Quantization of $r-Z$-quasi-Poisson manifolds and related modified classical dynamical r-matrices

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Abstract

Le X be a C^∞-manifold and $\mathfrak{g}$ be a finite dimensional Lie algebra acting freely on X. Let $r \in \Lambda^2(\mathfrak{g})$ be such that $Z = [r,r] \in \Lambda^3(\mathfrak{g})^{\mathfrak{g}}$. In this paper we prove that every quasi-Poisson $(\mathfrak{g},Z)$-manifold can be quantized. This is a generalization of the existence of a twist quantization of coboundary Lie bialgebras ([EH]) in the case $X = G$ (where G is the simply connected Lie group corresponding to $\mathfrak{g}$). We deduce our result from a generalized formality theorem. In the case $Z = 0$, we get a new proof of the existence of (equivariant) formality theorem and so (equivariant) quantization of Poisson manifold (*cf.* [Ko, Do]). As a consequence of our results, we get quantization of modified classical dynamical r-matrices over abelian bases in the reductive case.

0. Introduction

Throughout this paper, the ground field will be $\mathbb{R}$. Let $\mathfrak{g}$ be a finite dimensional Lie algebra with a fixed element $r \in \Lambda^3(\mathfrak{g})$ such that $[r,r] = Z \in \Lambda^3(\mathfrak{g})^{\mathfrak{g}}$. In [AK, AKM], quasi-Poisson manifolds were introduced as a generalization of Poisson $\mathfrak{g}$-manifolds with Poisson bracket satisfying the Jacobi identity up to an invariant trivector corresponding to Z. More precisely :

Definition 0.1. *A quasi-Poisson $(\mathfrak{g},Z)$-manifold is a $\mathfrak{g}$-manifold X with an invariant bivector π such that the Schouten bracket $[\pi,\pi]_S$ equals $\gamma^{\otimes 3}(Z)$, where $\gamma : \mathfrak{g} \to \mathrm{Vect}(X)$ is the action homomorphism.*

The Schouten bracket will be descibed later. Thus the Poisson bracket $\{-,-\}$ associated to π satisfies

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = m_0(\gamma^{\otimes 3}(Z)(f \otimes g \otimes h)),$$

where m_0 is the usual multiplication. In the framework of deformation quantization (see [BFFLS1, BFFLS2]), Enriquez and Etingof defined the quantization of quasi-Poisson manifolds in [EE1] : let $\hbar$ be a formal parameter and $\Phi = 1 + \frac{\hbar^2}{6}Z + O(\hbar^3) \in (U(\mathfrak{g})^{\otimes 3})^{\mathfrak{g}}[[\hbar]]$ be an associator for $\mathfrak{g}$ (Drinfeld proved in [Dr], Proposition 3.10, that such an associator always exists).

Definition 0.2. *A quantization of X associated to Φ is an invariant star-product $\star$ on X, i.e. an invariant bidifferential operator on $C^\infty(X)$, which satisfies $f \star g = fg + O(\hbar)$ and the equation*

$$f \star g - g \star f = \hbar\{f,g\} + O(\hbar^2),$$

and is associative in the tensor category of $(U(\mathfrak{g})[[\hbar]], \Phi)$-modules. This means,

$$m_\star(m_\star \otimes 1) = m_\star(1 \otimes m_\star)\gamma^{\otimes 3}(\Phi),$$

on $C^\infty(X)^{\otimes 3}$, where $m_\star(f \otimes g) = f \star g$.

They also conjectured that such quantizations always exist when the action of $\mathfrak{g}$ on the quasi-Poisson manifold X is free. Note that when the action is not free, Fronsdal ([Fr]) gave in 1978 a counter-example where such quantization is impossible even in the symplectic case. From now on, we will suppose that the manifold X is a G-bundle over a manifold M, where G is the simply connected Lie group corresponding to $\mathfrak{g}$. In the case $G = \{e\}$, the conjecture is equivalent to the existence of star-products and was proved by Kontsevich ([Ko]). In the case $Z = 0$, the conjecture follows from the equivariant formality theorem of Dolgushev ([Do]).

In the general case, $\gamma^{\otimes 3}(Z)$ commutes with all the left invariant polyvector fields in the following sense :

$$[\gamma^{\otimes 3}(Z), X]_S = 0, \quad \text{for all invariant polyvector fields } X. \tag{0.1}$$

Moreover, for Φ an associator, $\gamma^{\otimes 3}(\Phi)$ commutes with all the invariant differential operators in the following sense:

$$[\gamma^{\otimes 3}(\Phi), C]_G = 0, \quad \text{for all invariant differential operator } C \tag{0.2}$$

(the Gerstenhaber bracket $[-,-]_G$ will be described later in this paper). From now on, if $s \in \Lambda^k(\mathfrak{g})$, we will denote s instead of $\gamma^{\otimes k}(s)$ when no confusion is possible.

In this paper, we prove that there exists (a least) one associator for $\mathfrak{g}$ such that Enriquez-Etingof's conjecture is true:

Theorem 0.3. *Let $r \in \Lambda^3(\mathfrak{g})$ such that $[r,r] = Z \in \Lambda^3(\mathfrak{g})^{\mathfrak{g}}$. There exists $\Phi = 1 + \frac{\hbar^2}{6}Z + O(\hbar^3) \in (U(\mathfrak{g})^{\otimes 3})^{\mathfrak{g}}[[\hbar]]$ and a deformation $\mathfrak{g}_\hbar$ of the Lie algebra $\mathfrak{g}$ such that for every invariant bivector π satisfying $[\pi,\pi]_S = \gamma^{\otimes 3}(Z)$, the quasi-Poisson manifold (X,π) admits a quantization associated to $(\Phi, \mathfrak{g}_\hbar)$ i.e. a multiplication associative in the tensor category of $(U(\mathfrak{g}_\hbar)[[\hbar]], \Phi)$-modules.*

To prove this theorem, we will construct a formality between invariant polyvector and polydifferential operator as stated in Theorem 7.3. We first prove a local version of this theorem in the case $X = \mathbb{R}^n \times \mathfrak{g}$. Using Fedosov's resolutions we will be able to get a global version. We then get the wanted invariant star-product on the manifold X and classification of such deformations. We will then discuss the relation with quantization of modified classical dynamical r-matrices.

Remark 0.4. As a particular case, our results give a new proof of Kontsevich (and Dolgushev for equivariant) formality theorem. One can see this approach as related to Merkulov's work (see [Me]) for quantization of Lie bialgebras. In our work the use of a graded version of Etingof-Kazhdan theorem was a crucial step to go from quantization of Lie bialgebra to quantization of Poisson manifolds.

The paper is organized as follows:
- In Section 1, we recall definitions of L_∞-structures and formality morphisms.
- In Section 2, we give a graged version of quantization of Lie bialgebras: in particular, we get differential graded Etingof-Kazhdan quantization/dequantization functors.
- In Sections 3 and 4, we construct two useful functors between Lie and Gerstenhaber algebras "up to homotopy" and prove the existence of two resolutions for those algebras.
- In Section 5, we prove the existence of L_∞-morphisms between DG Lie bialgebras and the Gerstenhaber algebra of their Etingof-Kazhdan quantization
- In Section 6, we transpose the algebra structures into the category of $(U(\mathfrak{g})[[\hbar]],\Phi)$-modules. We define the graded Lie bialgebra $\tilde{\mathfrak{g}} = \mathbb{R}\oplus V[1]\oplus V^*\oplus \mathfrak{g}$, the direct sum of the Eisenberg Lie algebra $E = \mathbb{R}\oplus V[1]\oplus V^*$ and the Lie bialgebra $(\mathfrak{g},[r,-])$ which corresponds locally to the algebra of invariant poly-vectors. We prove the existence of the local wanted L_∞-morphism.
- In Section 7, we show that this L_∞-morphism can be globalized and prove our main theorem.
- In Section 8, we discuss relation between our quantization and quantization of modified classical dynamical r-matrices.

