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# CREPANT RESOLUTIONS OF WEIGHTED PROJECTIVE SPACES AND QUANTUM DEFORMATIONS

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Abstract. We compare the Chen-Ruan cohomology ring of the weighted projective spaces $\mathbb{P}(1,3,4,4)$ and $\mathbb{P}(1,...,1,n)$ with the cohomology ring of their crepant resolutions.
In both cases, we prove that the Chen-Ruan cohomology ring is isomorphic to the quantum corrected cohomology ring of the crepant resolution after suitable evaluation of the quantum parameters. For this, we prove a formula for the Gromov-Witten invariants of the resolution of a transversal $A_3$ singularity.

## 1. Introduction

Given a complex orbifold $\mathcal{X}$, Chen and Ruan defined the so called Chen-Ruan cohomology ring of $\mathcal{X}$ [CR04], it is denoted by $H^\star_{\mathrm{CR}}(\mathcal{X})$ (see [AGV02] and [AGV06] for the definition in the algebraic case). The *Cohomological Crepant Resolution Conjecture*, as proposed by Y. Ruan [Rua06], predicts the existence of an isomorphism between the Chen-Ruan cohomology ring of a Gorenstein orbifold $\mathcal{X}$ and the quantum corrected cohomology ring of any crepant resolution $\rho : Z \to X$ of the coarse moduli space $X$ of $\mathcal{X}$. The quantum corrected cohomology ring of $Z$ is the ring obtained from the small quantum cohomology of $Z$ after specialization of some quantum parameters and is denoted by $H^\star_\rho(Z)(q_1,...,q_m)$ (see Sec. 3). The conjecture belongs to the so called generalized McKay correspondence.

The main examples used to study the conjecture are the following:

- the Hilbert scheme of $r$ points on a projective surface $S$ (see [LQ02] for $r=2$; [ELQ03] and [LL07] for $S=\mathbb{P}^2$ $r=3$; [FG03], [Uri05] and [QW02] for $r$ general and $S$ with numerically trivial canonical class; we remark that the proofs of [FG03] and [Uri05] use the computation of $H^\star(\mathrm{Hilb}^r(S))$ done in [LS03], while [QW02] is self-contained);
- the Hilbert scheme of $r$ points on a quasi-projective surface $S$ carrying a holomorphic symplectic form (see [LQW04], [LS01] and [Vas01]);
- the quotient $V/G$, where $V$ is a complex symplectic vector space and $G$ is a finite subgroup of $\mathrm{Sp}(V)$ (see [GK04]);
- orbifolds with transversal $A_n$ singularities (see [Per07]);
- wreath product orbifolds (see [Mat06]).

The main result of the present paper concerns $\mathbb{P}(1,3,4,4)$. The singular locus of its coarse moduli space $|\mathbb{P}(1,3,4,4)|$ is the disjoint union of an isolated singularity of type $\frac{1}{3}(1,1,1)$ (we use Reid's notation [Rei87]) and a transversal $A_3$ singularity. The quantum corrected cohomology ring of the crepant resolution $\rho : Z \to X$ has four quantum parameters: $q_1, q_2, q_3$ and $q_4$; the parameters $q_1, q_2, q_3$ come from the

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resolution of the transversal singularity and $q_4$ comes from the isolated singularity. We compute the 3-points genus zero Gromov-Witten invariants of curves which are contracted by $\rho$ and which are contained in the part of the exceptional divisor that contracts to the transversal singularity (Theorem 5.1). Then we prove that, for $(q_1, q_2, q_3, q_4) \in \{(\mathrm{i}, \mathrm{i}, \mathrm{i}, 0), (-\mathrm{i}, -\mathrm{i}, -\mathrm{i}, 0)\}$ there is a ring isomorphism (Theorem 3.3)

$$H^{\star}_{\rho}(Z; \mathbb{C})(q_1, q_2, q_3, q_4) \cong H^{\star}_{\mathrm{CR}}(\mathbb{P}(1, 3, 4, 4); \mathbb{C}).$$

In the case of $\mathbb{P}(1, ..., 1, n)$, we prove that the Chen-Ruan cohomology ring is isomorphic to the cohomology ring of the crepant resolution $\rho : Z \to |\mathbb{P}(1, ..., 1, n)|$ (Proposition 6.1). In this case we will always assume that the dimension of $\mathbb{P}(1, ..., 1, n)$ is $n$, therefore the orbifold is Gorenstein.

The paper is organized as follows. In Section 2 we recall some general facts about weighted projective spaces. In Section 3 we state the main result (Thm. 3.3) and we write a recipe which we follow for the proof. We prove Theorem 3.3 in Section 4 using the computation of the Gromov-Witten invariants done in Section 5. To compute those invariants we use the theory of the deformations of surfaces with rational double points and the deformation invariance property of the Gromov-Witten invariants. In Section 6 we prove that, for $\mathbb{P}(1, \ldots, 1, n)$, the Chen-Ruan cohomology ring is isomorphic to the cohomology ring of its crepant resolution (Proposition 6.1).

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## 2. Weighted projective spaces

In this section we recall some basic facts about weighted projective spaces.

Let $n \geq 1$ be an integer and $w = (w_0, \ldots, w_n)$ a sequence of integers greater or equal than one. Consider the action of the multiplicative group $\mathbb{C}^{\star}$ on $\mathbb{C}^{n+1} - \{0\}$ given by:

$$\lambda \cdot (x_0, \ldots, x_n) := (\lambda^{w_0} x_0, \ldots, \lambda^{w_n} x_n).$$

The *weighted projective space* $\mathbb{P}(w)$ is defined as the quotient stack $[\mathbb{C}^{n+1} - \{0\}/\mathbb{C}^{\star}]$. It is a smooth Deligne-Mumford stack whose coarse moduli space, denoted $|\mathbb{P}(w)|$, is a projective variety of dimension $n$.

According to [BCS05], $\mathbb{P}(w)$ is a toric stack associated to the following stacky fan:

$$N := \mathbb{Z}^{n+1} / \sum_{i=0}^{n} w_i v_i, \quad \beta : \mathbb{Z}^{n+1} \to N, \quad \Sigma, \tag{2.1}$$

where $v_0, ..., v_n$ is the standard basis of $\mathbb{Z}^{n+1}$, $\beta$ is the canonical projection, and $\Sigma \subset N \otimes_{\mathbb{Z}} \mathbb{Q}$ is the fan whose cones are generated by any proper subset of $\{\beta(v_0) \otimes 1, ..., \beta(v_n) \otimes 1\}$.

The weighted projective space $\mathbb{P}(w)$ comes with a natural invertible sheaf $\mathcal{O}_{\mathbb{P}(w)}(1)$ defined as follows: for any scheme $Y$ and any morphism $Y \to \mathbb{P}(w)$ given by a principal $\mathbb{C}^{\star}$-bundle $P \to Y$ and a $\mathbb{C}^{\star}$-equivariant morphism $P \to \mathbb{C}^{n+1} - \{0\}$, $\mathcal{O}_{\mathbb{P}(w)}(1)_Y$ is the sheaf of sections of the associated line bundle of $P$.

We will use the following

**Notation 2.2.** An *orbifold* is a smooth algebraic Deligne-Mumford stack over $\mathbb{C}$ with generically trivial stabilizers. The orbifold is said to be *Gorenstein* if all its ages are integers.

**Proposition 2.3.** *(1) The Deligne-Mumford stack $\mathbb{P}(w)$ is an orbifold if and only if the greatest common divisor of $w_0, \dots, w_n$ is 1.*
*(2) The orbifold $\mathbb{P}(w)$ is Gorenstein if and only if $w_i$ divides $\sum_{j=0}^n w_j$ for any $i$.*

*Proof.* (1) In general, the generic stabilizer of a toric Deligne-Mumford stack which is associated to the stacky fan $(N, \beta, \Sigma)$ is isomorphic to the torsion part of $N$. This implies the first part.

(2) For any $i \in \{0, \dots, n\}$, set

$$U_i := \{(x_0, \dots, x_n) \in \mathbb{C}^{n+1} - \{0\} \mid x_i = 1\}.$$

The trivial $\mathbb{C}^\star$-bundle on $U_i$ and the $\mathbb{C}^\star$-equivariant morphism $U_i \times \mathbb{C}^\star \to \mathbb{C}^{n+1} - \{0\}$, $(x, \lambda) \mapsto \lambda \cdot x$, define an étale morphism $\varphi_i : U_i \to \mathbb{P}(w)$ such that $\sqcup_i \varphi_i : \sqcup_i U_i \to \mathbb{P}(w)$ is a covering. The group $\mathcal{U}_{w_i} \subset \mathbb{C}^\star$ of $w_i$-th roots of the unity acts linearly and diagonally on $U_i$ with weights $w_0, ..., w_n$.
Let now $x \in U_i$, and let $\exp\left(\frac{2\pi \mathrm{i} k}{w_i}\right) \in \mathcal{U}_{w_i}$ be an element that fixes $x$. Then

$$\text{age}\left(x, \exp\left(\frac{2\pi \mathrm{i} k}{w_i}\right)\right) = \sum_{j=0}^{n} \left\{\frac{w_j k}{w_i}\right\}, \tag{2.4}$$

where the brackets $\{\}$ means the fractional part of a rational number. The claim follows. □

In dimension 1, the only weighted projective space which is Gorenstein is $\mathbb{P}(1,1) \cong \mathbb{P}^1$. In dimension 2 and 3, the complete list of Gorenstein weighted projective spaces is given by the following weights :

(2.5)

| Dimension 2 | Dimension 3 | | | |
|---|---|---|---|---|
| $(1,1,1)$ | $(1,1,1,1)$ | $(1,2,2,5)$ | $(2,3,3,4)$ | $(2,3,10,15)$ |
| $(1,1,2)$ | $(1,1,1,3)$ | $(1,1,4,6)$ | $(1,2,6,9)$ | $(1,6,14,21)$ |
| $(1,2,3)$ | $(1,1,2,2)$ | $(1,2,3,6)$ | $(1,4,5,10)$ | |
| | $(1,3,4,4)$ | $(1,1,2,4)$ | $(1,3,8,12)$ | |

In dimension $n$, the problem of determining all Gorenstein $\mathbb{P}(w)$ is equivalent to the problem of *Egyptian fractions*, *i.e.* the number of solutions of $1 = \frac{1}{x_0} + \dots + \frac{1}{x_n}$ with $1 \le x_0 \le \ldots \le x_n$ (see [Slo]). Hence, there is a finite number of such $\mathbb{P}(w)$.

