

# DEFORMATION OF LINEAR POISSON ORBIFOLD

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ABSTRACT. Let  $\Gamma$  be a finite group acting faithfully and linearly on a vector space  $V$ . Let  $T(V)$  ( $S(V)$ ) be the tensor (symmetric) algebra associated to  $V$  which has a natural  $\Gamma$  action. We study generalized quadratic relations on the tensor algebra  $T(V) \rtimes \Gamma$ . We prove that the quotient algebras of  $T(V) \rtimes \Gamma$  by such relations satisfy PBW property. Such a quotient algebra can be viewed as a quantization of a linear or constant Poisson structure on  $S(V) \rtimes \Gamma$ , and is a natural generalization of symplectic reflection algebra.

## 1. INTRODUCTION

Poisson structure on a manifold is a bivector field  $\pi$  whose Schouten-Nijenhuis bracket with itself vanishes, i.e.  $\pi \in \Gamma(\wedge^2 TM)$ , and  $[\pi, \pi] = 0$ . The problem of deformation quantization of a Poisson manifold was solved by Kontsevich in his semina paper [9]. In this paper, we study quantization problem of Poisson structures on an orbifold following [8].

The study of the first and third author in [8] starts with the idea that Poisson structures on an algebra  $A$  should correspond to infinitesimal deformations of  $A$ . According to Gerstenhaber's theory, an infinitesimal deformation of an algebra is classified by a second Hochschild cohomology class in  $H^2(A, A)$  whose Gerstenhaber bracket with itself is zero. This type of cohomology class is called a Poisson structure on  $A$ . Applying this idea to orbifold, we can represent an orbifold  $X$  by a proper étale groupoid  $\mathcal{G}$  [10] (different representations of a same orbifold are Morita equivalent as Lie groupoids). We consider the smooth groupoid algebra  $C_c^\infty(\mathcal{G})$  associated to  $\mathcal{G}$ . We studied in [8] Poisson structures on  $C_c^\infty(\mathcal{G})$ . We find that Poisson structures on  $C_c^\infty(\mathcal{G})$  are richer than we naturally expect from geometry. On an orbifold, multivector fields and Schouten-Nijenhuis bracket are well defined. Accordingly, we can consider bivector fields on  $X$  whose Schouten-Nijenhuis bracket with themselves vanish. We found that [8][Theorem 4.1] there are many more Poisson structures on  $C_c^\infty(\mathcal{G})$  than the above type of bivector fields on  $X$ . For example, in the case of a finite group  $\Gamma$  acting on a symplectic vector space  $V$ , we [8][Corollary 4.2] find Poisson structures on  $S(V^*) \rtimes \Gamma$  which have supports on codimension 2 fixed point subspaces, where  $S(V^*)$  is the algebra of real coefficients symmetric polynomials on the dual vector space  $V^*$ .

In this paper, we continue our study of Poisson structures in the above frame work. We will study Poisson structures in a neighborhood of a point in a reduced orbifold. Locally, a reduced orbifold can always be viewed as a quotient of a finite group acting faithfully and linearly on an open set of  $\mathbb{R}^n$ . This leads us to study the following data. Let  $\Gamma$  be a finite group acting on a vector space  $V$  faithfully, and  $S(V^*)$  be the algebra of polynomials on  $V^*$ . The  $\Gamma$  action on  $V^*$  defines the crossed product algebra  $S(V^*) \rtimes \Gamma$ . According to [12], the second Hochschild cohomology  $H^2(S(V^*) \rtimes \Gamma, S(V^*) \rtimes \Gamma)$  is equal to

$$(S(V^*) \otimes \wedge^2 V)^\Gamma \oplus \left( \sum_{\gamma \in \Gamma, l(\gamma)=2} S(V^{\gamma^*}) \otimes \wedge^2 N^\gamma \right)^\Gamma.$$

In the above equation, we have that  $l(\gamma)$  is the codimension of the  $\gamma$  fixed point subspace  $V^\gamma$  and  $N^\gamma$  consists of vectors in  $V$  vanishing on  $V^{\gamma^*}$  (the fixed point subspace of  $\gamma$  action on  $V^*$ ), and  $\Gamma$  acts on the set  $S := \{\gamma \in \Gamma, l(\gamma) = 2\}$  by conjugation.

We study two types of Poisson structures on  $S(V^*) \rtimes \Gamma$  which are of the forms

$$i) \text{ Hom}(\wedge^2 V^*, \mathbb{R}\Gamma), \quad ii) \text{ Hom}(\wedge^2 V^*, V^* \otimes_{\mathbb{R}} \mathbb{R}\Gamma).$$

The first type of Poisson structure can be viewed as constant value Poisson structures, and the second type can be viewed as linear Poisson structures, which defines a generalized "Lie" algebra structure on

$V^* \otimes_{\mathbb{R}} \mathbb{R}\Gamma$ . Our main theorems for these Poisson structures are that the quotient algebras of  $T(V^*) \rtimes \Gamma$  by the relations defined by the above two types of Poisson structures satisfy PBW property. The way we prove such theorems is using the Braverman-Gaitsgory conditions [2] for PBW property. However, the proof for the second type is quite involved. We need to use properties of finite subgroups of  $GL(2, \mathbb{C})$ , which is closely related to McKay correspondence. A new and interesting phenomena we found in the proof of PBW property is that we have to have a nontrivial coboundary term for the bracket  $[\pi, \pi]$  of the linear Poisson structure  $\pi$ . This kind of term never shows up in the study of PBW property for Lie algebras and symplectic reflection algebras. The PBW property of the quotient algebras shows that they define quantizations of the Poisson structures on  $S(V^*) \rtimes \Gamma$ . This confirms that any constant or linear Poisson structures on  $S(V^*) \rtimes \Gamma$  can be quantized, and gives a strong evidence that the deformation theory of the convolution algebra  $S(V^*) \rtimes \Gamma$  is formal.

The second part of this paper is dedicated to studying various properties and examples of the above two types of Poisson structures and their quantizations. We mention a few of them here. Firstly, using Poisson cohomology computation, we are able to give a new proof of Hochschild cohomology of a symplectic reflection algebra [5][Theorem 1.8]. The advantage of our work is that our result works for Laurent series of  $\hbar$  so that we can drop the assumption "except possibly a countable set" in [5][Theorem 1.8]. Secondly, assuming a finite group  $\Gamma$  act faithfully and linearly on a Lie algebra  $\mathfrak{g}$ , we compute the Hochschild cohomology of  $H^\bullet(\mathcal{U}(\mathfrak{g}) \rtimes \Gamma, \mathcal{U}(\mathfrak{g}) \rtimes \Gamma)$  with  $\mathcal{U}(\mathfrak{g})$  the universal enveloping algebra of  $\mathfrak{g}$ . This is a natural generalization of results [1] for finite group action on a symplectic vector space. Our result shows that the Hochschild cohomology  $H^\bullet(\mathcal{U}(\mathfrak{g}) \rtimes \Gamma, \mathcal{U}(\mathfrak{g}) \rtimes \Gamma)$  is computed by the noncommutative Poisson cohomology associated to the  $\Gamma$  action and Lie-Poisson structure  $\pi$  on  $\mathfrak{g}$ . Thirdly, if  $V$  is equipped with a  $\Gamma$  invariant Lie Poisson structure, we introduce a large class of examples of linear Poisson structures on  $S(V^*) \rtimes \Gamma$ . This class of linear Poisson structures should be viewed as a natural generalization of symplectic reflection algebras. Analogous to the symplectic case, we are able to prove that the Poisson cohomology of such a general linear Poisson structure is determined by the data supported at identity of  $\Gamma$  in the case that  $\Gamma$  is abelian.

In the third part of this paper, we restrict ourselves to  $\mathbb{R}^2$  with a cyclic group  $\mathbb{Z}_n$  action. Let  $\omega$  be the standard symplectic form on  $\mathbb{R}^2$  and  $\pi$  be the corresponding Poisson structure. We study the quantization of the Poisson structure  $\pi_\gamma : x \wedge y \rightarrow \pi(x, y)\gamma$  on  $S(\mathbb{R}^2) \rtimes \Gamma$  (We identify  $\mathbb{R}^{2*}$  with  $\mathbb{R}^2$ ). Nadaud [11] gave a Moyal type formula for such a deformation quantization. Here, we use this formula to study the center of the quantization. Our computation shows that the center of the quantization of  $\pi_\gamma$  is not isomorphic to the center of the algebra  $S(\mathbb{R}^2) \rtimes \mathbb{Z}_n$ . Instead, the center of the quantization is a nontrivial deformation of the center of  $S(\mathbb{R}^2) \rtimes \mathbb{Z}_n$ . This suggests that there is no analog of Duflo's isomorphism for the quantization of the Poisson structure  $\pi_\gamma$ , and also that deformation quantization of  $\pi_\gamma$  on  $S(\mathbb{R}^2) \rtimes \mathbb{Z}_n$  is closely connected to the deformation of the underlying orbifold singularity. We plan to study the relation between the deformation of  $S(V^*) \rtimes \Gamma$  and the deformation of the underlying orbifold  $V/\Gamma$  in the near future.

This paper is organized as follows. In the second section, we review some results about Hochschild cohomology  $H^\bullet(S(V^*) \rtimes \Gamma, S(V^*) \rtimes \Gamma)$  in [12] and [8], and also the Braverman-Gaitsgory conditions for PBW property [2]; in the third section, we prove that constant and linear Poisson structures on  $S(V^*) \rtimes \Gamma$  can be quantized; in the fourth section, we study various properties and examples of constant and linear Poisson structures on  $S(V^*) \rtimes \Gamma$ ; in the fifth section, using Nadaud's formula, we study the center of quantization of some Poisson structures on  $S(\mathbb{R}^2) \rtimes \mathbb{Z}_n$ .

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## 2. PRELIMINARIES AND NOTATIONS

In this whole paper,  $\Gamma$  is a finite group, acting faithfully and linearly on a finite dimensional real vector space  $V$ . We fix on  $V$  a  $\Gamma$ -invariant metric. We denote  $C(\Gamma)$  the set of conjugacy classes of  $\Gamma$ . For any

element  $\gamma$  in  $\Gamma$ , let  $V^\gamma$  be the  $\gamma$ -invariant subspace of  $V$ ,  $N^\gamma$  be the subspace of  $V$  orthogonal to  $V^\gamma$  which is the direct sum of all nontrivial representations of  $\Gamma$ ,  $l(\gamma)$  be the real codimension of  $V^\gamma$ , and  $Z(\gamma)$  the centralizer of  $\gamma$  in  $\Gamma$ . In this paper, we always work with the field  $\mathbb{R}$ . All dimensions, algebras, and tensor products if not specified are over the field  $\mathbb{R}$ . For the convenience of proofs, we are many times using the following complexification trick,

$$(1) \quad S_{\mathbb{R}}(V^*) \rtimes_{\mathbb{R}} \mathbb{C}\Gamma \cong S_{\mathbb{C}}(V^* \otimes \mathbb{C}) \rtimes_{\mathbb{C}} \mathbb{C}\Gamma \cong \left( S_{\mathbb{R}}(V^*) \rtimes_{\mathbb{R}} \mathbb{R}\Gamma \right) \otimes_{\mathbb{R}} \mathbb{C}.$$

This helps us to deduce the results in  $\mathbb{R}$  from their complex versions where  $\gamma$  action is diagonalizable for any  $\gamma \in \Gamma$ . Many results in this paper hold true for field  $\mathbb{C}$  and even more general field with characteristic 0 in which the order of  $\Gamma$  is invertible.

**2.1. The Koszul complex and the Hochschild cohomology of  $S(V^*) \rtimes \Gamma$ .** The algebra  $S(V^*) \rtimes \Gamma$  is generated by  $V^*$  and  $\Gamma$  with the quadratic relations :

$$x \otimes y \otimes \gamma - y \otimes x \otimes \gamma, \quad \gamma \otimes x - {}^\gamma x \otimes \gamma,$$

for all  $x$  and  $y$  in  $V^*$  and  $\gamma$  in  $\Gamma$ , and  ${}^\gamma x$  is the image of  $x$  under the  $\gamma$  action. Moreover,  $S(V^*) \rtimes \Gamma$  is a Koszul algebra over the semi-simple algebra  $\mathbb{R}\Gamma$ . The general theory of Koszul algebras over a semi-simple algebra gives therefore a small complex which calculates the Hochschild cohomology of  $S(V^*) \rtimes \Gamma$  :

$$CK^\bullet(S(V^*) \rtimes \Gamma) = \left( \bigoplus_{\gamma \in \Gamma} S(V^*) \otimes \wedge^\bullet V \right)^\Gamma.$$

A  $n$ -cochain  $f$  of this complex splits in a sum of maps  $f_\gamma$  in  $S(V^*) \otimes \wedge^n V$ . The  $\Gamma$ -invariance can be written :

$$(2) \quad {}^g f_\gamma(g^{-1}x_1, \dots, g^{-1}x_n) = f_{g\gamma g^{-1}}(x_1, \dots, x_n),$$

which explains that  $CK^\bullet(S(V^*) \rtimes \Gamma)$  splits in a sum of sub-complexes :

$$CK^\bullet(S(V^*) \rtimes \Gamma) = \bigoplus_{\gamma \in C(\Gamma)} (S(V^*) \otimes \wedge^\bullet V)^{Z(\gamma)}$$

with the boundary

$$\partial_\gamma(f)(x_0, \dots, x_n) = \sum_{i=0}^n (-1)^i f(x_0, \dots, \widehat{x}_i, \dots, x_n)(x_i - {}^\gamma x_i),$$

for  $x_0, \dots, x_n \in V^*$ .

Using this small complex, Neumaier, Pflaum, Posthuma and the third author calculated in [12] the Hochschild cohomology of  $S(V^*) \rtimes \Gamma$  :

$$(3) \quad H^\bullet(S(V^*) \rtimes \Gamma, S(V^*) \rtimes \Gamma) = \bigoplus_{\gamma \in C(\Gamma)} \left( S(V^{\gamma^*}) \otimes \wedge^{\bullet-l(\gamma)} V^\gamma \otimes \wedge^{l(\gamma)} N^\gamma \right)^{Z(\gamma)}.$$

This statement in  $\mathbb{R}$  is easily deduced from its complex version [12] using the trick (1).

We point out that the projection  $\text{pr}_\gamma : (S(V^*) \otimes \wedge^\bullet V)^{Z(\gamma)} \rightarrow (S(V^{\gamma^*}) \otimes \wedge^{\bullet-l(\gamma)} V^\gamma \otimes \wedge^{l(\gamma)} N^\gamma)^{Z(\gamma)}$  and the embedding  $\iota : (S(V^{\gamma^*}) \otimes \wedge^{\bullet-l(\gamma)} V^\gamma \otimes \wedge^{l(\gamma)} N^\gamma)^{Z(\gamma)} \rightarrow (S(V^*) \otimes \wedge^\bullet V)^{Z(\gamma)}$  are inverse quasi-isomorphisms of complexes. Another useful remark is that if  $l(\gamma)$ =dimension of  $N^\gamma$  is odd, the determinant of  $\gamma$  action on  $N^\gamma$  is -1 (otherwise  $\gamma$  has an eigenvalue 1 as  $\gamma$  is an isometry). Therefore  $S(V^{\gamma^*}) \otimes \wedge^{\bullet-l(\gamma)} V^\gamma \otimes \wedge^{l(\gamma)} N^\gamma$  has no  $\gamma$  invariant element if  $l(\gamma)$  is odd. Therefore, Poisson brackets on  $S(V^*) \rtimes \Gamma$  do not contain  $\gamma$ -component for  $l(\gamma) = 1$ . Furthermore, when  $\Gamma$  acts faithfully, the identity of  $\Gamma$  is the only group element with  $l(\gamma) = 0$ .

We will say that a cocycle is constant if it is in  $(\wedge^2 V)^\Gamma \oplus \left( \bigoplus_{\gamma \in \Gamma, l(\gamma)=2} \wedge^2 N^\gamma \right)^\Gamma$ . Similarly, we will say that a cocycle is linear if it is in  $(V^* \otimes \wedge^2 V)^\Gamma \oplus \left( \bigoplus_{\gamma \in \Gamma, l(\gamma)=2} V^{\gamma^*} \otimes \wedge^2 N^\gamma \right)^\Gamma$ .