Notations

We use the standard notation for the coproduct-insertion maps: we say that an ordered set is a pair of a finite set S and a bijection $\{1,\dots,|S|\}\to S$. For $I_1,\dots,I_m$ disjoint ordered subsets of $\{1,\dots,n\}$, (U,Δ) a Hopf algebra and $a\in U^{\otimes m}$, we define

$$a^{I_1,\dots,I_m} = \sigma_{I_1,\dots,I_m}\circ(\Delta^{(|I_1|)}\otimes\cdots\otimes\Delta^{(|I_m|)})(a),$$

with $\Delta^{(1)} = \mathrm{id}$, $\Delta^{(2)} = \Delta$, $\Delta^{(n+1)} = (\mathrm{id}^{\otimes n-1}\otimes\Delta)\circ\Delta^{(n)}$, and $\sigma_{I_1,\dots,I_m} : U^{\otimes\sum_i |I_i|}\to U^{\otimes n}$ is the morphism corresponding to the map $\{1,\dots,\sum_i |I_i|\}\to\{1,\dots,n\}$ taking $(1,\dots,|I_1|)$ to I_1, $(|I_1|+1,\dots,|I_1|+|I_2|)$ to I_2, etc. When U is cocommutative, this definition depends only on the sets underlying $I_1,\dots,I_m$.

Until the end of this paper, although we will often omit to mention it, we will always deal with graded structures.

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1 L_∞-structures

1.1 Definitions

Let us recall definitions of L_∞-algebras and L_∞-morphisms. Let A be a graded vector space. We denote $T_+A = T_+(A[-1])$ the free tensor algebra (without unit) of A which, equipped with the coshuffle coproduct, is a bialgebra. We also denote $C(A) = S(A[-1])$ the free graded commutative algebra generated by $A[-1]$, seen as a quotient of T_+A. The coshuffle coproduct is still well defined on $C(A)$ which becomes a cofree cocommutative coalgebra on $A[-1]$. We also denote $\Lambda A = S(A[1])$, the analogous graded commutative algebra generated by $A[1]$ (in particular, for $A_1, A_2 \in A$, $A_1 \Lambda A_2$ stands for the corresponding quotient of $A_1[1] \otimes A_2[1]$ in ΛA). We will use the notations $T_+^n A$, $\Lambda^n A$ and $C^n(A)$ for the elements of degree n.

Definition 1.1. A vector space A is endowed with a L_∞-algebra (Lie algebra "up to homotopy") structure if there are degree one linear maps $d^{1,\dots,1}\colon \Lambda^k A \to A[1]$ such that the associated coderivations (extended with respect to the cofree cocommutative structure on ΛA) d: $\Lambda A \to \Lambda A$, satisfy $d \circ d = 0$ where d is the coderivation

$$d = d^1 + d^{1,1} + \cdots + d^{1,\dots,1} + \cdots .$$

In particular, a differential Lie algebra $(A, b, [-,-])$ is a L_∞-algebra with structure maps $d^1 = b[1]$, $d^{1,1} = [-,-][1]$ and $d^{1,\dots,1}\colon \Lambda^k A \to A[1]$ are 0 for $k \geq 3$. One can now define the generalization of Lie algebra morphisms:

Definition 1.2. A L_∞-morphism between two L_∞-algebras $(A_1, d_1 = d_1^1 + \cdots)$ and $(A_2, d_2 = d_2^1 + \cdots)$ is a morphism of codifferential cofree coalgebras, of degree 0,

$$\varphi : \ (\Lambda A_1, d_1) \to (\Lambda A_2, d_2).$$

In particular $\varphi \circ d_1 = d_2 \circ \varphi$. As φ is a morphism of cofree cocommutative coalgebras, φ is determined by its image on the cogenerators, i.e., by its components: $\varphi^{1,\dots,1} : \Lambda^k A_1 \to A_2[1]$.

Let E be a graded vector space. Let us denote ${}^cT(E)$ the cofree tensor coalgebra of E with coproduct Δ'. Equipped with the shuffle product $\bullet$ (defined on the cogenerators ${}^cT(E) \otimes {}^cT(E) \to E$ as $\mathrm{pr} \otimes \varepsilon + \varepsilon \otimes \mathrm{pr}$, where $\mathrm{pr} : {}^cT(E) \to E$ is the projection and ε is the counit), it is a bialgebra. Let ${}^cT_+(E)$ be the augmentation ideal. We denote $\underline{{}^cT}(E) = {}^cT_+(E)/({}^cT_+(E) \bullet {}^cT_+(E))$ the quotient by the shuffles. It has a graded cofree Lie coalgebra structure (with coproduct $\delta = \Delta' - \Delta'^{\mathrm{op}}$). Then $S(\underline{{}^cT}(E)[1])$ has a structure of cofree coGerstenhaber algebra (i.e. equipped with cofree coLie and cofree cocommutative coproducts satisfying compatibility condition). We use the notation $\underline{{}^cT^n}(E)$ for the elements of degree n.

Remark 1.3. One could also define G_∞-structures. Most of the L_∞-morphism constructed in this paper are also G_∞-morphisms between corresponding G_∞-structures. Definitions and extensions to G_∞-structures can be found in [Ha].

2 Etingof-Kazhdan functors

2.1 QUE and QFSH algebras

We recall some facts from [Dr] (proofs and definitions can be found in [Gav]). Let us denote by **QUE** the category of quantized universal enveloping (QUE) algebras and by **QFSH** the category of quantized formal series Hopf (QFSH) algebras. Let us recall the definition of FSH and QFSH algebras:

Definition 2.1. A FSH algebra is a Hopf algebra of power series isomorphic as an algebra to $\mathbb{K}[[\{u_i | i \in J\}]]$ (for some set J).

There is an equivalence of categories between the category of FSH algebra and the category of Lie coalgebra (LC algebra), sending $\mathscr{O}_{\mathfrak{h}}$ to $\mathfrak{h} = \mathscr{O}_{\mathfrak{h}+}/\mathscr{O}_{\mathfrak{h}+}^2$ where $\mathscr{O}_{\mathfrak{h}+}$ is the maximal ideal of $\mathscr{O}_{\mathfrak{h}}$.

Definition 2.2. A QFSH algebra is a Hopf algebra H, which is a topologically free $\mathbb{K}[[\hbar]]$-module, such that $H_0 := H/\hbar H$ is isomorphic to a FSH algebra

Let us give an example of a FSH algebra, very important in this paper: let V be a vector space and ${}^cTV$ (defined in the previous section) the cofree coalgebra, equipped with the shuffle product. Let us now complete ${}^cTV$. The algebra ${}^cTV$ is a graded algebra with V being the set of elements of degree 1. Let us denote $\mathscr{M}_{{}^cTV}$ the set of elements of degree ≥ 1. Finally, we denote $\widehat{{}^cTV}$ the commutative cofree bialgebra, $\mathscr{M}_{{}^cTV}$-adic completion of ${}^cTV$.