## 3. The main result

In this section we state our main result, Theorem 3.3. Then we present a recipe that we will follow during the proof.

Recall that, given a Gorenstein orbifold $\mathcal{X}$, a resolution of singularities $\rho : Z \to X$ is crepant if $\rho^* K_X \cong K_Z$.

In order to state the theorem, we recall the definition of the quantum corrected cohomology ring [Rua06].

Let $\mathcal{X}$ be a Gorenstein orbifold with projective coarse moduli space $X$. Let $\rho : Z \to X$ be a crepant resolution such that $Z$ is projective. Let $\mathrm{N}^+(\mathrm{Z}) \subset \mathrm{A}_1(\mathrm{Z};\mathbb{Z})$ be the monoid of effective 1-cycles in $Z$, and set

$$\mathrm{M}_\rho(\mathrm{Z}) := \mathrm{Ker}(\rho_\star) \cap \mathrm{N}^+(\mathrm{Z}),$$

where $\rho_\star : \mathrm{A}_\star(Z;\mathbb{Z}) \to \mathrm{A}_\star(X;\mathbb{Z})$ is the morphism of Chow groups induced by the map $\rho$.

We assume that $\mathrm{M}_\rho(\mathrm{Z})$ is generated by a finite number of classes of rational curves which are linearly independent over $\mathbb{Q}$. We fix a set of such generators: $\Gamma_1, ..., \Gamma_m$. Then, any $\Gamma \in \mathrm{M}_\rho(\mathrm{Z})$ can be written in a unique way as:

$$\Gamma = \sum_{\ell=1}^{m} d_\ell \Gamma_\ell,$$

for some non negative integers $d_\ell$.

The *quantum corrected cohomology ring* of $Z$ is defined as follows. We assign a formal variable $q_\ell$ for each $\Gamma_\ell$, hence $\Gamma = \sum_{\ell=1}^{m} d_\ell \Gamma_\ell \in \mathrm{M}_\rho(\mathrm{Z})$ corresponds to the monomial $q_1^{d_1} \cdots q_m^{d_m}$. The *quantum* 3*-points function* is by definition:

$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle_{\mathrm{q}}(q_1, ..., q_m) := \sum_{d_1,...,d_m > 0} \Psi_\Gamma^Z(\alpha_1, \alpha_2, \alpha_3) q_1^{d_1} \cdots q_m^{d_m}, \tag{3.1}$$

where $\alpha_1, \alpha_2, \alpha_3 \in H^\star(Z;\mathbb{C})$ and $\Psi_\Gamma^Z(\alpha_1, \alpha_2, \alpha_3)$ is the Gromov-Witten invariant of $Z$ of genus zero, homology class $\Gamma$ and three marked points.

We assume that (3.1) defines an analytic function of the variables $q_1, ..., q_m$ on some region of the complex space $\mathbb{C}^m$, it will be denoted by $\langle \alpha_1, \alpha_2, \alpha_3 \rangle_{\mathrm{q}}$. In the following, when we evaluate $\langle \alpha_1, \alpha_2, \alpha_3 \rangle_{\mathrm{q}}$ on a point $(q_1, ..., q_m)$, we will implicitly assume that it is defined on such a point.

The *quantum corrected triple intersection* is defined by:

$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle_\rho(q_1, ..., q_m) := \int_Z \alpha_1 \cup \alpha_2 \cup \alpha_3 + \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{\mathrm{q}}(q_1, ..., q_m).$$

The *quantum corrected cup product* $\alpha_1 *_\rho \alpha_2$ of two classes $\alpha_1, \alpha_2 \in H^*(Z;\mathbb{C})$ is defined by requiring that:

$$\int_Z (\alpha_1 *_\rho \alpha_2) \cup \alpha = \langle \alpha_1, \alpha_2, \alpha \rangle_\rho(q_1, ..., q_m) \qquad \forall \alpha \in H^\star(Z;\mathbb{C}).$$

The following result holds.

**Proposition 3.2.** *For any $(q_1, ..., q_m)$ belonging to the domain of the quantum 3-points function, the quantum corrected cup product $*_\rho$ satisfies the following properties:*

- Associativity: *it is associative on $H^\star(Z;\mathbb{C})$, moreover it has a unit which coincides with the unit of the usual cup product of $Z$.*
- Skewsymmetry: $\alpha_1 *_\rho \alpha_2 = (-1)^{\deg\alpha_1 \cdot \deg\alpha_2} \alpha_2 *_\rho \alpha_1$ *for any* $\alpha_1, \alpha_2 \in H^\star(Z;\mathbb{C})$.
- Homogeneity: *for any* $\alpha_1, \alpha_2 \in H^\star(Z;\mathbb{C})$, $\deg\,(\alpha_1 *_\rho \alpha_2) = \deg\,\alpha_1 + \deg\,\alpha_2$.

For any $(q_1, ..., q_m)$ belonging to the domain of the quantum corrected 3-point function, the resulting ring $(H^\star(Z;\mathbb{C}), *_\rho)$ is the *quantum corrected cohomology ring* with the quantum parameters specialized at $(q_1, ..., q_m)$, and it will also be denoted as $H_\rho^\star(Z;\mathbb{C})(q_1, ..., q_m)$.

Let now $\mathcal{X} := \mathbb{P}(1,3,4,4)$, let $\rho : Z \to X$ be the crepant resolution defined in Sec.4 (1) and let $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 \in \mathrm{M}_\rho(Z)$ be defined in Sec. 4 (3).
Then we have the following

**Theorem 3.3.** *For* $(q_1, q_2, q_3, q_4) \in \{(\mathrm{i}, \mathrm{i}, \mathrm{i}, 0), (-\mathrm{i}, -\mathrm{i}, -\mathrm{i}, 0)\}$ *there is a ring isomorphism*

$$H^\star_\rho(Z;\mathbb{C})(q_1, q_2, q_3, q_4) \cong H^\star_{\mathrm{CR}}(\mathbb{P}(1,3,4,4);\mathbb{C})$$

*which is an isometry with respect to the Poincaré pairing on* $H^\star_\rho(Z;\mathbb{C})(q_1, q_2, q_3, q_4)$ *and the Chen-Ruan pairing on* $H^\star_{\mathrm{CR}}(\mathbb{P}(1,3,4,4);\mathbb{C})$.
*Furthermore, in each case an explicit isomorphism is given by the linear map* (4.3) *and* (4.4) *respectively.*

**Remark 3.4.** In general, the *Cohomological Crepant Resolution Conjecture* asserts that there exist roots of the unity $c_1, ..., c_m$ and a ring isomorphism (see [Rua06])

$$H^\star_\rho(Z;\mathbb{C})(c_1, ..., c_m) \cong H^\star_{\mathrm{CR}}(\mathcal{X};\mathbb{C}).$$

When the orbifold $\mathcal{X}$ satisfies the hard Lefschetz condition, Ruan's conjecture has been generalized by Bryan and Graber [BG06]: the Frobenius manifolds associated to the big quantum cohomology of $Z$ and of $\mathcal{X}$ are analytic continuations of each other. Bryan and Graber's conjecture is called the *Crepant Resolution Conjecture*, see [BG06], [BGP05], [CCIT06], [CCIT07], [Wis07] for more details and its verification in some examples.

Surprisingly, we obtain that some quantum parameters can be put to zero. This result is strange in regard to the Conjecture. One can observe that in our computations for $\mathbb{P}(1,3,4,4)$ the quantum parameters corresponding to the transversal $\mathrm{A}_3$ singularity are evaluated at primitive 4-th roots of the unity, as predicted by the conjecture, and the quantum parameter evaluated to zero corresponds to the singular point $\frac{1}{3}(1,1,1)$. Note that, also in the case of $\mathbb{P}(1,...,1,n)$ we set the quantum parameter to zero (see Prop. 6.1) and the singularity is of type $\frac{1}{n}(1,...,1)$.

We present below the recipe that we will follow. We write it for a general Gorenstein orbifold $\mathbb{P}(w)$ because it could be used in a more general case.

**Notation 3.5.** For what concerns Chow rings, homology and cohomology of Deligne-Mumford stacks, we refer the reader to [AGV06] Sec. 2; we follow the same notations.

We begin with a general result.

**Lemma 3.6.** *Let* $\mathbb{P}(w)$ *be a Gorenstein orbifold and let* $\rho : Z \to |\mathbb{P}(w)|$ *be a crepant resolution associated to a subdivision* $\Sigma'$ *of* $\Sigma$ *and the identity morphism of* $N$. *Then the cone* $\mathrm{M}_\rho(\mathrm{Z})$ *is polyhedral.*

*Proof.* Let $\Sigma'(n-1)$ be the set of $(n-1)$-dimensional cones of $\Sigma'$. Then

$$\mathrm{M}_\rho(\mathrm{Z}) = \left\{ \sum_{\nu \in \Sigma'(\mathrm{n}-1)} \gamma_\nu \, [\mathrm{V}(\nu)] \;\middle|\; \gamma_\nu \in \mathbb{N},\, \rho_\star \left( \sum_{\nu \in \Sigma'(\mathrm{n}-1)} \gamma_\nu \, [\mathrm{V}(\nu)] \right) = 0 \right\},$$

where, for any $\nu \in \Sigma'(n-1)$, $\mathrm{V}(\nu)$ denotes the rational curve in $Z$ stable under the torus action which is associated to $\nu$ [Ful93], $[\mathrm{V}(\nu)]$ is the induced Chow class.

Let now $L \in \mathrm{Pic}(|\mathbb{P}(w)|)$ be an ample line bundle. From standard intersection theory we have (see e.g. [Ful98]):

$$\rho_\star\left(\sum_{\nu\in\Sigma'(n-1)}\gamma_\nu\,[\mathrm{V}(\nu)]\right)=0\quad\text{if and only if}\quad c_1\left(\rho^\star L\right)\cap\left(\sum_{\nu\in\Sigma'(n-1)}\gamma_\nu\,[\mathrm{V}(\nu)]\right)=0.$$

Since $c_1\left(\rho^\star L\right)\cap[\mathrm{V}(\nu)]\geq 0$ for any $\nu$, it follows that

$$\mathrm{M}_\rho(\mathrm{Z})=\left\{\sum_{\nu\in\Sigma'(\mathrm{n}-1)}\gamma_\nu\,[\mathrm{V}(\nu)]\ \mid\ \gamma_\nu\in\mathbb{N},\ \rho_\star\left([\mathrm{V}(\nu)]\right)=0\right\},$$

hence the claim. □

The steps that we will follow during the proof of our results are the following.