**2.2. The Braverman-Gaitsgory conditions for PBW.** Let  $T(V^*) \rtimes \Gamma$  be the free  $\mathbb{R}\Gamma$ -algebra generated by the bimodule  $V^* \rtimes \Gamma$ , and  $A$  be its quotient by the relations :

$$x \otimes y - y \otimes x - \sum_{\gamma} \pi_{\gamma}(x, y)\gamma - \sum_{\gamma} b_{\gamma}(x, y)\gamma$$

where  $\pi$  and  $b$  are  $\Gamma$ -invariant elements in  $\bigoplus_{\gamma \in \Gamma} V^* \otimes \wedge^2(V)$  and  $\bigoplus_{\gamma \in \Gamma} \wedge^2 V$ . As before,  $\pi$  and  $b$  split into sums of  $\gamma$ -components, and the  $\Gamma$ -invariance is expressed in the same way as in (2). The algebra  $A$  is clearly filtered by the length of words. Following Braverman and Gaitsgory [2], the associated graded algebra  $Gr(A)$  is isomorphic to  $S(V^*) \rtimes \Gamma$  if and only if :

$$(4) \quad \partial_{\gamma}(\pi_{\gamma}) = 0,$$

$$(5) \quad \sum_{\alpha\beta=\gamma} \pi_{\alpha}(\pi_{\beta}(x, y), z + {}^{\beta}z) + \pi_{\alpha}(\pi_{\beta}(y, z), x + {}^{\beta}x) + \pi_{\alpha}(\pi_{\beta}(z, x), y + {}^{\beta}y) = \partial_{\gamma}(b_{\gamma}),$$

$$(6) \quad \sum_{\alpha\beta=\gamma} b_{\alpha}(\pi_{\beta}(x, y), z + {}^{\beta}z) + b_{\alpha}(\pi_{\beta}(y, z), x + {}^{\beta}x) + b_{\alpha}(\pi_{\beta}(z, x), y + {}^{\beta}y) = 0$$

for all  $\gamma$  in  $\Gamma$  and  $x, y, z$  in  $V^*$ .

When the three conditions above are satisfied, the algebra  $A$  gives a quantization of the algebra  $S(V^*) \rtimes \Gamma$  for the same reason as is explained in [8](Proposition 4.5). This will be our method to obtain the quantization results of the next section.

For our purpose, let us denote  $[[\pi, \pi]]_{\gamma} \in V^* \otimes \wedge^3 V$  and  $[[b, \pi]]_{\gamma} \in \wedge^3 V$  defined by :

$$[[\pi, \pi]]_{\gamma}(x, y, z) := \sum_{\alpha\beta=\gamma} \pi_{\alpha}(\pi_{\beta}(x, y), z + {}^{\beta}z) + \pi_{\alpha}(\pi_{\beta}(y, z), x + {}^{\beta}x) + \pi_{\alpha}(\pi_{\beta}(z, x), y + {}^{\beta}y),$$

$$[[b, \pi]]_{\gamma}(x, y, z) := \sum_{\alpha\beta=\gamma} b_{\alpha}(\pi_{\beta}(x, y), z + {}^{\beta}z) + b_{\alpha}(\pi_{\beta}(y, z), x + {}^{\beta}x) + b_{\alpha}(\pi_{\beta}(z, x), y + {}^{\beta}y)$$

for all  $x, y, z$  in  $V^*$ .

**2.3. The Gerstenhaber bracket on  $H^{\bullet}(S(V^*) \rtimes \Gamma, S(V^*) \rtimes \Gamma)$  and Poisson structures.** The Gerstenhaber bracket on  $H^{\bullet}(S(V^*) \rtimes \Gamma, S(V^*) \rtimes \Gamma)$  was explicitly calculated by the first and last authors in [8]. We only recall here the results in the cases we will need and refer to [8] for a complete description.

Firstly, the Gerstenhaber bracket of two constant cocycles is zero.

Secondly, let  $b$  be a constant cocycle and  $\pi$  be a linear cocycle of  $H^2(S(V^*) \rtimes \Gamma, S(V^*) \rtimes \Gamma)$ . Let  $\text{pr}_{\gamma}$  be the projection from  $S(V^*) \otimes \wedge^{\bullet} V$  onto  $S(V^{\gamma^*}) \otimes \wedge^{\bullet-l(\gamma)} V \otimes \wedge^{l(\gamma)} N^{\gamma}$ . Then the  $\gamma$ -component of their Gerstenhaber bracket is :

$$(7) \quad [b, \pi]_{\gamma} = \text{pr}_{\gamma} \circ [[b, \pi]]_{\gamma}.$$

Moreover, the  $\gamma$ -component of the Gerstenhaber bracket of  $\pi$  with itself is obtained by

$$(8) \quad [\pi, \pi]_{\gamma} = \text{pr}_{\gamma} \circ [[\pi, \pi]]_{\gamma}.$$

Let  $\alpha$  and  $\beta$  be two elements of  $\Gamma$ ,  $f_{\alpha}$  be an element of  $(S(V^{\alpha^*}) \otimes \wedge^{\bullet-l(\alpha)} V^{\alpha} \otimes \wedge^{l(\alpha)} N^{\alpha})^{Z(\alpha)}$  and  $g_{\beta}$  be an element of  $(S(V^{\beta^*}) \otimes \wedge^{\bullet-l(\beta)} V^{\beta} \otimes \wedge^{l(\beta)} N^{\beta})^{Z(\beta)}$ . If  $l(\gamma) \neq l(\alpha) + l(\beta)$ , then the  $\gamma$  component of  $[f_{\alpha}, g_{\beta}]$  vanishes. Suppose that any elements in  $\langle \alpha \rangle$  commutes with any elements in  $\langle \beta \rangle$ , where  $\langle \alpha \rangle, \langle \beta \rangle$  are subsets of  $\Gamma$  of elements conjugate to  $\alpha$  and  $\beta$ . Then the Gerstenhaber bracket of  $f_{\alpha}$  and  $g_{\beta}$  is :

$$(9) \quad [f_{\alpha}, g_{\beta}]_{\gamma} = \sum_{\substack{\alpha' \in \langle \alpha \rangle, \beta' \in \langle \beta \rangle, \alpha'\beta' = \gamma \\ l(\gamma) = l(\alpha'\beta') = l(\alpha) + l(\beta)}} \text{pr}_{\gamma} \circ \{f_{\alpha'}, g_{\beta'}\},$$

where  $\{f_{\alpha'}, g_{\beta'}\}$  is the usual Schouten-Nijenhuis bracket of  $f_{\alpha'}$  and  $g_{\beta'}$ .

The first and third author defined that a Poisson structure  $\Pi$  on  $S(V^*) \rtimes \Gamma$  is a sum of elements like

$$\pi_{\gamma} \in \left( S(V^{\gamma^*}) \otimes \wedge^{2-l(\gamma)} V^{\gamma} \otimes \wedge^{l(\gamma)} N^{\gamma} \right)^{Z(\gamma)}, \quad \gamma \in C(\Gamma),$$

with  $l(\gamma) = 0, 2$  satisfying  $[\Pi, \Pi] = 0$ .

### 3. PBW PROPERTY FOR CONSTANT AND LINEAR POISSON STRUCTURES

In this section, we prove that constant and linear Poisson structures have PBW property.

**3.1. Quantization of constant Poisson brackets.** Following (3), a Poisson bracket  $\pi$  on  $S(V^*) \rtimes \Gamma$  splits into a sum of  $\pi_\gamma$ ,

$$\pi_0 + \sum_{\gamma} \pi_\gamma \in (S(V^*) \otimes \wedge^2 V)^\Gamma \oplus \left( \bigoplus_{\gamma \in \Gamma, l(\gamma)=2} S(V^{\gamma^*}) \otimes \wedge^2 N^\gamma \right)^\Gamma.$$

We say that  $\pi_0 + \sum_{\gamma} \pi_\gamma$  is a constant Poisson structure if  $\pi_0 \in \wedge^2 V$  and  $\pi_\gamma \in \wedge^2 N^\gamma$ . We notice that in this constant case the Braverman-Gaitsgory conditions 2.2 reduce to only one condition (5), which means that  $\pi_\gamma$  has to be a cocycle. But this is automatically satisfied as we know from (3) that an element in  $(S(V^*) \otimes \wedge^2 V)^\Gamma \oplus \left( \bigoplus_{\gamma \in \Gamma, l(\gamma)=2} S(V^{\gamma^*}) \otimes \wedge^2 N^\gamma \right)^\Gamma$  is closed respect the differential  $b_\gamma$  and  $b_0 = 0$ .

**Theorem 3.1.** *Any constant Poisson structure of  $S(V^*) \rtimes \Gamma$  is quantizable.*

*Proof.* According to the above explanation, we know that any constant Poisson structure satisfies the Braverman-Gaitsgory conditions (4)-(6). This implies that the quotient algebra

$$H_\pi := T(V^*) \rtimes \Gamma[[\hbar]] / \langle x \otimes y - y \otimes x - \hbar(\pi_0(x, y) + \sum_{\gamma, l(\gamma)=2} \pi_\gamma(x, y)\gamma) \rangle$$

has PBW property, which defines a deformation quantization of the algebra  $S(V^*) \rtimes \Gamma[[\hbar]]$  respect the Poisson structure  $\pi = \pi_0 + \sum_{\gamma \in \Gamma, l(\gamma)=2} \pi_\gamma$ .  $\square$

We remark that the PBW property of the algebra  $H_\pi$  is known to Etingof-Ginzburg [5]. Our proof is evident by using the results from [12].

**3.2. Quantization of linear Poisson brackets - the abelian case.** In this subsection, we will assume that  $\Gamma$  is an abelian group which acts faithfully on  $V$ . According to representation theory of a finite abelian group,  $V$  is decomposed into a direct sum of 1 or 2 real dimensional subspaces where  $\Gamma$  acts irreducibly.  $\gamma$  acts on 1 dimensional subspace with eigenvalue 1 or -1, and on 2 dimensional subspace by rotation of finite order.

Let  $\pi$  be a Poisson structure on  $S(V^*) \rtimes \Gamma$ , and denote  $\pi_0$  its identity component. The  $\gamma$ -components of  $\pi$  are null whenever  $l(\gamma) \neq 0, 2$  and take values in  $V^\gamma$ . As we have remarked that if  $l(\gamma) = 1$ , then  $\gamma$  action on  $N^\gamma$  has eigenvalue -1. There will not be any nonzero element in  $V^{\gamma^*} \otimes V^\gamma \otimes N^\gamma$  invariant under  $\gamma$ . Therefore, we have  $\pi_\gamma$  is possible nonzero only when  $l(\gamma) = 2$ . And the eigenvalues of  $\gamma$  action on  $N^\gamma$  are either -1 with multiplicity 2 or roots of unity.

**Lemma 3.2.** *The identity component  $\pi_0$  of  $\pi$  defines a Lie bracket on  $V$ .*

*Proof.* It is a consequence of the fact that  $\pi_\gamma$  takes values in  $V^\gamma$  which is in the kernel of  $\pi_{\gamma^{-1}}$ . Therefore, the condition  $[[\pi, \pi]]_0 = 0$  reduces to the Jacobi identity of  $\pi_0$ .  $\square$

**Lemma 3.3.** *Let  $\alpha$  and  $\beta$  be elements of  $\Gamma$  with  $l(\alpha) = l(\beta) = 2$ . Then,  $[[\pi_\alpha, \pi_\beta]] = 0$ .*

*Proof.* Let  $x$  and  $y$  be the coordinates on  $N^\alpha$ . It follows from the  $\Gamma$ -invariance for a Poisson structure that  $\pi_\alpha$  has to be  $\beta$ -invariant as  $\alpha$  commutes with  $\beta$  for any  $\beta \in \Gamma$ :

$$(10) \quad \pi_\alpha({}^\beta x, {}^\beta y) = {}^\beta \pi_\alpha(x, y).$$

We observe  $\beta$  preserves on  $V^\alpha$  and  $N^\alpha$  as  $\beta$  commutes with  $\alpha$ . If  $l(\beta) = 2$ , there are three possibilities, 1)  $N^\alpha \cap N^\beta = \{0\}$ , 2)  $\dim(N^\alpha \cap N^\beta) = 1$ , 3)  $N^\alpha = N^\beta$ .

If  $N^\alpha \cap N^\beta = \{0\}$ , then  $x, y$  are  $\beta$  invariant for  $x, y$  in  $N^{\alpha^*}$ . Hence by Equation (10),  $\pi_\alpha(x, y) = {}^\beta \pi_\alpha(x, y)$ , which shows  $\pi_\alpha(x, y)$  is  $\beta$  invariant. Similarly, we know that  $\pi_\beta$  takes value in  $V^\alpha$ . This shows that  $[[\pi_\alpha, \pi_\beta]] = 0$ .

If  $\dim(N^\alpha \cap N^\beta) = 1$ , then we know that both  $\alpha$  and  $\beta$  preserves  $N^{\alpha, \beta} =: N^\alpha + N^\beta$  which is of 3 dimension. Furthermore, we conclude that  $\beta$ 's ( $\alpha$ 's) action on  $N^\alpha$  ( $N^\beta$ ) has eigenvalue 1 and -1.

Hence,  $N^{\alpha, \beta}$  is decomposed into a direct sum of  $N_1 \oplus N_2 \oplus N_3$  such that  $\alpha$  acts on  $N_1$  and  $N_2$  by  $-1$ , and  $N_3$  by  $1$ , and  $\beta$  acts on  $N_1$  and  $N_3$  by  $-1$ , and  $N_2$  by  $1$ . By Equation (10), we know that  $\pi_\alpha(\beta x, \beta y) = -\pi_\alpha(x, y) = \beta \pi_\alpha(x, y)$ . This shows that  $\beta$  acts on  $\pi_\alpha(x, y)$  by  $-1$ . Similarly  $\alpha$  acts on the image of  $\pi_\beta$  by  $-1$ . This shows that  $\pi_\alpha \in N_3^* \otimes N_1 \otimes N_2$  and  $\pi_\beta \in N_2^* \otimes N_1 \otimes N_3$ . It is straightforward to check that  $\llbracket \pi_\alpha, \pi_\beta \rrbracket = 0$  in  $V^* \otimes \wedge^3 V$ .

If  $N^\alpha = N^\beta$ ,  $\beta$  acts on  $N^\alpha = N^\beta$  with determinant  $1$ . This shows that  $\pi_\alpha(\beta x, \beta y) = \pi_\alpha(x, y)$ . By Equation (10), we see that  $\pi_\alpha$  takes value in  $V^\beta$ , and similarly  $\pi_\beta$  takes value in  $V^\alpha$ . Direct computation shows that  $\llbracket \pi_\alpha, \pi_\beta \rrbracket = 0$ .  $\square$

**Theorem 3.4.** *Let  $\Gamma$  be a finite abelian group which acts faithfully on a finite dimensional vector space  $V$ . Then any linear Poisson structure  $\pi$  of  $S(V^*) \rtimes \Gamma$  is quantizable.*

*Proof.* Following the above lemmas,  $\llbracket \pi, \pi \rrbracket_\gamma$  reduces to :

$$\begin{aligned} \llbracket \pi, \pi \rrbracket_\gamma(x, y, z) &= \pi_0(\pi_\gamma(x, y), z + \gamma z) + \pi_0(\pi_\gamma(y, z), x + \gamma x) + \pi_0(\pi_\gamma(z, x), y + \gamma y) \\ &\quad + 2 \times (\pi_\gamma(\pi_0(x, y), z) + \pi_\gamma(\pi_0(y, z), x) + \pi_\gamma(\pi_0(z, x), y)). \end{aligned}$$

This expression is zero whenever  $l(\gamma) \neq 2$ . Suppose now that  $l(\gamma) = 2$ . If  $x, y, z$  are all in  $V^\gamma$ , we get  $0$  since  $V^\gamma$  is the kernel of  $\pi_\gamma$ . If two of  $x, y, z$  are in  $V^\gamma$ , we also get  $0$  for the same reason and because  $\pi_0$  is  $\gamma$ -invariant. Suppose now that  $x$  and  $y$  are in  $N^\gamma$  and that  $z$  is  $\gamma$ -invariant. Then,  $\llbracket \pi, \pi \rrbracket_\gamma(x, y, z)$  lies in  $V^\gamma$ , and it is zero from the fact that  $\pi$  is a Poisson bracket (Section 2.3):  $[\pi, \pi]_\gamma = 0$ .  $\square$

**3.3. An important example.** In order to prove the quantization theorem for general linear Poisson structures, we consider an important example in this subsection.