Proposition 2.3. *[Ha] $\widehat{{}^cTV}$ is the FSH algebra $\mathscr{O}_{\underline{{}^cTV}}$ associated with the Lie coalgebra $\underline{{}^cTV} = {}^cT_+V/({}^cT_+V)^2$, which is the cofree Lie coalgebra over V.*

We have covariant functors **QUE** $\to$ **QFSH**, $U \mapsto U'$ and **QFSH** $\to$ **QUE**, $\mathscr{O} \mapsto \mathscr{O}^\vee$. These functors are also inverse to each other.

U' is a subalgebra of U defined as follows: for any ordered subset $\Sigma = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$ with $i_1 < \cdots < i_k$, define the morphism $j_\Sigma : U^{\otimes k} \longrightarrow U^{\otimes n}$ by $j_\Sigma(a_1 \otimes \cdots \otimes a_k) := b_1 \otimes \cdots \otimes b_n$ with $b_i := 1$ if $i \notin \Sigma$ and $b_{i_m} := a_m$ for $1 \leq m \leq k$; then set $\Delta_\Sigma := j_\Sigma \circ \Delta^{(k)}$, $\Delta_\emptyset := \Delta^{(0)}$, and

$$\delta_\Sigma := \sum_{\Sigma' \subset \Sigma} (-1)^{n-|\Sigma'|} \Delta_{\Sigma'}\,, \qquad \delta_\emptyset := \varepsilon\,.$$

We shall also use the notation $\delta^{(n)} := \delta_{\{1,2,\ldots,n\}}$, $\delta^{(0)} := \delta_\emptyset$, and the useful formula

$$\delta^{(n)} = (id_U - \varepsilon)^{\otimes n} \circ \Delta^{(n)}\,.$$

Finally, we define

$$U' := \left\{ a \in U \,\middle|\, \delta^{(n)}(a) \in h^n U^{\otimes n} \right\} \quad (\subseteq U)$$

and endow it with the induced topology.

On the other way, $\mathscr{O}^\vee$ is the $\hbar$-adic completion of $\sum_{k\geq 0} \hbar^{-k}\mathscr{M}^k \subset \mathscr{O}[1/\hbar]$ (here $\mathscr{M} \subset \mathscr{O}$ is the maximal ideal).

2.2 The functor DQ

In [GH], a generalization of Etingof-Kazhdan theorem ([EK]) was proved in an appendix by Enriquez and Etingof:

Theorem 2.4. *We have an equivalence of categories*

$$DQ_\Phi \ : \ \mathrm{DGQUE} \to \mathrm{DGLBA}_h$$

from the category of differential graded quantized universal enveloping super-algebras to that of differential graded Lie super-bialgebras such that if $U \in \mathrm{Ob}(\mathrm{DGQUE})$ and $\mathfrak{a} = DQ(U)$, then $U/hU = \mathbb{U}(\mathfrak{a}/h\mathfrak{a})$, where $\mathbb{U}$ is the universal algebra functor, taking a differential graded Lie super-algebra to a differential graded super-Hopf algebra.

Here Φ is a Drinfeld associator. We will use any of these functors and denote it DQ.

3 Two functors

3.1 Functor L-G

Let $(\mathfrak{h}, \delta, d)$ be a differential Lie bialgebra. Let $C(\mathfrak{h}) = S(\mathfrak{h}[-1])$ be the free graded commutative algebra generated by $\mathfrak{h}$. Recall from the previous subsection that $C(\mathfrak{h})$ is also a cofree coalgebra and that coderivations $C(\mathfrak{h}) \to C(\mathfrak{h})$ are defined by their images in $\mathfrak{h}$. Thus, one easily checks that the coderivation $[-,-]$: $C(\mathfrak{h}) \to C(\mathfrak{h})$ extending the Lie bracket (with degree shifted by one) defines a Lie (even Gerstenhaber) algebra structure on $C(\mathfrak{h})$. Moreover, one can extend maps d: $\mathfrak{h} \to \mathfrak{h}$ and δ: $\mathfrak{h} \to S^2(\mathfrak{h}[-1])$ on the free commutative algebra $C(\mathfrak{h})$ so that $(C(\mathfrak{h}), [-,-], \wedge, d+\delta)$ is a differential Gerstenhaber algebra. The differential δ is actually the Chevalley Eilenberg differential: the space $C(\mathfrak{h}) = S^*(\mathfrak{h}[-1])$ is isomorphic to the standard complex $(\Lambda^*(\mathfrak{h}))[-*]$ and δ is simply the differential given by the underlying Lie coalgebra structure of $\mathfrak{h}$.

Proposition 3.1. *[Ha] Any DGLA morphism $f : \mathfrak{h}_1 \to \mathfrak{h}_2$ can be extended into a DGLA (and even differential graded Gerstenhaber) morphism $C(f) : C(\mathfrak{h}_1) \to C(\mathfrak{h}_2)$ of free commutative algebras. This defines an exact functor* L-G *from differential Lie bialgebras to differential Gerstenhaber algebras which sends $\mathfrak{h}$ to $C(\mathfrak{h})$. Quasi-isomorphisms $(\mathfrak{h}_1, d_1) \to (\mathfrak{h}_2, d_2)$ induce a quasi-isomorphisms $(C(\mathfrak{h}_1), d_1, \delta_1) \to (C(\mathfrak{h}_2), d_2, \delta_2)$.*

3.2 Functor L-G$_\infty$

Consider now the category CFDLB of differential Lie bialgebras which are cofree as a Lie coalgebra. In other words we are interested in cofree Lie coalgebra ${}^c\underline{T}(E)$ on a graded vector space E together with a differential ℓ and a cobracket L on ${}^c\underline{T}(E)$ that makes it a differential Lie bialgebra. As ${}^c\underline{T}(E)$ is cofree, the differential is uniquely determined by its restriction to cogenerators l^p: ${}^c\underline{T^p}(E) \to E$. Similarly, the Lie bracket is uniquely determined by maps L^{p_1,p_2}: ${}^c\underline{T^{p_1}}(E)\Lambda{}^c\underline{T^{p_2}}(E) \to E$.

Proposition 3.2. *[Ha] Restriction map ${}^c\underline{T^p}(E) \to E$ defines an exact functor* L-G$_\infty$ *from* CFDLB *to the category of G_∞ (and so Lie)-algebras.*

Until the end of the paper, we will use the notations TE for $T(E[-1])$ and ${}^c\underline{T}E$ for $\underline{T}(E[1])$.

4 Two resolutions

4.1 bialgebra structure on ${}^cTT_+U$

Here, we will define a bialgebra structure on ${}^cTT_+U$. One can construct a bialgebra structure on the space of Hochschild cochains of an algebra using the brace operations. In our case, we will firstly generalize the definition of brace operations for a general Hopf algebra. More precisely, let $(H, \Delta_\hbar, \times)$ be a Hopf algebra (in our case H will be the Etingof-Kazhdan quantization $U_\hbar(\mathfrak{a})$ of the Lie bialgebra $\mathfrak{a}$). We will define a brace structure on the cofree tensor coalgebra ${}^cTT_+H$ of the free tensor algebra $T(H[-1])$ without unit. To distinguish the two tensor products, we denote $\otimes$ the tensor product on T_+H and $\boxtimes$ the tensor product on ${}^cTT_+H$.