(1) Let $(N,\beta,\Sigma)$ be the stacky fan (2.1), choose a subdivision $\Sigma'$ of $\Sigma$ such that the morphism $\rho: Z\to|\mathbb{P}(w)|$ associated to $\Sigma'$ and the identity of N is a crepant resolution, where $Z$ is the toric variety associated to $\Sigma'$.
In dimension 2, this is the classical Hirzebruch-Jung algorithm. In dimension 3, this is always possible (see [CR02]).

(2) Set
$$H:=c_1\left(\mathcal{O}_{\mathbb{P}(w)}(1)\right)\in H^2(\mathbb{P}(w);\mathbb{C}),$$
and set
$$h:=\rho^\star H\in H^2(Z;\mathbb{C}).$$
Note that we identify $H^2(\mathbb{P}(w);\mathbb{C})$ with $H^2(|\mathbb{P}(w)|;\mathbb{C})$. Let us denote by $b_i\in H^2(Z;\mathbb{C})$ (by $e_1,...,e_d\in H^2(Z;\mathbb{C})$ resp.) the first Chern class of the line bundle associated to the torus invariant divisor corresponding to the ray of $\Sigma'$ generated by $\beta(v_i)$ (the rays in $\Sigma'(1)-\Sigma(1)$ resp.), for any $i\in\{0,...,n\}$.
Then compute a presentation of $H^\star(Z;\mathbb{C})$ as a quotient of $\mathbb{C}[h,e_1,\dots,e_d]$, e.g. following [Ful93] Sec. 5.2.
Note that, since $\rho$ is crepant, we have the following equality:
$$h=\frac{1}{\sum_{i=0}^n w_i}\left(\sum_{i=0}^n b_i+\sum_{j=1}^d e_j\right).\tag{3.7}$$

(3) Find the generators of $\mathrm{M}_\rho(Z)$. Here is a possible way: $H$ is an ample line bundle (see e.g. [Ful93] Sec. 3.4), then, following the proof of Lemma 3.6, we see that $\mathrm{M}_\rho(Z)$ is generated by the set
$$\left\{[V(\nu)]\ \mid\ h\cap[V(\nu)]=0,\ \nu\in\Sigma'(n-1)-\Sigma(n-1)\right\}.\tag{3.8}$$
We remark that, given the fans $\Sigma$ and $\Sigma'$ and using (3.7), the determination of the set (3.8) is straightforward using for instance the results in [Ful93] Sec. 5.2.

(4) Compute the Gromov-Witten invariants in a basis of $H^\star(Z;\mathbb{C})$ and express the quantum corrected cohomology ring $H^\star_\rho(Z;\mathbb{C})(q_1,\dots,q_m)$ in such a basis.

(5) Compute a presentation of $H^{\star}_{\mathrm{CR}}(\mathbb{P}(w);\mathbb{C})$. Note that using the combinatorial model presented in [BMP07], or results from [Man05], one obtains directly a basis and the multiplicative table of $H^{\star}_{\mathrm{CR}}(\mathbb{P}(w);\mathbb{C})$. Equivalently the results from [BCS05] and [CCLT06] can be used to obtain a presentation of the Chen-Ruan cohomology ring of weighted projective spaces.

(6) Find a suitable evaluation of the quantum parameters and a linear map $H^{\star}_{\rho}(Z;\mathbb{C})(c_1,\dots,c_m) \to H^{\star}_{\mathrm{CR}}(\mathbb{P}(w);\mathbb{C})$ that induces a ring isomorphism.

**Example 3.9.** As an example we work out the steps above for $\mathcal{X} := \mathbb{P}(1,1,2,2)$. Note that $\mathbb{P}(1,1,2,2)$ is an orbifold with transversal $\mathrm{A}_1$ singularities, hence Ruan's conjecture in this case follows from [Per07].

(1) Identify the stacky fan $(N,\beta,\Sigma)$ with $(\mathbb{Z}^3, \{\Lambda(\beta(v_i))\}_{i\in\{0,1,2,3\}}, \Sigma)$, where the $v_i$ are defined in (2.1) and $\Lambda : N \to \mathbb{Z}^3$ is the isomorphism defined by sending $v_0 \mapsto (-1,-2,-2)$, $v_1 \mapsto (1,0,0)$, $v_2 \mapsto (0,1,0)$ and $v_3 \mapsto (0,0,1)$.

The subdivision $\Sigma'$ of $\Sigma$ is obtained by adding the ray generated by

$$\mathrm{P} := (0,-1,-1) = \frac{1}{2}\left(\Lambda(\beta(v_0)) + \Lambda(\beta(v_1))\right)$$

as shown in Figure 1.

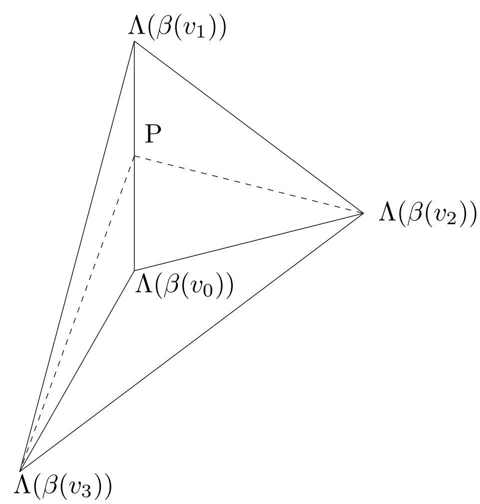


FIGURE 1. Polytope of $\mathbb{P}(1,1,2,2)$ and a crepant resolution.

(2) We have $H^{\star}(Z;\mathbb{C}) \cong \mathbb{C}[h,e]/\langle h^2 + \frac{1}{4}e^2 - he, h^2 e\rangle$.

(3) The new effective curves are $\mathrm{PD}(eb_0)$ and $\mathrm{PD}(eb_2)$ but only $\mathrm{PD}(eb_2)$ is contracted by $\rho$, so $\mathrm{M}_{\rho}(Z)$ is generated by $\Gamma_1 := \mathrm{PD}(eb_2)$. Note that $\mathrm{PD}(eb_2) = \mathrm{PD}(2he)$.

(4) Since $\dim\left([\overline{\mathcal{M}}_{0,3}(Z,d\Gamma_1)]^{\mathrm{vir}}\right) = 3$, the Gromov-Witten invariants $\Psi^{Z}_{d\Gamma_1}(\alpha_1,\alpha_2,\alpha_3)$ are not zero only when all the $\alpha_i$ have degree 2. By the Divisor Axiom we have:

$$\Psi^{Z}_{d\Gamma_1}(\alpha_1,\alpha_2,\alpha_3) = \left(\int_{\Gamma_1}\alpha_1\right)\left(\int_{\Gamma_1}\alpha_2\right)\left(\int_{\Gamma_1}\alpha_3\right)\cdot d^3 \cdot \Psi^{Z}_{\Gamma_1}(\cdot).$$

From [Man05, Proposition IV.3.13] we get:

$$\int_Z h^3 = \int_Z \rho^\star c_1(\mathcal{O}_{\mathbb{P}(w)}(1))^3 = \int^{\mathrm{CR}}_{\mathbb{P}(1,1,2,2)} c_1(\mathcal{O}_{\mathbb{P}(w)}(1))^3 = \frac{1}{4}.$$

This gives:

$$\Psi^Z_{d\Gamma_1}(h,h,h) = 0 \qquad \Psi^Z_{d\Gamma_1}(h,h,e) = 0$$
$$\Psi^Z_{d\Gamma_1}(h,e,e) = 0 \qquad \Psi^Z_{d\Gamma_1}(e,e,e) = -2^3 d^3 \cdot \Psi^Z_{\Gamma_1}(\cdot)$$

From [Per07] we get:

$$\Psi^Z_{d\Gamma_1}(\cdot) = \frac{2}{d^3}\int_{|\mathbb{P}(2,2)|} c_1(\mathcal{O}_{\mathbb{P}(2,2)}(1)) = \frac{1}{d^3}.$$

This gives the quantum correction:

$$e *_\rho e = -4h^2 + \left(4 + \frac{8q_1}{1-q_1}\right) he,$$

hence:

$$H^\star_\rho(Z;\mathbb{C})(q_1) \cong \mathbb{C}((q_1))[h,e]/\langle h^2 e, h^2 + \frac{1}{4}e^2 - he - \frac{2q_1}{1-q_1}he\rangle. \tag{3.10}$$

(5) The Chen-Ruan cohomology ring has the following presentation (see [BMP07] or Sec. 4 for more details):

$$H^\star_{\mathrm{CR}}(\mathbb{P}(1,1,2,2);\mathbb{C}) \cong \mathbb{C}[H,E]/\langle H^2 - E^2, H^2 E\rangle.$$

(6) Set $q_1 = -1$ in (3.10) and define the following map:

$$\begin{array}{rcl} H^\star_{\mathrm{CR}}(\mathbb{P}(1,1,2,2)) & \to & H^\star_\rho(Z)(-1) \\ H & \mapsto & h, \\ E & \mapsto & \frac{\mathrm{i}}{2}e. \end{array} \tag{3.11}$$

Then (3.11) is a ring isomorphism.

## 4. Proof of Theorem 3.3

In this section we prove Theorem 3.3 by following the steps above for $\mathcal{X} = \mathbb{P}(1,3,4,4)$.

We remark that this result confirms for the transversal $A_3$ singularity case Conjecture 1.9 in [Per07] regarding the values of the $q_i$'s, secondly, the change of variables is inspired from those of [NW03] (see also [BGP05] and [CCIT07]).

The coarse moduli space of $\mathbb{P}(1,3,4,4)$ has a transversal $A_3$ singularity on the line $[0:0:x_2:x_3]$ and an isolated singularity of type $\frac{1}{3}(1,1,1)$ at the point $[0:1:0:0]$.

