\omega_J\}_{J \subset S_e}$ is a set of representatives for Λ_W/Λ_R. For each J, there is a unique $w_J \in W$ such that $w_J t_{-p\pi^{-1}\omega_J} \in W_+$.*

Note that the content of Lemma 2.2 can be rephrased by saying that the map

$$\begin{aligned}\Lambda_R \times W_+ &\longrightarrow \Lambda_W\\ (\nu, wt_{-p\pi^{-1}\omega}) &\longmapsto (\nu+\omega)\end{aligned}$$

is a bijection.

Let μ be $\sum_{i=0}^{f-1}\mu_i^{(i)}$ where μ_i are dominant generic p-restricted weights. By Lemma 2.2 we define a map

$$\begin{aligned}\mathfrak{t}'_\mu : \Lambda_W &\to X^*(T)\\ \omega+\nu &\mapsto wt_{\nu+\omega-p\pi^{-1}\omega}\cdot(\mu-\eta)\end{aligned}$$

where ν is a root, $\omega = \omega_J$ for some $J \subset S_e$, and $wt_{-p\pi^{-1}\omega} \in W_+$ is uniquely determined by ω. Note that on the right hand side the elements ω, ν are considered as element of $X^*(T)$ via our chosen section $\Lambda_W \to X^*(T)$

We define $\mathfrak{t}_\mu$ as be the composition of $\mathfrak{t}'_\mu$ with the projection map $X^*(T) \to X^*(T)/(p-\pi)X^0(T)$ and set

$$\Lambda_W^\mu = \big\{\omega \in \Lambda_W : \mathfrak{t}_\mu(\omega) \in C_0 + (p-\pi)X^0(T)\big\}$$

We establish some properties of $\mathfrak{t}_\mu$. From now on we will use the same notation to denote the restriction of $\mathfrak{t}_\mu$ to Λ_W^μ. This shall cause no confusion.

Proposition 2.3. *The map $\mathfrak{t}_\mu$ is injective.*

Proof. Let $\omega'_1,\ \omega'_2 \in \Lambda_W^\mu$. For each $i \in \{1,2\}$ we have a unique decomposition $\omega'_i = \omega_{J_i} + \nu_i$ where $\nu_i \in \Lambda_R$ and ω_{J_i} defined as in (2.1) (Lemma 2.2). Let $\Sigma : X^*(T) \to X^*(Z)$ be the natural map induced by the inclusion $Z \hookrightarrow T$. It is π-equivariant. The choice of our section $\Lambda_W \hookrightarrow X^*(T)$ identifies $\Lambda_R = \ker(\Sigma)$. Moreover $\Sigma(X^0(T)) \subseteq 2X^*(Z)$. Hence, if $\mathfrak{t}_\mu(\omega'_1) = \mathfrak{t}_\mu(\omega'_2)$, we obtain:

$$\Sigma(\omega_{J_1} - \omega_{J_2}) - p\pi^{-1}\Sigma(\omega_{J_1} - \omega_{J_2}) \in 2(p-\pi)X^*(Z)$$

or, in other words,

$$(p-\pi)\big(\Sigma(\omega_{J_1} - \omega_{J_2}) - 2n\big) = 0$$

for some $n \in X^*(Z)$. As $(p-\pi)$ is injective on $X^*(Z)$ we conclude that

$$\Sigma(\omega_{J_1}) \equiv \Sigma(\omega_{J_2}) \mod 2X^*(Z)$$

which in turn implies $\omega_{J_1} = \omega_{J_2}$ since $\langle \omega_{J_i}, \alpha\rangle \in \{0,1\}$ for all positive roots $\alpha \in R$. By Lemma 2.2 we conclude that $w_{J_1} = w_{J_2}$ hence finally $\nu_1 = \nu_2$.

□

The following proposition gives symmetries of the extension graph.

Proposition 2.4. *Suppose that $\lambda - \eta = \mathfrak{t}_\mu(\omega+\nu)$ where $wt_{-p\pi^{-1}\omega} \in W_+$, $\eta \in \Lambda_R$ (and $\omega+\nu \in \Lambda_W^\mu$). Then $\mathfrak{t}_\lambda(\omega') = \mathfrak{t}_\mu(w^{-1}(\omega') + \omega + \nu)$ for $\omega' \in \Lambda_W^\lambda$.*

Proof. This can be checked by direct computation. □

We now recall the definition of deepness for a weight.

Definition 2.5. Let $\lambda \in X^*(T)$ be a dominant weight and let $n \in \mathbf{N}$. We say that λ lies n-deep in its alcove if for each positive coroot $\alpha^\vee$ there exist integers $m_\alpha \in \mathbf{Z}$ such that $pm_\alpha + n < \langle \lambda+\eta, \alpha^\vee\rangle < p(m_\alpha+1) - n$.

Definition 2.6. Let $\mu = \sum_i \mu_i^{(i)} \in X^*(T)$ be a p-restricted character where $\mu_i = (a_i, b_i) \in \mathbf{Z}^2$. We say that μ is *generic* if μ is 0-deep in alcove C_0, that is to say $2 \leq a_i - b_i \leq p-2$ for all i.

Following [LLHLM16] we can introduce the notion of adjacency in the extension graph.

Definition 2.7. Two elements $\omega,\ \omega' \in \Lambda_W^\mu$ are said to be *adjacent* if $\omega - \omega' \in \{\pm\omega_j\}$ for some index j.

The following proposition justifies the name extension graph. Recall that Γ denotes the group $G(\mathbf{F}_p) \cong \mathrm{GL}_2(\mathbf{F}_q)$.

Proposition 2.8. *Let $\omega, \omega' \in \Lambda_W^\mu$ such that $\lambda - \eta := \mathfrak{t}_\mu(\omega)$ and $\lambda' - \eta := \mathfrak{t}_\mu(\omega')$ are generic. Then*

$$\dim \mathrm{Ext}^1_\Gamma(F(\lambda-\eta), F(\lambda'-\eta)) = \dim \mathrm{Ext}^1_\Gamma(F(\lambda'-\eta), F(\lambda-\eta)) \leq 1$$

with equality if and only if ω and ω' are adjacent in the graph Λ_W^μ.

Proof. By Proposition 2.4, we can assume without loss of generality that $\omega = 0$. Then the extensions of $\sigma := F(\mu-\omega)$ are given by the first layer of the cosocle filtration of the projective envelope of σ. The proposition now follows from Propositions 3.2 and 3.6 (which do not depend on this proposition). □

We next show that a set of modular Serre weights forms a hypercube in the extension graph. Recall that:

Definition 2.9. We say that a weight $\lambda \in X^*(T)$ is *regular* if $0 \leq \langle \lambda, \alpha \rangle < p-1$ for all positive roots $\alpha \in \Lambda_R$. (This is equivalent to ask $\lambda \in X^*(T)$ to be 0-deep.) We write $X_{\mathrm{reg}}(T) \subseteq X_1(T)$ for the set of regular weights.

We write $\mathscr{W} = X_1(T)/(p-\pi)X^0(T)$ for the set of Serre weights, and $\mathscr{W}_{\mathrm{reg}} = X_{\mathrm{reg}}(T)/(p-\pi)X^0(T)$ for the set of regular Serre weights. We have a bijection $\mathcal{R} : X^*(T) \to X^*(T)$ (also called *Herzig reflection*) defined by $\lambda \mapsto w_0 t_{-p\eta} \cdot \lambda$. It induces a bijection $\mathcal{R} : \mathscr{W}_{\mathrm{reg}} \to \mathscr{W}_{\mathrm{reg}}$.

For $s \in W$ and a character $\mu \in X^*(T)$, we denote the corresponding Deligne–Lusztig representation as in [Her09, Lemma 4.2] by $R_s(\mu)$. It is easy to see that if $\mu - \eta$ is n-deep then any weight $F(\lambda-\eta) \in \mathrm{JH}(\overline{R}_s(\mu))$ is $n-1$-deep. In particular, if $\mu - \eta$ is 1-deep, then $\#\,\mathrm{JH}(\overline{R}_s(\mu)) = 2^f$ and all the Jordan-Hölder constituents in $\mathrm{JH}(\overline{R}_s(\mu))$ are generic in the sense of Definition 2.6. Following [GHS15, §9.1] an L-parameter for G is, in our context, a continuous morphism $I_{F_v} \to \mathrm{GL}_2(\mathbf{F})$ which extends to G_{F_v}. Given an inertial L-parameter τ we can associate a Deligne-Lusztig representation $V_\phi(\tau)$ following [GHS15, Proposition 9.2.1]. We define the set $W^?(\tau)$ as

$$W^?(\tau) = \{\mathcal{R}(F),\ F \in \mathrm{JH}(\overline{V_\phi(\tau)})\}.$$

Proposition 2.10. *Suppose that τ is an inertial L-parameter such that $V_\phi(\tau) = R_s(\mu)$. Assume that $\mu - \eta$ is 1-deep. Then $W^?(\tau) = F(\mathfrak{t}_\mu(\{s\omega_J : J \subset S_e\}))$.*

Proof. The obvious crystalline lifts, in the sense of [GHS15, §7.1], have Hodge–Tate weights $wt_{s\pi\omega - p\omega}(\mu)$, where $wt_{-p\omega}$ ranges over all elements of $\widetilde{W}$ fixing the dominant base alcove. Observe that ω ranges over ω_J with $J \subset S_e$. Noting that

$$wt_{s\omega - p\pi^{-1}\omega}(\mu) - \eta = wt_{s\omega - p\pi^{-1}\omega}(\mu - \eta) = \mathfrak{t}_\mu(s\omega),$$

we have that, in the notation of [GHS15], $W_{\mathrm{obv}}(\tau) = F(\mathfrak{t}_\mu(\{s\omega_J : J \subset S_e\}))$. Finally, we have $W^?(\tau) = W_{\mathrm{obv}}(\tau)$ (see [Gee11a, §4.2]). □

3. Generic $\mathrm{GL}_2(\mathbf{F}_q)$-projective envelopes

In this section, we describe the submodule structure of generic $\mathrm{GL}_2(\mathbf{F}_q)$-projective envelopes. Recall that Γ is the group $G(\mathbf{F}_p) \cong \mathrm{GL}_2(\mathbf{F}_q)$ and if R is a Γ-representation, we write $R^{(i)}$ to denote its i-th Frobenius twist. In what follows we set $\mu = \sum_{i=0}^{f-1} \mu_i \in X^*(T)$ where $\mu_i = (a_i, b_i) \in \mathbf{Z}^2$.