Let  $\rho = \exp(\frac{2\pi i}{2n+1})$ . Denote  $\alpha_k = \begin{pmatrix} \rho^k & 0 \\ 0 & \rho^{-k} \end{pmatrix}$ , and  $\beta_k = \begin{pmatrix} 0 & \rho^k \\ \rho^{-k} & 0 \end{pmatrix}$ , and  $\Gamma_n = \{\alpha_k, \beta_l : 0 \leq k, l \leq 2n\}$ .  $\Gamma_n$  is a finite group of order  $4n+2$  acting faithfully on  $V = \mathbb{C}^2$ , a complex 2-dim and real 4-dim vector space.  $\alpha_k$ 's eigenvalues are  $\rho^k$  and  $\rho^{-k}$ , and  $\beta_k$ 's eigenvalues are  $\pm 1$ . Let  $z_1, z_2$  be complex coordinate functions on  $V$ .

We consider linear Poisson structures on  $S(V^*) \rtimes \Gamma_n$ . We first look at the Poisson structure on the identity component. As  $\alpha_k$  acts on  $V$  diagonally,  $\alpha_k$  acts on  $V^* \otimes \wedge^2 V$  also diagonally with eigenvalues  $\rho^{3k}, \rho^k, \rho^{-k}$ , and  $\rho^{-3k}$ . If  $\rho^{3k} \neq 1$ , there is no non zero linear bivector field on  $V$ , which is  $\alpha_k$  invariant. Accordingly, if  $\rho^3 \neq 1$ , there is no  $\Gamma_n$ -invariant linear Poisson structure  $\pi_0$  on  $V$ . If  $\rho^3 = 1$ , then  $\pi_0$  is a linear combination of  $z_1 \bar{\partial}_1 \wedge \partial_2$ ,  $z_2 \partial_1 \wedge \bar{\partial}_2$ , and  $\bar{z}_1 \partial_1 \wedge \bar{\partial}_2$ ,  $\bar{z}_2 \bar{\partial}_1 \wedge \partial_1$ . If we assume that  $\pi_0$  to be real, then we have

$$\pi_0 = a z_1 \bar{\partial}_1 \wedge \partial_2 + \bar{a} \bar{z}_1 \partial_1 \wedge \bar{\partial}_2 + b z_2 \partial_1 \wedge \bar{\partial}_2 + \bar{b} \bar{z}_2 \bar{\partial}_1 \wedge \partial_2.$$

Furthermore by invariance respect the  $\beta_k$ 's action, we have  $a = -b$  and  $\bar{a} = -\bar{b}$  in the above equation, i.e.

$$\pi_0 = a (z_1 \bar{\partial}_1 \wedge \partial_2 - z_2 \partial_1 \wedge \bar{\partial}_2) + \bar{a} (\bar{z}_1 \partial_1 \wedge \bar{\partial}_2 - \bar{z}_2 \bar{\partial}_1 \wedge \partial_2).$$

Observe  $\beta_k$  has real codimension=2 fixed point subspace, while  $\alpha_k$  only fixes the origin of  $V$ .  $V^k := V^{\beta_k}$  is determined by  $\rho^k z_1 - z_2 = \rho^{-k} \bar{z}_1 - \bar{z}_2 = 0$ . The normal subspace  $N^k$  to  $V^k$  is determined by  $\rho^k z_1 + z_2 = \rho^{-k} \bar{z}_1 + \bar{z}_2 = 0$ . Vector fields along  $N^k$  are spanned by  $\rho^{-k} \partial_1 - \partial_2$  and  $\rho^k \bar{\partial}_1 - \bar{\partial}_2$ . Therefore, the Poisson structure at  $\beta_k$  component can be written as

$$\Pi_k = [c_k(\rho^k z_1 + z_2) - \bar{c}_k(\rho^{-k} \bar{z}_1 + \bar{z}_2)] (\rho^{-k} \partial_1 - \partial_2) \wedge (\rho^k \bar{\partial}_1 - \bar{\partial}_2).$$

Furthermore, as  $\alpha_l \beta_k \alpha_l^{-1} = \alpha_{k+2l}$ , by invariance of  $\Pi_k$  respect the conjugation action of  $\Gamma_k$ , we have that  $c_{2k} = c_0 \rho^{-k}$ ,  $0 \leq k \leq 2n$ . Therefore, we have

$$\begin{aligned} \Pi_{2k} &= [c_0(\rho^k z_1 + \rho^{-k} z_2) - \bar{c}_0(\rho^{-k} \bar{z}_1 + \rho^k \bar{z}_2)] \\ &\quad (\partial_1 \wedge \bar{\partial}_1 - \rho^{-2k} \partial_1 \wedge \bar{\partial}_2 - \rho^{2k} \partial_2 \wedge \bar{\partial}_1 + \partial_2 \wedge \bar{\partial}_2), \quad 0 \leq k \leq 2n. \end{aligned}$$

By the same reason as in the proof of Theorem 3.4, we conclude that if we assume  $[\pi_0, \Pi_k] = 0$ , then  $\llbracket \pi_0, \Pi_k \rrbracket + \llbracket \Pi_k, \pi_0 \rrbracket = 0$  for any  $k$ .

From Equation (5), we see that as  $\dim(N^\beta) = 2$  the  $\llbracket \pi_\alpha, \pi_\beta \rrbracket(x, y, z) = 0$  if  $x, y, z$  are all along the normal direction of  $N^\beta$ . Furthermore, as  $\pi_\beta$  is a multiple of the highest wedge power of the normal

direction  $N^\beta$ , to have non-zero outcome two of the three  $x, y, z$  have to be from the normal direction  $N^\beta$ . This implies that  $[[\pi_\alpha, \pi_\beta]]$  as an element in  $V^* \otimes \wedge^3 V$  is equal to the Schouten-Nijenhuis bracket  $[\pi_\alpha, \pi_\beta]$ .

We use this observation to compute  $[[\Pi_{2k}, \Pi_{2l}]]$ . A long but straightforward computation leads to the following result at  $\alpha_{2k-2l}$ ,

$$\begin{aligned} [[\Pi_{2k}, \Pi_{2l}]] &= [c_0(\rho^k z_1 + \rho^{-k} z_2) - \bar{c}_0(\rho^{-k} \bar{z}_1 + \rho^k \bar{z}_2)] \\ &\quad \times \left[ (2\bar{c}_0 \rho^l - \bar{c}_0 \rho^{-l+2k} - \bar{c}_0 \rho^{3l-2k}) \partial_1 \wedge \partial_2 \wedge \bar{\partial}_1 \right. \\ &\quad + (-2\bar{c}_0 \rho^{-l} + \bar{c}_0 \rho^{l-2k} + \bar{c}_0 \rho^{-3l+2k}) \partial_1 \wedge \partial_2 \wedge \bar{\partial}_2 \\ &\quad + (2c_0 \rho^{-l} - c_0 \rho^{l-2k} - c_0 \rho^{-3l+2k}) \bar{\partial}_1 \wedge \partial_1 \wedge \bar{\partial}_2 \\ &\quad \left. + (-2c_0 \rho^l + c_0 \rho^{-l+2k} + c_0 \rho^{3l-2k}) \bar{\partial}_1 \wedge \partial_2 \wedge \bar{\partial}_2 \right]. \end{aligned}$$

Define for  $0 \leq k \leq 2n$ ,

$$(11) \quad B_{2k} := (2n+1)(\rho^k - \rho^{-k}) [-|c_0|^2 \partial_2 \wedge \bar{\partial}_2 + |c_0|^2 \partial_1 \wedge \bar{\partial}_1 + (\bar{c}_0)^2 \partial_1 \wedge \partial_2 - c_0^2 \bar{\partial}_1 \wedge \bar{\partial}_2].$$

With a long but straightforward computation, we are able to prove

$$\sum_{p-q=2k, 0 \leq q \leq 2n} [[\Pi_{2p}, \Pi_{2q}]] = \partial^{\alpha_{2k}} B_{2k}.$$

And it is not difficult to compute that

$$(12) \quad [[B_{2k}, \Pi_{2l}]] = 0, \quad 0 \leq k, l \leq 2n;$$

and

$$[[B_{2k}, \pi_0]] = 0, \quad 0 \leq k \leq 2n, \quad n = 3.$$

Therefore, we conclude with the following proposition

**Proposition 3.5.** *For  $\Gamma_n$  action on  $V = \mathbb{C}^2$ , any linear Poisson structures on  $S(V^*) \rtimes \Gamma_n$  can be quantized.*

*Proof.* By the above computation, we see that the relation defining

$$\begin{aligned} H_\Pi &:= T(V) \rtimes \Gamma_n[[\hbar]] \\ &\left\langle x \otimes y - y \otimes x - \hbar \left( \pi_0(x, y) + \sum_{0 \leq k \leq 2n} \Pi_{2k}(x, y) \beta_{2k} \right) - \hbar^2 \left( \sum_{1 \leq k \leq 2n} B_{2k}(x, y) \alpha_{2k} \right) \right\rangle \end{aligned}$$

satisfies the Braverman-Gaitsory conditions (4)-(6). This implies that  $H_\Pi$  has PBW property, which shows that  $H_\Pi$  is a deformation quantization of  $S(V^*) \rtimes \Gamma_n$  along the direction defined by  $\Pi$ .  $\square$

**Remark 3.4.** *We point out that in the proof of Proposition 3.5, there have to be nonzero terms  $B_{2k}$  for  $0 \leq k \leq 2n$  as  $[\pi, \pi]$  is not zero. This is different from the standard PBW theorem for Lie algebras where  $B_{2k}$  can be chosen to be zero. We will see in the following subsection that this example is essentially the only case that  $B_{2k}$  has to be nonzero.*

**3.5. Quantization of linear Poisson structures-general case.** In this subsection  $\Gamma$  is a finite group (not necessary abelian) acting faithfully on a vector space  $V$ . We assume that  $V$  is equipped with a  $\Gamma$ -invariant complex structure. We prove the following theorem.

**Theorem 3.6.** *Let  $\Gamma$  be a finite group acting faithfully on a complex vector space  $V$ . Any real linear Poisson structures on  $S(V^*) \rtimes \Gamma$  is quantizable.*

The proof of this theorem consists of several steps. We start with recalling some results about finite subgroups of  $GL(2, \mathbb{C})$ .

**Lemma 3.7.** *A nonabelian finite subgroup  $G$  of  $SL(2, \mathbb{C})$  must contain the element  $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in SL(2, \mathbb{C})$ .*

*Proof.* We notice that the canonical action of  $G$  on  $\mathbb{C}^2$  is irreducible. Otherwise,  $G$  will be a subgroup of  $GL(1, \mathbb{C}) \times GL(1, \mathbb{C})$ , which is abelian. Therefore, according to [13][Chapter 6, Proposition 17], the order of the center of  $G$  is divisible by 2. Therefore, there is an element in  $G$  of order 2. As  $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$  is the unique element in  $SL(2, \mathbb{C})$  of order 2, we conclude that if  $G$  is not abelian,  $G$  contains  $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ .  $\square$

The following lemma is a corollary of [4][§26, Theorem 26].

**Lemma 3.8.** *Let  $\Gamma$  be a nonabelian finite subgroup of  $GL(2, \mathbb{C})$ . If  $\Gamma$  does not contain any matrix of the form  $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$  for  $a \neq 1$ , then there is a natural number  $n$  such that  $\Gamma$  is conjugate to the group  $\Gamma_n$  as is introduced in subsection 3.3.*

*Proof.* We start by considering the intersection  $G = \Gamma \cap SL(2, \mathbb{C})$ . If  $G$  is trivial, then there is an injective group homomorphism

$$\Gamma \rightarrow GL(2, \mathbb{C})/SL(2, \mathbb{C}) \cong \mathbb{C} - \{0\}.$$

This shows that  $\Gamma$  is abelian, which contradicts the assumption that  $\Gamma$  is not abelian. Therefore,  $G$  is a nontrivial subgroup of  $SL(2, \mathbb{C})$ . The following discussion is divided into two parts according to whether  $G$  is abelian.

- $G$  is not abelian. Then by Lemma 3.7,  $G$  contains the element  $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ , which is in the center of  $G$ . This contradicts to the assumption of this lemma.
- $G$  is abelian. By conjugation with an invertible matrix, we can assume that  $G$  contains a diagonal element like  $A = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$  with  $a \neq -1$ . (We remark that any element in  $\Gamma$  is diagonalizable as  $\Gamma$  is of finite order.) Furthermore, we recall the fact that if  $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$  commutes with  $A$ , then  $\beta = \gamma = 0$ . Hence, we have any element in  $G$  is of the form  $B = \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}$  for  $\beta \in \mathbb{C} - \{0\}$ . Therefore, we conclude that  $G$  is a cyclic subgroup of  $SL(2, \mathbb{R})$  isomorphic to  $\left\{ \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} : \rho^{2n+1} = 1 \right\}$  for some  $n \in \mathbb{N}$ . (If  $\rho^{2n} = 1$ , then  $\rho^n = -1$  and  $G$  contains the element  $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ .)

We observe that if  $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL(2, \mathbb{C})$  is a normalizer of  $G$ , then  $\alpha\beta = \gamma\delta = 0$ . Therefore,  $B = \begin{pmatrix} \alpha & 0 \\ 0 & \gamma \end{pmatrix}$  or  $\begin{pmatrix} 0 & \beta \\ \delta & 0 \end{pmatrix}$ . This shows that any element in  $\Gamma$  is either diagonal or of the form  $\begin{pmatrix} 0 & \beta \\ \delta & 0 \end{pmatrix}$ . As we have assumed that  $\Gamma$  is not an abelian group, there has to be a nonzero  $B$  in  $\Gamma$  of the form  $\begin{pmatrix} 0 & \beta \\ \delta & 0 \end{pmatrix}$ .

Compute  $B^2 = \begin{pmatrix} \beta\delta & 0 \\ 0 & \delta\beta \end{pmatrix}$ . By the assumption of  $\Gamma$ ,  $\beta\delta = 1$ . Therefore, we have  $B = \begin{pmatrix} 0 & a \\ a^{-1} & 0 \end{pmatrix} \in \Gamma$ . Now choose  $U = \begin{pmatrix} 0 & a^{\frac{1}{2}} \\ a^{-\frac{1}{2}} & 0 \end{pmatrix}$ , and consider the group  $\tilde{\Gamma} = U^{-1}\Gamma U$  which is again not abelian and does not contain any matrix of the form  $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ . Under this isomorphism, we see that  $\tilde{G} = G = \left\{ \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} : \rho^{2n+1} = 1 \right\}$  and  $\tilde{\Gamma}$  contains a matrix  $\beta_0 = U^{-1}BU = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \tilde{\Gamma}$ .

Now if there is any other element  $C$  in  $\tilde{\Gamma}$  of the form  $\begin{pmatrix} 0 & \beta' \\ \delta' & 0 \end{pmatrix}$  then by the same arguments as  $B$ , we know that  $\beta' = 1/\delta'$ . Furthermore, as  $\beta_0 B = \begin{pmatrix} \delta' & 0 \\ 0 & \delta'^{-1} \end{pmatrix} \in \tilde{G} = G$ . This implies that  $B = \begin{pmatrix} 0 & \rho \\ \rho^{-1} & 0 \end{pmatrix}$  with  $\rho^{2n+1} = 1$ .

Next if there is any element  $D = \begin{pmatrix} \alpha & 0 \\ 0 & \gamma \end{pmatrix}$  in  $\tilde{\Gamma}$ , compute  $\beta_0 D \beta_0 D = \begin{pmatrix} \alpha\gamma & 0 \\ 0 & \alpha\gamma \end{pmatrix}$ . As  $\tilde{\Gamma}$  has no element like  $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$  with  $a \neq 1$ , we conclude that  $\alpha = 1/\gamma$  and  $D$  belongs to  $G$ .

Summarizing the above analysis, we have seen that there exists  $n \in \mathbb{N}$ , such that  $\tilde{\Gamma} = \left\{ \begin{pmatrix} 0 & \rho^i \\ \rho^{-i} & 0 \end{pmatrix}, \begin{pmatrix} \rho^i & 0 \\ 0 & \rho^{-i} \end{pmatrix} : 0 \leq i \leq 2n, \rho^{2n+1} = 1 \right\}$ .

□

*Proof of theorem 3.6:*

With Theorem 3.4, it is sufficient to work with  $\Gamma$  which is nonabelian. We choose a  $\Gamma$ -invariant hermitian metric on  $V$  which always exists as  $\Gamma$  is finite. We prove that there exists a choice for  $B_\gamma$  such that the Braverman-Gaitsory conditions (4)-(6) are satisfied.