Definition 4.1. We define brace operations on ${}^cTT_+H$ by extending the following maps given on the cogenerators of the cofree coalgebra ${}^cTT_+H$:

1. $B^0 = 0$,
2. $B^1 = b_{\mathrm{cH}}$ (the coHochschild coboundary on T_+H),
3. $B^2 : \alpha \boxtimes \beta \mapsto \alpha \otimes \beta$,
4. $B^n = 0$ for $n > 2$,
5. $B^{0,1} = B^{1,0} = \mathrm{id}$,
6. $B^{0,n} = B^{n,0} = 0$ for $n \geq 1$,
7. $B^{1,n} : (\alpha, \beta_1 \boxtimes \cdots \boxtimes \beta_n) \mapsto$

$$\sum_{\substack{0 \leq i_1, \ldots, i_m + k_m \leq n \\ i_l + k_l \leq i_{l+1}}} (-1)^\varepsilon \alpha^{1,\ldots,i_1+1\cdots i_1+k_1,\ldots,i_m+1\cdots i_m+k_m,\ldots,n} \times$$
$$1^{\otimes i_1} \otimes \beta_1 \otimes 1^{\otimes i_2-(i_1+k_1)} \otimes \beta_2 \otimes \cdots \otimes \beta_n \otimes 1^{\otimes n-(i_m+k_m)},$$

   where $k_s = |\beta_s|$, $n = |\alpha| + \sum_s k_s - m$ and $\varepsilon = \sum_s (k_s - 1) i_s$,
8. $B^{k,l} = 0$ for $k > 1$.

Operations $(2), (3)$ and (4) define a differential d and $(5), (6), (7)$ and (8) define a product $\star$ deforming the shuffle product.

Note that, when $H = U(\mathfrak{a})$, the enveloping algebra of a Lie algebra $\mathfrak{a}$, $T(H[-1])$ can be seen as the space of invariant polydifferential operators over the Lie group corresponding to $\mathfrak{a}$ and in that case, our definition coincides with usual braces operations.

We have:

Theorem 4.2. *[Ha] The brace operations of Definition 4.1 define a differential bialgebra structure on the cofree tensor coalgebra ${}^cTT_+H$, with product $\star$ extending $\sum B^{p_1,p_2}$ and differential d extending $\sum B^p$.*

Let us now complete ${}^cTT_+H$ as in section 2 with $V = T_+H$. We get a commutative cofree bialgebra $\widehat{{}^cTT_+H}$, the $\mathscr{M}_{{}^cTT_+H}$-adic completion of ${}^cTT_+H$ (where $\mathscr{M}_{{}^cTT_+H}$ is the maximal ideal of ${}^cTT_+H$). Let us consider the free $\mathbb{K}[[\nu]]$-module $\widehat{{}^cTT_+H}[[\nu]]$. One can now replace the operations $B^{p,q}$ of Definition 4.1 with $\mathbb{K}[[\nu]]$-linear operations $\nu^{p+q-1}B^{p,q}$. Those operations are well defined on the completion $\widehat{{}^cTT_+H}[[\nu]]$ as this space is complete for the grading induced by the degree in ${}^cTT_+H = {}^cTV$ plus the $\hbar$-adic valuation and because the operations we just defined are homogeneous for this grading. Thus we get a morphism of differential bialgebra

$$\begin{aligned} I_\nu:\ ({}^cTT_+H,\star,\Delta,d) &\to ({}^cTT_+H[[\nu]][\nu^{-1}],\star_\nu,\Delta_\nu,d_\nu)\\ x &\mapsto \nu^{-|x|}x, \end{aligned} \tag{4.3}$$

where $|x|$ is the degree in ${}^cT$. The morphism I_ν extends to $I_\nu:({}^cTT_+H[[\nu]],\star,\Delta,d)\to({}^cTT_+H[[\nu]][\nu^{-1}],\star_\nu,\Delta_\nu,d_\nu)$ which restricts to

$$I'_\nu:\ (\widehat{\bigoplus}\nu^{n\,c}T^nT_+H[[\nu]],\star,\Delta,d)\to(\widehat{{}^cT}T_+H[[\nu]][\nu^{-1}],\star_\nu,\Delta_\nu,d_\nu) \tag{4.4}$$

We have:

Proposition 4.3. *[Ha] The algebra $(\widehat{{}^cT}T_+H[[\nu]],\star_\nu,\Delta_\nu,d_\nu)$ is a QFSHA. The underlying differential Lie bialgebra structure on ${}^c\underline{T}T_+H$ is given by the Gerstenhaber bracket*

$$[\alpha,\beta]_G=B^{1,1}(\alpha,\beta)-(-1)^{(|\alpha|-1)(|\beta|-1)}B^{1,1}(\beta,\alpha)$$

and coHochschild differential

$$b_{\mathrm{cH}}(\alpha)=[1\otimes 1,\alpha]_G,$$

for $\alpha,\beta\in TH$ and then naturally extended on ${}^c\underline{T}T_+H$ using the cofree Lie cobracket.

Remark 4.4. Let now H be the QUE algebra $U=U_\hbar(\mathfrak{a})$. We have proved that T_+U can be equipped with a G_∞-structure. Since the cofree Lie coalgebras are rigid, the differential Lie bialgebra corresponding to $\widehat{{}^cT}T_+U[[\nu]]$ through Etingof-Kazhdan dequantization functor DQ is isomorphic to ${}^c\underline{T}T_+U[[\nu]]$ as a $\mathbb{K}[[\nu]]$-Lie coalgebra, and is therefore free.

4.2 A bialgebra quasi-isomorphism $\varphi_{\mathrm{alg}}: U\to(\widehat{{}^cT}T_+U)^\vee$

We have:

Proposition 4.5. *Let U be a QUE algebra. One can define a bialgebra quasi-isomorphism ϕ_{alg}: $U\to\widehat{{}^cT}T_+U$ from the bialgebra $(U,\Delta_\hbar,\times)$ to the bialgebra $({}^cTT_+U,\Delta,\star)$ whose structure was described in the previous section.*

Let $U'\subset U$ (see section 2).

Proposition 4.6. *[Ha] We have a bialgebra quasi-isomorphism $\varphi_{\mathrm{alg}}:(U',\times)\to(\widehat{{}^cT}T_+U,\star_\hbar)$ of QFSH algebra, where $(\widehat{{}^cT}T_+U,\star_\hbar)$ is $(\widehat{{}^cT}T_+U[[\nu]],\star_\nu)/(\nu=\hbar)$ ($\widehat{{}^cT}T_+U[[\nu]]$ is the free $\mathbb{K}[[\hbar]]$-module defined in the previous section: we d the operations $B^{p,q}$ into $\nu^{p+q-1}B^{p,q}$).*

Finally, applying to φ_{alg} the derived Drinfeld functor $(-)^\vee$, we get a bialgebra quasi-isomorphism φ_{alg}: $U\to(\widehat{{}^cT}T_+U)^\vee$.