The following known theorem gives a coarse description of generic Γ-projective envelopes.

Theorem 3.1. *Suppose that for all i, $1 \leq a_i - b_i \leq p-1$. The projective (and injective) envelope of the weight $F(\mu-\eta)$ is of the form $R_\mu = \otimes_{i=0}^{f-1} R_{\mu_i}^{(i)}$, where*

(1) R_{μ_i} is a Γ-representation with a filtration $\mathrm{Fil}^0 R_{\mu_i} = R_{\mu_i}$, $\mathrm{Fil}^1 R_{\mu_i} \cong V(t_{(p,-p)} w_0 \cdot (\mu_i - \omega_i))$, $\mathrm{Fil}^2 R_{\mu_i} \cong F(\mu_i - \omega_i)$, and $\mathrm{Fil}^3 R_{\mu_i} = 0$, and
(2) $\mathrm{gr}^0 R_{\mu_i}$ and $\mathrm{gr}^2 R_{\mu_i}$ are isomorphic to $F(\mu_i - \omega_i)$ and $\mathrm{gr}^1 R_{\mu_i}$ is isomorphic to $F(w_0 t_{-p\omega_i} \cdot (\mu_i - \omega_i)) \otimes F(\omega_i)^{(1)}$.

Proof. See [BP12, §3, Lemmas 3.4, 3.5]. □

The filtrations on R_{μ_i} induce a tensor multifiltration on R_μ. More precisely, the set $\{0,1,2\}^f$ has a partial order so that $(k_i)_i = \mathbf{k} \leq \mathbf{k}' = (k'_i)_i$ if $k_i \leq k'_i$ for all $i \in \mathbf{Z}/f$. We write $\mathbf{k} < \mathbf{k}'$ if $\mathbf{k} \leq \mathbf{k}'$ and $\mathbf{k} \neq \mathbf{k}'$. For $\mathbf{k} = (k_i)_i \in \{0,1,2\}^f$, let $\mathrm{Fil}^{\mathbf{k}} R_\mu := \otimes_i \mathrm{Fil}^{k_{i+1}} R_{\mu_i}^{(i)}$. Then $\mathrm{Fil}^{\mathbf{k}'} R_\mu \subsetneq \mathrm{Fil}^{\mathbf{k}} R_\mu$ if and only if $\mathbf{k} < \mathbf{k}'$. Let $\mathrm{Fil}^{>\mathbf{k}} R_\mu = \sum_{\mathbf{k}<\mathbf{k}'} \mathrm{Fil}^{\mathbf{k}'} R_\mu$. Let $\mathrm{gr}^{\mathbf{k}} R_\mu = \mathrm{Fil}^{\mathbf{k}} R_\mu / \mathrm{Fil}^{>\mathbf{k}} R_\mu$. To ease notation, we will also denote $\mathrm{gr}^{\mathbf{k}} R_\mu$ by $W_{\mathbf{k}}$. For $\mathbf{k} = (k_i)_i \in \{0,1,2\}^f$, let $|\mathbf{k}| = k = \sum_i k_i$. There is also the tensor filtration $\mathrm{Fil}^k_\otimes R_\mu = \sum_{|\mathbf{k}|=k} \mathrm{Fil}^{\mathbf{k}} R_\mu$. Note in particular that for all $\mathbf{k} \in \{0,1,2\}^f$ we have a natural surjection $R_\mu / \mathrm{Fil}^{>\mathbf{k}} R_\mu \twoheadrightarrow R_\mu / \mathrm{Fil}^k_\otimes R_\mu$ whose restriction to $W_{\mathbf{k}} \subseteq R_\mu / \mathrm{Fil}^{>\mathbf{k}} R_\mu$ is injective.

Proposition 3.2. $\mathrm{gr}^k_\otimes R_\mu = \oplus_{|\mathbf{k}|=k} W_{\mathbf{k}}$.

Proof. This follows from general facts about tensor products of filtered objects. □

To describe the representations $W_{\mathbf{k}}$, we will need the following translation principles.

Proposition 3.3. *Let $\lambda-\eta$, $\omega \in X^*(T)$ be dominant weights in alcove C_0. Assume that the weight space of $F(\omega)$ is multiplicity free and that for all weights ν in $F(\omega)$, $\lambda - \eta + \nu$ is still in alcove C_0. Then we have isomorphisms*

$$F(\lambda-\eta) \otimes F(\omega) \cong \oplus_{\nu \in F(\omega)} F(\lambda - \eta + \nu).$$

Assume moreover that $\lambda - \eta + \nu \neq 0$ for all $\nu \in F(\omega)$ and let $R_\lambda = \otimes_i R_{\lambda_i}^{(i)}$ denote the projective envelope of $F(\lambda-\eta)$ (cf. Theorem 3.1). Then

$$R_\lambda \otimes F(\omega) \cong \oplus_{\nu \in F(\omega)} R_{\lambda+\nu} \cong \oplus_{\nu \in F(\omega)} \otimes_{i=0}^{r-1} R_{\lambda_i+\nu_i}^{(i)}$$

where we have written $\nu = \sum_{i=0}^{f-1} \nu_i^{(i)}$ for all $\nu \in F(\omega)$.

Remark 3.4. In the statement of the Proposition 3.3 assume that $\langle \omega, \alpha^\vee \rangle \leq 1$ for all positive coroots $\alpha^\vee$. It is then easy to check that the proposition applies as soon as $\lambda - \eta$ is 1-deep and $\langle \lambda - \eta, \alpha^\vee \rangle > 1$ for at least one positive coroot $\alpha^\vee$.

Proof. We prove the analogous results for G^{der}. By [Pil93, Lemma 5.1(i)] we have a G^{der}-decomposition $L(\lambda)\otimes L(\omega)\cong\oplus_{\nu\in L(\omega)}L(\lambda+\nu)$ and the first statement for G^{der} follows by restriction to the finite group $G^{\mathrm{der}}(\mathbf{F}_p)$.

As for the second statement, we need to recall some standard facts about injective envelopes of Frobenius kernels. Let T_{SL_2} be the standard torus of $\mathrm{SL}_{2/\mathbf{F}_p}$. For any $r\geq 1$ we let $(\mathrm{SL}_2)_r$ denote the r-th Frobenius kernel of SL_2 and, for any weight $\lambda\in X_r(T_{\mathrm{SL}_2})$ we write $Q_r(\lambda)$ for the injective envelope of $L(\lambda)|_{(\mathrm{SL}_2)_rT_{\mathrm{SL}_2}}$. Under our assumption on p the $(\mathrm{SL}_2)_rT_{\mathrm{SL}_2}$-module $Q_r(\lambda)$ has a unique SL_2-module structure, as well as a SL_2-equivariant decomposition:

$$Q_r(\lambda)\cong\otimes_{i=0}^{r-1}Q_1(\lambda_i)^{(i)} \tag{3.1}$$

if λ decomposes as $\lambda=\sum_{i=0}^{r-1}p^i\lambda_i$ with each $\lambda_i\in X^*(T_{\mathrm{SL}_2})$ being p-restricted.

Assume now that $\omega\in X_r(T_{\mathrm{SL}_2})$ is such that $L(\omega)$ is multiplicity free and $\lambda+\nu$ lies in the same alcove as λ for any weight $\nu\in L(\omega)$. By [Pil93, Lemma 5.1(ii)] and 3.1 we have a decomposition

$$Q_r(\lambda)\otimes L(\omega)\cong\oplus_{\nu\in L(\omega)}Q_r(\lambda+\nu)\cong\oplus_{\nu\in L(\omega)}\otimes_{i=0}^{r-1}Q_1(\lambda_i+\nu_i)^{(i)} \tag{3.2}$$

where we have written $\nu=\sum_{i=0}^{r-1}p^i\nu_i$ with $\nu_i\in X_1(T_{\mathrm{SL}_2})$ for all $\nu\in L(\omega)$.

By [AJL83, Lemma 4.1] if the weight $\lambda\in X_r(T_{\mathrm{SL}_2})$ satisfies $\lambda\neq 0$ then $Q_r(\lambda)$ (endowed with its SL_2-module structure) restricts to the injective envelope of the $\mathrm{SL}_2(\mathbf{F}_p^r)$-representation $F(\lambda)$. The second statement of the Propostion for G^{der} follows now from 3.2.

The statements for G are now deduced from the previous results on G^{der} by a formal argument, cf. for instance [LLHLM16], Theorem 4.1.3. □

From now on we assume that $\mu-\eta$ is 1-deep.

Definition 3.5. Let $S=\{\pm\omega_i\}_i$, and $\mathfrak{J}$ be the set of subsets of S. For $J\in\mathfrak{J}$ define $\omega_J:=\sum_{\omega\in J}\omega\in\Lambda_W$ and $\sigma_J:=F(\mathfrak{t}_\mu(\omega_J))$. Finally, let $\mathbf{k}(J):=(k_i(J))_i\in\{0,1,2\}^f$ where $k_{i+1}(J):=\#\{\pm\omega_i\}\cap J$, and let $k(J):=\#J=|\mathbf{k}(J)|$.

The following key multiplicity one result allows one to give a reasonable description of the submodule structure of generic Γ-projective envelopes.

Proposition 3.6. *Let* $\mathbf{k}\in\{0,1,2\}^f$. *Then* $W_{\mathbf{k}}\cong\bigoplus_{J\in\mathfrak{J},\ \mathbf{k}(J)=\mathbf{k}}\sigma_J$. *Moreover, this sum is multiplicity free.*

Proof. By definition and Theorem 3.1(2) we have $W_{\mathbf{k}}\cong\otimes_i(F(\lambda_i-\omega_i)\otimes F(\nu_i))^{(i)}$ where $\lambda_i-\omega_i=w_0t_{-p\omega_i}\cdot(\mu_i-\omega_i)$ if $k_{i+1}=1$ and $\lambda_i-\omega_i=\mu_i-\omega_i$ otherwise, and $\nu_i=\omega_i$ if $k_i=1$ and $\nu_i=0$ otherwise. Note that $\lambda_i-\omega_i$ is n-deep in alcove C_0 if and only if $\mu_i-\omega_i$ is n-deep in alcove C_0. By Proposition 3.3, $F(\lambda_i-\omega_i)\otimes F(\omega_i)\cong F(\lambda_i)\oplus F(\lambda_i+(-1,1))$. In particular, $W_{\mathbf{k}}$ is semisimple and of length 2^δ, where $\delta=\#\{i:k_i=1\}$.