**Step I:** The Braverman-Gaitsory condition (4) is satisfied automatically by the assumption on  $\pi_\alpha$ .

**Step II:** In the following, we make a proper choice for  $B_\gamma$  with  $l(\gamma) = 4$  such that the Braverman-Gaitsory condition (5) is satisfied.

As  $\Gamma$  is acting on a complex vector space, the fixed subspace of any group element  $\gamma$  is of even real codimension. Let  $\pi_0$  be the linear Poisson structure at the identity component,  $\pi_\alpha$  be the linear Poisson structure at the  $\alpha$  component with  $l(\alpha) = 2$ .

We look at  $[[\pi_0, \pi_\alpha]]$ ,  $[[\pi_\alpha, \pi_0]]$ , and  $[[\pi_\alpha, \pi_\beta]]$  with  $l(\alpha) = l(\beta) = 2$ . By the same arguments as in the proof of Theorem 3.4, we have that for any  $\alpha$  with  $l(\alpha) = 2$ ,  $[[\pi_0, \pi_\alpha]] + [[\pi_\alpha, \pi_0]] = 0$  as  $\pi_0 + \sum_\alpha \pi_\alpha$  is a Poisson structure. Therefore, we are reduced to look at  $[[\pi_\alpha, \pi_\beta]]$ .

We observe that if  $V^\alpha = V^\beta$ , then at each point of  $V^\alpha = V^\beta$ , then  $\wedge^2 N^\alpha = \wedge^2 N^\beta$  vanishes on functions depending only on variables in  $V^\alpha = V^\beta$ . It is easy to compute that  $[[\pi_\alpha, \pi_\beta]] = 0$ . This reduces us to the situation that  $V^\alpha \neq V^\beta$ .

When  $V^\alpha \neq V^\beta$ , we have the equation  $V^\alpha + V^\beta = V$  as both  $V^\alpha$  and  $V^\beta$  are complex subspaces of  $V$  of complex codimension 1. Furthermore  $V^\alpha + V^\beta = V$  implies that  $V^{\alpha\beta} = V^\alpha \cap V^\beta$  which is of complex codimension 2 and real codimension 4. We notice that in this case both  $\alpha$  and  $\beta$  fix every point in  $V^{\alpha\beta}$ , and also preserve the normal direction  $N^{\alpha\beta}$ . We are interested in is  $[[\pi_\alpha, \pi_\beta]]$ , which is now at  $\alpha\beta$  component. As  $[[\pi_\alpha, \pi_\beta]]$  is only a tri-vector field supported at  $\alpha$  with  $l(\alpha\beta) = 4$ , we know that  $[[\pi_\alpha, \pi_\beta]]$  is 0 in the Hochschild cohomology of  $S(V) \rtimes \Gamma$ .

Now we fix an element  $\gamma$  with  $l(\gamma) = 4$ , then we know from the previous paragraph that if  $[[\pi_\alpha, \pi_\beta]]$  is nonzero at  $\gamma = \alpha\beta$ , then  $\alpha$  and  $\beta$  acts on  $V$  preserving  $N^\gamma$  and fixing every element in  $V^\gamma$ . This leads us to look at the subgroup  $\Gamma_\gamma$  of  $\Gamma$  whose elements act trivially on  $V^\gamma$ .  $\Gamma_\gamma$  contains all  $\alpha$  such that  $V^\gamma \subset V^\alpha$ , which acts on  $N^\gamma$  faithfully.

Let  $\alpha \in \Gamma_\gamma$  with  $l(\alpha) = 2$ . By the assumption on  $\pi_\alpha$ , it is an element in  $V^{\alpha*} \otimes \wedge^2 N^\alpha$ , which can be written as a sum of two terms  $\pi_\alpha^1 + \pi_\alpha^2$  as  $V^{\alpha*} = V^{\gamma*} \oplus (N^{\gamma*} \cap V^{\alpha*})$ . We easily see that  $\pi_\alpha^1$  will not contribute to  $[[\pi_\alpha, \pi_\beta]]$  as  $N^\alpha$  and  $N^\beta$  are orthogonal to  $V^\gamma$ . This shows that to study  $[[\pi_\alpha, \pi_\beta]]$ , it is enough to assume that  $\pi_\alpha$  is taking value in  $N^{\gamma*} \otimes \wedge^2 N^\alpha$ . Furthermore, if  $n_\alpha$  is the holomorphic vector along  $N^\alpha$  and  $v_\alpha$  is the holomorphic vector along  $V^{\alpha*}$  in  $N^\gamma$ , then we can write  $\pi_\alpha = (c_\alpha v_\alpha - \bar{c}_\alpha \bar{v}_\alpha) n_\alpha \wedge \bar{n}_\alpha$  for some complex number  $c_\alpha$  by the fact that  $\pi_\alpha$  is real.

If  $\Gamma_\gamma$  is abelian, then by Theorem 3.4, we know that  $[[\pi_\alpha, \pi_\beta]] = 0$  for any  $\alpha, \beta$ , and therefore we set  $B_\delta = 0$  for  $\delta \in \Gamma_\gamma$ . In the following, we assume that  $\Gamma_\gamma$  is not abelian. If  $\Gamma_\gamma$  contains an element  $\nu$  which acts on  $N^\gamma$  of the form  $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$  with a unitary number  $a \neq 1$ , then it is easy to check

$\nu_*(\pi_\alpha) = (a^{-1}c_\alpha v_\alpha - \bar{a}^{-1}\bar{c}_\alpha \bar{v}_\alpha)n_\alpha \wedge \bar{n}_\alpha$  is not invariant under  $\nu$  unless  $c_\alpha = 0$ . This shows that  $\pi_\alpha$  has to be zero if it is invariant under  $\nu$  and therefore  $[[\pi_\alpha, \pi_\beta]] = 0$  in this case. We choose  $B_\gamma = 0$  for this type of  $\gamma$ . Therefore, for nonzero  $B_\gamma$ , we only need to consider the situation that  $\Gamma_\gamma$  is not abelian and contains no element of the form  $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$  with  $a \neq 1$ . By Lemma 3.8,  $\Gamma_\gamma$  action on  $N^\gamma$  is isomorphic to the situation studied in the previous subsection. And we can choose  $B_\gamma$  as Equation (11).

**Step III:** we prove that  $\sum_{\alpha, \gamma, l(\alpha) \leq 2, l(\gamma) = 4} [[B_\gamma, \pi_\alpha]] = 0$  with the choices of  $B_\gamma$  introduced in Step II.

We decompose the above sum into 2 parts

- (1)  $\sum_{\alpha, \gamma, l(\alpha) \leq 2, l(\gamma) = 4, l(\gamma\alpha) \geq 4} [[B_\gamma, \pi_\alpha]]$ ;
- (2)  $\sum_{\alpha, \gamma, l(\alpha) \leq 2, l(\gamma) = 4, l(\gamma\alpha) = 2} [[B_\gamma, \pi_\alpha]]$ .

We consider the part of sum with  $l(\gamma\alpha) \geq 4$ . Let  $\delta = \gamma\alpha$ . Define  $C_\delta = \sum_{\gamma\alpha = \delta, l(\alpha) \leq 2, l(\gamma) = 4} [[B_\gamma, \pi_\alpha]]$ , which is an element in  $\wedge^3 V$ . We notice that  $C_\delta$  is  $\partial^\delta$  closed as

$$\begin{aligned} \partial\left(\sum_{\alpha, \gamma} [[B_\gamma, \pi_\alpha]]\right) &= \sum_{\alpha, \gamma} [\partial B_\gamma, \pi_\alpha] \\ &= \sum_{\alpha, \gamma = \beta\lambda} [[[\pi_\beta, \pi_\lambda], \pi_\alpha]] \\ &= 0. \end{aligned}$$

According to Equation (3), we see that  $C_\gamma$  in  $\wedge^3 V$  is a zero cocycle as  $l(\gamma) \geq 4$ . On the other hand, we see that  $C_\gamma$  is in  $\wedge^3 V$ . If we define elements in  $V$  of degree -1 and elements in  $V^*$  of degree 1, then  $C_\gamma$  is of degree -3, and  $\partial^\gamma$  is of degree 0. We see that  $S(V^*) \otimes \wedge^2 V$  has degree greater or equal to -2. Therefore,  $\partial^\gamma(S(V^*) \otimes \wedge^2 V)$  has no term with degree less than -2 since  $\deg(\partial^\gamma) = 0$ . This shows that if  $C_\gamma \neq 0$ , it cannot be a coboundary of  $\partial^\gamma$ . This shows that  $C_\gamma$  has to be zero as  $C_\gamma$  is a trivial cocycle by Equation (3).

We consider the part of sum with  $l(\gamma\alpha) \leq 2$ . This implies that  $V^\gamma \subset V^\alpha$  because otherwise  $V^\gamma + V^\alpha = V$  and  $V^{\gamma\alpha} = V^\gamma \cap V^\alpha$  which is of real codimension 6. In this case, we know that  $\gamma\alpha$  is also in  $\Gamma_\gamma$ . Therefore, we can use Equation (12) to conclude that  $[[B_\gamma, \pi_\alpha]] = 0$ .

In conclusion, Steps I-III show that Braverman-Gaiitsgory conditions (4)-(6) are satisfied for any linear Poisson structures on  $S(V^*) \rtimes \Gamma$  and a proper choice of  $B_\gamma$ . Therefore, by PBW property, we see that the algebra

$$\begin{aligned} H_{\Pi} &:= T(V) \rtimes \Gamma[[\hbar]] \\ &\left\langle x \otimes y - y \otimes x - \hbar(\pi_0(x, y) + \sum_{\alpha} \pi_\alpha(x, y)\alpha) - \hbar^2\left(\sum_{\gamma} B_\gamma(x, y)\gamma\right) \right\rangle \end{aligned}$$

defines a deformation quantization of  $S(V^*) \rtimes \Gamma$  along the direction of  $\pi_0 + \sum_{\alpha} \pi_\alpha$ .  $\square$

#### 4. HOCHSCHILD COHOMOLOGY AND NON COMMUTATIVE POISSON COHOMOLOGY

In this section, we would like to study various properties and examples of the Poisson structures and algebras we constructed in the previous section. To state our results, we fix some convention. Note that if  $\alpha$  and  $\beta$  are conjugate to each other inside  $\Gamma$ , then  $l(\alpha) = l(\beta)$ . Therefore, it is legitimate to define **codimension** of a conjugacy class of  $\Gamma$  by the codimension of an element  $\alpha \in \Gamma$ . Define  $c_k$  to be the number of conjugacy class of  $\Gamma$  with codimension  $k$ . In the following, we use  $\mathbb{R}((\hbar))$  to stand for the algebra of Laurent Polynomials of  $\hbar$ . For any algebra  $A$  over the ring  $\mathbb{R}[[\hbar]]$ , we use  $A((\hbar))$  to stand for the extension  $A \otimes_{\mathbb{R}[[\hbar]]} \mathbb{R}((\hbar))$ .

**Proposition 4.1.** *The periodic cyclic homology of the algebra  $H_\pi$  for any constant or linear Poisson structures on  $S(V^*) \rtimes \Gamma$  is equal to*

$$\begin{aligned} HP_0(H_\pi((\hbar))) &= \sum_k \mathbb{R}((\hbar))^{\times c_k} = \mathbb{R}((\hbar))^{\times |C(\Gamma)|} \\ HP_1(H_\pi((\hbar))) &= 0. \end{aligned}$$

*Proof.* By Getzler-Goodwillie [6], we know that as the periodic cyclic homology is invariant under deformation,  $HP_\bullet(H_\pi((\hbar)))$  is equal to  $HP_\bullet(S(V^*) \rtimes C(\Gamma))((\hbar))$ . By the computation in [3],  $HP_\bullet(S(V^*) \rtimes \Gamma)$  is equal as is stated above.  $\square$

**4.1. Hochschild cohomology of symplectic reflection algebra.** In this subsection, we restrict ourselves to quantization of a special type of constant Poisson structures. These are called symplectic reflection algebras. Let  $V$  be a symplectic vector space with standard symplectic 2-form  $\omega$ . Let  $\pi$  be the associated Poisson structure of  $\omega$ . For  $\gamma$  with  $l(\gamma) = 2$ , consider the restriction  $\omega_\gamma$  of  $\omega$  to  $N^\gamma$ .  $\omega_\gamma$  is an invertible bilinear operation on  $N^\gamma$ . Define  $\pi_\gamma$  be the inverse of  $\omega_\gamma$ . We choose  $c_\gamma \in \mathbb{R}$  for all  $\gamma$  with  $l(\gamma) = 2$  with  $c_{\alpha\gamma\alpha^{-1}} = c_\gamma$  for all  $\alpha, \gamma \in \Gamma$ . Define a constant Poisson structure  $\Pi$  on  $S(V^*) \rtimes \Gamma$  by

$$\Pi = \pi + \sum_{\gamma, l(\gamma)=2} c_\gamma \pi_\gamma.$$

By Thm 3.1,  $S(V^*) \rtimes \Gamma$  has a deformation quantization with respect to  $\Pi$ . This algebra can be written as

$$H_{\omega, c} = T(V^*) \otimes \Gamma[[\hbar]] / \langle x \otimes y - y \otimes x - \hbar(\pi(x, y) + \sum_{\gamma, l(\gamma)=2} c_\gamma \pi_\gamma(x, y)\gamma) \rangle.$$

This algebra is called symplectic reflection algebra by Etingof and Ginzburg [5].

To compute the Hochschild cohomology of  $H_{\omega, c}$ , we compute the Poisson cohomology of  $\Pi$  first. We recall that the Poisson cohomology of a Poisson structure  $\Pi$  on an algebra  $A$  is defined to be the cohomology of the complex  $H^\bullet(A, A)$  with the differential  $\partial(a) = [\Pi, a]$ , where  $[\ , \ ]$  is the Gerstenhaber bracket on  $H^\bullet(A, A)$ .

**Proposition 4.2.** *The Poisson cohomology  $H_\Pi^\bullet(S(V^*) \rtimes \Gamma)$  is equal to*

$$H_\Pi^\bullet(S(V^*) \rtimes \Gamma) = \mathbb{R}^{\times c_\bullet}.$$

*Proof.* We introduce a grading on

$$H^\bullet(S(V^*) \rtimes \Gamma, S(V^*) \rtimes \Gamma) = \left( \bigoplus_{\gamma \in \Gamma} S(V^{\gamma^*}) \otimes \wedge^{\bullet - l(\gamma)} V^\gamma \otimes \wedge^{l(\gamma)} N^\gamma \right)^\Gamma$$

by setting elements in  $S(V^{\gamma^*}) \otimes \wedge^{\bullet - l(\gamma)} V^\gamma \otimes \wedge^{l(\gamma)} N^\gamma$  of degree  $l(\gamma)$ .

By Poisson cohomology with respect to  $\Pi$ , we mean the cohomology on  $H^\bullet(S(V^*) \rtimes \Gamma, S(V^*) \rtimes \Gamma)$  with the differential  $d_\Pi$  defined by taking the generalized Schouten-Nijenhuis bracket  $d_\Pi f = [\Pi, f]$  for  $f \in H^\bullet(S(V^*) \rtimes \Gamma, S(V^*) \rtimes \Gamma)$ . According to [8][Thm. 3.4], the Poisson differential  $d_\Pi$  is compatible with the above filtration respect  $l(\gamma)$  as if  $\deg(f) = i$ ,  $(d_\Pi f)_\gamma = 0$  if  $l(\gamma) \neq i$  or  $i + 2$ . Therefore, we can use spectral sequence associated to the filtration defined by the grading  $l$  to compute the Poisson cohomology  $H_\Pi$ .