(1) We identify the stacky fan $(N,\beta,\Sigma)$ with $(\mathbb{Z}^3, \{\Lambda(\beta(v_i))\}_{i\in\{0,1,2,3\}}, \Sigma)$, where the $v_i$ are defined in (2.1) and $\Lambda : N \to \mathbb{Z}^3$ is the isomorphism defined by sending $v_0 \mapsto (-3,-4,-4)$, $v_1 \mapsto (1,0,0)$, $v_2 \mapsto (0,1,0)$ and $v_3 \mapsto (0,0,1)$.

A crepant resolution of $|\mathbb{P}(1,3,4,4)|$ can be constructed using standard methods in toric geometry. More precisely, consider the integral points

$$\begin{aligned}
&\mathrm{P}_1 := (0,-1,-1) = \frac{3}{4}\Lambda(\beta(v_1)) + \frac{1}{4}\Lambda(\beta(v_0)),\\
&\mathrm{P}_2 := (-1,-2,-2) = \frac{1}{2}\Lambda(\beta(v_1)) + \frac{1}{2}\Lambda(\beta(v_0)),\\
&\mathrm{P}_3 := (-2,-3,-3) = \frac{1}{4}\Lambda(\beta(v_1)) + \frac{3}{4}\Lambda(\beta(v_0)),\\
\text{and}\quad &\mathrm{P}_4 := (-1,-1,-1) = \frac{1}{3}\Lambda(\beta(v_0)) + \frac{1}{3}\Lambda(\beta(v_2)) + \frac{1}{3}\Lambda(\beta(v_3)),
\end{aligned}$$

then subdivide $\Sigma$ inserting the rays generated by $\mathrm{P}_1$, $\mathrm{P}_2$, $\mathrm{P}_3$ and $\mathrm{P}_4$ as shown in Figure 2. Let $\Sigma'$ be the fan obtained after this subdivision, let $Z$ be the toric variety and $\rho : Z \to |\mathbb{P}(1,3,4,4)|$ be the morphism associated to the identity on $\mathbb{Z}^3$. Then $Z$ is smooth and $\rho$ is crepant. The crepancy of $\rho$ follows from the existence of a continuous piecewise linear function $|\Sigma| \to \mathbb{R}$ which is linear when restricted to each cone of $\Sigma$ and associates the value $-1$ to the minimal lattice points of the rays of $\Sigma'$ (see [Ful93], Sec. 3.4).

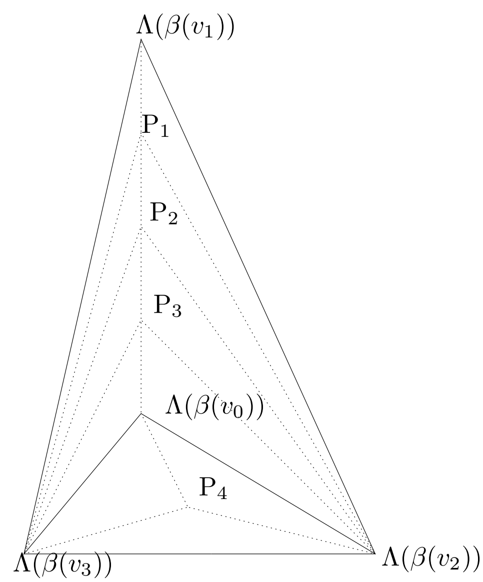


FIGURE 2. Polytope of $\mathbb{P}(1,3,4,4)$ and a crepant resolution

(2) We denote by $b_i \in H^2(Z;\mathbb{C})$ ($e_j$ resp.) the first Chern class of the line bundle associated to the torus invariant divisor corresponding to the ray generated by $\Lambda(\beta(v_i))$ ($\mathrm{P}_j$ resp.), for $i \in \{0,1,2,3\}$ ($j \in \{1,2,3,4\}$ resp.), furthermore we define

$$h = \frac{1}{12}\left(\sum_{i=0}^{3} b_i + \sum_{j=1}^{4} e_j\right).$$

The cohomology ring of $Z$ has the following presentation: it is isomorphic to the quotient of the polynomial ring $\mathbb{C}[h, e_1, e_2, e_3, e_4]$ by the ideal generated by

$$\begin{gathered}3he_4,\ e_1e_3,\ e_1e_4,\ e_2e_4,\ e_3e_4,\\ e_1^2 - 10he_1 - 4he_2 - 2he_3 + 24h^2,\\ e_1e_2 + 3he_1 + 2he_2 + he_3 - 12h^2,\\ e_2^2 - 6he_1 - 12he_2 - 2he_3 + 24h^2,\\ e_2e_3 + 3he_1 + 6he_2 + he_3 - 12h^2,\\ e_3^2 - 6he_1 - 12he_2 - 14he_3 + 24h^2,\\ 16h^2e_1,\ 16h^2e_2,\ 16h^2e_3, 16h^3 - \frac{1}{27}e_4^3.\end{gathered}$$

We fix the following basis of the vector space $H^\star(Z;\mathbb{C})$:

$$1,\ h,\ e_1,\ e_2,\ e_3,\ e_4,\ h^2,\ he_1,\ he_2,\ he_3,\ e_4^2,\ h^3.$$

(3) As explained in Sec. 3, the cone $\mathrm{M}_\rho(Z)$ can be directly determined from the combinatorial data $\Sigma$ and $\Sigma'$. In our case $\mathrm{M}_\rho(Z)$ is generated by $\Gamma_1 := \mathrm{PD}(4\mathrm{he}_1)$, $\Gamma_2 := \mathrm{PD}(4\mathrm{he}_2)$, $\Gamma_3 := \mathrm{PD}(4\mathrm{he}_3)$ and $\Gamma_4 := \mathrm{PD}(-\frac{1}{3}\mathrm{e}_4^2)$, where PD means Poincaré dual.

(4) Here we give a presentation of the quantum corrected cohomology ring $H^\star_\rho(Z;\mathbb{C})(q_1, q_2, q_3, 0)$.

We first notice that any curve of homology class $d_4\Gamma_4$ is disjoint from any other curve of class $d_1\Gamma_1 + d_2\Gamma_2 + d_3\Gamma_3$; in other words, $\overline{\mathcal{M}}_{0,0}(Z,\Gamma)$ is empty if $\Gamma = \sum_{\ell=1}^4 d_\ell\Gamma_\ell$ with $d_4\cdot(d_1+d_2+d_3) \neq 0$. From the degree axiom it follows that we need to consider only Gromov-Witten invariants $\Psi^Z_\Gamma(\alpha_1,\alpha_2,\alpha_3)$ with $\alpha_i \in H^2(Z;\mathbb{C})$, $i \in \{1,2,3\}$. Finally, applying the divisor axiom we deduce the following expression for the quantum 3-point function:

$$\begin{aligned}&\langle\alpha_1,\alpha_2,\alpha_3\rangle_{\mathrm{q}}(q_1,q_2,q_3,q_4)\\ =\ & \sum_{d_1,d_2,d_3>0}\left(\prod_{i=1}^3\int_{\sum_{\ell=1}^3 d_\ell\Gamma_\ell}\alpha_i\right)\deg\left[\overline{\mathcal{M}}_{0,0}(Z,\sum_{\ell=1}^3 d_\ell\Gamma_\ell)\right]^{\mathrm{vir}} q_1^{d_1}q_2^{d_2}q_3^{d_3}\\ +\ & \sum_{d_4>0}\left(\prod_{i=1}^3\int_{d_4\Gamma_4}\alpha_i\right)\deg[\overline{\mathcal{M}}_{0,0}(Z,d_4\Gamma_4)]^{\mathrm{vir}}q_4^{d_4}.\end{aligned}$$

Since $\int_{\Gamma_\ell} h = 0$ for any $\ell \in \{1,2,3,4\}$,

$$h *_\rho \alpha = h\alpha \qquad \text{for any} \quad \alpha \in H^\star(Z;\mathbb{C}),$$

similarly

$$e_i *_\rho e_4 = \begin{cases} e_ie_4 = 0, & \text{if} \quad i \neq 4;\\ \epsilon(q_4)e_4^2, & \text{otherwise,}\end{cases}$$

for some function $\epsilon(q_4)$ such that $\epsilon(0) = 1$.

As in the isomorphism of rings that we will define later we will put $q_4 = 0$, we only consider classes $\Gamma = d_1\Gamma_1 + d_2\Gamma_2 + d_3\Gamma_3$ for $d_i \in \mathbb{N}$. We set $\Gamma_{\mu\nu} := \Gamma_\mu + \cdots + \Gamma_\nu$

for $\mu, \nu \in \{1,2,3\}$ and $\mu \leq \nu$. Using Theorem 5.1 in Section 5 we get:

$$\deg[\overline{\mathcal{M}}_{0,0}(Z,\Gamma)]^{\mathrm{vir}} = \begin{cases} 1/d^3 & \text{if} \quad \Gamma = d\Gamma_{\mu\nu} \quad \text{for} \quad \mu \leq \nu \in \{1,2,3\} \quad \text{and} \quad d \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Hence the remaining part of the multiplicative table of $H^\star_\rho(Z;\mathbb{C})(q_1,q_2,q_3,0)$ is as follows:

$$\begin{aligned} e_1 *_\rho e_1 = -24h^2 &+ \left(10 + 16\frac{q_1}{1-q_1} + 4\frac{q_1q_2}{1-q_1q_2} + 4\frac{q_1q_2q_3}{1-q_1q_2q_3}\right) he_1 \\ &+ \left(4 + 4\frac{q_2}{1-q_2} + 4\frac{q_1q_2}{1-q_1q_2} + 4\frac{q_2q_3}{1-q_2q_3} + 4\frac{q_1q_2q_3}{1-q_1q_2q_3}\right) he_2 \\ &+ \left(2 + 4\frac{q_2q_3}{1-q_2q_3} + 4\frac{q_1q_2q_3}{1-q_1q_2q_3}\right) he_3, \end{aligned}$$

$$\begin{aligned} e_1 *_\rho e_2 = 12h^2 &+ \left(-3 - 8\frac{q_1}{1-q_1} + 4\frac{q_1q_2}{1-q_1q_2}\right) he_1 \\ &+ \left(-2 - 8\frac{q_2}{1-q_2} + 4\frac{q_1q_2}{1-q_1q_2} - 4\frac{q_2q_3}{1-q_2q_3}\right) he_2 \\ &+ \left(-1 - 4\frac{q_2q_3}{1-q_2q_3}\right) he_3, \end{aligned}$$