Suppose that $J\in\mathfrak{J}$ such that $\mathbf{k}(J)=\mathbf{k}$. Then $\mathfrak{t}_\mu(\omega_J)_i=(\lambda_i-\omega_i+\omega_i')$ where $\omega_i'=0$ if $k_i\neq 1$ and is ω_i or $w_0\omega_i$ otherwise. By the last paragraph, there is an inclusion $\sigma_J\hookrightarrow W_{\mathbf{k}}$. One easily checks that $\#\{J\in\mathfrak{J}:\mathbf{k}(J)=\mathbf{k}\}=\#\{\omega_J:\mathbf{k}(J)=\mathbf{k}\}=2^\delta$. Since $\#\{\omega_J:\mathbf{k}(J)=\mathbf{k}\}=\#\{\sigma_J:\mathbf{k}(J)=\mathbf{k}\}$ by Proposition 2.3 and $W_{\mathbf{k}}$ is semisimple and of length 2^δ, we are done. □

By abuse of notation, σ_J will often denote the σ_J-isotypic component of $W_{\mathbf{k}(J)}$, which is isomorphic to σ_J by Proposition 3.6.

In what follows we fix $\mathbf{k} \in \{0,1,2\}^f$ and let $k = |\mathbf{k}|$. Let

$$W_{\mathbf{k},\mathbf{k}+1} := \operatorname{Fil}^{\mathbf{k}} R_\mu / \big(\operatorname{Fil}_\otimes^{k+2} R_\mu \cap \operatorname{Fil}^{\mathbf{k}} R_\mu \big) \subset \operatorname{Fil}_\otimes^k R_\mu / \operatorname{Fil}_\otimes^{k+2} R_\mu.$$

The module $W_{\mathbf{k},\mathbf{k}+1}$ is endowed with the induced filtration from $\operatorname{Fil}_\otimes^k R_\mu / \operatorname{Fil}_\otimes^{k+2}$. This is a two step filtration with associated graded pieces described as follows. We have $\operatorname{gr}^k W_{\mathbf{k},\mathbf{k}+1} = W_{\mathbf{k}}$ and $\operatorname{gr}^{k+1} W_{\mathbf{k},\mathbf{k}+1} = \oplus_{\mathbf{k}'} W_{\mathbf{k}'}$ where the direct sum ranges over the elements $\mathbf{k}' \in \{0,1,2\}^f$ satisfying $\mathbf{k} \leq \mathbf{k}'$ and $k' - k = 1$. We have the following refinement of Proposition 3.6.

Lemma 3.7. *Keep the previous hypotheses and notation. The graded piece*

$$\operatorname{gr}^{k+1} W_{\mathbf{k},\mathbf{k}+1} \subset \operatorname{gr}_\otimes^{k+1} R_\mu$$

is multiplicity free.

Proof. Let $\sigma \in \mathrm{JH}(\operatorname{gr}^{k+1} W_{\mathbf{k},\mathbf{k}+1})$ be a constituent appearing with multiplicity. By Proposition 3.6, we deduce the existence of $J_1, J_2 \in \mathfrak{J}$ with $\mathbf{k}(J_1) \neq \mathbf{k}(J_2)$, $\sigma_{J_1} \cong \sigma \cong \sigma_{J_2}$, and $\mathbf{k}(J_1)$, $\mathbf{k}(J_2)$ are of the form $\mathbf{k}'$ above. In what follows, we write $(k_{1,i})_i = \mathbf{k}_1 := \mathbf{k}(J_1)$ and similarly $\mathbf{k}_2 := \mathbf{k}(J_2)$. Let $j_1, j_2 \in \{0, \dots, f-1\}$ be such that $k_{j_1+1} + 1 = k_{1,j_1+1}$ and $k_{j_2+1} + 1 = k_{2,j_2+1}$. Then $j_1 \neq j_2$, and hence $k_{j_2+1} + 1 = k_{1,j_1+1}$, from which we see that the j_1 component of ω_{J_1} and ω_{J_2} must differ. By Proposition 2.3, we conclude that $\sigma_{J_1} \ncong \sigma_{J_2}$, a contradiction. □

Let now $\mathbf{k}' \in \{0,1,2\}^f$ be as above and let $j \in \mathbf{Z}/f$ be such that $k_{i+1} = k'_{i+1}$ for $i \neq j$ and $k_{j+1} + 1 = k'_{j+1}$. We define

$$W_{\mathbf{k},\mathbf{k}'} := \otimes_{i \neq j} \operatorname{gr}^{k_{i+1}} R_{\mu_i}^{(i)} \otimes (\operatorname{Fil}^{k_{j+1}} R_{\mu_j} / \operatorname{Fil}^{k_{j+1}+2} R_{\mu_j})^{(j)},$$

which is a quotient of $W_{\mathbf{k},\mathbf{k}+1}$. We endow $W_{\mathbf{k},\mathbf{k}'}$ with the induced quotient filtration from $W_{\mathbf{k},\mathbf{k}+1}$; it is a two step filtration with graded pieces $\operatorname{gr}^k W_{\mathbf{k},\mathbf{k}'} = W_{\mathbf{k}}$ and $\operatorname{gr}^{k+1} W_{\mathbf{k},\mathbf{k}'} = W_{\mathbf{k}'}$.

From now on we assume that $\mu - \eta$ is 1-deep and moreover that there exist positive coroots $\alpha^\vee$, $\beta^\vee$ such that $\langle \mu - \eta, \alpha^\vee \rangle > 1$, $\langle \mu - \eta, \beta^\vee \rangle < p - 3$.

Proposition 3.8. *Suppose that $J \subset J'$ and $\# J' \setminus J = 1$. Let $\mathbf{k} = \mathbf{k}(J)$ and $\mathbf{k}' = \mathbf{k}(J')$. Then there is a subquotient of $W_{\mathbf{k},\mathbf{k}'}$ which is the unique up to isomorphism nontrivial extension of σ_J by $\sigma_{J'}$.*

Proof. Suppose that $J' \setminus J \subset \{\pm \omega_j\}$ and that $k_{j+1} = 0$ (resp. $k_{j+1} = 1$). It suffices to show that $\sigma_{J'}$ (resp. σ_J) is not in the cosocle (resp. the socle) of $W_{\mathbf{k},\mathbf{k}'}$. Indeed, this would show that the image of the extension $W_{\mathbf{k},\mathbf{k}'}$ under the map (canonically defined up to scalar) $\operatorname{Ext}^1_\Gamma(W_{\mathbf{k}}, W_{\mathbf{k}'}) \to \operatorname{Ext}^1_\Gamma(W_{\mathbf{k}}, \sigma_{J'})$ (resp. $\operatorname{Ext}^1_\Gamma(W_{\mathbf{k}}, W_{\mathbf{k}'}) \to \operatorname{Ext}^1_\Gamma(\sigma_J, W_{\mathbf{k}'})$) is nonzero. Since by Proposition 2.8, the map (canonically defined up to scalar) $\operatorname{Ext}^1_\Gamma(W_{\mathbf{k}}, \sigma_{J'}) \to \operatorname{Ext}^1_\Gamma(\sigma_J, \sigma_{J'})$ (resp. $\operatorname{Ext}^1_\Gamma(\sigma_J, W_{\mathbf{k}'}) \to \operatorname{Ext}^1_\Gamma(\sigma_J, \sigma_{J'})$) is an isomorphism, we would be done. We show the following: if $k_{j+1} = 0$ (resp. $k_{j+1} = 1$), then the cosocle (resp. the socle) of $W_{\mathbf{k},\mathbf{k}'}$ is isomorphic to $W_{\mathbf{k}}$ (resp. $W_{\mathbf{k}'}$).

Assume that $k_{j+1} = 0$. We freely use the notation in the proof of Proposition 3.6. Hence

$$\operatorname{cosoc}(\operatorname{Fil}^{k_{j+1}} R_{\mu_j} / \operatorname{Fil}^{k_{j+1}+2} R_{\mu_j}) = F(\lambda_j - \omega_j) \otimes F(\nu_j)$$

and we get a surjection $R_{\lambda_j} \otimes F(\nu_j) \twoheadrightarrow F(\lambda_j - \omega_j) \otimes F(\nu_j)$. Noting that $\lambda_i - \omega_i$ is 1-deep for all i the further assumption on $\mu - \eta$ guarantees that Proposition 3.3 applies and hence the latter surjection is actually the projective envelope of the semisimple representation $F(\lambda_j-\omega_j)\otimes F(\nu_j)$. By projectivity we can complete the diagram:

$$\begin{array}{ccc} R_{\lambda_j}\otimes F(\nu_j) & & \\ \vdots & \searrow & \\ \downarrow & & \\ \operatorname{Fil}^{k_{j+1}} R_{\mu_j}/\operatorname{Fil}^{k_{j+1}+2} R_{\mu_j} & \twoheadrightarrow & F(\lambda_j-\omega_j)\otimes F(\nu_j))^{(j)} \end{array}$$

and the dotted arrow is actually *surjective* (as it is surjective on the cosocles by construction). We easily deduce a surjection $\otimes_i(R_{\lambda_i}\otimes F(\nu_i))^{(i)}\twoheadrightarrow W_{\mathbf{k},\mathbf{k}'}$ such that the composite

$$\otimes_i\,(R_{\lambda_i}\otimes F(\nu_i))^{(i)}\twoheadrightarrow W_{\mathbf{k},\mathbf{k}'}\twoheadrightarrow W_{\mathbf{k}} \tag{3.3}$$

is surjective. By Propositions 3.3 and 3.6 we deduce that the domain of (3.3) is actually the projective envelope of $W_{\mathbf{k}}$. We conclude that the cosocle of $W_{\mathbf{k},\mathbf{k}'}$ is $W_{\mathbf{k}}$, as desired.