The  $E_0$  of the spectral sequence associated to the above filtration is the Poisson cohomology respect the Poisson structure  $\pi$  which is the component of  $\Pi$  supported at identity of the group  $\Gamma$  on the graded complex  $\text{Gr}(H^\bullet(S(V^*) \rtimes \Gamma, S(V^*) \rtimes \Gamma))$ , i.e.

$$\text{Gr}^p(H^\bullet(S(V^*) \rtimes \Gamma, S(V^*) \rtimes \Gamma)) = \left( \bigoplus_{\gamma, l(\gamma)=p} S(V^{\gamma^*}) \otimes \wedge^{\bullet - l(\gamma)} V^\gamma \otimes \wedge^{l(\gamma)} N^\gamma \right)^\Gamma.$$

As  $\omega$  is a symplectic form, the Poisson cohomology of  $\pi$  can be computed easily

$$E_1^{p, q} = \begin{cases} H_\pi^p(S(V^*) \rtimes \Gamma) = \mathbb{R}^{\times c_p}, & q = 0, \\ \{0\}, & q \neq 0. \end{cases}$$

We notice that the codimension of any group element  $\gamma$  is even because  $\gamma$  preserves the symplectic structure. Therefore,  $E_1^{p,q} = 0$  if and only if  $p + q$  is odd. This implies that the spectral sequence degenerates at  $E_1$ . Therefore we have

$$H_{\Pi}^{\bullet}(S(V^*) \rtimes \Gamma) = H_{\pi}^{\bullet}(S(V^*) \rtimes \Gamma) = \mathbb{R}^{\times c \bullet}.$$

□

**Theorem 4.3.**

$$H^{\bullet}(H_{\omega,c}((\hbar)), H_{\omega,c}((\hbar))) = \mathbb{R}((\hbar))^{\times c \bullet}$$

*Proof.* The proof of this theorem is by applying the spectral sequence with respect to the  $\hbar$ -filtration. The  $E_0$  terms are the Poisson cohomology. We notice that  $H_{\Pi}^{\bullet}(S(V^*) \rtimes \Gamma)$  is trivial when  $\bullet$  is odd, and conclude that the spectral sequence degenerated at  $E_1$ . Therefore, we have that the Hochschild (co)homology equals the Poisson (co)homology. □

**Remark 4.2.** *Theorem 4.3 is a generalization of [5][Theorem 1.8 (i)]. Here, with more information about the generalized Schouten-Nijenhuis bracket, we are able to avoid the assumption in [5] on “except possibly a countable set”.*

In Thm 4.3, we see that for a constant Poisson structure  $\Pi$ , if the Poisson structure at the identity component  $\pi_0$  is the inverse of a symplectic structure, then its Poisson cohomology and Hochschild cohomology are determined by  $\pi_0$  completely.

In the following we show one example that if the Poisson structure  $\pi_0$  is degenerated, then the Poisson cohomology of  $\Pi$  depends also on the information of  $\Pi$  at other conjugacy classes.

Consider  $(V = \mathbb{R}^2, \omega = dx \wedge dy)$ , and  $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z} = \{1, e\}$  acts on  $\mathbb{R}^2$  by  $e(x, y) = (-x, -y)$ . Denote  $\pi = -\partial_x \wedge \partial_y$ . Observe that 0 is the only fixed point of  $e$ . For any constant  $c$ , consider  $\Pi = c\pi|_0$  which defines a constant Poisson structure on  $S(V^*) \rtimes \mathbb{Z}_2$ . In the following, we show that the Poisson cohomology of  $\Pi$  does distinguish  $\Pi$  from the trivial Poisson structure and also those Poisson structures in Proposition 4.2.

**Proposition 4.4.**

$$\begin{aligned} H_{\Pi}^0 &= (S(V^*))^{\mathbb{Z}_2} \\ H_{\Pi}^1 &= \{f_1 \partial_x + f_2 \partial_y \in (S(V^*) \otimes V)^{\mathbb{Z}_2} : \partial_x f_1(0) + \partial_y f_2(0) = 0\} \\ H_{\Pi}^2 &= (S(V^*) \otimes \wedge^2 V)^{\mathbb{Z}_2}. \end{aligned}$$

*Proof.* The computation for  $H^0$  are trivial as there is only degree=2 elements supported at  $e$  by (3).

For  $H^1$ , we first observe that the image of  $d^{\Pi}$  on  $(S(V^*))^{\mathbb{Z}_2}$  is trivial by (3). For the kernel of  $d^{\Pi}$ , we compute

$$[f_1 \partial_x + f_2 \partial_y, c\pi|_0] = c(\partial_x(f_1) + \partial_y(f_2))\partial_x \wedge \partial_y|_0.$$

Therefore,  $H_{\Pi}^1 = \{f_1 \partial_x + f_2 \partial_y \in (S(V^*) \otimes V)^{\mathbb{Z}_2} : \partial_x f_1(0) + \partial_y f_2(0) = 0\}$ .

For  $H_{\Pi}^2$ , we notice that  $d^{\Pi c}$  vanishes as there is no higher degree terms. For the image of  $d^{\Pi}$ , from the previous computation, we see that  $Im(d^{\Pi}) = \mathbb{R}\partial_x \wedge \partial_y|_0$ . Therefore we conclude from (3) that  $H_{\Pi c}^2 = (S(V^*) \otimes \wedge^2 V)^{\mathbb{Z}_2}$ . □

**4.3. Hochschild cohomology of  $\mathcal{U}(\mathfrak{g}) \rtimes \Gamma$ .** Let  $V = \mathfrak{g}$  be a Lie algebra such that its bracket is  $\Gamma$  invariant. The Lie bracket on  $\mathfrak{g}$  defines a Poisson structure on  $V^*$ , which also defines a Poisson structure on  $S(V) \rtimes \Gamma$ . Then,  $\mathcal{U}(\mathfrak{g}) \rtimes \Gamma$  is a quantization of  $\pi$  on  $S(V) \rtimes \Gamma$ . We notice that the Poisson bracket  $\pi$  does not have any other  $\gamma$ -component for  $\gamma$  different from the unity, i.e.  $\pi = \pi_0$ .

Now,  $\mathcal{U}(\mathfrak{g}) \rtimes \Gamma$  is a filtered Koszul algebra over  $\mathbb{R}\Gamma$ . Therefore, it has a Koszul resolution and a small complex which calculates its Hochschild cohomology. This complex splits into a direct sum of subcomplexes:

$$(CK^{\bullet}(\mathcal{U}(\mathfrak{g}) \rtimes \Gamma), \partial) = \bigoplus_{\gamma \in C(\Gamma)} (CK_{\gamma}^{\bullet}, \partial_{\gamma})$$

where  $CK_\gamma^\bullet = (\mathcal{U}(\mathfrak{g}) \otimes \wedge^\bullet \mathfrak{g}^*)^{Z(\gamma)}$  and :

$$\begin{aligned} \partial_\gamma(f)(x_0, \dots, x_n) &= \sum_{i=0}^n (-1)^i f(x_0, \dots, \widehat{x}_i \dots, x_n) (x_i -^\gamma x_i) \\ &+ \sum_{i=0}^n (-1)^i [x_i, f(x_0, \dots, \widehat{x}_i \dots, x_n)] + \sum_{i < j} (-1)^{j-i-1} f(x_0, \dots, x_{i-1}, [x_i, x_j], x_{i+1}, \dots, \widehat{x}_j \dots, x_n). \end{aligned}$$

The brackets in the above formula stand for the Lie bracket of  $\mathfrak{g}$  and for the action of  $\mathfrak{g}$  on  $\mathcal{U}(\mathfrak{g})$ . Notice that the PBW property of  $\mathcal{U}(\mathfrak{g})$  implies that the symmetrization map from  $S\mathfrak{g}$  to  $\mathcal{U}(\mathfrak{g})$  is an isomorphism of  $\mathfrak{g}$ -modules as well as  $\Gamma$ -modules.

According to 2.3, the Poisson complex splits as well in a direct sum of subcomplexes

$$(C_\pi^\bullet(\mathcal{U}(\mathfrak{g}) \rtimes \Gamma), \partial^\pi) = \bigoplus_{\gamma \in C(\Gamma)} (C_\gamma^\bullet, \partial_\gamma^\pi),$$

where  $C_\gamma^\bullet = (S(\mathfrak{g}^\gamma) \otimes \wedge^{\bullet-l(\gamma)} \mathfrak{g}^{\gamma*} \otimes \wedge^{l(\gamma)} N^{\gamma*})^{Z(\gamma)}$  with the differential :

$$\begin{aligned} \partial_\gamma^\pi(f)(x_0, \dots, x_{n-l(\gamma)}, y_1, \dots, y_{l(\gamma)}) &= \\ &\sum_{0 \leq i \leq n-l(\gamma)} (-1)^i [x_i, f(x_0, \dots, \widehat{x}_i \dots, x_{n-l(\gamma)}, y_1, \dots, y_{l(\gamma)})] \\ &+ \sum_{i < j \leq n-l(\gamma)} (-1)^{j-i-1} f(x_0, \dots, x_{i-1}, [x_i, x_j], x_{i+1}, \dots, \widehat{x}_j \dots, x_{n-l(\gamma)}, y_1, \dots, y_{l(\gamma)}) \\ &+ \sum_{i \leq n-l(\gamma), j} (-1)^{n-l(\gamma)-i+j-1} f(x_0, \dots, \widehat{x}_i \dots, x_{n-l(\gamma)}, y_1, \dots, [x_i, y_j], \dots, y_{l(\gamma)}) \end{aligned}$$

where the  $x$ -variables belong to  $\mathfrak{g}^\gamma$  and the  $y$ 's belong to  $N^\gamma$ . Notice that, as the bracket of  $\mathfrak{g}$  is  $\Gamma$ -invariant,  $N^\gamma$  is a  $\mathfrak{g}^\gamma$ -submodule of  $\mathfrak{g}$ . This defines the last summand of  $\partial_\gamma^\pi$ .

We define a map  $\psi$  from the Poisson complex  $C_\gamma^\bullet$  to the Koszul complex  $CK_\gamma^\bullet$ . For any  $f$  in  $C_\gamma^n$ , we put :

$$\psi(f) := \Lambda^n \mathfrak{g} \longrightarrow \Lambda^{n-l(\gamma)} \mathfrak{g}^\gamma \otimes \wedge^{l(\gamma)} N^\gamma \xrightarrow{f} S\mathfrak{g}^\gamma \xrightarrow{Sym} \mathcal{U}(\mathfrak{g}^\gamma) \longrightarrow \mathcal{U}(\mathfrak{g})$$

where the first and last map are the usual projection and injection, and  $Sym$  is the classical symmetrization map from  $S\mathfrak{g}^\gamma$  to  $\mathcal{U}(\mathfrak{g}^\gamma)$ .

**Lemma 4.5.** *The map  $\psi^*$  is a morphism of complexes.*

*Proof.* We have to check that  $\psi$  commutes with the differentials. For this purpose, let us decompose  $\partial_\gamma$  into a sum of three terms,  $\partial_\gamma^1 + \partial_\gamma^2 + \partial_\gamma^3$  corresponding to the three components of its definition. We also use the following decomposition of  $\Lambda^{n+1} \mathfrak{g}$  coming from the direct sum  $\mathfrak{g} = \mathfrak{g}^\gamma \oplus N^\gamma$  :

$$\Lambda^{n+1} \mathfrak{g} = \bigoplus_{p=0}^{l(\gamma)} \Lambda^{n+1-p} \mathfrak{g}^\gamma \otimes \Lambda^p N^\gamma$$

First, as  $\psi(f)$  needs  $l(\gamma)$  independent variables in  $N^\gamma$ , it follows easily that  $\partial_\gamma^1(\psi(f)) = 0$ . For the same reason, we check that  $\partial(\psi(f))$  is null on the  $\Lambda^{n+1-p} \mathfrak{g}^\gamma \otimes \Lambda^p N^\gamma$  whenever  $p < l(\gamma)$ . Therefore,  $\partial(\psi(f))$  reduces to a map from  $\Lambda^{n+1-l(\gamma)} \mathfrak{g}^\gamma \otimes \wedge^{l(\gamma)} N^\gamma$  to  $\mathcal{U}(\mathfrak{g}^\gamma)$ .

To complete the proof, we check that the two formulas of the differentials agree on  $\Lambda^{n+1-l(\gamma)} \mathfrak{g}^\gamma \otimes \wedge^{l(\gamma)} N^\gamma$  thanks to the fact that the symmetrization map  $Sym$  is a  $\mathfrak{g}$ -morphism.  $\square$

**Theorem 4.6.** *The map  $\psi$  is a quasi-isomorphism. Therefore the Hochschild cohomology of  $\mathcal{U}(\mathfrak{g}) \rtimes \Gamma$  is isomorphic, as a graded vector space, to the Poisson cohomology of  $S(\mathfrak{g}) \rtimes \Gamma$  with the Poisson bracket induced by the Lie bracket of  $\mathfrak{g}$  :*

$$H^\bullet(\mathcal{U}(\mathfrak{g}) \rtimes \Gamma, \mathcal{U}(\mathfrak{g}) \rtimes \Gamma) \simeq H_\pi^\bullet(S(\mathfrak{g}) \rtimes \Gamma).$$

*Proof.* Let us introduce a formal parameter  $\hbar$  and work over the formal power series  $\mathbb{R}[[\hbar]]$ . Introduce the  $\mathbb{R}[[\hbar]]$ -Lie algebra  $\mathfrak{g}_\hbar$  whose underlying  $\mathbb{R}[[\hbar]]$ -module is  $\mathfrak{g}[[\hbar]]$  with the Lie bracket given by :

$$[x, y]_\hbar = \hbar[x, y],$$

for  $x$  and  $y$  in  $\mathfrak{g}$ . Then, the Koszul complex  $CK^\bullet(\mathcal{U}(\mathfrak{g}_\hbar) \rtimes \Gamma)$  specializes to that of  $S(\mathfrak{g}) \rtimes \Gamma$  for  $\hbar = 0$ , and, for  $\hbar = 1$  to that of  $\mathcal{U}(\mathfrak{g}) \rtimes \Gamma$ .

Similarly, the Poisson complex  $C_\pi^\bullet(\mathcal{U}(\mathfrak{g}_\hbar) \rtimes \Gamma)$  has a zero differential for  $\hbar = 0$  and specializes to  $C_\pi^\bullet(\mathcal{U}(\mathfrak{g}) \rtimes \Gamma)$  for  $\hbar = 1$ .

The map  $\psi$  extends to the  $\mathbb{R}[[\hbar]]$ -context, and defines a morphism of  $\mathbb{R}[[\hbar]]$ -complexes. It follows from 2.1 that  $\psi$  specializes to a quasi-isomorphism for  $\hbar = 0$ . The result is then a consequence of the following standard lemma, which can be found in [7].  $\square$

**Lemma 4.7.** *Let  $C_1^\bullet[[\hbar]]$  and  $C_2^\bullet[[\hbar]]$  be two topologically free  $\mathbb{R}[[\hbar]]$ -complexes, and  $\psi$  a morphism of  $\mathbb{R}[[\hbar]]$ -complexes. Suppose that  $\psi$  specializes to a quasi-isomorphism for  $\hbar = 0$ . Then  $\psi$  is a quasi-isomorphism.*

We remark that in this paper we have always worked with real coefficient  $\mathbb{R}$ . Theorem 4.6 is true for a general field  $\mathbb{K}$  with characteristic 0.

**4.4. Examples of linear Poisson structures.** In this section, we provide a large class of linear Poisson structures coming from invariant Lie algebra structures.

We assume that  $\mathfrak{g}$  be a Lie algebra and  $\Gamma$  be a finite group acting on  $\mathfrak{g}$  preserving its Lie bracket. We choose a  $\Gamma$  invariant metric on  $\mathfrak{g}$ . Let  $V$  be the dual of  $\mathfrak{g}$  with the linear Poisson structure  $\pi$  from the Lie bracket. Accordingly,  $\Gamma$  acts on  $V$  preserving the Poisson structure  $\pi$  with  $l(\gamma)$  even for all  $\gamma$ .

For any  $\gamma \in \Gamma$  with  $l(\gamma) = 2$ , let  $N^\gamma$  be the subspace of  $V$  normal to  $V^\gamma$ . As  $V = V^\gamma \oplus N^\gamma$  and  $V^* = V^{\gamma*} \oplus N^{\gamma*}$ . One can decompose  $\pi = V^* \otimes \wedge^2 V = (V^{\gamma*} \oplus N^{\gamma*}) \otimes (N^\gamma \wedge N^\gamma \oplus N^\gamma \otimes V^\gamma \oplus V^\gamma \otimes N^\gamma \oplus V^\gamma \wedge V^\gamma)$ . We define  $\pi_\gamma$  to be the projection of  $\pi$  onto the component  $(V^\gamma)^* \otimes \wedge^2 N^\gamma$ .