4.3 A Lie bialgebra quasi-isomorphism φ_{Lie} : ${}^c\underline{T}A \to {}^c\underline{T}C({}^c\underline{T}A)$

Let A be a vector space. Suppose now that the cofree Lie coalgebra ${}^c\underline{T}A$ has a structure $({}^c\underline{T}A, \delta, [-,-], d)$ of a differential Lie bialgebra. Using the functor L-G (see section 3), one gets a differential Gerstenhaber algebra $(C({}^c\underline{T}A), [-,-], \wedge, d+\delta)$. One can extend the structure maps on the cofree Lie coalgebra ${}^c\underline{T}C({}^c\underline{T}A)$ and one gets a differential cofree Lie bialgebra $({}^c\underline{T}C({}^c\underline{T}A), \delta', [-,-], d+\delta+\wedge)$ (we will set $d^1 = d+\delta$ and $d^2 = \wedge$).

Proposition 4.7. *[Ha] Let φ_{Lie} be the composition map $\varphi_{\text{Lie}} = {}^cTi \circ \bar{\delta}$ of a map*

$$\begin{array}{rl} \bar{\delta} : {}^c\underline{T}A & \to {}^c\underline{T}({}^c\underline{T}A), \\ x & \mapsto x + \sum_{k\geq 2} \bar{\delta}_k(x), \end{array}$$

where $\bar{\delta}_k$ is built using iterates of δ, with ${}^cTi$: ${}^c\underline{T}({}^c\underline{T}A) \to {}^c\underline{T}C({}^c\underline{T}A)$ which is ${}^c\underline{T}$ of the inclusion i: ${}^c\underline{T}A[-1] \to C({}^c\underline{T}A)$. Then φ_{Lie} is a differential Lie bialgebra quasi-isomorphism φ_{Lie} : ${}^c\underline{T}A \to {}^c\underline{T}C({}^c\underline{T}A)$.

5 L_∞-morphism for Lie bialgebras

5.1 A Lie bialgebra quasi-isomorphism φ'_{Lie}: $\mathfrak{a} \to {}^c\underline{T}T_+U$

Let $(\mathfrak{a}, \delta_\hbar)$ be a graded Lie bialgebra. We write $\delta_\hbar = \hbar\delta_1 + \hbar^2\delta_2 + \cdots$. Let $(U_\hbar(\mathfrak{a}), \Delta_\hbar)$ be the Etingof-Kazhdan canonical quantization of $(\mathfrak{a}, \delta_\hbar)$. We denote $U = U_\hbar(\mathfrak{a})$ for short. In section 4, we proved the existence of a bialgebra structure on ${}^cTT_+U$ and a bialgebra quasi-isomorphism φ_{alg}: $U \to ({}^cTT_+U)^\vee$. Thanks to Etingof-Kazhdan dequantization functor (see section 2), and the fact that $({}^cTT_+U)^\vee$ is a QUE algebra quantizing ${}^c\underline{T}T_+U$ (see section 4), we get a Lie bialgebra quasi-isomorphism φ'_{Lie}: $\mathfrak{a} \to {}^c\underline{T}T_+U$.

5.2 Inversion of formality morphisms

Let us recall Theorem 4.4 of Kontsevich ([Ko]):

Theorem 5.1. *Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two L_∞-algebras and $\mathscr{F}$ be a L_∞-morphism from $\mathfrak{g}_1$ to $\mathfrak{g}_2$. Assume that $\mathscr{F}$ is a quasi-isomorphism. Then there exists an L_∞-morphism from $\mathfrak{g}_2$ to $\mathfrak{g}_1$ inducing the inverse isomorphism between associated cohomology of complexes.*

Remark 5.2. We know the existence of a similar G_∞-version of this theorem. This result would imply the existence of corresponding G_∞-morphisms.

5.3 L_∞-morphism for Lie bialgebras

Let us summarize functors and quasi-isomorphisms constructed in the previous sections in the following diagram:

$$
\begin{array}{ccccccccc}
{}^c\underline{T}C({}^c\underline{T}T_+U) & & C({}^c\underline{T}T_+U)[1] & = & C({}^c\underline{T}T_+U)[1] & & {}^c\underline{T}T_+U & & (\widehat{{}^cT}T_+U)^\vee \\
\uparrow \varphi_{\mathrm{Lie}} & \xrightarrow{\text{L-G}_\infty} & \uparrow \varphi_{\mathrm{G}_\infty} & & \uparrow \varphi_{\mathrm{Ger}_\infty} & \xleftarrow{\text{L-G}} & \uparrow \varphi'_{\mathrm{Lie}} & \xleftarrow{\text{DQ}} & \uparrow \varphi_{\mathrm{alg}} \\
{}^c\underline{T}T_+U & & T_+U[1] & & C(\mathfrak{a})[1] & & \mathfrak{a} & & U = U_\hbar(\mathfrak{a}).
\end{array}
$$

Thus, thanks to section 5.2, the composition $\varphi\colon C(\mathfrak{a}) \to T_+U$ of $\varphi_{\mathrm{Ger}_\infty}$ with the inverse of $\varphi_{\mathrm{G}_\infty}$ gives the wanted quasi-isomorphism.

Theorem 5.3. *[Ha] the map φ $C(\mathfrak{a}) \to T_+U$ is a L_∞-quasi-isomorphism that maps $v \in C(\mathfrak{a})$ to* $\mathrm{Alt}(v) \in T_+U \bmod \hbar$.

5.4 L_∞-morphism for $X = \mathbb{R}^n \times \mathfrak{g}$

We will now consider $X = \mathbb{R}^n \times \mathfrak{g}$ and $r \in \mathfrak{g} \wedge \mathfrak{g}$ such that $[r,r] = Z$. So $(\mathfrak{g}, [r,-])$ is a Lie bialgebra. Let us set $V = \mathbb{R}$. From now on we will consider the graded Lie bialgebra $\tilde{\mathfrak{g}} = \mathbb{R} \oplus V[1] \oplus V^* \oplus \mathfrak{g}$, the direct sum of the Eisenberg Lie algebra $E = \mathbb{R} \oplus V[1] \oplus V^*$ and the Lie bialgebra $(\mathfrak{g}, [r,-])$. We will now deduce our main result from:

Proposition 5.4. *There exists a L_∞-quasi-isomorphism $\varphi_\hbar$ between $(C(\tilde{\mathfrak{g}}), [-,-], [r,-])$ and $(T_+\tilde{U}, [-,-]_\hbar, [1 \otimes 1, -]_\hbar)$. Here $\tilde{U}$ is the Etingof-Kazhdan quantization of $\tilde{\mathfrak{g}}$ and so $\tilde{U} = U(E) \otimes U_\hbar(\mathfrak{g})$ where $U_\hbar(\mathfrak{g})$ is the Etingof-Kazhdan quantization of $(\mathfrak{g}, [r,-])$. The bracket $[-,-]_\hbar$ denotes the Gerstenhaber bracket constructed in Section 4 corresponding to the coproduct of $\tilde{U}$.*

6 Deformed structures and local L_∞-morphism

6.1 Deformed structures

Suppose we are given $\Phi \in (U((\mathfrak{g})^{\otimes 3})^{\mathfrak{g}}[[\hbar]]$ an associator. In particular, $\gamma^{\otimes 3}(\Phi)$ commutes with all the invariant differential operator. We have in fact:

$$[C, \gamma^{\otimes 3}(\Phi)]_G = 0 \text{ for all } C \in U(\tilde{\mathfrak{g}})[[\hbar]]).$$