If $k_{j+1}=1$, one makes the dual argument using the injection $W_{\mathbf{k}'}\hookrightarrow\otimes_i(R_{\lambda_i}\otimes F(\nu_i))^{(i)}$. □

Fix $J\in\mathfrak{J}$. Recall that by Proposition 3.6, there is a unique submodule of $W_{\mathbf{k}(J)}\subset R_\mu/\operatorname{Fil}^{>\mathbf{k}(J)}R_\mu$ isomorphic to σ_J. Let us write $\mathbf{k}:=\mathbf{k}(J)$ and $k:=|\mathbf{k}|$ in what follows. Let P_{σ_J} be a projective envelope of σ_J. Then $\operatorname{Hom}_\Gamma(P_{\sigma_J},\operatorname{gr}^{k+1}_\otimes R_\mu)\cong\operatorname{Hom}_\Gamma(\sigma_J,\operatorname{gr}^{k+1}_\otimes R_\mu)=0$ by Proposition 3.6 and the fact that $\sigma_J\cong\sigma_{J'}$ implies that $\omega_J=\omega_{J'}$ by Proposition 2.3, which implies that $\#J\equiv\#J' \mod 2$. Then since P_{σ_J} is projective, the natural map

$$\operatorname{Hom}_\Gamma(P_{\sigma_J},\operatorname{Fil}^k_\otimes R_\mu/\operatorname{Fil}^{k+2}_\otimes R_\mu)\to\operatorname{Hom}_\Gamma(P_{\sigma_J},\operatorname{gr}^k_\otimes R_\mu)$$

is an isomorphism. Thus the morphism $P_{\sigma_J}\twoheadrightarrow\sigma_J\subset W_{\mathbf{k}}\subset\operatorname{gr}^k_\otimes R_\mu$, uniquely lifts to a morphism $\overline{\psi}_J:P_{\sigma_J}\to\operatorname{Fil}^k_\otimes R_\mu/\operatorname{Fil}^{k+2}_\otimes R_\mu$. Note that since $\sigma_J\subset W_{\mathbf{k}}$, we could also take a lift of $P_{\sigma_J}\twoheadrightarrow\sigma_J\subset W_{\mathbf{k}}$ in $\operatorname{Hom}_\Gamma(P_{\sigma_J},W_{\mathbf{k},\mathbf{k}+1})$, which must coincide with $\overline{\psi}_J$ by uniqueness. We conclude that the image of $\overline{\psi}_J$ lies in $W_{\mathbf{k},\mathbf{k}+1}$. Let $\overline{V}_J$ be the image of $\overline{\psi}_J$, which obtains a filtration from $W_{\mathbf{k},\mathbf{k}+1}$. The following describes the structure of $\overline{V}_J$.

Proposition 3.9. *We have that $\operatorname{gr}^k_\otimes\overline{V}_J=\sigma_J$ and $\operatorname{gr}^{k+1}_\otimes\overline{V}_J=\oplus_{J'}\sigma_{J'}$ where the sum runs over J' such that $J\subset J'$ and $\#J'-\#J=1$.*

Proof. Since $\overline{V}_J$ has irreducible cosocle isomorphic to σ_J, $\operatorname{gr}^k_\otimes\overline{V}_J=\sigma_J$. By Proposition 3.8, for every J' as in the statement of the theorem there is a subquotient of $W_{\mathbf{k},\mathbf{k}+1}$, and therefore of $\overline{V}_J$, which is a nontrivial extension of σ_J by $\sigma_{J'}$. Hence $\operatorname{gr}^{k+1}_\otimes\overline{V}_J$ contains a submodule isomorphic to $\sigma_{J'}$. Since $\sigma_{J'}$ appears with multiplicity one in $\operatorname{gr}^{k+1}_\otimes W_{\mathbf{k},\mathbf{k}+1}$ by Lemma 3.7, $\operatorname{gr}^{k+1}_\otimes\overline{V}_J$ contains $\sigma_{J'}$. We see that $\oplus_{J'}\sigma_{J'}\subset\operatorname{gr}^{k+1}_\otimes\overline{V}_J$.

Since $\operatorname{gr}^{k+1}_\otimes W_{\mathbf{k},\mathbf{k}+1}$ is multiplicity free by Lemma 3.7, it suffices to show that if $\sigma_{J'}\subset\operatorname{gr}^{k+1}_\otimes W_{\mathbf{k},\mathbf{k}+1}$ is a Jordan–Hölder factor of $\operatorname{gr}^{k+1}_\otimes\overline{V}_J$ then J' has the above form. Since $\overline{V}_J$ has Loewy length two and cosocle isomorphic to σ_J, if $\sigma_{J'}$ is a Jordan–Hölder factor of $\operatorname{gr}^{k+1}_\otimes\overline{V}_J$, $\overline{V}_J$ must have as a quotient a nontrivial extension of σ_J by $\sigma_{J'}$. Hence $\omega_{J'}-\omega_J=\pm\omega_j$ for some j by Proposition 2.8. Since $\sigma_{J'}\subset\operatorname{gr}^{k+1}_\otimes W_{\mathbf{k},\mathbf{k}+1}$, we deduce, from Proposition 3.6 and the description of $\operatorname{gr}^{k+1}_\otimes W_{\mathbf{k},\mathbf{k}+1}$, that $\mathbf{k}(J') \geq \mathbf{k}(J)$ and $|\mathbf{k}(J')| = |\mathbf{k}(J)| + 1$; in particular $k_i(J') - k_i(J) = \delta_{ij}$. So if $i \neq j$, then $J \cap \{\pm\omega_i\} = J' \cap \{\pm\omega_i\}$. While if $i = j$, then $J' \cap \{\pm\omega_j\} = J \cap \{\pm\omega_j\} \sqcup \{\omega'_J - \omega_J\}$. Hence J' is of the above form. □

Fix $J \in \mathfrak{J}$. Recall that by Proposition 3.6, there is a unique submodule of $W_{\mathbf{k}(J)} \subset R_\mu / \mathrm{Fil}^{>\mathbf{k}(J)} R_\mu$ isomorphic to σ_J. If P_{σ_J} is a projective envelope of σ_J, then the morphism $P_{\sigma_J} \twoheadrightarrow \sigma_J \subset W_{\mathbf{k}(J)} \subset R_\mu / \mathrm{Fil}_\otimes^{>\mathbf{k}(J)} R_\mu$ lifts to a map $\psi_J : P_{\sigma_J} \to R_\mu$. We let V_J be the image of ψ_J. The following proposition partially describes the graded pieces of V_J.

Proposition 3.10. *Let $J \in \mathfrak{J}$. The filtration $\mathrm{Fil}^{\mathbf{k}}$ on R_μ induces a filtration on the submodule V_J. Then for all J' such that $J \subset J'$, $\sigma_{J'} \subset \mathrm{gr}^{\mathbf{k}(\mathbf{J}')} V_J$.*

Proof. We proceed by induction on $k = k(J')$. Suppose that $k < k(J)$. Then $J \not\subset J'$, and there is no J' as in the statement of the theorem. Thus the theorem holds in this case.

If $k = k(J)$, then $J \subset J'$ implies that $J' = J$. By construction, $\sigma_J \subset \mathrm{gr}^{\mathbf{k}(J)} V_J$, and so the theorem holds in this case.

Now assume that $k > k(J)$ and that the theorem holds for $\mathrm{gr}_\otimes^{k-1} V_J$. Suppose that $J' \in \mathfrak{J}$ such that $J \subset J'$ and $k(J') = k$. Then there exists a $J'' \in \mathfrak{J}$ such that $J \subsetneq J'' \subset J'$ and $\#J'' = k - 1$. By the inductive hypothesis $\sigma_{J''} \subset \mathrm{gr}^{\mathbf{k}(J'')} V_J \subset \mathrm{gr}_\otimes^{k-1} V_J$. We thus obtain a nonzero map $P_{\sigma_{J''}} \to \mathrm{Fil}_\otimes^{k-1} V_J / \mathrm{Fil}_\otimes^{k+1} V_J$ which lifts the map $P_{\sigma_{J''}} \twoheadrightarrow \sigma_{J''} \subset W_{\mathbf{k}(J'')}$, and therefore must be $\overline{\psi}_{J''}$. By definition, the image of $\overline{\psi}_{J''}$ is $\overline{V}_{J''}$. By Proposition 3.9, $\sigma_{J'} \subset \mathrm{gr}^{\mathbf{k}(J')} \overline{V}_{J''} \subset \mathrm{gr}^{\mathbf{k}(J')} V_J$. □

We need the following two formal lemmas about tensor products of filtered vector spaces for Proposition 3.13.

Lemma 3.11. *Let $\mathbf{k}$ and $\mathbf{k}' \in \{0,1,2\}^f$. Then $\mathrm{Fil}^{\mathbf{k}} R_\mu \cap \mathrm{Fil}^{\mathbf{k}'} R_\mu = \mathrm{Fil}^{\mathbf{k}''} R_\mu$ where $k''_i = \max(k_i, k'_i)$.*

Proof. Clearly, $\mathrm{Fil}^{\mathbf{k}''} R_\mu \subset \mathrm{Fil}^{\mathbf{k}} R_\mu \cap \mathrm{Fil}^{\mathbf{k}'} R_\mu$. For each $i \in \mathbf{Z}/f$, choose a basis for R_{μ_i} compatible with the filtration and consider the corresponding tensor basis for R_μ. Then the elements of the tensor basis in $\mathrm{Fil}^{\mathbf{k}} R_\mu$ (resp. $\mathrm{Fil}^{\mathbf{k}'} R_\mu$) form a basis for $\mathrm{Fil}^{\mathbf{k}} R_\mu$ (resp. $\mathrm{Fil}^{\mathbf{k}'} R_\mu$). Thus the elements of the tensor basis in $\mathrm{Fil}^{\mathbf{k}} R_\mu \cap \mathrm{Fil}^{\mathbf{k}'} R_\mu$ form a basis for $\mathrm{Fil}^{\mathbf{k}} R_\mu \cap \mathrm{Fil}^{\mathbf{k}'} R_\mu$. These elements are in $\mathrm{Fil}^{\mathbf{k}''} R_\mu$, and so $\mathrm{Fil}^{\mathbf{k}} R_\mu \cap \mathrm{Fil}^{\mathbf{k}'} R_\mu \subset \mathrm{Fil}^{\mathbf{k}''} R_\mu$ □

For $I \subseteq \{0,1,2\}^f$, let $\mathrm{Fil}^I R_\mu := \sum_{\mathbf{k} \in I} \mathrm{Fil}^{\mathbf{k}} R_\mu$.