**Proposition 4.8.** *The collection  $\Pi = \pi + \sum_{\gamma \in l(\gamma)=2} c_\gamma \pi_\gamma$  with constant  $c_\gamma$  satisfying  $c_{\alpha\gamma\alpha^{-1}} = c_\gamma$  for any  $\alpha \in \Gamma$  defines a linear Poisson structure on  $S(V^*) \rtimes \Gamma$ .*

*Proof.* As is explained in subsection 2.1, there is no  $Z(\gamma)$  invariant section in  $S(V^{\gamma*}) \otimes \wedge^\bullet V^\gamma \otimes \wedge^{l(\gamma)} N^\gamma$  with odd  $l(\gamma)$ . Furthermore, by (3), any tri-vector field on a  $\gamma$ -component with  $l(\gamma) = 4$  is a trivial cocycle. This implies that

$$[\pi_\alpha, \pi_\beta] = 0,$$

if neither  $\alpha$  nor  $\beta$  is the identity of  $\Gamma$ .

To prove that  $\Pi$  is a Poisson structure, it is sufficient to prove

$$[\pi, \pi_\gamma] = 0, \quad \gamma \in \Gamma.$$

When  $\gamma$  is identity, the above equation is from the fact that  $\pi$  is the Lie Poisson structure.

When  $\gamma \neq 0$ , we can decompose  $\pi$  according to the decomposition  $V = V^\gamma \oplus N^\gamma$ ,  $V^* = V^{\gamma*} \oplus N^{\gamma*}$ . As  $\pi \in V^* \otimes V \wedge V$  is  $\gamma$  invariant, we have that  $\pi$  is a sum of the following terms

$$\begin{aligned} \pi_{111} &\in V^{\gamma*} \otimes V^\gamma \wedge V^\gamma, & \pi_{122} &\in V^{\gamma*} \otimes N^\gamma \wedge N^\gamma, \\ \pi_{221} &\in N^{\gamma*} \otimes N^\gamma \wedge V^\gamma, & \pi_{222} &\in N^{\gamma*} \otimes N^\gamma \wedge N^\gamma. \end{aligned}$$

We compute

$$\begin{aligned} [\pi_{111}, \pi_{111}] &\in V^{\gamma*} \otimes V^\gamma \wedge V^\gamma \wedge V^\gamma, & [\pi_{111}, \pi_{122}] &\in V^{\gamma*} \otimes V^\gamma \wedge N^\gamma \wedge N^\gamma \\ [\pi_{111}, \pi_{221}] &\in N^{\gamma*} \otimes N^\gamma \wedge V^\gamma \wedge V^\gamma, & [\pi_{111}, \pi_{222}] &= 0 \\ [\pi_{122}, \pi_{122}] &= 0, & [\pi_{122}, \pi_{222}] &\in V^{\gamma*} \otimes N^\gamma \wedge N^\gamma \wedge N^\gamma \\ [\pi_{122}, \pi_{221}] &\in (V^\gamma)^* \otimes N^\gamma \wedge N^\gamma \wedge V^\gamma \oplus N^{\gamma*} \otimes N^\gamma \wedge N^\gamma \wedge N^\gamma, \\ [\pi_{221}, \pi_{221}] &\in N^{\gamma*} \otimes N^\gamma \wedge V^\gamma \wedge V^\gamma, & [\pi_{221}, \pi_{222}] &\in N^{\gamma*} \otimes N^\gamma \wedge N^\gamma \wedge V^\gamma \\ [\pi_{222}, \pi_{222}] &\in N^{\gamma*} \otimes N^\gamma \wedge N^\gamma \wedge N^\gamma. \end{aligned}$$

According to the fact that  $[\pi, \pi] = 0$  and  $N^\gamma \wedge N^\gamma \wedge N^\gamma = 0$  for  $l(\gamma) = 2$ , we see

$$\begin{aligned} [\pi_{111}, \pi_{111}] &= 0, \\ [\pi_{111}, \pi_{221}] + [\pi_{221}, \pi_{221}] &= 0, \\ [\pi_{111} + \pi_{221}, \pi_{122}] &= 0, \\ [\pi_{122}, \pi_{222}] &= 0, \\ [\pi_{122}, \pi_{122}] &= 0, \\ [\pi_{221}, \pi_{221}] &= 0. \end{aligned}$$

In particular, this implies  $[\pi, \pi_{122}] = 0$ . Noticing that  $\pi_\gamma = \pi_{122}$ , we conclude that the Schouten bracket of  $[\pi, \pi_\gamma] = 0$ .  $\square$

In the following, we construct explicitly examples of linear Poisson structures using Proposition 4.8 on  $\mathbb{R}^3$  with the  $\mathbb{Z}/2\mathbb{Z} = \{1, e\}$  action by  $e(x, y, z) = (-x, -y, z)$ . In this case, the fixed point subspace of  $e$  is  $\{x = y = 0\}$ , which is of codimension 2.

**Example 4.4.1.** Denote  $\pi_1 = z\partial_x \wedge \partial_y$ . One can easily check that  $[\pi_1, \pi_1] = 0$ . And the Poisson structure constructed from Proposition 4.8 is  $\Pi_1 = \pi_1 + \pi_1|_{x=y=0}e = z\partial_x \wedge \partial_y + z\partial_x \wedge \partial_y|_{x=y=0}e$ .

**Example 4.4.2.** Denote  $\pi_2 = z\partial_x \wedge \partial_y + x\partial_x \wedge \partial_z - y\partial_y \wedge \partial_z$ . Noticing that  $[\partial_x \wedge \partial_y, x\partial_x - y\partial_y] = 0$ , we have  $[\pi_2, \pi_2] = 0$ . And the Poisson structure constructed from Proposition 4.8 is  $\Pi_2 = \pi_2 + \pi_2|_{x=y=0}e = z\partial_x \wedge \partial_y + x\partial_x \wedge \partial_z - y\partial_y \wedge \partial_z + z\partial_x \wedge \partial_y|_e$ .

We compute the 0-th Poisson cohomology of  $\Pi_1$  and  $\Pi_2$  to distinguish them.

- (1) If  $f \in (S\mathbb{R}^3)^{\mathbb{Z}/2\mathbb{Z}}$  is a 0-th Poisson cocycle with respect to  $\Pi_1$ , i.e.  $\Pi_1(f) = 0$ , then  $[\pi, f] = 0$  i.e.  $\partial_x(f) = \partial_y(f) = 0$ , which is also a sufficient condition. Therefore,  $H_{\Pi_1}^0 = \{f \in (S\mathbb{R}^3)^{\mathbb{Z}/2\mathbb{Z}} : \partial_x(f) = \partial_y(f) = 0\}$ , which means  $f$  is a polynomial depending only on  $z$ .
- (2) If  $f \in (S\mathbb{R}^3)^{\mathbb{Z}/2\mathbb{Z}}$  is a 0-th Poisson cocycle with respect to  $\Pi_2$ , i.e.  $\Pi_2(f) = 0$ , then  $\pi_2(f) = 0$  and  $\pi_1(f)|_{x=y=0} = 0$ . This equivalent to  $zf_x + yf_z = 0, zf_y + xf_z = 0, xf_x - yf_y = 0$ , and  $f_x|_{x=y=0} = f_y|_{x=y=0} = 0$ . If we write  $f = \sum_{k,m,n} c_{kmn}x^k y^m z^n$ , we have that

$$\begin{aligned} (k+1)c_{k+1mn-1} + (n+1)c_{km-1n+1} &= 0 \\ (m+1)c_{km+1n-1} + (n+1)c_{k-1mn+1} &= 0 \\ kc_{kmn} - mc_{kmn} &= 0 \\ c_{10n} = c_{01n} &= 0. \end{aligned}$$

From the above third equation, we have that  $c_{kmn} = 0$  if  $k \neq m$ . From the above first and second equation, we have that  $(k+1)c_{k+1k+1n} + (n+2)c_{kkn+2} = 0$ . From this we can quickly conclude that  $f$  is a polynomial of  $xy - 1/2z^2$ , and  $H_{\Pi_2}^0$  consists of polynomials on  $xy - 1/2z^2$ .

**Proposition 4.9.** *If the group  $\Gamma$  is abelian, the Poisson cohomology of  $\Pi$  is isomorphic to the Poisson cohomology of  $\pi$ .*

*Proof.* We construct a quasi-isomorphism

$$\Psi : \left( \left( \bigoplus_{\gamma \in \Gamma} S(V^{\gamma*}) \otimes \wedge^{\bullet-l(\gamma)} V^\gamma \otimes \wedge^{l(\gamma)} N^\gamma \right)^\Gamma, [\pi, ] \right) \longrightarrow \left( \left( \bigoplus_{\gamma \in \Gamma} S(V^{\gamma*}) \otimes \wedge^{\bullet-l(\gamma)} V^\gamma \otimes \wedge^{l(\gamma)} N^\gamma \right)^\Gamma, [\Pi, ] \right).$$

Given  $X = \sum_\gamma X_\gamma \in \left( \bigoplus_{\gamma \in \Gamma} S(V^{\gamma*}) \otimes \wedge^{\bullet-l(\gamma)} V^\gamma \otimes \wedge^{l(\gamma)} N^\gamma \right)^\Gamma$ , we define  $\Psi(X)$  as a sum of  $\sum_\gamma \Psi(X_\gamma)$ . Define  $\Psi(X_\gamma)$  as a sum of  $1/k! \psi_{\gamma, \alpha_1, \dots, \alpha_k}(X_\gamma)$  for  $k = 0, 1, 2, \dots$ . If  $l(\gamma\alpha_1 \cdots \alpha_k) \neq l(\gamma) + 2k$ , then define  $\psi_{\gamma, \alpha_1, \dots, \alpha_k}(X_\gamma) = 0$ ; if  $l(\gamma\alpha_1 \cdots \alpha_k) = l(\gamma) + 2k$ , then we define  $\psi_{\gamma, \alpha_1, \dots, \alpha_k}(X_\gamma)$  to be the projection of  $X_\gamma$  down to the component

$$\left( S(V^{\gamma\alpha_1 \cdots \alpha_k*}) \otimes \wedge^{\bullet-l(\gamma\alpha_1 \cdots \alpha_k)} V^{\gamma\alpha_1 \cdots \alpha_k} \otimes \wedge^{l(\gamma)} N^\gamma \otimes \wedge^2 N^{\alpha_1} \otimes \dots \otimes \wedge^2 N^{\alpha_k} \right)^\Gamma.$$

We notice that since  $l(\gamma\alpha_1 \cdots \alpha_k) \leq \dim(V)$ , there are only a finite number of occasions that  $\psi_{\gamma, \alpha_1, \dots, \alpha_k}$  is not zero. Therefore, the map  $\Psi(X_\gamma)$  is well-defined.

To prove that  $\Psi$  is a chain map, we prove the following equation for  $\psi_{\gamma, \alpha_1, \dots, \alpha_k}$  in  $H^\bullet(S(V^*) \rtimes \Gamma, S(V^*) \rtimes \Gamma)$  at component  $\gamma\alpha_1 \cdots \alpha_k$ ,

$$(13) \quad \sum_{\alpha_1 \cdots \alpha_k = \delta} \psi_{\gamma, \alpha_1, \dots, \alpha_k}([\pi, X_\gamma]) = \sum_{\alpha_1 \cdots \alpha_k = \delta} [\pi_{\alpha_k}, k\psi_{\gamma, \alpha_1, \dots, \alpha_{k-1}}(X_\gamma)] + [\pi, \psi_{\gamma, \alpha_1, \dots, \alpha_{k-1}, \alpha_k}(X_\gamma)].$$

Equation (13) can be proved by induction. When  $k = 0$ , the identity is trivial. Assume that Equation (13) is true for  $k = n$ . We look at the case when  $k = n + 1$ . Using the induction assumption, we have

$$\begin{aligned} \sum_{\alpha_1 \cdots \alpha_{n+1} = \delta} \psi_{\gamma, \alpha_1, \dots, \alpha_{n+1}}([\pi, X_\gamma]) &= \sum_{\alpha_1 \cdots \alpha_{n+1} = \delta} \psi_{\gamma, \alpha_1, \dots, \alpha_{n+1}}(\psi_{\gamma, \alpha_1, \dots, \alpha_n}([\pi, X_\gamma])) \\ &= \sum_{\alpha_1 \cdots \alpha_{n+1} = \delta} \psi_{\gamma, \alpha_1, \dots, \alpha_{n+1}}\left(n[\pi_{\alpha_n}, \psi_{\gamma, \alpha_1, \dots, \alpha_{n-1}}(X_\gamma)] + [\pi, \psi_{\gamma, \alpha_1, \dots, \alpha_n}(X_\gamma)]\right). \end{aligned}$$

Looking at the contribution of  $[\pi, \psi_{\gamma, \alpha_1, \dots, \alpha_n}(X_\gamma)]$  at  $\gamma\alpha_1 \cdots \alpha_{n+1}$ , we decompose  $\psi_{\gamma, \alpha_1, \dots, \alpha_n}(X_\gamma)$  and  $\pi$  according to  $V = V^{\alpha_{n+1}} \oplus N^{\alpha_{n+1}}$ . As both  $\pi$  and  $\psi_{\gamma, \alpha_1, \dots, \alpha_n}(X_\gamma)$  are  $\alpha_{n+1}$  invariant, we can write  $\pi = \pi_0 + \pi_1 + \pi_2$  and  $\psi_{\gamma, \alpha_1, \dots, \alpha_n}(X_\gamma) = X_0 + X_1 + X_2$  such that  $\pi_i, X_i$  contain  $i$  number of vector fields along  $N^{\alpha_{n+1}}$ . It is easy to see that  $[\pi_0 + \pi_1, X_0 + X_1]$  will contribute zero at  $\gamma\alpha_1 \cdots \alpha_{n+1}$  by the facts that  $\pi$  is a linear Poisson structure and  $[\pi_0 + \pi_1, X_0 + X_1]$  at  $\gamma\alpha_1 \cdots \alpha_{n+1}$  is the restriction to  $V^{\gamma\alpha_1 \cdots \alpha_{n+1}}$  of the component

$$S(V^{\gamma\alpha_1 \cdots \alpha_{n+1}*}) \otimes \wedge^\bullet V^{\gamma\alpha_1 \cdots \alpha_{n+1}} \otimes \wedge^{l(\gamma)} N^\gamma \otimes \wedge^2 N^{\alpha_1} \otimes \cdots \otimes \wedge^2 N^{\alpha_{n+1}}$$

in  $[\pi_0 + \pi_1, X_0 + X_1]$ . Furthermore,  $[\pi_2, X_2]$  vanishes as it is a 3 vector field normal to  $V^{\alpha_{n+1}}$ , which is of codimension 2. Therefore, we have that  $[\pi, \psi_{\gamma, \alpha_1, \dots, \alpha_n}(X_\gamma)]$  at  $\gamma\alpha_1 \cdots \alpha_{n+1}$  is equal to

$$[\pi, X_2] + [\pi_2, X] = [\pi, \psi_{\gamma, \alpha_1, \dots, \alpha_{n+1}}(X_\gamma)] + [\pi_{\alpha_{n+1}}, \psi_{\gamma, \alpha_1, \dots, \alpha_n}(X_\gamma)].$$

Next, we observe that  $\psi_{\gamma, \alpha_1, \dots, \alpha_{n+1}}([\pi_{\alpha_n}, \psi_{\gamma, \alpha_1, \dots, \alpha_{n-1}}(X_\gamma)])$  is equal to  $[\pi_{\alpha_n}, \psi_{\gamma, \alpha_1, \dots, \alpha_{n-1}, \alpha_{n+1}}(X_\gamma)]$  as the space  $N^{\alpha_n}$  and  $N^{\alpha_{n+1}}$  are orthogonal to each other by the assumption that  $l(\gamma) + 2n + 2 = l(\gamma\alpha_1 \cdots \alpha_{n+1})$ . Hence, we have the above Equation (13) for  $k = n + 1$ .

Using Equation (13), we can easily check that  $\Psi$  is a chain map. To prove that  $\Psi$  is a quasi-isomorphism, we look at the filtration respect to the grading  $l(\gamma)$  as is used in the proof of Proposition 4.2. It is straight forward to see that the induced chain map of  $\Psi$  is identity at  $E_1$  as for  $k = 0$ ,  $\psi_\gamma(X_\gamma) = X_\gamma$ , which implies that  $\Psi$  is a quasi-isomorphism.  $\square$

## 5. EXAMPLES OF QUANTIZATION

In this section, we study some examples of quantization of constant Poisson structures. We look at  $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$  action on  $\mathbb{R}^2$  by the rotation

$$\gamma : (x, y) \mapsto \left(\cos\left(\frac{2\pi}{n}\right)x - \sin\left(\frac{2\pi}{n}\right)y, \sin\left(\frac{2\pi}{n}\right)x + \cos\left(\frac{2\pi}{n}\right)y\right), \quad \gamma^n = 1,$$

where  $\gamma$  is the generator of  $\mathbb{Z}/n\mathbb{Z}$ . If we introduce complex coordinates  $z = x + iy, \bar{z} = x - iy$ , then the above action is diagonalized

$$\gamma : (z, \bar{z}) \mapsto \left(\exp\left(\frac{2\pi i}{n}\right)z, \exp\left(-\frac{2\pi i}{n}\right)\bar{z}\right).$$

We study the Poisson structure of the following form  $\pi : \wedge^2 \mathbb{R}^2 \rightarrow \mathbb{R}\mathbb{Z}_n$  by  $\pi(x, y) = \gamma$ . In complex coordinates  $\pi(z, \bar{z}) = -i/2\gamma$ . By [8][Corollary 4.2],  $\pi$  defines a noncommutative structure on  $S(\mathbb{R}^2) \rtimes \mathbb{Z}_n$ , and by Proposition 4.2,  $\pi$  can be quantized. In the following two sections, we study properties of quantization of  $\pi$ .