From now on, we will consider the tensor category of $(U(\mathfrak{g})[[\hbar]], \Phi)$-modules (in which we want to construct an associative star-product). Let us define the "deformed" Gerstenhaber bracket as the Bracket defined in Section 4 but in the new tensor category. We get a new Lie algebra structure on $U(\tilde{\mathfrak{g}})[[\hbar]]$ given by the bracket $[-,-]_\Phi$ defined, for $D, E \in U(\tilde{\mathfrak{g}})[[\hbar]]$, by

$$[D,E]_\Phi = \{D|E\}_\Phi - (-1)^{|E||D|}\{E|D\}_\Phi,$$

where for $D \in U(\tilde{\mathfrak{g}})^{\otimes d}$ and $E \in U(\tilde{\mathfrak{g}})^{\otimes e}$,

$$\{D|E\} = \sum_{i \geq 0} (-1)^{(e-1)\cdot i} \tilde{\Phi} D^{1,\dots,i,i+1\cdots i+e,i+e+1,\cdots} E^{i+1,\dots,i+e}.$$

$\tilde{\Phi}$ corresponds to the obvious change of parenthesis in the tensor category of $(U(\mathfrak{g})[[\hbar]],\Phi)$-modules. For example, if A and B are two 2-cochains in $U(\tilde{\mathfrak{g}})[[\hbar]]$, one has

$$\{A,B\}_\Phi = A^{12,3}B^{1,2} - \Phi^{-1}A^{1,23}B^{2,3}.$$

Remark 6.1. One could also define a deformed bialgebra structure on $(\widehat{{}^cT}T_+U(\tilde{g})[[\hbar]])^\vee$ and so using Etingof-Kazhdan dequantization a G_∞-structure on $U(\tilde{\mathfrak{g}})[[\hbar]]$ (proof can be copied from [Ta] or [GH]).

6.2 Twist quantization of coboundary Lie bialgebras

Let us recall results from [EH]:

Theorem 6.2. *[Ha] Let $(\mathfrak{a},[r,-])$ be a coboundary Lie bialgebra. There exists a coboundary quantization of it: $(U_\hbar(\mathfrak{a}),\Delta_\hbar,R_\hbar)$.*

Then, following [Dr], it was proved in [EH]:

Theorem 6.3. *There exists a deformation $\mathfrak{a}_\hbar$ of $\mathfrak{a}$ in the category of topologically free $\mathbb{R}[[\hbar]]$-Lie algebras, $J=1+\hbar r/2+O(\hbar^2)\in U(\mathfrak{a}_\hbar)^{\otimes 2}$ and $\Phi_0\in(U(\mathfrak{a}_\hbar)^{\otimes 3})^{\mathfrak{a}_\hbar}$ such that the coboundary Hopf algebra $(U_\hbar(\mathfrak{a}),\Delta_\hbar,R_\hbar)$ is twist equivalent through J to the coboundary quasi-Hopf algebra $(U(\mathfrak{a}_\hbar),\Delta_0,1,\Phi_0)$ and we have, in $U_\hbar(\mathfrak{a})$,*

$$J\Delta_0 J^{-1}=\Delta_\hbar \text{ and } J^{1,2}J^{12,3}=J^{2,3}J^{1,23}\Phi_0.$$

6.3 local L_∞-morphism

Let us keep the notation of the previous section for $\mathfrak{a}=\tilde{g}$, the Lie algebra define in Section 5.4. Let us set $F=J^{-1}$ and $\Phi=\Phi_0$. We can now prove the existence of a L_∞-morphism for our structures, in the local case:

Theorem 6.4. *There exists a L_∞-quasi-isomorphism φ_{loc} between the differential Lie algebra $(\widehat{S}(V)\otimes\Lambda(V^*\oplus\mathfrak{g}),[-,-],[r,-])$ (corresponding to local invariant polyvector fields) and the Lie algebra $(T_{\widehat{S}(V)}(U(V^*\oplus\mathfrak{g})[[\hbar]]),[-,-]_\Phi,[F,-]_\Phi)$ (corresponding to invariant polydifferential operators).*

PROOF. Let us consider the Lie bialgebra $\tilde{g}$ defined in Section 5. Let $\tilde{U}$ be its Etingof-Kazhdan quantization. We know from Section 5 that there exists a L_∞-quasi-isomorphism $\varphi_\hbar$: $(C(\tilde{\mathfrak{g}}),[-,-],[r,-])=(\widehat{S}(V)\otimes\Lambda(V^*\oplus\mathfrak{g}),[-,-],[r,-])\to(T_+\tilde{U},[-,-]_\hbar,[1\otimes 1,-]_\hbar)$.

Let now define φ_F: $T_+U_\hbar(\tilde{g})\to T_+U(\tilde{g}_\hbar)[[\hbar]]$ to be the map defined as follows: for $x\in U_\hbar(\tilde{g})^{\otimes n}$,

$$\varphi_F(x)=F^{12\cdots n-1,n}\cdots F^{12,3}F^{1,2}\cdot x,$$

in $U(\tilde{g}_\hbar)^{\otimes n}[[\hbar]]$. It is clear that φ_F is an isomorphism of differential Lie algebras sending the bracket $[-,-]_\hbar$ to $[-,-]_\Phi$ and the differential $[1\otimes 1,-]_\hbar$ to $[F,-]_\Phi$. Composing $\varphi_\hbar$ with φ_F we get a L_∞-quasi-isomorphism:

$$(\widehat{S}(V)\otimes\Lambda(V^*\oplus\mathfrak{g}),[-,-],[r,-])\to(T_+U(\tilde{g}_\hbar),[-,-]_\Phi,[F,-]_\Phi).$$

This gives the result as one can identify $(T_+U(\tilde{g}_\hbar),[-,-]_\Phi,[F,-]_\Phi)$ with $(T_{\widehat{S}(V)}(U(V^*\oplus\mathfrak{g})[[\hbar]]),[-,-]_\Phi,[F,-]_\Phi)$ as differential Lie algebras. □

Remark 6.5. Construction of the Lie algebras isomorphism φ_F can be generalized to differential Lie algebras between any two twist equivalent quasi-Hopf algebras (H_1,Δ_1,Φ_1) and (H_1,Δ_1,Φ_1) as far as the associators Φ_1 and Φ_2 are invariant (so that one can define corresponding Lie algebras on T_+H_i).

7 Globalization and Proof of Theorem 0.3

7.1 Globalization

In this section X is a principal G-bundle over a manifold M. We will use Kontsevich globalization procedure as described in [Do]. One can deduce global version of the local formality theorem (proved in Section 6) to a global one using Fedosov resolution as described in [Do]. The only things one has to check are the extra conditions that the L_∞-quasi-isomorphism φ_{loc} has to fulfill:

1. The L_∞-quasi-isomorphism φ_{loc} is equivariant with respect to linear transformations of coordinates.
2. $\varphi(v_1, v_2) = 0$ for any formal vector fields v_1 and v_2.
3. If $n \geq 2$ and v is a linear vector filed in the coordinates on $\mathbb{R}^n$, then for any set of polyvector fields $\gamma_2, \dots, \gamma_n$ we have $\varphi(v, \gamma_2, \dots, \Lambda\gamma_n) = 0$.