Lemma 3.12. *Let I and $I' \subseteq \{0,1,2\}^f$. Then*

$$\mathrm{Fil}^I R_\mu \cap \mathrm{Fil}^{I'} R_\mu = \sum_{\mathbf{k} \in I, \mathbf{k}' \in I'} \mathrm{Fil}^{\mathbf{k}} R_\mu \cap \mathrm{Fil}^{\mathbf{k}'} R_\mu.$$

Proof. Clearly, $\sum_{\mathbf{k} \in I, \mathbf{k}' \in I'} \mathrm{Fil}^{\mathbf{k}} R_\mu \cap \mathrm{Fil}^{\mathbf{k}'} R_\mu \subset \mathrm{Fil}^I R_\mu \cap \mathrm{Fil}^{I'} R_\mu$. For each $i \in \mathbf{Z}/f$, choose a basis for R_{μ_i} compatible with the filtration and consider the corresponding tensor basis for R_μ. Since the elements of the tensor basis in $\mathrm{Fil}^{\mathbf{k}} R_\mu$ span $\mathrm{Fil}^{\mathbf{k}} R_\mu$ for any $\mathbf{k}$, the elements of the tensor basis in $\mathrm{Fil}^I R_\mu$ (resp. $\mathrm{Fil}^{I'} R_\mu$) span $\mathrm{Fil}^I R_\mu$ (resp. $\mathrm{Fil}^{I'} R_\mu$) and thus form a basis for $\mathrm{Fil}^I R_\mu$ (resp. $\mathrm{Fil}^{I'} R_\mu$). Thus the elements of the tensor basis in $\mathrm{Fil}^I R_\mu \cap \mathrm{Fil}^{I'} R_\mu$ form a basis for $\mathrm{Fil}^I R_\mu \cap \mathrm{Fil}^{I'} R_\mu$.

It is easy to see that a basis element is in $\mathrm{Fil}^I R_\mu$ (resp. $\mathrm{Fil}^{I'} R_\mu$) if and only it is in $\mathrm{Fil}^{\mathbf{k}} R_\mu$ (resp. $\mathrm{Fil}^{\mathbf{k}'} R_\mu$) for some $\mathbf{k} \in I$ (resp. $\mathbf{k}' \in I'$). Thus $\mathrm{Fil}^I R_\mu \cap \mathrm{Fil}^{I'} R_\mu \subset \sum_{\mathbf{k}\in I, \mathbf{k}'\in I'} \mathrm{Fil}^{\mathbf{k}} R_\mu \cap \mathrm{Fil}^{\mathbf{k}'} R_\mu$. □

The following proposition shows that V_J does not depend on the choice of lift ψ_J, but rather just on $J \in \mathfrak{J}$.

Proposition 3.13. *Let $J \in \mathfrak{J}$. Let ψ'_J be a lift of the map $P_{\sigma_J} \twoheadrightarrow \sigma_J \subset W_{\mathbf{k}(J)} \subset R_\mu/\mathrm{Fil}^{>\mathbf{k}(J)} R_\mu$. Then the image of ψ'_J lies in V_J. In other words, V_J does not depend on the choice of ψ_J.*

Proof. We recursively define maps $\phi^k : P_{\sigma_J} \to \mathrm{Fil}^k_\otimes R_\mu \cap \mathrm{Fil}^{>\mathbf{k}(J)} R_\mu$ and $\psi^k : P_{\sigma_J} \to \mathrm{Fil}^k_\otimes V_J$ for $k > k(J)$. Let $\phi^{k(J)+1} = \psi'_J - \psi_J : P_{\sigma_J} \to \mathrm{Fil}^{k(J)}_\otimes R_\mu \cap \mathrm{Fil}^{>\mathbf{k}(J)} R_\mu$. Since ψ'_J and ψ_J coincide modulo $\mathrm{Fil}^{k(J)+1}_\otimes R_\mu \cap \mathrm{Fil}^{>\mathbf{k}(J)} R_\mu$, we see that the image of $\phi^{k(J)+1}$ lies in $\mathrm{Fil}^{k(J)+1}_\otimes R_\mu \cap \mathrm{Fil}^{>\mathbf{k}(J)} R_\mu$.

We now define ϕ^{k+1} and ψ^k in terms of ϕ^k. We first claim that the σ_J-isotypic part of $\mathrm{gr}^k_\otimes \mathrm{Fil}^{>\mathbf{k}(J)} R_\mu$ lies in $\mathrm{gr}^k_\otimes V_J$ for all k. Indeed, by Lemmas 3.11 and 3.12, $\mathrm{Fil}^k_\otimes \mathrm{Fil}^{>\mathbf{k}(J)} R_\mu$ (resp. $\mathrm{Fil}^{k+1}_\otimes \mathrm{Fil}^{>\mathbf{k}(J)} R_\mu$) is the sum $\sum_{\mathbf{k}>\mathbf{k}(J), |\mathbf{k}|\geq k} \mathrm{Fil}^{\mathbf{k}} R_\mu$ (resp. $\sum_{\mathbf{k}>\mathbf{k}(J), |\mathbf{k}|\geq k+1} \mathrm{Fil}^{\mathbf{k}} R_\mu$). From this, we see that $\mathrm{gr}^k_\otimes \mathrm{Fil}^{>\mathbf{k}(J)} R_\mu = \oplus_{\mathbf{k}>\mathbf{k}(J), |\mathbf{k}|=k} W_{\mathbf{k}}$, which is $\oplus_{J'}\sigma_{J'}$ where the sum runs over J' such that $\mathbf{k}(J') > \mathbf{k}(J)$ and $k(J') = k$ by Proposition 3.6. If additionally $\sigma_{J'} \cong \sigma_J$, then $\omega_{J'} = \omega_J$ by Proposition 2.3. The properties $\mathbf{k}(J') > \mathbf{k}(J)$ and $\omega_{J'} = \omega_J$ imply that for each $i \in \mathbf{Z}/f$, either $J' \cap \{\pm\omega_i\} = J \cap \{\pm\omega_i\}$ or $J \cap \{\pm\omega_i\}$ is empty. In any case, $J \subset J'$. We conclude that $\sigma_{J'} \subset \mathrm{gr}^k_\otimes V_J$ by Proposition 3.10.

Thus the image of ϕ^k in $\mathrm{gr}^k_\otimes \mathrm{Fil}^{>\mathbf{k}(J)} R_\mu$, which is σ_J-isotypic, lies in $\mathrm{gr}^k_\otimes V_J$. Let $\psi^k : P_{\sigma_J} \to \mathrm{Fil}^k_\otimes V_J$ be a lift of the map $P_{\sigma_J} \to \mathrm{gr}^k_\otimes V_J$ induced by ϕ^k. Let $\phi^{k+1} = \phi^k - \psi^k : P_{\sigma_J} \to \mathrm{Fil}^k_\otimes R_\mu \cap \mathrm{Fil}^{>\mathbf{k}(J)} R_\mu$. Since ϕ^k and ψ^k coincide modulo $\mathrm{Fil}^{k+1}_\otimes R_\mu \cap \mathrm{Fil}^{>\mathbf{k}(J)} R_\mu$, the image of ϕ^{k+1} lies in $\mathrm{Fil}^{k+1}_\otimes R_\mu \cap \mathrm{Fil}^{>\mathbf{k}(J)} R_\mu$.

Then by construction, $\psi'_J = \psi_J + \sum_{k=k(J)+1}^{2f} \psi^k$. Thus $\mathrm{im}\,\psi'_J \subset \mathrm{im}\,\psi_J + \sum_{k=k(J)+1}^{2f} \mathrm{im}\,\psi^k \subset V_J$. □

The following is the main submodule structure theorem for generic Γ-projective envelopes.

Theorem 3.14. *Let $\mu \in X^*(T)$. Assume that $\mu - \eta$ is 1-deep and that there exist positive coroots $\alpha^\vee$, $\beta^\vee$ such that $\langle \mu - \eta, \alpha^\vee\rangle > 1$, $\langle \mu - \eta, \beta^\vee\rangle < p - 3$.*

Let J' and $J \in \mathfrak{J}$ and let $V_{J'}$ and V_J be the submodules of R_μ defined above Theorem 3.10. If $J \subset J'$ then $V_{J'} \subset V_J$.

Proof. Suppose that $J \subset J'$. First note that $\sigma_{J'} \subset R_\mu/\mathrm{Fil}^{>\mathbf{k}(J')} R_\mu$ is contained in $V_J/\mathrm{Fil}^{>\mathbf{k}(J')} V_J$ by Proposition 3.10. Let $\psi'_{J'} : P_{\sigma_{J'}} \to V_J$ be a lift of the composition $P_{\sigma_{J'}} \twoheadrightarrow \sigma_{J'} \subset V_J/\mathrm{Fil}^{>\mathbf{k}(J')} V_J \subset R_\mu/\mathrm{Fil}^{>\mathbf{k}(J')} R_\mu$. Then $\mathrm{im}\,\psi'_{J'} = V_{J'}$ by Proposition 3.13. We conclude that $V_{J'} \subset V_J$. □

Recall that for $J \in \mathfrak{J}$, we defined maps $\psi_J : P_{\sigma_J} \twoheadrightarrow V_J \subset R_\mu$ above Proposition 3.10. The following lemma will be useful for multiplicity computations.