**5.1. A Moyal type formula.** In this subsection, we provide an explicit formula for quantization of  $\pi$ , which is a generalization of Moyal product. We would like to point out that many of the following formulas appear already in Nadaud's paper [11]. We prove that in the case of finite group, this product is convergent. We start with introducing several operators on  $S(\mathbb{R}^2)$ . We work with complex coordinates  $z = x + iy$ ,  $\bar{z} = x - iy$ .

Define  $D_z, D_{\bar{z}} : S(\mathbb{R}^2) \rightarrow S(\mathbb{R}^2)$  as

$$D_z(f) = \frac{f(z, \bar{z}) - f(e^{\frac{2\pi i}{n}} z, \bar{z})}{(1 - e^{\frac{2\pi i}{n}})z} \quad D_{\bar{z}}(f) = \frac{f(z, \bar{z}) - f(z, e^{-\frac{2\pi i}{n}} \bar{z})}{(1 - e^{-\frac{2\pi i}{n}})\bar{z}}.$$

Define  $\sigma_z, \sigma_{\bar{z}} : S(\mathbb{R}^2) \rightarrow S(\mathbb{R}^2)$  as

$$\sigma_z(f)(z, \bar{z}) = f(e^{\frac{2\pi i}{n}} z, \bar{z}) \quad \sigma_{\bar{z}}(f)(z, \bar{z}) = f(z, e^{-\frac{2\pi i}{n}} \bar{z}).$$

Let  $q = \exp(2\pi i/n)$ . Define  $[k]_q = 1 + q + \dots + q^{k-1}$ . Define the following star product  $\star$  on  $S(\mathbb{R}^2) \rtimes \mathbb{Z}_n$ . Define  $f\gamma^k \star g\gamma^l$  by

$$f\gamma^k \star g\gamma^l = \sum_{j=0}^{\infty} \frac{(\frac{i\hbar}{2})^j}{[j]_q!} (D_z)^j(f)(\sigma_z D_z)^j(\gamma^k(g))\gamma^{j+k+l}$$

To prove the associativity of  $\star$ , we study properties of  $D_z, D_{\bar{z}}$ .

**Lemme 5.1.**

$$D_z^k(fg) = \sum_{i=0}^k \frac{[k]_q!}{[k-i]_q![i]_q!} D_z^i(f)\sigma_z^i D_z^{k-i}(g), \quad k \geq 0.$$

*Proof.* We prove this by induction. When  $k = 0$ , this identity is trivial.

Assume that this identity holds for  $k$ . For  $k + 1$ , we compute

$$\begin{aligned} D_z^{k+1}(fg) &= D_z\left(\sum_{i=0}^k \frac{[k]_q!}{[k-i]_q![i]_q!} D_z^i(f)\sigma_z^i D_z^{k-i}(g)\right) \\ &= \sum_{i=0}^k \frac{[k]_q!}{[k-i]_q![i]_q!} D_z(D_z^i(f)\sigma_z^i D_z^{k-i}(g)) \\ &= \sum_{i=0}^k \left( \frac{[k]_q!}{[k-i]_q![i]_q!} D_z^{i+1}(f)\sigma_z^{i+1} D_z^{k-i}(g) + \frac{[k]_q!}{[k-i]_q![i]_q!} D_z^i(f)D_z\sigma_z^i D_z^{k-i}(g) \right) \\ &= D_z^{k+1}(f)\sigma_z^{k+1}(g) + fD_z^{k+1}(g) \\ &+ \sum_{i=1}^k \left( \frac{[k]_q!}{[k-i+1]_q![i-1]_q!} D_z^i(f)\sigma_z^i D_z^{k-i+1}(g) + \frac{[k]_q!}{[k-i]_q![i]_q!} q^i D_z^i(f)\sigma_z^i D_z^{k-i+1}(g) \right) \\ &= D_z^{k+1}(f)\sigma_z^{k+1}(g) + \sum_{i=1}^k \frac{[k+1]_q!}{[k-i+1]_q![i]_q!} D_z^i(f)\sigma_z^i D_z^{k-i+1}(g) + fD_z^{k+1}(g) \\ &= \sum_{i=0}^{k+1} \frac{[k+1]_q!}{[k-i+1]_q![i]_q!} D_z^i(f)\sigma_z^i D_z^{k+1-i}(g). \end{aligned}$$

□

We start to prove the associativity of  $\star$ .

$$\begin{aligned} &(f \star g) \star h \\ &= \sum_k \left(\frac{i\hbar}{2}\right)^k \frac{1}{[k]_q!} D_z^k(f)\sigma_z^k D_z^k(g)\gamma^k \star h \\ &= \sum_{k,l} \left(\frac{i\hbar}{2}\right)^{k+l} \frac{1}{[l]_q![k]_q!} D_z^l \left( D_z^k(f)\sigma_z^k D_z^k(g) \right) \sigma_z^l D_z^l(\gamma^k(h))\gamma^{k+l} \end{aligned}$$

Using Lemma 5.1, we have that the above product is equal to

$$\begin{aligned}
&= \sum_{k,l} \left(\frac{i\hbar}{2}\right)^{k+l} \frac{1}{[l]_q! [k]_q! [l-m]_q! [m]_q!} D_z^{m+k}(f) \sigma_z^m D_z^{l-m} \sigma_z^k D_{\bar{z}}^k(g) \sigma_z^l D_{\bar{z}}^l(\gamma^k(h)) \gamma^{k+l} \\
&= \sum_{s,t,k} \left(\frac{i\hbar}{2}\right)^{s+t+l} \frac{1}{[k]_q! [s]_q! [t]_q!} D_z^{s+k}(f) q^{kt} \sigma_z^{s+k} D_z^t D_{\bar{z}}^k(g) \sigma_z^{s+t} D_{\bar{z}}^{s+t}(\gamma^k(h)) \gamma^{k+s+t} \\
&= \sum_{s,t,k} \left(\frac{i\hbar}{2}\right)^{s+t+l} \frac{1}{[k]_q! [s]_q! [t]_q!} D_z^{s+k}(f) q^{kt} \sigma_z^{s+k} D_z^t D_{\bar{z}}^k(g) \sigma_z^{s+t} D_{\bar{z}}^{s+t} \sigma_z^k \sigma_{\bar{z}}^k(h) \gamma^{k+s+t} \\
&= \sum_{s,t,k} \left(\frac{i\hbar}{2}\right)^{s+t+l} \frac{1}{[k]_q! [s]_q! [t]_q!} D_z^{s+k}(f) q^{kt} \sigma_z^{s+k} D_z^t D_{\bar{z}}^k(g) \sigma_z^{s+t+k} \sigma_{\bar{z}}^k q^{-k(s+t)} D_{\bar{z}}^{s+t}(h) \gamma^{k+s+t}
\end{aligned}$$

On the other hand, we compute

$$\begin{aligned}
&f \star (g \star h) \\
&= \sum_k \left(\frac{i\hbar}{2}\right)^k \frac{1}{[k]_q!} f \star (D_z^k(g) \sigma_z^k D_{\bar{z}}^k(h) \gamma^k) \\
&= \sum_{k,l} \left(\frac{i\hbar}{2}\right)^{k+l} \frac{1}{[l]_q! [k]_q!} D_z^l(f) \sigma_z^l D_{\bar{z}}^l \left( D_z^k(g) \sigma_z^k D_{\bar{z}}^k(h) \right) \gamma^{k+l}
\end{aligned}$$

Applying the similar formula for  $D_z$  as Lemma 5.1, we have

$$\begin{aligned}
&= \sum_{k,s,t} \left(\frac{i\hbar}{2}\right)^{k+s+t} \frac{1}{[k]_q! [s+t]_q! [s]_{q-1}! [t]_{q-1}!} D_z^{s+t}(f) \sigma_z^{s+t} D_{\bar{z}}^s D_z^t D_{\bar{z}}^k(g) \sigma_z^{s+t} \sigma_{\bar{z}}^s D_{\bar{z}}^t \sigma_z^k D_{\bar{z}}^k(h) \gamma^{k+s+t} \\
&= \sum_{k,s,t} \left(\frac{i\hbar}{2}\right)^{k+s+t} \frac{1}{[k]_q! [s+t]_q! [s]_{q-1}! [t]_{q-1}!} D_z^{s+t}(f) \sigma_z^{s+t} D_{\bar{z}}^s D_z^t D_{\bar{z}}^k(g) \sigma_z^{s+t+k} \sigma_{\bar{z}}^s D_{\bar{z}}^{t+k}(h) \gamma^{k+s+t}
\end{aligned}$$

In the above equation, we make the change  $s \mapsto k, t \mapsto s, k \mapsto t$ , then we have

$$\sum_{k,s,t} \left(\frac{i\hbar}{2}\right)^{k+s+t} \frac{1}{[t]_q! [k+s]_q! [k]_{q-1}! [s]_{q-1}!} D_z^{k+s}(f) \sigma_z^{k+s} D_{\bar{z}}^k D_z^t(g) \sigma_z^{s+t+k} \sigma_{\bar{z}}^k D_{\bar{z}}^{s+t}(h) \gamma^{k+s+t}$$

which is identified with the above expression of  $(f \star g) \star h$ . We remark that  $[k]_q = [k]_{q-1} q^{k-1}$  and  $[k]_q! = [k]_{q-1}! q^{(k-1)k/2}$ . Therefore, we conclude that  $\star$  defines an associative deformation of  $S(\mathbb{R}^2) \rtimes \mathbb{Z}_n$ , whose  $\hbar$  component is equal to  $i/2 D_z \otimes \sigma_z D_{\bar{z}}$ , which is cohomologous to the Poisson structure  $\pi$ .

We remark that our proof of associativity of  $\star$  is slightly different from [11]. One can view  $f \star g$  as an extension of the Moyal product as follows

$$f \star g = m \circ \exp_q \left( \frac{i\hbar}{2} (D_z \otimes \sigma_z D_{\bar{z}} \otimes \gamma) \right) (f \otimes g \otimes 1).$$

Nadaud proved the associativity of  $\star$  analogous to the associativity of Moyal product by using the property of the  $q$ -exponential. Here our proof is more straightforward, and it leads to the following more precise formula for  $\star$ .

We have the following property for the operator  $D_z$ .

**Proposition 5.2.**

$$D_z^m(f) = \frac{\sum_{i=0}^m (-1)^{m-i} \frac{[m]_q!}{[m-i]_q! [i]_q!} q^{i(i-1)/2} \sigma_z^{m-i} f}{(1-q)^m q^{m(m-1)/2} z^m}.$$

In particular, when  $m = n$ , we have that  $D_z^n(f) = 0$  for any  $f$ . And this implies that

$$f \star g = \sum_{j=0}^{n-1} \left(\frac{i\hbar}{2}\right)^j \frac{1}{[j]_q!} D_z^j(f) \sigma_z^j D_{\bar{z}}^j(g) \gamma^j.$$

*Proof.* We prove the identity by induction. When  $m = 1$ , we have

$$D_z(f) = \frac{f(z, \bar{z}) - f(qz, \bar{z})}{z - \gamma(z)} = \frac{f(z, \bar{z}) - \sigma_z(f)(z, \bar{z})}{(1-q)z}.$$

Assume that the above identity holds for  $m$ . Then for  $m + 1$ ,

$$\begin{aligned} D_z^{m+1}(f) &= D_z(D_z^m(f)) = D_z\left(\frac{\sum_{i=0}^m (-1)^{m-i} \frac{[m]_q!}{[m-i]_q! [i]_q!} q^{i(i-1)/2} \sigma_z^{m-i} f}{(1-q)^m q^{m(m-1)/2} z^m}\right) \\ &= \sum_{i=0}^m (-1)^{m-i} \frac{[m]_q!}{[m-i]_q! [i]_q!} \frac{q^{i(i-1)/2}}{(1-q)^m q^{m(m-1)/2}} D_z\left(\frac{\sigma_z^{m-i} f}{z^m}\right) \\ &= \sum_{i=0}^m (-1)^{m-i} \frac{[m]_q!}{[m-i]_q! [i]_q!} \frac{q^{i(i-1)/2}}{(1-q)^m q^{m(m-1)/2}} \frac{\frac{\sigma_z^{m-i} f}{z^m} - \frac{\sigma_z^{m-i+1}(f)}{\gamma(z^m)}}{(z - \gamma(z))} \\ &= \sum_{i=0}^m (-1)^{m-i} \frac{[m]_q!}{[m-i]_q! [i]_q!} \frac{q^{i(i-1)/2}}{(1-q)^m q^{m(m-1)/2}} \frac{q^m \sigma_z^{m-i}(f) - \sigma_z^{m-i+1}(f)}{q^m (1-q) z^{m+1}} \\ &= \frac{1}{q^{m(m+1)/2} (1-q)^{m+1} z^{m+1}} \sum_{i=0}^m (-1)^i \frac{[m]_q!}{[m-i]_q! [i]_q!} q^{(m-i)(m-i-1)/2} (q^m \sigma_z^i(f) - \sigma_z^{i+1}(f)) \\ &= \frac{1}{q^{m(m+1)/2} (1-q)^{m+1} z^{m+1}} \left( q^{m(m+1)/2} f + (-1)^{m+1} \sigma_z^{m+1}(f) + \right. \\ &\quad \left. + [m]_q! \sum_{i=1}^m (-1)^i \left( \frac{q^m q^{(m-i)(m-i-1)/2}}{[m-i]_q! [i]_q!} + \frac{q^{(m+1-i)(m-i)/2}}{[m-i+1]_q! [i-1]_q!} \right) \sigma_z^i(f) \right) \\ &= \frac{1}{q^{m(m+1)/2} (1-q)^{m+1} z^{m+1}} \left( q^{m(m+1)/2} f + (-1)^{m+1} \sigma_z^{m+1}(f) + \right. \\ &\quad \left. + [m]_q! \sum_{i=1}^m (-1)^i \frac{q^{(m+1-i)(m-i)/2} (q^i [m-i+1]_q! + [i]_q!)}{[m+1-i]_q! [i]_q!} \sigma_z^i(f) \right) \\ &= \frac{1}{q^{m(m+1)/2} (1-q)^{m+1} z^{m+1}} \sum_{i=0}^{m+1} (-1)^i q^{(m+1-i)(m-i)/2} \frac{[m+1]_q!}{[m+1-i]_q! [i]_q!} \sigma_z^i(f). \end{aligned}$$

We have proved the identity of  $D_z^m(f)$  by induction. To conclude that  $D_z^n(f) = 0$ . We see that by the above formula of  $D_z^n(f)$ , as  $[n]_q = 0$ ,

$$D_z^n(f) = \frac{1}{q^{n(n-1)/2} (1-q)^n z^n} (q^{(n-1)n/2} f + (-1)^n \sigma_z^n(f)).$$

Since  $\sigma_z^n(f) = f$ , we have

$$\frac{1}{q^{n(n-1)/2} (1-q)^n z^n} (q^{(n-1)n/2} + (-1)^n)(f).$$

The statement follows from the identity  $q^{(n-1)n/2} + (-1)^n = 0$ .  $\square$

We conclude from Proposition 5.2 that the star product  $\star$  on  $S(\mathbb{R}^2) \rtimes \mathbb{Z}_n$  is convergent for any value of  $\hbar$ .