Proposition 7.1. *The L_∞-quasi-isomorphism φ_{loc} can be built so that it satisfies those three conditions*

PROOF. Let us recall that the map φ_{loc} was built from two differential Lie algebra morphism: $\varphi_{\mathrm{Ger}_\infty}\colon C(\tilde{g}_\hbar) \to C({}^c\underline{T}T_+\tilde{U})$ and $\varphi_{\mathrm{G}_\infty}\colon T_+\tilde{U} \to C({}^c\underline{T}T_+\tilde{U})$. Recall also that $\varphi_{\mathrm{G}_\infty}$ was built as a resolution of $T_+\tilde{U}$. Now, instead of using Theorem 5.1 we will construct directly the composition of $\varphi_{\mathrm{Ger}_\infty}$ with the inverse of $\varphi_{\mathrm{G}_\infty}$. More precisely, we will construct $\tilde{\varphi}_{\mathrm{Ger}_\infty}$ a L_∞-quasi-isomorphism deforming $\varphi_{\mathrm{Ger}_\infty}$ so that the image of $\tilde{\varphi}_{\mathrm{Ger}_\infty}$ is contained in the space of cocycle of $C({}^c\underline{T}T_+\tilde{U})$, and satisfy conditions of the theorem. This will then give the result. Let us now recall a useful lemma that can be found in [Do]:

Lemma 7.2. *Let ϕ be a L_∞-quasi-isomorphism between two DGLAs $(\mathfrak{g}_1, d_1)$ and $(\mathfrak{g}_2, d_2)$. Let $\phi^n\colon S^n(\mathfrak{g}_1[1]) \to \mathfrak{g}_2$ be the structure maps of ϕ. Let $m \geq 1$. Then it is possible to construct a deformed L_∞-quasi-isomorphism $\tilde{\phi}$ satisfying*

- $\tilde{\phi}(\gamma_1, \dots, \gamma_n) = \phi(\gamma_1, \dots, \gamma_n)$, *for $n < m$.*
- $\tilde{\phi}(\gamma_1, \dots, \gamma_m) = \phi(\gamma_1, \dots, \gamma_m) + d_2 V(\gamma_1, \dots, \gamma_m)$
  $- \sum_{1 \leq l \leq m} (-1)^{l + |\gamma_1| + \cdots + |\gamma_{l-1}|} V(\gamma_1, \dots, d_1\gamma_l, \dots, \gamma_m)$,
  where $V\colon S^m(\mathfrak{g}_1[1]) \to \mathfrak{g}_2$ is an arbitrary polylinear map.

Moreover one has explicit computation of $\tilde{\phi}$ from ϕ and V: let $D_1 = d_1 + d_1^{1,1}$ and $D_2 = d_2 + d_2^{1,1}$ be the structure maps of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ and Δ_1, Δ_2 the associated free comultiplications (see Section 1). Then, for $x \in C(\mathfrak{g}_1[1])$,

$$\tilde{\phi}(x) = \phi(x) + D_2 V(x) + V(D_1 x),$$

where V is extended as follows:

$$\Delta_2 D_2 V(x) = \left(\phi \otimes V + V \otimes \phi + \frac{1}{2}(V \otimes D_2 V + D_2 V \otimes V) + \frac{1}{2}(V \otimes V D_1 + V D_1 \otimes V) \right) \Delta_1(x).$$

Let us denote by ∂ the differential in $C({}^c\underline{T}T_+\tilde{U})$ and ϕ will be the map $\varphi_{\mathrm{Ger}_\infty}$. We know that the complex $(C({}^c\underline{T}T_+\tilde{U}), \partial)$ is acyclic except for elements of $T_+\tilde{U}$. We will write $(x)_0$ for the component in $T_+\tilde{U}$ of an element $x \in C({}^c\underline{T}T_+\tilde{U})$. So, as $\partial\phi^1(x) = 0$, there exists a linear map $V\colon C(\tilde{g}_\hbar) \to C({}^c\underline{T}T_+\tilde{U})$ such that for every element $v \in C(\tilde{g}_\hbar)$, $\phi^1(x) = (\phi^1(x))_0 + \partial V(x)$. Note that for degree reasons, when x is a vector field, $(V(x))_0$

is a function and so can be chosen to be zero. Moreover, the map ϕ is equivariant with respect to change of coordinates (see [Ha2], Section 5.2). So we can assume that V is also equivariant. So one can define $\tilde{\phi}$ as in the lemma and $\tilde{\phi}$ satisfies the first condition of the theorem. Moreover $(\tilde{\phi})_0$ clearly satisfy the second condition and the third is again a consequence of equivariance with respect to change of coordinates. Let us replace ϕ with $\tilde{\phi}$. We will know proceed by induction and suppose that the first condition of the theorem is true the structure maps of ϕ, that the second and third conditions a true for their $(-)_0$ parts, and are also true for the structure maps of $\phi^i = (\phi^i)_0$ for $i \leq n$. Using the induction hypothesis and the fact that ϕ is a L_∞-morphism, we get that $(\partial \phi^{n+1})_0 = 0$. So there exists W such that $\phi^{n+1} = (\phi^{n+1})_0 + \partial W$. Again W can be chosen equivariant. $(W)_0$ can be chosen to satisfy the last condition (the second is automatic for degree reason) as $(\phi^{n+1})_0$ satisfies it. Then again thanks to equivariance, one checks that $\tilde{\phi}$ obtained from ϕ and W satisfies the hypothesis of the induction. This concludes the proof. □

7.2 Proof of Theorem 0.3

Let us summarize what we have done so far:

Theorem 7.3. *Let X be a principal G-bundle over a manifold M. Let $r \wedge^2 \mathfrak{g}$ such that $[r,r] = Z \in (\wedge^3 \mathfrak{g})^{\mathfrak{g}}$. There exists $\Phi = 1 + \frac{\hbar^2}{6} Z + O(\hbar^3) \in (U(\mathfrak{g})^{\otimes 3})^{\mathfrak{g}}$, $J = 1 + \hbar r + O(\hbar^2) \in U(\mathfrak{g})^{\otimes 2}[[\hbar]]$ such that $J^{1,2} J^{12,3} = J^{2,3} J^{1,23} \Phi$, a deformation $\mathfrak{g}_\hbar$ of the Lie algebra $\mathfrak{g}$ and φ a L_∞-quasi-isomorphism*

$$\varphi : \; (T^{\mathrm{inv}}_{\mathrm{poly}}(M), [-,-]_S, [r,-]_S) \to (D^{\mathrm{inv}}_{\mathrm{poly}}(M), [-,-]_\Phi, [J,-]_\Phi),$$

where $T^{\mathrm{inv}}_{\mathrm{poly}}(M)$ and $D^{\mathrm{inv}}_{\mathrm{poly}}(M)$ are respectively the spaces of invariant polyvectors fields an M or polydifferential operators on M and $[-,-]_\phi$ is the deformed Gerstenhaber bracket in the tensor category of $(U(\mathfrak{g}_\hbar)[[\hbar]], \Phi)$-modules.