Lemma 3.15. *Let σ be a Serre weight and P_σ a projective envelope of σ. The vector space $\mathrm{Hom}_\Gamma(P_\sigma, R_\mu)$ is spanned by the set $\{\psi_J : \sigma_J \cong \sigma\}$.*

Proof. Since P_σ is a projective Γ-module, $\mathrm{Hom}_\Gamma(P_\sigma, R_\mu) \xrightarrow{\sim} \oplus_{\mathbf{k}} \mathrm{Hom}_\Gamma(P_\sigma, \mathrm{gr}^{\mathbf{k}} R_\mu)$. Since $\mathrm{gr}^{\mathbf{k}} R_\mu$ is semisimple, $\mathrm{Hom}_\Gamma(P_\sigma, \mathrm{gr}^{\mathbf{k}} R_\mu) \xrightarrow{\sim} \mathrm{Hom}_\Gamma(\sigma, \mathrm{gr}^{\mathbf{k}} R_\mu)$. The space $\mathrm{Hom}_\Gamma(\sigma, \mathrm{gr}^{\mathbf{k}} R_\mu)$ is one-dimensional if there exists a $J \in \mathfrak{J}$ with $\mathbf{k}(J) = \mathbf{k}$ so that $\sigma \cong \sigma_J$ and is otherwise zero by Proposition 3.6. In the case that $\mathrm{Hom}_\Gamma(\sigma, \mathrm{gr}^{\mathbf{k}} R_\mu)$ is nonzero, it is spanned by the image of ψ_J. □

4. The Breuil–Paškūnas construction

In this section, we use the results of Section 3 to give two distinct characterizations of a Γ-module constructed in [BP12].

Let $F_v/\mathbf{Q}_p$ be an unramified extension. Fix a tamely ramified representation $\overline{\rho} : \mathbf{G}_{F_v} \to \mathrm{GL}_2(\mathbf{F})$, and let $R_w(\mu) = V_\phi(\overline{\rho}^\vee(1))$ where $\mu = (\mu_i)_i \in X^*(T)$ and $w \in W = (S_2)^f$.

Definition 4.1. We say that $\overline{\rho}$ is 1-*generic* if for all possible choices of μ we have that $\mu - \eta$ is 1-deep in alcove C_0 and moreover $\mu - \eta \notin \{\eta,\ (p-3)\eta\}$. (We have 2^f possible choices for μ, *a posteriori*.)

Concretely, if $\mu = \sum_i \mu_i^{(i)}$ where $\mu_i = (a_i, b_i) \in \mathbf{Z}^2$ then μ is 1-generic iff $2 \leq a_i - b_i \leq p-2$ for all i and moreover $(a_i - b_i)_i \notin \{(2, \ldots, 2),\ (p-2, \ldots, p-2)\}$.

Note that if $\overline{\rho}$ is 1-generic then for any $F(\mu-\eta) \in W^?(\overline{\rho}^\vee(1))$ the corresponding projective envelope R_μ satisfies the hypotheses of Theorem 3.14. Moreover, if $\overline{\rho}$ is 1-generic as in Definition 4.1, then it is in particular generic in the sense of [BP12, Definition 11.7] and [EGS15, Definition 2.1.1].

We assume throughout that $\overline{\rho}$ is 1-generic. Let $\sigma := F(\mu - \eta) \in W^?(\overline{\rho}^\vee(1))$. Recall that the Weyl group W acts naturally on Λ_W. Let $S_w = w(S_e)$. Then $W^?(\overline{\rho}^\vee(1)) = F(\mathfrak{t}_\mu(\{\omega_J : J \subset S_w\}))$ by Proposition 2.10.

Definition 4.2. Let $\overline{\rho}$ be 1-generic and let $\sigma := F(\mu-\eta) \in W^?(\overline{\rho}^\vee(1))$. We define the Γ-representation $D_0^\vee(\sigma, \overline{\rho})$ as

$$D_0^\vee(\sigma, \overline{\rho}) = R_\mu / \Big(\sum_{\substack{J \subset S_w \\ \# J = 1}} V_J \Big).$$

Lemma 4.3. *With the hypotheses of Definition 4.2, the space*

$$\mathrm{Hom}_\Gamma\Bigg(\bigoplus_{\kappa \,\in\, W^?(\overline{\rho}^\vee(1))} P_\kappa, D_0^\vee(\sigma, \overline{\rho}) \Bigg)$$

has dimension at most one and is nonzero if and only if $\kappa \cong \sigma$.

Proof. Let $J_0 \in \mathfrak{J}$ be such that $\sigma_{J_0} = \kappa \in W^?(\overline{\rho}^\vee(1))$. Recall from §3 that for any $J \in \mathfrak{J}$ we have defined a morphism $\psi_J : P_{\sigma_J} \to R_\mu$ with image V_J. By Lemma 3.15, we see that the space $\mathrm{Hom}_\Gamma(P_{\sigma_{J_0}}, R_\mu)$, and hence its quotient $\mathrm{Hom}_\Gamma(P_{\sigma_{J_0}}, D_0^\vee(\sigma, \overline{\rho}))$, is spanned by the image of $\{\psi_J : \omega_{J_0} = \omega_J\}$. Thus it suffices to show that the image of ψ_J in $\mathrm{Hom}_\Gamma(P_{\sigma_{J_0}}, D_0^\vee(\sigma, \overline{\rho}))$ is zero unless $J = \emptyset$ since $\sigma_\emptyset \cong \sigma$.

Let $J \in \mathfrak{J}$. If $w\omega_j \in J$ for some j, then $V_J \subset V_{\{w\omega_j\}}$ by Theorem 3.14, and we conclude that the image of ψ_J in

$$\mathrm{Hom}_\Gamma(P_{\sigma_J}, D_0^\vee(\sigma, \overline{\rho}))$$

is 0. Thus if the image of ψ_J is nonzero, then $w\omega_j \notin J$ for all j.

If $w\omega_j \notin J$ for all j, then $J \subset S_{w_0 w}$ where $w_0 \in W$ is the longest element. Hence ω_J is in the closed $w_0 w$-chamber in $X^*(T)$, and is 0 if and only if $J = \emptyset$. Since $\sigma_J \in W^?(\overline{\rho}^\vee(1))$, ω_J is in the closed w-chamber in $X^*(T)$. Since the intersection of these two closed chambers is $\{0\}$, $\omega_J = 0$ and $J = \emptyset$. Of course, the image of $\psi_\emptyset$ in $\mathrm{Hom}_\Gamma(P_{\sigma_J}, D_0^\vee(\sigma, \overline{\rho}))$ is nonzero. □

Let $D_0^\vee(\overline{\rho}) = \oplus_{\sigma \in W^?(\overline{\rho}^\vee(1))} D_0^\vee(\sigma, \overline{\rho})$. Let $D_0(\overline{\rho})$ be $(D_0^\vee(\overline{\rho}))^\vee$ (where $(\cdot)^\vee$ denotes the Pontrjagin duality). The following proposition gives a characterization of $D_0^\vee(\overline{\rho})$, which is key for multiplicity one.

Recall that in [BP12, Theorem 13.8] a Γ-representation $D_0(\rho)$ is attached to a *generic* continuous Galois representation $\rho : G_{\mathbf{Q}_{p^f}} \to \mathrm{GL}_2(\mathbf{F})$. For the sake of readibility, we denote this Γ-representation by $D_0^{\mathrm{BP}}(\rho)$.

Proposition 4.4. *Assume that $\overline{\rho} : \mathbf{G}_{F_v} \to \mathrm{GL}_2(\mathbf{F})$ is 1-generic. Then $D_0(\overline{\rho}) \cong D_0^{\mathrm{BP}}(\overline{\rho})$. In particular the Jordan–Hölder factors of $D_0(\overline{\rho})$ appear with multiplicity one.*

Proof. The cosocle of $D_0^\vee(\overline{\rho})$ is isomorphic to $\oplus_{\sigma \in W^?(\overline{\rho}^\vee(1))} \sigma$, and for $\sigma \in W^?(\overline{\rho}^\vee(1))$, σ appears with multiplicity one in $D_0^\vee(\overline{\rho})$ by Lemma 4.3. We will show that there is a surjection from $D_0^\vee(\overline{\rho})$ to any representation with these properties.

Indeed, assume that Q is any Γ-representation with cosocle $\oplus_{\sigma \in W^?(\overline{\rho}^\vee(1))} \sigma$ and such that any $\sigma \in W^?(\overline{\rho}^\vee(1))$ appears with multiplicity one in Q. Fix $\sigma \in W^?(\overline{\rho}^\vee(1))$ and write $\sigma = F(\mu - \eta)$. We have a map $R_\mu \to Q$ whose composite with $Q \twoheadrightarrow \mathrm{cosoc}(Q)$ is non-zero. Let $J \in \mathfrak{J}$ be such that $J \subset w(S_e)$ and $\#J = 1$ (we follow the notations as in the beginning of this section) and write Q_J for the image of $V_J \subseteq R_\mu$ in Q. For any J as above, if $Q_J = 0$ then $V_J \subset \ker(R_\mu \to Q)$. If $Q_J = 0$ for all J as above, then the map $R_\mu \to Q$ would factor through $D_0^\vee(\sigma, \overline{\rho})$. If for all $\sigma \in W^?(\overline{\rho}^\vee(1))$, $Q_J = 0$ for all J as above, then we would obtain a surjection $D_0^\vee(\overline{\rho}) \twoheadrightarrow Q$. Assume for the sake of contradiction that for some σ and some J as above, $Q_J \neq 0$. Then the modular weight σ_J would appear as a Jordan–Hölder factor of the radical of Q. However, σ_J is a Jordan–Hölder factor of the cosocle of Q, contradicting the multiplicity one assumption.

To conclude, note that $\sigma \in W^?(\overline{\rho}^\vee(1))$ if and only if $\sigma^\vee \in W^?(\overline{\rho})$ (cf. e.g. [Her09, Proposition 6.23]). Hence by duality, $D_0(\overline{\rho})$ satisfies hypothesis [BP12, Theorem 13.8(iii)]. □

The following proposition is an alternative characterization of $D_0^\vee(\overline{\rho})$. Recall that $W(\mathbf{F})$ denotes the ring of Witt vectors of $\mathbf{F}$.

Proposition 4.5. *Suppose that $D_0^\vee$ is a Γ-representation such that $\dim \mathrm{Hom}_K(D_0^\vee, \sigma)$ is 1 if $\sigma \in W^?(\overline{\rho}^\vee(1))$ and 0 otherwise. Assume moreover that for any tame type $\sigma(\tau)$ and for any $W(\mathbf{F})$-lattice $\sigma^0(\tau) \subseteq \sigma(\tau)$ such that $\mathrm{soc}(\overline{\sigma}^0(\tau))$ is irreducible, one has*

$$\dim \mathrm{Hom}_K(D_0^\vee, \overline{\sigma}^0(\tau)) \leq 1.$$

Then there is a Γ-surjection $D_0^\vee(\overline{\rho}) \twoheadrightarrow D_0^\vee$.