$$\begin{aligned} e_1 *_\rho e_3 = &\left(-4\frac{q_1q_2}{1-q_1q_2} + 4\frac{q_1q_2q_3}{1-q_1q_2q_3}\right) he_1 \\ &+ \left(4\frac{q_2}{1-q_2} - 4\frac{q_1q_2}{1-q_1q_2} - 4\frac{q_2q_3}{1-q_2q_3} + 4\frac{q_1q_2q_3}{1-q_1q_2q_3}\right) he_2 \\ &+ \left(-4\frac{q_2q_3}{1-q_2q_3} + 4\frac{q_1q_2q_3}{1-q_1q_2q_3}\right) he_3, \end{aligned}$$

$$\begin{aligned} e_2 *_\rho e_2 = -24h^2 &+ \left(6 + 4\frac{q_1}{1-q_1} + 4\frac{q_1q_2}{1-q_1q_2}\right) he_1 \\ &+ \left(12 + 16\frac{q_2}{1-q_2} + 4\frac{q_1q_2}{1-q_1q_2} + 4\frac{q_2q_3}{1-q_2q_3}\right) he_2 \\ &+ \left(2 + 4\frac{q_3}{1-q_3} + 4\frac{q_2q_3}{1-q_2q_3}\right) he_3, \end{aligned}$$

$$\begin{aligned} e_2 *_\rho e_3 = 12h^2 &+ \left(-3 - 4\frac{q_1q_2}{1-q_1q_2}\right) he_1 \\ &+ \left(-6 - 8\frac{q_2}{1-q_2} - 4\frac{q_1q_2}{1-q_1q_2} + 4\frac{q_2q_3}{1-q_2q_3}\right) he_2 \\ &+ \left(-1 - 8\frac{q_3}{1-q_3} + 4\frac{q_2q_3}{1-q_2q_3}\right) he_3, \end{aligned}$$

$$e_3 *_\rho e_3 = -24h^2 + \left(6 + 4\frac{q_1q_2}{1-q_1q_2} + 4\frac{q_1q_2q_3}{1-q_1q_2q_3}\right) he_1$$
$$+ \left(12 + 4\frac{q_2}{1-q_2} + 4\frac{q_1q_2}{1-q_1q_2} + 4\frac{q_2q_3}{1-q_2q_3} + 4\frac{q_1q_2q_3}{1-q_1q_2q_3}\right) he_2$$
$$+ \left(14 + 16\frac{q_3}{1-q_3} + 4\frac{q_2q_3}{1-q_2q_3} + 4\frac{q_1q_2q_3}{1-q_1q_2q_3}\right) he_3.$$

(5) To compute the Chen-Ruan cohomology ring $H^\star_{\mathrm{CR}}(\mathcal{X};\mathbb{C})$ we follow [BMP07]. The twisted sectors are indexed by the set $\mathrm{T} := \left\{\exp(2\pi\mathrm{i}\gamma) \,|\, \gamma \in \left\{0, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right\}\right\}$. For any $g \in \mathrm{T}$, $\mathcal{X}_{(g)}$ is a weighted projective space: set $\mathrm{I}(g) := \{i \in \{0,1,2,3\} \mid g^{w_i} = 1\}$, then $\mathcal{X}_{(g)} = \mathbb{P}(w_{\mathrm{I}(g)})$, where $(w_{\mathrm{I}(g)}) = (w_i)_{i\in\mathrm{I}(g)}$. The inertia stack is the disjoint union of the twisted sectors:

$$\mathrm{I}\mathcal{X} = \sqcup_{g\in\mathrm{T}}\mathbb{P}(w_{\mathrm{I}(g)}).$$

As a vector space, the Chen-Ruan cohomology is the cohomology of the inertia stack; the graded structure is obtained by shifting the degree of the cohomology of any twisted sector by twice the corresponding age (see (2.4)). We have

$$(4.1)\quad \begin{aligned} H^p_{\mathrm{CR}}(\mathcal{X};\mathbb{C}) &= \oplus_{g\in\mathrm{T}} H^{p-2\mathrm{age}(g)}(\mathbb{P}(w_{\mathrm{I}(g)});\mathbb{C}) \\ &= H^p(\mathbb{P}(1,3,4,4);\mathbb{C}) \oplus H^{p-2}(\mathbb{P}(3);\mathbb{C}) \oplus H^{p-4}(\mathbb{P}(3);\mathbb{C}) \oplus \\ &\quad\; H^{p-2}(\mathbb{P}(4,4);\mathbb{C}) \oplus H^{p-2}(\mathbb{P}(4,4);\mathbb{C}) \oplus H^{p-2}(\mathbb{P}(4,4);\mathbb{C}). \end{aligned}$$

A basis of $H^\star_{\mathrm{CR}}(\mathcal{X};\mathbb{C})$ is easily obtained in the following way: set

$$H, E_1, E_2, E_3, E_4 \in H^\star_{\mathrm{CR}}(\mathcal{X};\mathbb{C})$$

be the image of $c_1(\mathcal{O}_\mathcal{X}(1)) \in H^2(\mathcal{X};\mathbb{C})$, $1 \in H^0(\mathcal{X}_{(\exp(\pi\mathrm{i}/2))};\mathbb{C})$, $1 \in H^0(\mathcal{X}_{(\exp(\pi\mathrm{i}))};\mathbb{C})$, $1 \in H^0(\mathcal{X}_{(\exp(\pi\mathrm{i}3/2))};\mathbb{C})$ and $1 \in H^0(\mathcal{X}_{(\exp(2\pi\mathrm{i}/3))};\mathbb{C})$ respectively, under the inclusion $H^\star(\mathbb{P}(w_{\mathrm{I}(g)})) \to H^\star_{\mathrm{CR}}(\mathcal{X})$ determined by the decomposition (4.1). As a $\mathbb{C}$-algebra, the Chen-Ruan cohomology ring is generated by $H, E_1, E_2, E_3, E_4$ with relations (see [BMP07]):

$$HE_4,\ E_1E_1 - 3HE_2,\ E_1E_2 - 3HE_3,\ E_1E_3 - 3H^2,$$
$$E_2E_2 - 3H^2,\ E_2E_3 - HE_1,\ E_3E_3 - HE_2,\ 16H^3 - E_4^3,$$
$$H^2E_1,\ H^2E_2,\ H^2E_3,\ E_1E_4,\ E_2E_4,\ E_3E_4.$$

We see that the following elements form a basis of $H^\star_{\mathrm{CR}}(\mathcal{X};\mathbb{C})$ which we fix for the rest of the paper:

$$1,\ H,\ E_1,\ E_2,\ E_3,\ E_4,\ H^2,\ HE_1,\ HE_2,\ HE_3,\ E_4^2,\ H^3.$$

**Remark 4.2.** Note that the elements of our basis are different from those used in [BMP07] by a combinatorial factor.

Other methods are suitable in order to compute the Chen-Ruan cup product of weighted projective spaces, here are a few: the results in [BCS05] provide a presentation of the Chen-Ruan cohomology ring for toric Deligne-Mumford stack; results from [Man05] and from [CCLT06].

(6) We have shown in (4) that, for cohomology classes $\alpha_1$ and $\alpha_2$, the product $\alpha_1 *_\rho \alpha_2 \in H^\star_\rho(Z;\mathbb{C})(q_1,q_2,q_3,0)$ differs from the usual cup product only if $\alpha_1, \alpha_2 \in$

$\{e_1, e_2, e_3\}$. We now set $q_1 = q_2 = q_3 = \mathrm{i}$ and write $e_i *_\rho e_j$ with respect to the basis of $H^\star(Z;\mathbb{C})$ fixed in point (2), we have:

$$\begin{aligned}
e_1 *_\rho e_1 &= -24h^2 + (-2+6\mathrm{i})he_1 - 4he_2 + (-2-2\mathrm{i})he_3,\\
e_1 *_\rho e_2 &= 12h^2 + (-1-4\mathrm{i})he_1 + (2-4\mathrm{i})he_2 + he_3,\\
e_1 *_\rho e_3 &= -2\mathrm{i}he_1 - 2\mathrm{i}he_3,\\
e_2 *_\rho e_2 &= -24h^2 + (2+2\mathrm{i})he_1 + 8\mathrm{i}he_2 + (-2+2\mathrm{i})he_3,\\
e_2 *_\rho e_3 &= 12h^2 - he_1 + (-2-4\mathrm{i})he_2 + (1-4\mathrm{i})he_3,\\
e_3 *_\rho e_3 &= -24h^2 + (2-2\mathrm{i})he_1 + 4he_2 + (2+6\mathrm{i})he_3.
\end{aligned}$$

We define a linear map

$$H^\star_\rho(Z;\mathbb{C})(\mathrm{i},\mathrm{i},\mathrm{i},0) \to H^\star_{\mathrm{CR}}(\mathbb{P}(1,3,4,4);\mathbb{C}) \tag{4.3}$$

as follows: we send

$$\begin{pmatrix} h \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{2} & -2\mathrm{i} & \sqrt{2} & 0 \\ 0 & -\mathrm{i}\sqrt{2} & 2\mathrm{i} & -\mathrm{i}\sqrt{2} & 0 \\ 0 & \sqrt{2} & -2\mathrm{i} & -\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 3\exp\left(\frac{2\pi\mathrm{i}}{3}\right) \end{pmatrix} \begin{pmatrix} H \\ E_1 \\ E_2 \\ E_3 \\ E_4 \end{pmatrix},$$

the image of the other elements of the basis is uniquely determined by requiring that (4.3) is a ring isomorphism.

A direct computation shows that (4.3) is a ring isomorphism and that it is an isometry with respect to the inner products given by the Poincaré duality and the Chen-Ruan pairing respectively.

The case where $q_1 = q_2 = q_3 = -\mathrm{i}$ and $q_4 = 0$ is analogous to the previous one. We define a linear map

$$H^\star_\rho(Z;\mathbb{C})(-\mathrm{i},-\mathrm{i},-\mathrm{i},0) \to H^\star_{\mathrm{CR}}(\mathbb{P}(1,3,4,4);\mathbb{C}) \tag{4.4}$$

by sending

$$\begin{pmatrix} h \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{2} & 2\mathrm{i} & \sqrt{2} & 0 \\ 0 & \mathrm{i}\sqrt{2} & -2\mathrm{i} & \mathrm{i}\sqrt{2} & 0 \\ 0 & \sqrt{2} & 2\mathrm{i} & -\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 3\exp\left(\frac{2\pi\mathrm{i}}{3}\right) \end{pmatrix} \begin{pmatrix} H \\ E_1 \\ E_2 \\ E_3 \\ E_4 \end{pmatrix}$$

and extending to the remaining part of the basis in the unique way such that the resulting map is a ring isomorphism.