Suppose now that (X, π, r, Z) is a quasi-(r,Z)-Poisson manifold. Set $\pi = \pi' + r$. Then π is a Maurer-Cartan in the DGLA $(T^{\mathrm{inv}}_{\mathrm{poly}}(M), [-,-]_S, [r,-]_S)$. Thanks to Theorem 7.3, we know that those Maurer-Cartan elements, up to Gauge transform are in one to one correspondence with Maurer-Cartan elements of the DGLA $(D^{\mathrm{inv}}_{\mathrm{poly}}(M), [-,-]_\Phi, [J,-]_\Phi)$, up to Gauge transform. If $m'_\star$ is such a Maurer-Cartan element, set $m_\star = m'_\star + J$, $m_\star$ is a quantization of the quasi-(r,Z)-Poisson manifold (X, π, r, Z). We have prove:

Theorem 7.4. *Let (X, π, r, Z) be a quasi-(r,Z)-Poisson manifold, quantization of X, up to equivalence, are in one to one correspondence with*

$$\{\pi_\hbar = \hbar \pi + O(\hbar^2) \text{ such that } [r, \pi_\hbar] + \frac{1}{2}[\pi_\hbar, \pi_\hbar] = 0\}.$$

8 Quantization of modified dynamical Yang-Baxter r-matrices

We know that modified classical dynamical Yang-Baxter r-matrices provide examples of quasi-Poisson manifolds. Let us recall their definition: let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$. Let ρ be a $\mathfrak{h}$-equivariant map $\rho : \mathfrak{h}^* \to \Lambda^2(\mathfrak{g})$, solution of the modified classical dynamical Yang-Baxter equation:

$$-\mathrm{Alt}(d\rho) + \mathrm{CYB}(\rho) = Z,$$

where

$$\mathrm{CYB}(\rho) = [\rho^{1,2}, \rho^{1,3}] + [\rho^{1,2}, \rho^{2,3}] + [\rho^{1,3}, \rho^{2,3}]$$

and

$$\mathrm{Alt}(d\rho) = \sum_i h_i^1 \frac{\partial \rho^{2,3}}{\partial \lambda_i} - \sum_i h_i^2 \frac{\partial \rho^{1,3}}{\partial \lambda_i} + \sum_i h_i^3 \frac{\partial \rho^{1,2}}{\partial \lambda_i}.$$

Using a quasi-Poisson generalization of a construction of Xu [Xu], Enriquez and Etingof [EE1] built a quasi-Poisson manifold X_ρ associated to ρ (for which the action of the corresponding group G is free). They then prove (following [Xu]) that any twist quantization J associated to an associator Φ (i.e.

- $J \in \mathrm{Mer}(\mathfrak{h}^*, U(\mathfrak{g})^{\otimes 2}[[\hbar]]$, $\mathfrak{h}$-invariant, such that $J(\lambda) = 1 + O(\hbar)$,
- $J^{12,3}(\lambda) \star J^{1,2}(\lambda + \lambda \hbar^3) = \Phi^{-1} J^{1,23}(\lambda) \star J^{2,3}(\lambda)$,
- $Z = \mathrm{Alt}\left(\frac{\Phi - 1}{\hbar^2}\right) \bmod \hbar$,
- $\rho(\lambda) = \left(\frac{J(\lambda)-1}{\hbar}\right) - \left(\frac{J(\lambda)-1}{\hbar}\right)^{2,1} \bmod \hbar$)

gives rise to a quantization of the quasi-Poisson manifold X_ρ. Our result provides us with a quantization of the manifolds X_ρ when Z satisfies our conditions but unfortunately, we don't know whether this quantization provides us with a twist quantization of the modified dynamical r-matrix ρ.

Let us write, according to [Xu], the Poisson bracket associated π_ρ to a dynamical r-matrix ρ: in the decomposition of $T_{\mathrm{poly}}^{\mathrm{inv}} = \wedge T\mathfrak{h}^* \otimes \wedge \mathfrak{g}$,

$$\pi_\rho = \pi_{\mathfrak{h}^*} + \sum_i \frac{\partial}{\partial \lambda_i} \wedge \mathfrak{h}_i + \rho \in \wedge^2 T\mathfrak{h}^* \oplus T\mathfrak{h}^* \wedge \mathfrak{g} \oplus \wedge^2 \mathfrak{g},$$

where h_i and λ_i are basis and dual basis of $\mathfrak{h}$ and $\pi_{\mathfrak{h}^*}$ is the Kostant-Kirilov-Souriau Poisson bracket (which we will denote $\star_{\mathfrak{h}^*}$). Now, still following [Xu], a quantization $\star$ of X_ρ corresponds to a quantization of ρ is an only if: it satisfies the following conditions:

1. for any $f, g \in C^\infty(\mathfrak{h}^*)$, $f(\lambda) \star g(\lambda) = f \star_{\mathfrak{h}^*} g$,
2. for any $f \in C^\infty(G)$ and $g \in C^\infty(\mathfrak{h}^*)$, $f(x) \star g(\lambda) = f(x) g(\lambda)$,
3. for any $f \in C^\infty(\mathfrak{h}^*)$ and $g \in C^\infty(G)$, $f(\lambda) \star g(x) = \sum_{k \geq 0} \frac{\hbar^k}{k!} \frac{\partial^k f}{\partial \lambda^{i_1} \cdots \partial \lambda^{i_k}} h_{i_1} \cdots h_{i_k} g$,
4. for any $f, g \in C^\infty(G)$, $f(x) \star g(x) = R(f,g)$, where R would be the quantization of ρ.

Let us notice that our quantization of X_ρ will not satisfiy those conditions as, for symmetry conditions, conditions 2 and 3 will not be fulfilled. So we will use a trick proved by Alekseev and Calaque ([AC]). Let us first recall their definition of strongly $\mathfrak{g}$-invariant quantization of quasi-Poisson manifold.

Definition 8.1. Let $\star$ be a quantization of a quasi-Poisson manifold (X, π, Z). Suppose μ: $X \to \mathfrak{h}^*$ is a momentum map for which the map $M = U(\mu^*) \circ$ sym: $(\mathscr{O}_{\mathfrak{h}^*}[[\hbar]], \star_{PBW}) \to (\mathscr{O}_X[[\hbar]], \star)$ is an algebra morphsism satisfying $[M(x), f]_\star = \hbar\{\mu^* x, f\}$ for any $f \in \mathscr{O}_X$ and any $x \in \mathfrak{g}$, then we say that the quantization $\star$ is strongly $\mathfrak{g}$-invariant.

Proposition 8.2. *[AC] Assume that $\star$ is a strongly $\mathfrak{g}$-invariant quantization of X_ρ. Then there exists a gauge equivalent quantization $\star'$ of X_ρ such that corresponds to a quantization of ρ.*

Thus we only need to prove that we can construct a strongly $\mathfrak{g}$-invariant quantization of X_ρ.

Theorem 8.3. *Suppose that $\mathfrak{h}$ is an abelian subalgebra of $\mathfrak{g}$ and that there exists a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. Then the modified classical dynamical Yang-Baxter r-matrice $(r - Z)$ can be quantized.*

PROOF. Using Proposition 8.2, we need to prove that there exists a a strongly $\mathfrak{g}$-invariant quantization of the quasi-Poisson manifold X_ρ. To do so, we can copy the proof of Proposition 7.1 to get adapted product (we use the fact that $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, so the quantization is a sub algebra, and then use that $[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{g}$). □

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