Proof. By properties of projective envelopes, there is a Γ-surjection $\oplus_{\kappa \in W^?(\overline{\rho}^\vee(1))} P_\kappa \twoheadrightarrow D_0^\vee$. Fix $\sigma \in W^?(\overline{\rho}^\vee(1))$, and let $\sigma = F(\mu - \eta)$. It suffices to show that $V_J \subseteq R_\mu$ is in the kernel of the map $R_\mu \to D_0^\vee$ for all $J \in \mathfrak{J}$ such that $\#J = 1$ and $\sigma_J \in W^?(\overline{\rho}^\vee(1))$. Suppose for the sake of contradiction that V_J is not in the kernel. Then by Proposition 3.8 we have a surjection $D_0^\vee \twoheadrightarrow E$, where E is the unique up to isomorphism nontrivial extension of σ by σ_J. Choose a type $\sigma(\tau)$ such that $\overline{\sigma}(\tau)$ contains both σ and σ_J as Jordan–Hölder factors (for example the type with Jordan–Hölder factors given by the set $W^?(\overline{\rho}^\vee(1)))$. There exists a unique up to homothety lattice $\sigma^0(\tau)$ such that $\mathrm{soc}(\overline{\sigma}^0(\tau)) \cong \sigma_J$ (see [EGS15, Proposition 4.1.1]). There is an injection $E \hookrightarrow \overline{\sigma}^0(\tau)$ by [EGS15, Theorem 5.1.1]. Then the maps $D_0^\vee \twoheadrightarrow E \hookrightarrow \overline{\sigma}^0(\tau)$ and $D_0^\vee \twoheadrightarrow \sigma_J \hookrightarrow \overline{\sigma}^0(\tau)$ are linearly independent, so that $\dim \mathrm{Hom}_K(D_0^\vee, \overline{\sigma}^0(\tau)) > 1$, a contradiction. □

5. Global applications

In this section, we deduce our main theorem on cohomology of Shimura curves at full congruence level. We are going to follow closely [BD14], §3.2, 3.5 and 3.6, and [EGS15], §6.5.

Recall that F is a totally real field where p is unramified. We write Σ_p (resp. Σ_∞) the set of places of F above p (resp. above ∞). We write $\mathbf{A}_F$ to denote the ring of adèles of F. We fix a continuous Galois representation $\overline{r} : G_F \to \mathrm{GL}_2(\mathbf{F})$ which satisfies the following conditions:

(i) $\overline{r}$ is modular;
(ii) $\overline{r}|_{G_{F(\zeta_p)}}$ is absolutely irreducible;
(iii) if $p = 5$ then the image of $\overline{r}(G_{F(\zeta_p)})$ in $\mathbf{PGL}_2(\mathbf{F})$ is not isomorphic to A_5;
(iv) $\overline{r}|_{G_{F_w}}$ is *generic* (in the sense of [EGS15], Definition 2.1.1) for all $w \in \Sigma_p$.

We write $\Sigma_{\overline{r}}$ for the ramification set of $\overline{r}$ and we fix a continuous character $\psi : G_F \to \mathbf{F}^\times$ defined by $\psi := \omega \det \overline{r}$.

Let D be a quaternion algebra with center F and let Σ_D be the set of places where D ramifies. We assume that:

- $\#(\Sigma_\infty \setminus \Sigma_D) \leq 1$;
- $\Sigma_p \cap \Sigma_D = \emptyset$.

We define $S := \Sigma_p \cup \Sigma_D \cup \Sigma_{\overline{r}}$. We note that the condition $p > 3$ (coming from the genericity assumption on $\overline{r}|_{G_{F_w}}$) guarantees the existence of a place $w_1 \notin S$ such that:

- $\mathbf{N}(w_1) \not\equiv 1$ modulo p;
- the ratio of the eigenvalues of $\overline{r}(\mathrm{Frob}_{w_1})$ is not in $\{1,\ \mathbf{N}(w_1)^{\pm 1}\}$; and
- if ℓ is a prime such that $[F(\sqrt[\ell]{1}) : F] \leq 2$, then $w_1 \nmid \ell$

(cf. [BD14], item (iv) in the proof of Lemma 3.6.2). If ℓ is the unique prime number which is divisible by w_1, we then define $K_{w_1} \leq (\mathcal{O}_D)_{w_1}^\times$ as the pro-ℓ-Iwahori of $(\mathcal{O}_D)_{w_1}^\times$. The conditions on w_1 et K_{w_1} guarantee that for any open compact subgroup $K^{w_1} \leq (D \otimes_F \mathbf{A}_F^{\infty, w_1})^\times$, the subgroup $K_{w_1}K^{w_1}$ is sufficiently small in the sense of [GK14], §2.1.2.

We define $K^S := \prod'_{w \notin S} K_w$ where $K_w := (\mathcal{O}_D)_w^\times$ for all $w \notin S \cup \{w_1\}$. We now follow the procedure of [EGS15], §6.5 to obtain a space of algebraic automorphic forms with minimal tame level. We fix once and for all a place $v \in \Sigma_p$ and assume moreover that

(v) for all $w \in \Sigma_D$, $\overline{r}|_{G_{F_w}}$ is non-scalar.

Let $S' \subseteq \Sigma_p \cup \Sigma_D$ be the subset of places $w \in \Sigma_p \cup \Sigma_D$ such that $\overline{r}|_{G_{F_w}}$ is reducible. Write $W(\mathbf{F})$ for the ring of Witt vectors of $\mathbf{F}$. Following [EGS15], §6.5 (which is in turn based on [BD14], §3.3 and the proof of Proposition 3.5.1 in *loc. cit.*), we fix for each $w \in S \setminus \{v\}$ the following data:

(1) if $\overline{r}|_{G_{F_w}}$ is irreducible, the maximal compact $K_w := (\mathcal{O}_D)_w^\times$, an inertial type $\tau_w : I_w \to \mathrm{GL}_2(W(\mathbf{F}))$, and a $W(\mathbf{F})$-lattice $L_w \subseteq \sigma(\tau_w)$ (cf. also [BD14], Cas IV in §3.3);
(2) if $\overline{r}|_{G_{F_w}}$ is reducible and $w \notin S'$, a compact subgroup $K_w \leq (\mathcal{O}_D)_w^\times$ and a free $W(\mathbf{F})$-module L_w with a locally constant action of K_w (cf. also [BD14], Cas III at §3.3);
(3) if $w \in S'$, a compact subgroup $K_w \leq (\mathcal{O}_D)_w^\times$, a free $W(\mathbf{F})$-module L_w with a locally constant action of K_w and a scalar $\beta_w \in \mathbf{F}^\times$ (cf. also [BD14], Cas II at §3.3).

We further remark that the K_w-representation L_w has been chosen so that the center $F_w \cap K_w$ acts on L_w via $\psi|_{I_w}$. We define $K_S^v := \prod_{w\in S\setminus\{v\}} K_w$, $K^v := K_S^v K^S$ and $V^v := \bigotimes_{w\in S\setminus\{v\}} L_w$, which is a $W(\mathbf{F})$-module of finite type with a locally constant action of K_S^v, hence of K^v by inflation. We note that V^v is endowed with a central character obtained by restriction from ψ.

Let $\mathrm{Rep}_{\mathbf{F}}^{\psi}(K_v)$ be the category of $\mathbf{F}$-modules of finite type, endowed with an action of $K_v := (\mathcal{O}_D)_v^\times \cong \mathrm{GL}_2(\mathcal{O}_{F_v})$ and such that $K_v \cap F_v^\times$ acts via the character $\psi \circ \mathrm{Art}_{F_v}$. In particular, if $V_v \in \mathrm{Rep}_{\mathbf{F}}^{\psi}(K_v)$, the K_S-representation $V := V^v \otimes V_v$ extends to a representation of $K := K^v K_v$ by inflation, hence to a representation of $K(\mathbf{A}_F^\infty)^\times$ by letting $(\mathbf{A}_F^\infty)^\times$ act on V via ψ.

If $\#\big(\Sigma_\infty \setminus \Sigma_D\big) = 1$ we define the space of algebraic modular forms of level K, coefficients in V and central character ψ as:

$$S_\psi(K, V_v^\vee) := H^1_{\text{ét}}(X_K \otimes_F \overline{F}, \mathcal{F}_{V^\vee}) \tag{5.1}$$

where X_K is the smooth projective algebraic curve associated to K as in [BD14, §3.1] and $\mathcal{F}_{V^\vee}$ is the local system on $X_K \otimes_F \overline{F}$ associated to $V^\vee$ in the usual way (cf. [BD14, proof of Lemma 6.2]).

If $\#\big(\Sigma_\infty \setminus \Sigma_D\big) = 0$ we define the space of algebraic modular forms of level K, coefficients in V and central character ψ as:

$$S_\psi(K, V_v^\vee) := \left\{ \begin{array}{c} f : D^\times \backslash (D \otimes_F \mathbf{A}_F^\infty)^\times \to V^\vee,\ f \text{ continuous}, \\ f(gk) = k^{-1} f(g)\ \forall g \in (D \otimes_F \mathbf{A}_F^\infty)^\times, k \in K(\mathbf{A}_F^\infty)^\times \end{array} \right\}. \tag{5.2}$$

We have a variation of the previous spaces with "infinite level at v" defined as follows:

$$S_\psi(K^v, \mathbf{F}) := \varinjlim_{U_v \leq K_v} S_\psi(K^v U_v, \mathbf{F})$$

where U_v ranges among the compact open subgroups K_v.It is endowed with a smooth action of $D_v^\times \cong \mathrm{GL}_2(F_v)$.