Also in this case a direct computation shows that (4.4) is a ring isomorphism and it respects the inner pairings.

**Remark 4.5.** Note that the isomorphisms (4.3) and (4.4) are those conjectured in [Per07].

## 5. Gromov-Witten invariants of the resolution of $|\mathbb{P}(1,3,4,4)|$

In this section we compute the Gromov-Witten invariants of the crepant resolution of $|\mathbb{P}(1,3,4,4)|$ of genus 0, homology class $\Gamma = d_1\Gamma_1 + d_2\Gamma_2 + d_3\Gamma_3$ and without marked points. Our result confirms Conjecture 5.1 [Per07]. We follow the notations from Section 4.

**Theorem 5.1.** *Let $\rho : Z \to |\mathbb{P}(1,3,4,4)|$ be the crepant resolution of $|\mathbb{P}(1,3,4,4)|$ defined in Section 4 (1), and let $\Gamma = d_1\Gamma_1 + d_2\Gamma_2 + d_3\Gamma_3$. Then*

$$\deg[\overline{\mathcal{M}}_{0,0}(Z,\Gamma)]^{\mathrm{vir}} = \begin{cases} 1/d^3 & \text{if} \quad \Gamma = d\sum_{i=\mu}^{\nu}\Gamma_i, \quad \text{with} \quad \mu \le \nu \in \{1,2,3\}; \\ 0 & \text{otherwise.} \end{cases}$$

To prove Theorem 5.1 we use the deformation invariance property of the Gromov-Witten invariants. More precisely: we define an open neighborhood $V$ of the singular locus $|\mathbb{P}(4,4)| \subset |\mathbb{P}(1,3,4,4)|$, we construct an explicit deformation of $V$ and then a simultaneous resolution. This gives a deformation of $\rho^{-1}(V)$, a neighborhood of the component of the exceptional divisor which lies over $|\mathbb{P}(4,4)|$. We will denote this deformation by $\overline{\mathrm{Graph}(\mu)}_t$, $t \in \Delta$. Next we relate the Gromov-Witten invariants of $Z$ we are interested in with some Gromov-Witten invariants of $\overline{\mathrm{Graph}(\mu)}_t$ that we can explicitly compute.

5.a. **The neighborhood.** The transversal $A_3$-singularity is identified with $\mathbb{P}^1$ by the morphism $[z_0 : z_1] \mapsto [0 : 0 : z_0 : z_1]$. By abuse of notation we denote by the same symbol $\mathcal{O}(a)$ the sheaf $\mathcal{O}_{\mathbb{P}^1}(a)$ and the corresponding vector bundle, for any $a \in \mathbb{Z}$. Moreover we identify $\mathcal{O}(a) \otimes \mathcal{O}(b)$ with $\mathcal{O}(a+b)$ using the canonical isomorphism. For any vector bundle $E$, we denote by $\underline{0}_E$ its zero section. Finally, we set

$$V_i := \{[x_0 : x_1 : x_2 : x_3] \in |\mathbb{P}(1,3,4,4)| \text{ such that } x_i \neq 0\}$$

for any $i \in \{0,1,2,3\}$, and

$$V := V_2 \cup V_3 \subset |\mathbb{P}(1,3,4,4)|.$$

Consider the bundle morphism

$$\begin{aligned} \psi : \mathcal{O}(1) \oplus \mathcal{O}(3) \oplus \mathcal{O}(1) &\longrightarrow \mathcal{O}(4) \\ (\xi, \eta, \zeta) &\longmapsto \xi \otimes \eta - \zeta^{\otimes 4}, \end{aligned}$$

and the inverse image under $\psi$ of the zero section of $\mathcal{O}(4)$: $\psi^{-1}(\mathrm{Im}(\underline{0}_{\mathcal{O}(4)})) \subset \mathcal{O}(1) \oplus \mathcal{O}(3) \oplus \mathcal{O}(1)$. We have the following

**Lemma 5.2.** *The variety $V$ is isomorphic to $\psi^{-1}(\mathrm{Im}(\underline{0}_{\mathcal{O}(4)}))$.*

*Proof.* An easy computation shows that $V_2 \simeq \mathrm{Spec}\left(\mathbb{C}[s,u,v,w]/(uv - w^4)\right)$ and $V_3 \simeq \mathrm{Spec}\left(\mathbb{C}[t,x,y,z]/(xy - z^4)\right)$. The affine open subvarieties $V_2, V_3 \subset V$ glue together by means of the following ring isomorphism

$$\begin{aligned} \frac{\mathbb{C}[s, \frac{1}{s}, u, v, w]}{(uv - w^4)} &\longrightarrow \frac{\mathbb{C}[t, \frac{1}{t}, x, y, z]}{(xy - z^4)} \\ s &\longmapsto \frac{1}{t} \\ u &\longmapsto \frac{1}{t}x \\ v &\longmapsto \frac{1}{t^3}y \\ w &\longmapsto \frac{1}{t}z. \end{aligned}$$

On the other hand, consider a trivialization of the bundle $\mathcal{O}(1) \oplus \mathcal{O}(3) \oplus \mathcal{O}(1)$ on $W_0 = \{[z_0, z_1] \in \mathbb{P}^1 \mid z_0 \neq 0\}$. On such a trivialization, the morphism $\psi$ is given by

$$\begin{aligned} W_0 \times \mathbb{C}^3 &\longrightarrow W_0 \times \mathbb{C} \\ (s, v_1, v_2, v_3) &\longmapsto (s, v_1 v_2 - v_3^4). \end{aligned}$$

Hence we have that, over $W_0$, $\psi^{-1}(\mathrm{Im}(\underline{0}_{\mathcal{O}(4)}))$ is $\mathrm{Spec}\left(\mathbb{C}[s, v_1, v_2, v_3]/(v_1 v_2 - v_3^4)\right)$. If we do the same over $W_1 = \{[z_0, z_1] \in \mathbb{P}^1 \mid z_1 \neq 0\}$, we deduce that $V$ and $\psi^{-1}(\underline{0}_{\mathcal{O}(4)})$ are union of the same affine varieties with the same gluing. This proves that they are isomorphic. □

5.b. **The deformation.**

We construct a deformation of $V$. The construction is inspired by the theory of deformations of surfaces with $\mathrm{A}_n$ singularities, for this our reference is [Tyu70].

We define

$$f : V \to \mathbb{P}^1 \tag{5.3}$$

to be the composition of the isomorphism $V \xrightarrow{\cong} \psi^{-1}(\mathrm{Im}(\underline{0}_{\mathcal{O}(4)}))$ in Lemma 5.2, followed by the inclusion $\psi^{-1}(\mathrm{Im}(\underline{0}_{\mathcal{O}(4)})) \subset \mathcal{O}(1) \oplus \mathcal{O}(3) \oplus \mathcal{O}(1)$, and then the bundle map. The following remark is crucial.

**Remark 5.4.** The morphism (5.3) exhibits $V$ as a 3-fold fibered over $\mathbb{P}^1$ with fibers isomorphic to a surface $\mathrm{A}_3$ singularity, furthermore the fibration is locally trivial.

The aim is to extend some of the results of [Tyu70] to $V$, when viewed as a family of such surfaces with respect to (5.3).

Consider the bundle morphism

$$\begin{aligned} \chi : \mathcal{O}(1)^{\oplus 4} &\longrightarrow \mathcal{O}(1) \\ (\delta_1, \dots, \delta_4) &\longmapsto \delta_1 + \cdots + \delta_4, \end{aligned}$$

and set $\mathcal{F} := \chi^{-1}(\mathrm{Im}(\underline{0}_{\mathcal{O}(1)}))$. Then consider the bundle morphism

$$\begin{aligned} \pi : \mathcal{O}(1) \oplus \mathcal{O}(3) \oplus \mathcal{O}(1) \oplus \mathcal{F} &\longrightarrow \mathcal{O}(4) \\ (\xi, \eta, \zeta, \delta_1, \dots, \delta_4) &\longmapsto \xi \otimes \eta - \otimes_{i=1}^4 (\zeta + \delta_i) \end{aligned}$$

and set $\mathcal{V}_{\mathcal{F}} := \pi^{-1}(\mathrm{Im}(\underline{0}_{\mathcal{O}(4)}))$. We obtain the following Cartesian diagram

$$\begin{array}{ccc} V & \longrightarrow & \mathcal{V}_{\mathcal{F}} \\ \big\downarrow f & & \big\downarrow F \\ \mathbb{P}^1 & \xrightarrow{\underline{0}_{\mathcal{F}}} & \mathcal{F} \end{array}$$

where $f$ is defined in (5.3) and the vertical right hand side arrow is the composition of the inclusion $\mathcal{V}_{\mathcal{F}} \to \mathcal{O}(1) \oplus \mathcal{O}(3) \oplus \mathcal{O}(1) \oplus \mathcal{F}$ followed by the projection $\mathcal{O}(1) \oplus \mathcal{O}(3) \oplus \mathcal{O}(1) \oplus \mathcal{F} \to \mathcal{F}$.

Note that $F : \mathcal{V}_{\mathcal{F}} \to \mathcal{F}$ is a family of surfaces. We now construct a simultaneous resolution. Consider the rational map

$$\begin{aligned} \mu : \mathcal{V}_{\mathcal{F}} &\dashrightarrow \mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}(1)) \times \mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}(2)) \times \mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}(3)) \\ (\xi, \eta, \zeta, \delta_1, ..., \delta_4) &\longmapsto (\xi, \zeta + \delta_1) \times (\xi, (\zeta + \delta_1) \otimes (\zeta + \delta_2)) \times (\xi, \otimes_{i=1}^3 (\zeta + \delta_i)), \end{aligned}$$

and let $\mathrm{Graph}(\mu)$ be the graph of $\mu$.
Then take the closure of $\mathrm{Graph}(\mu)$ in $\mathcal{V}_{\mathcal{F}} \times \left(\times_{i=1}^{3}\mathbb{P}(\mathcal{O}(1)\oplus\mathcal{O}(i))\right)$:

$$\overline{\mathrm{Graph}(\mu)} \subset \mathcal{V}_{\mathcal{F}} \times \left(\times_{i=1}^{3}\mathbb{P}(\mathcal{O}(1)\oplus\mathcal{O}(i))\right).$$

Let $\mathcal{R} : \overline{\mathrm{Graph}(\mu)} \to \mathcal{V}_{\mathcal{F}}$ be the composition of the inclusion $\overline{\mathrm{Graph}(\mu)} \to \mathcal{V}_{\mathcal{F}} \times \left(\times_{i=1}^{3}\mathbb{P}(\mathcal{O}(1)\oplus\mathcal{O}(i))\right)$ followed by the projection on the first factor $\mathcal{V}_{\mathcal{F}} \times \left(\times_{i=1}^{3}\mathbb{P}(\mathcal{O}(1)\oplus\mathcal{O}(i))\right) \to \mathcal{V}_{\mathcal{F}}$.