The $\mathbf{F}$-modules $S_\psi(K, V_v^\vee)$, $S_\psi(K^v, \mathbf{F})$ are faithful modules over a certain Hecke algebra which is defined as follows. Consider the $\mathbf{F}$-polynomial algebra $\mathbf{T}^{S\cup\{w_1\}} := \mathbf{F}[T_w^{(i)},\ w \notin S \cup \{w_1\}]$. For all $w \notin S \cup \{w_1\}$, $1 \leq i \leq 2$ define the Hecke operator $T_w^{(i)}$ as the usual double classe operator acting on $S_\psi(K, V_v^\vee)$:

$$\left[\mathrm{GL}_2(\mathcal{O}_{F_w}) \begin{pmatrix} \varpi_w \mathrm{Id}_i & \\ & \mathrm{Id}_{2-i} \end{pmatrix} \mathrm{GL}_2(\mathcal{O}_{F_w})\right]$$

We then have an evident morphism of $\mathbf{F}$-algebras $\mathbf{T}^{S\cup\{w_1\}} \to \mathrm{End}_W(S_\psi(K, V_v^\vee))$ whose image will be denoted by $\mathbf{T}(V_v)$. From the hypothesis (i) there is a surjection

$\alpha_{\overline{r}}:\mathbf{T}(V_v)\to\mathbf{F}$ such that

$$\det\big(X\mathrm{Id}_2-\overline{\psi}\overline{r}(\mathrm{Frob}_w)\big)=X-\alpha_{\overline{r}}(T_w^{(1)})X+\mathbf{N}(w)\alpha_{\overline{r}}(T_w^{(2)})$$

for all $w\notin S\cup\{w_1\}$. We note $\mathfrak{m}_{\overline{r}}:=\ker(\alpha_{\overline{r}})$.

For $w\in S'\cup\{w_1\}$ we can define the Hecke operator $T_w^{(1)}$ acting on $S_\psi(K,V_v^\vee)_{\mathfrak{m}_{\overline{r}}}$ (cf. [EGS15] §6.5, cf. also [BD14], §3.3 Cas I et II), as well as scalars $\beta_w\in\mathbf{F}^\times$. We write $\mathbf{T}'(V_v)$ for the subalgebra of $\mathrm{End}_{\mathbf{T}(V_v)}(S_\psi(K,V_v^\vee)_{\mathfrak{m}_{\overline{r}}})$ generated by $\mathbf{T}(V_v)$ and the operators $T_w^{(1)}$, $w\in S'\cup\{w_1\}$. In particular $\mathbf{T}(V_v)_{\mathfrak{m}_{\overline{r}}}\subseteq\mathbf{T}'(V_v)$ is a finite extension of semi-local rings. If $\mathfrak{m}'_{\overline{r}}$ denotes the ideal of $\mathbf{T}'(V_v)$ above $\mathfrak{m}_{\overline{r}}$ and generated by the elements $T_w^{(1)}-\beta_w$, we easily see that $\mathfrak{m}'_{\overline{r}}$ is a maximal ideal in $\mathbf{T}'(V_v)$.

We now define $\pi(\overline{\rho}_v):=S_\psi(K^v,\mathbf{F})[\mathfrak{m}'_{\overline{r}}]$ and set $K_v(1):=\ker(K_v\twoheadrightarrow\Gamma)$. From the main results in [EGS15] we have the following statement:

Theorem 5.1 ([EGS15], Theorem 9.1.1 and 10.1.1)**.** *Let $\overline{r}:G_F\to\mathrm{GL}_2(\mathbf{F})$ be a continuous Galois representation satisfying the hypotheses (i)-(v) above. Then*

$$\mathrm{cosoc}_{K_v}(\pi(\overline{\rho}_v)^\vee)=\bigoplus_{\sigma\in W^?(\overline{\rho}_v(1)^\vee)}\sigma.$$

Let $\sigma(\tau)$ be a K_v-type and let $\sigma^0(\tau)$ a $W(\mathbf{F})$-lattice with irreducible socle. Then

$$\mathrm{Hom}_\Gamma((\pi(\overline{\rho}_v)^\vee)_{K_v(1)},\overline{\sigma}^0(\tau))$$

is at most one dimensional.

Proof. We let $M_\infty:\mathrm{Rep}^\psi_{\mathbf{F}}(K_v)\to\mathrm{Mod}^{\mathrm{f.t.}}(R^\psi_\infty)$ be the patching functor associated to $\overline{r}$ as in [EGS15], §6.5. (The local ring R^ψ_∞ being defined in [EGS15] §6.5, cf. also [BD14], §3.4; by abuse of notation we let $\mathfrak{m}_{\overline{r}}$ denote its maximal ideal.) By construction of the functor M_∞, for any representation $V_v\in\mathrm{Rep}^\psi_{\mathbf{F}}(K_v)$ we have an isomorphism

$$\big(M_\infty(V_v)/\mathfrak{m}'_{\overline{r}}\big)^\vee\cong S_\psi(K^vK_v,V_v^\vee)[\mathfrak{m}'_{\overline{r}}]$$

together with a compatible morphism of local rings $R^\psi_\infty\to\mathbf{T}'(V_v)_{\mathfrak{m}'_{\overline{r}}}$.

Since K^vU_v is sufficiently small for any choice of a compact open subgroup $U_v\le K_v$ and since $\mathfrak{m}'_{\overline{r}}$ is non-Eisenstein, a standard spectral sequence argument gives:

$$\big(S_\psi(K^v,\mathbf{F})[\mathfrak{m}'_{\overline{r}}]\big)^{K_v(1)}\cong S_\psi(K^vK_v(1),\mathbf{F})[\mathfrak{m}'_{\overline{r}}].$$

In particular if $K_v(1)$ acts trivially on $V_v\in\mathrm{Rep}^\psi_{\mathbf{F}}(K_v)$ we obtain

$$\begin{aligned}(5.3)\qquad \big(M_\infty(V_v)/\mathfrak{m}'_{\overline{r}}\big)^\vee&\cong S_\psi(K^vK_v,V_v^\vee)[\mathfrak{m}'_{\overline{r}}]\\&\cong\mathrm{Hom}_\Gamma(V_v,S_\psi(K^vK_v(1),\mathbf{F})[\mathfrak{m}'_{\overline{r}}])\\&\cong\mathrm{Hom}_{K_v}(V_v,\pi(\overline{\rho}_v)^{K_v(1)}).\end{aligned}$$

Let $\sigma\in\mathrm{Rep}^\psi_{\mathbf{F}}(K_v)$ be a Serre weight. From [EGS15], Theorem 9.1.1 we have $M_\infty(\sigma)\neq0$ if and only if $\sigma\in W^?(\overline{\rho}_v)$. Hence, by Nakayama's lemma and the isomorphism (5.3) we have that $\mathrm{soc}_{K_v}(\pi(\overline{\rho}_v))=\bigoplus_{\sigma\in W^?(\overline{\rho}_v)}\sigma$. The first claim in the theorem follows by Pontrjagin duality.

Similarly if $\sigma^0(\tau)$ is a lattice with irreducible cosocle in a tame type $\sigma(\tau)$, we now deduce from [EGS15, Theorem 10.1.1] that $\mathrm{Hom}_{K_v}(\overline{\sigma}^0(\tau),\pi(\overline{\rho}_v))$ is at most one dimensional. With $\sigma^0(\tau)$ as in the statement of the theorem, $\overline{\sigma}^0(\tau)^\vee$ is the reduction of a lattice in the dual type τ^* with irreducible cosocle and thus the second claim in the theorem follows by Pontrjagin duality. ☐

From now on, we assume that:

(vi) $\overline{\rho}_v := \overline{r}|_{G_{F_v}}$ is semisimple and 1-generic in the sense of Definition 4.1.

Proposition 5.2. *Let $\overline{r} : G_F \to \mathrm{GL}_2(\mathbf{F})$ be a continuous Galois representation satisfying the hypotheses (i)-(vi) above. There is a K_v-surjection $\pi(\overline{\rho}_v)^\vee \twoheadrightarrow D_0^\vee(\overline{\rho}_v)$.*

Proof. This is Pontrjagin dual to [Bre14, Proposition 9.3], noting that $D_0^\vee(\overline{\rho}_v) \cong (D_0^{\mathrm{BP}}(\overline{\rho}_v))^\vee$. ☐

Theorem 5.3. *Let $\overline{r} : G_F \to \mathrm{GL}_2(\mathbf{F})$ be a continuous Galois representation satisfying the hypotheses (i)-(vi) above. Then we have an isomorphism of Γ-modules $(\pi(\overline{\rho}_v)^\vee)_{K_v(1)} \cong D_0^\vee(\overline{\rho}_v)$.*

Proof. By Proposition 5.2, there is a surjection $(\pi^\vee)_{K_v(1)} \twoheadrightarrow D_0^\vee(\overline{\rho}_v)$. By Theorem 5.1, $(\pi^\vee)_{K_v(1)}$ satisfies the conditions for $D_0^\vee$ in Proposition 4.5. We conclude that there is a surjection $D_0^\vee(\overline{\rho}_v) \twoheadrightarrow (\pi^\vee)_{K_v(1)}$. The composition of these surjections is a surjective endomorphism of $D_0^\vee(\overline{\rho}_v)$, a finite length Γ-module, and is thus an isomorphism. ☐

We conclude with the main result of this paper:

Corollary 5.4. *Let $\overline{r} : G_F \to \mathrm{GL}_2(\mathbf{F})$ be a continuous Galois representation satisfying the hypotheses (i)-(vi) above. Then*

$$S_\psi(K^v K_v(1), \mathbf{F})[\mathfrak{m}'_{\overline{r}}] \cong D_0^{\mathrm{BP}}(\overline{\rho}_v).$$

In particular, the Γ-representation $S_\psi(K^v K_v(1), \mathbf{F})[\mathfrak{m}'_{\overline{r}}]$ only depends on $\overline{r}|_{I_v}$ and is multiplicity free.

Proof. Recall from the proof of Theorem 5.1 the isomorphism:

$$\left(S_\psi(K^v, \mathbf{F})[\mathfrak{m}'_{\overline{r}}]\right)^{K_v(1)} \cong S_\psi(K^v K_v(1), \mathbf{F})[\mathfrak{m}'_{\overline{r}}].$$

The isomorphism follows now from Proposition 4.4 and Theorem 5.3 after applying Pontrjagin duality. For the second statement, recall that $D_0(\overline{\rho}_v)$ was defined only in terms of $W^?(\overline{\rho}_v^\vee(1))$ and is multiplicity free by Proposition 4.4. ☐

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