**Lemma 5.5.** *The following diagram*

$$\begin{array}{ccc} \overline{\mathrm{Graph}(\mu)} & \xrightarrow{\mathcal{R}} & \mathcal{V}_{\mathcal{F}} \\ \downarrow{\scriptstyle F\circ\mathcal{R}} & & \downarrow{\scriptstyle F} \\ \mathcal{F} & \xrightarrow{\mathrm{id}} & \mathcal{F} \end{array} \tag{5.6}$$

*is a simultaneous resolution of $F : \mathcal{V}_{\mathcal{F}} \to \mathcal{F}$.*

**Proof.** The property of being a simultaneous resolution is local in $\mathcal{F}$. The diagram (5.6) is fibered over $\mathbb{P}^1$. If we restrict it to an open subset of $\mathbb{P}^1$ where $\mathcal{O}(1)$ is trivial, then the assertion is exactly the result of E. Brieskorn [Bri66]. $\square$

Set $\Delta := \mathbb{C}$. For any section $\theta \in H^0(\mathbb{P}^1, \mathcal{F})$, we get a deformation of $V$ parametrized by $\Delta$ as follows

$$\begin{array}{ccccc} V & \longrightarrow & \mathcal{V}_\theta & \longrightarrow & \mathcal{V}_{\mathcal{F}} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_\theta} & & \downarrow{\scriptstyle F} \\ \mathbb{P}^1 & \longrightarrow & \mathbb{P}^1\times\Delta & \xrightarrow{\Theta} & \mathcal{F} \end{array}$$

where $\Theta : \mathbb{P}^1 \times \Delta \to \mathcal{F}$ sends $([z_0 : z_1], t)$ to $t \cdot \theta([z_0 : z_1])$, and $\mathcal{V}_\theta$ is defined by the requirement that the diagram is Cartesian, the map $\mathbb{P}^1 \to \mathbb{P}^1 \times \Delta$ is the inclusion $[z_0 : z_1] \mapsto ([z_0 : z_1], 0)$. The pull-back of the diagram (5.6) with respect to $\Theta$ gives the following diagram

$$\begin{array}{ccc} \overline{\mathrm{Graph}(\mu)}_\theta & \xrightarrow{\rho_\theta} & \mathcal{V}_\theta \\ \downarrow & & \downarrow{\scriptstyle f_\theta} \\ \mathbb{P}^1\times\Delta & \xrightarrow{\mathrm{id}} & \mathbb{P}^1\times\Delta \end{array} \tag{5.7}$$

where $\rho_\theta$ is the pull-back of $\mathcal{R}$ in (5.6). We remark that (5.7) is a simultaneous resolution of $\mathcal{V}_\theta$ over $\mathbb{P}^1 \times \Delta$.

### 5.c. Computation of the invariants.

We specialize the previous construction in the case where $\theta$ is given as follows. Let $\delta \in H^0(\mathbb{P}^1, \mathcal{O}(1))$ be a nonzero section, and set

$$\delta_\ell := \exp\left(\frac{(2\ell+1)\pi\mathrm{i}}{4}\right)\cdot\delta, \quad \ell \in \{1, ..., 4\},$$

then set $\theta := (\delta_1, ..., \delta_4) \in H^0(\mathbb{P}^1, \mathcal{F})$. For any $t \in \Delta$, let us denote $\mathcal{V}_t := f_\theta^{-1}\left(\mathbb{P}^1 \times \{t\}\right)$, $f_t : \mathcal{V}_t \to \mathbb{P}^1 \times \{t\}$ the restriction of $f_\theta$, $\overline{\mathrm{Graph}(\mu)}_t := \rho_\theta^{-1}(\mathcal{V}_t)$

and $\rho_t : \overline{\mathrm{Graph}(\mu)}_t \to \mathcal{V}_t$ the restriction of $\rho_\theta$. We have the following commutative diagram

$$\begin{array}{ccccc}
\rho^{-1}(V) = \overline{\mathrm{Graph}(\mu)}_0 & \hookrightarrow & \overline{\mathrm{Graph}(\mu)}_\theta & \hookleftarrow & \overline{\mathrm{Graph}(\mu)}_t \\
\downarrow{\scriptstyle \rho_0=\rho} & & \downarrow{\scriptstyle \rho_\theta} & & \downarrow{\scriptstyle \rho_t} \\
V & \hookrightarrow & \mathcal{V}_\sigma & \hookleftarrow & \mathcal{V}_t \\
\downarrow{\scriptstyle f_0=f} & & \downarrow{\scriptstyle f_\theta} & & \downarrow{\scriptstyle f_t} \\
\mathbb{P}^1 \times \{0\} & \hookrightarrow & \mathbb{P}^1 \times \Delta & \hookleftarrow & \mathbb{P}^1 \times \{t\}
\end{array}$$

**Lemma 5.8.** *Let $\delta$ be a global section of $\mathcal{O}(1) \to \mathbb{P}^1$ that vanishes only at one point. Then, for $t \neq 0$, the variety $\overline{\mathrm{Graph}(\mu)}_t$ has only one connected nodal complete curve of genus 0 whose dual graph is of type $A_3$ and which is contracted by $\rho_t$ (see the diagram above).*

*Proof.* Without lost of generality we can assume that $\delta$ vanishes only at the point $[1:0]$. Let $W_0 := \{[z_0 : z_1] \in \mathbb{P}^1 \mid z_0 \neq 0\}$. As our bundles are trivial over $W_0$, the restriction of $V$ over $W_0$ is given by $W_0 \times \mathbb{V}(xy - z^4) \subset W_0 \times \mathbb{C}^3$. The choice of the $\delta_\ell$ implies that the 3-fold $\mathcal{V}_t$ is given by

$$W_0 \times \mathbb{V}(xy - \prod_{\ell=1}^{4}(z + \delta_\ell t)) = W_0 \times \mathbb{V}(xy - z^4 - (t\delta)^4) \subset W_0 \times \mathbb{C}^3.$$

By means of $f_t$, $\mathcal{V}_t$ is viewed a family of surfaces parametrized by $\mathbb{P}^1$. As $t \neq 0$ and $\delta([1:0]) = 0$, the only singular surface of the family is the surface $f_t^{-1}([1:0] \times \{t\})$, which is a surface with an isolated $A_3$-singularity. As $\rho_t : \overline{\mathrm{Graph}(\mu)}_t \to \mathcal{V}_t$ is a simultaneous resolution over $\mathbb{P}^1 \times \{t\}$, the fiber $\overline{\mathrm{Graph}(\mu)}_{([1:0],t)}$ is a smooth surface with only one complete connected curve of genus 0 whose dual graph is of type $A_3$ and which is contracted by $\rho_t$. For any $[z_0 : z_1] \neq [1:0]$, the fiber $\overline{\mathrm{Graph}(\mu)}_{([z_0:z_1],t)}$ is isomorphic to the smooth surface $f_t^{-1}([z_0 : z_1] \times \{t\})$. Hence, the exceptional locus of the resolution $\rho_t : \overline{\mathrm{Graph}(\mu)}_t \to \mathcal{V}_t$ has only one connected nodal complete curve of genus 0 whose dual graph is of type $A_3$ and which is contracted by $\rho_t$. □

Let $\Gamma_1, \Gamma_2, \Gamma_3 \in H_2(\overline{\mathrm{Graph}(\mu)}_t; \mathbb{Z})$ be the homology classes of the components of the connected nodal complete curve of genus zero whose dual graph is of type $\mathrm{A}_3$ and which is contracted by $\rho_t$. Let us assume that they are numbered in such a way that, if $\Gamma_i$ is the class of $\tilde{\Gamma}_i$, then the intersection $\tilde{\Gamma}_i \cap \tilde{\Gamma}_j$ is empty if $\mid i - j \mid > 1$. Then the previous Lemma implies that, for $t \neq 0$, $\overline{\mathrm{Graph}(\mu)}_t$ satisfies the hypothesis of Proposition 2.10 of [BKL01]. Therefore we deduce the following formula:

(5.9)

$$\deg[\overline{\mathcal{M}}_{0,0}(\overline{\mathrm{Graph}(\mu)}_{\mathrm{t}}, \Gamma)]^{\mathrm{vir}} = \begin{cases} 1/d^3 & \text{if } \Gamma = d(\Gamma_\mu + \Gamma_{\mu+1} + \ldots + \Gamma_\nu), \text{ for } \mu \leq \nu; \\ 0 & \text{otherwise.} \end{cases}$$

This formula together with the next lemma completes the proof of Theorem 5.1.

**Lemma 5.10.** *For any $t \in \Delta$, the following equality holds*

$$\deg[\overline{\mathcal{M}}_{0,0}(\overline{\mathrm{Graph}(\mu)}_{\mathrm{t}}, \Gamma)]^{\mathrm{vir}} = \deg[\overline{\mathcal{M}}_{0,0}(\mathrm{Z}, \Gamma)]^{\mathrm{vir}}.$$

*Proof.* Since $\Gamma$ is the homology class of a contracted curve, we have an isomorphism of moduli stacks (see [Per07, Lemma 7.1]):

$$\overline{\mathcal{M}}_{0,0}(Z,\Gamma) \simeq \overline{\mathcal{M}}_{0,0}(\rho^{-1}(V),\Gamma); \tag{5.11}$$

in particular the right hand side moduli stack is proper with projective coarse moduli space. The isomorphism (5.11) identifies the tangent-obstruction theories used to define the Gromov-Witten invariants, hence the virtual fundamental classes $[\overline{\mathcal{M}}_{0,0}(\rho^{-1}(V),\Gamma)]^{\mathrm{vir}}$ and $[\overline{\mathcal{M}}_{0,0}(Z,\Gamma)]^{\mathrm{vir}}$ have the same degree. Then it is enough to prove that, for any $t \in \Delta$,

$$\deg[\overline{\mathcal{M}}_{0,0}(\overline{\mathrm{Graph}(\mu)}_t,\Gamma)]^{\mathrm{vir}} = \deg[\overline{\mathcal{M}}_{0,0}(\rho^{-1}(\mathrm{V}),\Gamma)]^{\mathrm{vir}}. \tag{5.12}$$

Gromov-Witten invariants of projective varieties are invariant under deformation of the target variety. We now explain why this result holds for $\rho^{-1}(V)$ and $\overline{\mathrm{Graph}(\mu)}_t$ even if they are not projective.

Let $q_\theta : \overline{\mathrm{Graph}(\mu)}_\theta \to \Delta$ be the composition of $f_\theta \circ \rho_\theta$ in (5.7) followed by the projection $\mathbb{P}^1 \times \Delta \to \Delta$. The morphism $q_\theta$ is smooth as composition of smooth morphisms. Moreover $q_\theta$ factories through an embedding followed by a projective morphism. To see this, it is enough to prove the same statement for the morphism $F \circ \mathcal{R} : \overline{\mathrm{Graph}(\mu)} \to \mathcal{F}$ in (5.6). By construction, $\overline{\mathrm{Graph}(\mu)}$ is embedded in $\mathcal{O}(1) \oplus \mathcal{O}(3) \oplus \mathcal{O}(1) \oplus \mathcal{F} \times \times_{i=1}^3 \mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}(i))$, moreover $F \circ \mathcal{R}$ is the restriction of the projection $\mathcal{O}(1) \oplus \mathcal{O}(3) \oplus \mathcal{O}(1) \oplus \mathcal{F} \times \left(\times_{i=1}^3 \mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}(i))\right) \to \mathcal{F}$.
Let us consider now the projection $\mathcal{O}(1) \oplus \mathcal{O}(3) \oplus \mathcal{O}(1) \oplus \mathcal{F} \to \mathcal{F}$, it has a vector bundle structure over $\mathcal{F}$, then it can be seen as a subbundle of the projective bundle $\mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}(3) \oplus \mathcal{O}(1) \oplus \mathcal{F} \oplus \mathcal{O}_{\mathcal{F}}) \to \mathcal{F}$, therefore we have that $F \circ \mathcal{R}$ factors as the composition of an embedding followed by a projective morphism.

To finish the proof, let us consider the moduli stack which parameterizes relative stable maps to $q_\theta : \overline{\mathrm{Graph}(\mu)}_\theta \to \Delta$ of homology class $\Gamma$ and genus zero. We denote it by $\overline{\mathcal{M}}_{0,0}(\overline{\mathrm{Graph}(\mu)}_\theta/\Delta,\Gamma)$. As $\Gamma$ is the class of curves which are contracted by the resolution $\rho_\theta$ and $q_\theta : \overline{\mathrm{Graph}(\mu)}_\theta \to \Delta$ factories through an embedding followed by a projective morphism, Theorem 1.4.1 of [AV02] implies that the moduli space $\overline{\mathcal{M}}_{0,0}(\overline{\mathrm{Graph}(\mu)}_\theta/\Delta,\Gamma)$ is a proper Deligne-Mumford stack. Since the class $\Gamma$ is contracted by $\rho_\theta$, for any $t \in \Delta$ the fiber at $t$ of the natural morphism $\overline{\mathcal{M}}_{0,0}(\overline{\mathrm{Graph}(\mu)}_\theta/\Delta,\Gamma) \to \Delta$ is the proper Deligne-Mumford stack $\overline{\mathcal{M}}_{0,0}(\overline{\mathrm{Graph}(\mu)}_t,\Gamma)$. Then the same proof of Theorem 4.2 in [LT98] applies in this situation and we get (5.12). □

## 6. The case $\mathcal{X} = \mathbb{P}(1,\ldots,1,n)$

In this Section we will prove the following proposition. Here $\mathcal{X}$ denotes $\mathbb{P}(1,...,1,n)$.

**Proposition 6.1.** *Let $n \geq 2$ be an integer and consider the $n$-dimensional weighted projective space $\mathbb{P}(1,...,1,n)$. Let $Z$ be the crepant resolution of $|\mathbb{P}(1,...,1,n)|$ defined in point (1) below. Then, there is a ring isomorphism*

$$\mathrm{H}^\star(Z;\mathbb{C}) \cong \mathrm{H}^\star_{\mathrm{CR}}(\mathbb{P}(1,\ldots,1,n);\mathbb{C}).$$

*Proof.* We follow the steps described in Section 3.

The coarse moduli space $|\mathbb{P}(1,\ldots,1,n)|$ has an isolated singularity of type $\frac{1}{n}(1,\ldots,1)$ at the point $[0:\ldots:0:1]$.

(1) We identify the stacky fan $(N, \beta, \Sigma)$ defined in (2.1) with $(\mathbb{Z}^n, \{\Lambda(\beta(v_i))\}_{i=0}^n, \Sigma)$ by means of the isomorphism $\Lambda : N \to \mathbb{Z}^n$ defined by sending $v_0$ to $(-1, ..., -1, -n)$ and $v_i$ to the $i$-th vector of the standard basis of $\mathbb{Z}^n$, for $i \in \{1, ..., n\}$.

The crepant resolution is defined as follows: consider the ray $\langle P \rangle$ generated by $\mathrm{P} := (0, ..., 0, -1) = \frac{1}{n}\sum_{i=0}^{n-1} \Lambda(\beta(v_i))$, then let $\Sigma'$ be the fan obtained from $\Sigma$ by replacing the cone generated by $\Lambda(\beta(v_0)), \ldots, \Lambda(\beta(v_{n-1}))$ with the cones generated by $\Lambda(\beta(v_0)), \ldots, \widehat{\Lambda(\beta(v_i))}, \ldots, \Lambda(\beta(v_{n-1}))$ and P for any $i \in \{0, \ldots, n-1\}$. We draw as an example the polytope for the case $n = 3$ in Figure 3. Define $Z$ to be the toric variety associated to $\Sigma'$, and $\rho : Z \to X$ to be the morphism associated to the identity in $\mathbb{Z}^n$.

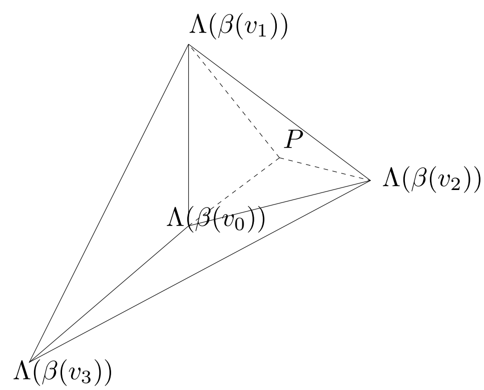


FIGURE 3. Polytope of $\mathbb{P}(1, 1, 1, 3)$ and a crepant resolution

(2) Let $b_i \in H^2(Z; \mathbb{C})$ ($e \in H^2(Z; \mathbb{C})$ resp.) be the first Chern class of the line bundle associated to the torus invariant divisor corresponding to the ray generated by $\Lambda(\beta(v_i))$ (P resp.) for any $i \in \{0, ..., n\}$. We have $H^\star(Z; \mathbb{C}) \cong \mathbb{C}[b_0, \ldots, b_n, e]/I$ where $I$ is generated by:

$$-b_0 + b_i \quad \text{for } 1 \leq i \leq n-1,$$
$$-nb_0 - e + b_n,\ eb_n,\ b_0 \cdots b_{n-1}.$$

Set $h := \frac{1}{2n}(b_0 + \ldots + b_n + e) = b_0 + \frac{1}{n}e$, then we get:

$$H^\star(Z; \mathbb{C}) \cong \mathbb{C}[h, e]/\langle h^n + (-1)^n \left(\frac{e}{n}\right)^n, he\rangle.$$

(3) $\mathrm{M}_\rho(Z)$ is generated by one class $\Gamma_1 := \mathrm{PD}\left(\left(h - \frac{e}{n}\right)^{n-2} e\right)$.

(4) We will set the quantum parameter $q_1 = 0$, then we do not have to compute any non trivial Gromov-Witten invariant.

(5) We follow the description given in [BMP07] of the Chen-Ruan cohomology ring. The twisted sectors are indexed by the set $\mathrm{T} = \left\{\exp\left(\frac{2\pi i k}{n}\right) \mid k \in \{0, ..., n-1\}\right\}$. For any $g \in \mathrm{T} - \{1\}$, $\mathcal{X}_{(g)} \cong \mathbb{P}(n)$, while $\mathcal{X}_{(1)} \cong \mathcal{X}$. As vector space we have

$$H^\star_{\mathrm{CR}}(\mathcal{X}; \mathbb{C}) := \oplus_{g \in T} H^\star(\mathcal{X}_{(g)}). \tag{6.2}$$

Let

$$H,\ E_1 \in H^2_{\mathrm{CR}}(\mathcal{X}; \mathbb{C})$$

be the image of $c_1\left(\mathcal{O}_{\mathcal{X}}(1)\right) \in H^2(\mathcal{X};\mathbb{C})$, $1 \in H^0(\mathcal{X}_{\left(\exp\left(\frac{2\pi i}{n}\right)\right)};\mathbb{C})$ respectively with respect to the inclusion $H^\star(\mathcal{X}_{(g)}) \to H^\star_{\mathrm{CR}}(\mathcal{X})$ determined by (6.2). Then we have the following presentation:

$$H^\star_{\mathrm{CR}}(\mathbb{P}(1,\dots,1,n);\mathbb{C}) \cong \mathbb{C}\left[H,E_1\right] / \left\langle H^n - (E_1)^n, HE_1\right\rangle.$$

(6) The ring isomorphism

$$H^\star_{\mathrm{CR}}(\mathbb{P}(1,\dots,1,n);\mathbb{C}) \xrightarrow{\sim} H^\star(Z;\mathbb{C})$$

is obtained by mapping $H \mapsto h$ and $E_1 \mapsto -\exp\left(\frac{i\pi}{n}\right)\frac{e}{n}$. □

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