In particular, when  $n = 2$ , we have the following explicit formula of a deformation on  $S(\mathbb{R}^2) \rtimes \mathbb{Z}_2$

$$f \star g = fh + \left(\frac{i\hbar}{2}\right) \frac{f(z, \bar{z}) - f(-z, \bar{z})}{2z} \frac{f(-z, \bar{z}) - f(-z, -\bar{z})}{2\bar{z}}.$$

**Remark 5.1.1.** Here our formula of product uses "normal ordering", by which we mean that  $D_z$  is contained only in the left component and  $D_{\bar{z}}$  is contained in the right component. We can also define product with "anti-normal ordering" or "symmetric ordering" as [11]. The similar property like Proposition 5.2 extends directly.

**Remark 5.1.2.** We observe that the formula for the star product  $\star$  on  $S(\mathbb{R}^2) \rtimes \mathbb{Z}_n[[\hbar]]$  works well for the algebra  $C_c^\infty(\mathbb{R}^2) \rtimes \mathbb{Z}_n[[\hbar]]$ . Again,  $\star$  product is convergent for any two smooth functions  $f$  and  $g$  on  $\mathbb{R}^2$ .

**5.2. Deformation of singularity.** In this subsection, we compute the center of the above quantized algebra  $(S(\mathbb{R}^2) \rtimes \mathbb{Z}_n[[\hbar]], \star)$ . We prove that the center is closely connected to the deformation of the underlying quotient space  $\mathbb{R}^2/\mathbb{Z}_n$ . We must say that this kind of idea is already in [5]. Here we are giving concrete examples about this idea.

We write an element in  $S(\mathbb{R}^2) \rtimes \mathbb{Z}_n[[\hbar]]$  by  $\sum_{i=0}^{n-1} f_i \gamma^i$  with  $f_i$  in  $S(\mathbb{R}^2)[[\hbar]]$ .

**Proposition 5.3.** *If  $f = \sum_{i=0}^{n-1} f_i \gamma^i$  is in the center of  $(S(\mathbb{R}^2) \rtimes \mathbb{Z}_n[[\hbar]], \star)$ , then  $f$  is completely determined by  $f_0$  by the following formula*

$$\gamma(f_0) = f_0, \quad f_j = \left(\frac{i\hbar}{2}\right)^j \frac{D_z^j(f_0)}{[j]_q!(1-q^{-1})^j \bar{z}^j} = \left(-\frac{i\hbar}{2}\right)^j q^{-j(j-1)/2} \frac{\sigma_z^j D_{\bar{z}}^j(f_0)}{(1-q)^j [j]_q! z^j}, \quad i = 1, \dots, n-1.$$

Therefore, as a vector space the center of the quantum algebra is isomorphic to  $S(\mathbb{R}^2)^{\mathbb{Z}_n}$ , the algebra of  $\mathbb{Z}_n$  invariant polynomials on  $V$ .

*Proof.* We need to first prove the above two expressions for  $f_i$  are same. We prove this using Proposition 5.2.

$$\begin{aligned} & \left(-\frac{i\hbar}{2}\right)^j q^{-j(j-1)/2} \frac{\sigma_z^j D_{\bar{z}}^j(f_0)}{(1-q)^j [j]_q! z^j} \\ &= \frac{\left(-\frac{i\hbar}{2}\right)^j q^{-j(j-1)/2}}{(1-q)^j [j]_q! z^j} \sigma_z^j \frac{\sum_{k=0}^j (-1)^k \frac{[j]_{q^{-1}}}{[j-k]_{q^{-1}}! [k]_{q^{-1}}!} q^{-(j-k)(j-k-1)/2} \sigma_{\bar{z}}^k(f_0)}{(1-q^{-1})^j q^{-j(j-1)/2} \bar{z}^j} \\ &= \frac{\left(\frac{i\hbar}{2}\right)^j q^{-j(j-1)/2} \sum_k (-1)^{j-k} \frac{1}{[j-k]_q! [j]_q!} q^{(j-k)(j-k-1)/2 + k(k-1)/2 - (j-k)(j-k-1)/2} \sigma_z^j \sigma_{\bar{z}}^k(f_0)}{(1-q)^j (1-q^{-1})^j z^j \bar{z}^j} \\ &= \frac{\left(\frac{i\hbar}{2}\right)^j q^{-j(j-1)/2} \sum_k (-1)^{j-k} \frac{[j]_q!}{[j-k]_q! [k]_q!} q^{k(k-1)} \sigma_z^{j-k} f_0}{[j]_q! (1-q^{-1})^j \bar{z}^j (1-q)^j z^j} \end{aligned}$$

where in the last line we have used that  $\sigma_z \sigma_{\bar{z}}(f_0) = \gamma(f_0) = f_0$ . By Proposition 5.2, we conclude that the above expression is equal to

$$\frac{\left(\frac{i\hbar}{2}\right)^j}{[j]_q! (1-q^{-1})^j \bar{z}^j} D_z^j(f_0).$$

Let  $f = f_0 + f_1 \gamma + \dots + f_{n-1} \gamma^{n-1}$  be an element in the center of  $S(\mathbb{R}^2) \rtimes \mathbb{Z}_n$ . We compute  $f \star z = \sum_i f_i \star \gamma^i(z) \gamma^i = \sum_i q^i f_i z \gamma^i$ , and  $z \star f = \sum_j z \star f_j \gamma^j = (\sum_j z f_j + \frac{i\hbar}{2} \sigma_z D_{\bar{z}}(f_j) \gamma) \gamma^j = \sum_j (z f_j + \frac{i\hbar}{2} \sigma_z D_{\bar{z}}(f_{j-1}) \gamma^j)$ . As  $f \star z = z \star f$ ,  $f_j = -\frac{i\hbar}{2} \sigma_z D_{\bar{z}}(f_{j-1}) / [(1-q)[j]_q! z]$ . And we can solve by induction to find that  $f_i$  has to be the form expressed in the Proposition.

To prove that the above defined  $f = f_0 + f_1 \gamma + \dots + f_{n-1} \gamma^{n-1}$  is in the center, we show in the following  $f \star g = g \star f$  for any  $g \in S(\mathbb{R}^2)$  and  $\gamma(f) = f$ .

For  $\gamma(f) = f$ , it is enough to prove that  $\gamma(f_i) = f_i$ . This is obvious from the following identity

$$f_j = \frac{\left(\frac{i\hbar}{2}\right)^j q^{-j(j-1)/2} \sum_k (-1)^{j-k} \frac{[j]_q!}{[j-k]_q! [k]_q!} q^{k(k-1)} \sigma_z^{j-k} f_0}{[j]_q! (1-q^{-1})^j \bar{z}^j (1-q)^j z^j}$$

and the fact that  $f_0$  is  $\gamma$  invariant.

For  $f \star g = g \star f$ , we compute the two sides of the equation separately.

$$\begin{aligned}
f \star g &= \sum_i f_i \gamma^i \star g = \sum_i f_i \star \gamma^i(g) \gamma^i \\
&= \sum_{i,j} \frac{(\frac{i\hbar}{2})^j}{[j]_q!} D_z^j(f_i) \sigma_z^j D_z^j(\gamma^i(g)) \gamma^{i+j} \\
&= \sum_{j,k} \frac{(\frac{i\hbar}{2})^j}{[j]_q!} D_z^j \left( \left( \frac{i\hbar}{2} \right)^k \frac{D_z^k(f_0)}{[k]_q!(1-q^{-1})^k \bar{z}^k} \right) \sigma_z^j D_z^j(\gamma^k(g)) \gamma^{j+k} \\
&= \sum_{j,k} \frac{(\frac{i\hbar}{2})^{j+k}}{[k]_q![j]_q!} \frac{D_z^{j+k}(f_0) \sigma_z^j D_z^j(\gamma^j(g)) \gamma^{j+k}}{(1-q^{-1})^k \bar{z}^k} \\
&= \sum_k \left( \frac{i\hbar}{2} \right)^k D_z^k(f_0) \sum_{j=0}^k \frac{\sigma_z^{k-j} D_z^{k-j}(\gamma^j(g)) \gamma^k}{[j]_q![k-j]_q!(1-q^{-1})^j \bar{z}^j}.
\end{aligned}$$

Applying Proposition 5.2 to the above  $D_z^{k-j}$ , we have that  $f \star g$  is equal to

$$\begin{aligned}
&= \sum_k \left( \frac{i\hbar}{2} \right)^k D_z^k(f_0) \sum_{j=0}^k \frac{1}{[j]_q![k-j]_q!(1-q^{-1})^j \bar{z}^j} \\
&\quad \sigma_z^{k-j} \left( \frac{\sum_{l=0}^{k-j} (-1)^l \frac{[k-j]_{q^{-1}}!}{[k-j-l]_{q^{-1}}![l]_{q^{-1}}!} q^{-(k-j-l)(k-j-l-1)/2} \sigma_z^l(\sigma_z^j \sigma_z^j g)}{(1-q^{-1})^{k-j} q^{-(k-j)(k-j-1)/2} \bar{z}^{k-j}} \right) \gamma^k \\
&= \sum_k \left( \frac{i\hbar}{2} \right)^k D_z^k(f_0) \sum_{j=0}^k \sum_{l=0}^j \frac{(-1)^{j-l} q^{(j-l)(j-l-1)/2} \sigma_z^k \sigma_z^j g}{[l]_q![k-j]_q![j-l]_q!(1-q^{-1})^k \bar{z}^k} \gamma^k.
\end{aligned}$$

It is not difficult to check that  $\sum_{i=0}^j \frac{(-1)^{j-l} q^{(j-i)(j-i-1)/2}}{[i]_q![j]_q!} = 0$  if  $j \neq 0$ , and  $= 1$  if  $j = 0$ . We replace this computation into the above line and have

$$f \star g = \sum \frac{(\frac{i\hbar}{2})^k}{[k]_q!(1-q^{-1})^k \bar{z}^k} D_z^k(f_0) \sigma_z^k g \gamma^k.$$

The computation of  $g \star f$  is similar to the above and we conclude that

$$f \star g = \sum \frac{(\frac{i\hbar}{2})^k}{[k]_q!(1-q^{-1})^k \bar{z}^k} D_z^k(f_0) \sigma_z^k g \gamma^k = g \star f.$$

□

**Remark 5.2.1.** *The above proof on  $f$  belonging to the center of quantum algebra can be simplified by checking that  $f$  commutes with the generators of  $S(\mathbb{R}^2) \rtimes \mathbb{Z}_n[[\hbar]]$ , which consists of  $z, \bar{z}, \gamma$ . We have taken the above proof because it extends to the algebra  $C_c^\infty(\mathbb{R}^2) \rtimes \mathbb{Z}_n[[\hbar]]$  directly.*

In the following, we study the algebraic structure on the center  $Z_{\mathbb{C}}^{\hbar}(\mathbb{R}^2, \mathbb{Z}_n)$  of the (complexified) quantum algebra  $(S(\mathbb{R}^2) \rtimes \mathbb{Z}_n[[\hbar]]) \otimes_{\mathbb{R}} \mathbb{C}$ . (The reason that we consider the complexified algebra is that it is relatively easy to write down a set of generators and their relations for the center. However, our following discussion also works for the real algebra.) It is easy to check that  $u = z^n, v = \bar{z}^n$  and  $w = z\bar{z} + \frac{i\hbar}{1-q^{-1}}\gamma$  are in the center of the complexified quantum algebra  $(S(\mathbb{R}^2) \rtimes \mathbb{Z}_n[[\hbar]]) \otimes_{\mathbb{R}} \mathbb{C}$ . According to the isomorphism as vector space between the center of the quantum algebra and  $S(\mathbb{R}^2)^{\Gamma}$ , we know that  $u = z^n, v = \bar{z}^n$  and  $w = z\bar{z} + i\hbar\gamma/(2(1-q^{-1}))$  generates the whole center  $Z_{\mathbb{C}}^{\hbar}(\mathbb{R}^2, \mathbb{Z}_n)$ . The relation

between these three generators is generated by

$$\begin{aligned}
z^n \star \bar{z}^n &= z \star \cdots \star z \star \bar{z} \star \cdots \star \bar{z} = z^{\star(n-1)}(z\bar{z} + \frac{i\hbar}{2}\gamma)\bar{z}^{\star(n-1)} \\
&= z^{\star(n-1)}(w + \frac{i\hbar}{1-q}\gamma)\bar{z}^{\star(n-1)} \\
&= z^{\star(n-1)} \star \bar{z}^{\star(n-1)} \star (w + \frac{i\hbar}{1-q}\gamma) \\
&= z^{\star(n-2)} \star \bar{z}^{\star(n-2)} \star (w + \frac{i\hbar}{1-q}\gamma) \star (w + \frac{i\hbar}{1-q}\gamma) \\
&\quad \cdots \\
&= (w + \frac{i\hbar}{1-q}\gamma) \star \cdots \star (w + \frac{i\hbar}{1-q}\gamma)
\end{aligned}$$

As  $w$  is in the center, the last line can be viewed as the expansion of  $w^{\star(n)} + \frac{(\frac{i\hbar}{2})^n q^{-\frac{n(n-1)}{2}}}{(1-q)^n} \gamma^n$ . Therefore, we have that  $Z_{\mathbb{C}}^{\hbar}(\mathbb{R}^2, \mathbb{Z}_n)$  is generated by  $u, v, w$  with the relation that  $u^n v^n = w^n + (\frac{i\hbar}{2})^n q^{-\frac{n(n-1)}{2}} / (1-q)^n$ . In particular, we defines a deformation of the cone  $\langle u, v, w \rangle / \{u^n v^n = w^n\}$ , which is the algebra of polynomials on the quotient  $V/\mathbb{Z}_n$ . Furthermore, we notice that for function  $F^{\hbar}(u, v, w) = u^n v^n - w^n - \frac{(\frac{i\hbar}{2})^n q^{-\frac{n(n-1)}{2}}}{(1-q)^n}$ ,  $(F_u^{\hbar}, F_v^{\hbar}, F_w^{\hbar})$  is a non-zero vector in  $\mathbb{C}^3$  if and only if  $u = v = w = 0$ , which is not on the surface determined by  $F^{\hbar} = 0$ . Therefore, we conclude that  $F^{\hbar} = 0$  determines a smooth surface when  $\hbar \neq 0$ , and  $Z_{\mathbb{C}}^{\hbar}(\mathbb{R}^2, \mathbb{Z}_n)$  is a nontrivial deformation of the cone  $\mathbb{C}^2/\mathbb{Z}_n$ .

On the other hand, we look at the 0-th Poisson cohomology of the Poisson structure  $\pi_{\gamma}$  on  $H^{\bullet}(S(\mathbb{R}^2) \times \mathbb{Z}_n, S(\mathbb{R}^2) \times \mathbb{Z}_n) \otimes_{\mathbb{R}} \mathbb{C}$ . It is not difficult to see that the 0-th Poisson cohomology  $H_{\pi_{\gamma}}^0$  is isomorphic to  $S(\mathbb{R}^2)^{\mathbb{Z}_n} \otimes_{\mathbb{R}} \mathbb{C}$  as an algebra.

We summarize the above study into the following corollary.

**Proposition 5.4.** *The center  $Z_{\mathbb{C}}^{\hbar}(\mathbb{R}^2, \mathbb{Z}_n)$  is not isomorphic to the Poisson center  $H_{\pi_{\gamma}}^0[[\hbar]] = S(\mathbb{R}^2)^{\mathbb{Z}_n}[[\hbar]]$ , but defines a nontrivial deformation.*

In particular, when  $n = 2$ , the center  $Z_{\mathbb{C}}^{\hbar}(\mathbb{R}^2, \mathbb{Z}_2)$  is equal to  $\langle u, v, w \rangle / \{uv = w^2 + \frac{\hbar^2}{16}\}$ . This is the algebra of polynomial functions on the hyperboloid (when  $\hbar$  is real, the surface is one-sheeted, when  $\hbar$  is imaginary, the surface is two-sheeted.).

**Remark 5.2.2.** *We can extend the above discussion of center to quantization of more general Poisson structures. For example, the same discussion holds true for the center of the quantization of the linear Poisson structure  $z\partial_x \wedge \partial_y$  on  $S(\mathbb{R}^3) \times \mathbb{Z}_n$  with  $\mathbb{Z}_n$  acting on  $\mathbb{R}^3$  by rotating the  $x, y$ -plane and fixing the  $z$  axis.*

In summary, we have seen that the center of the quantization of a Poisson structure  $\pi$  on an orbifold may not be isomorphic to the 0-th Poisson cohomology of  $\pi$  as an algebra. On the other hand, the well-known Duflo's isomorphism for a Lie algebra states that the center of the universal enveloping algebra  $\mathcal{U}(\mathfrak{g})$  is isomorphic to the Poisson center of  $S(\mathfrak{g})$  as an algebra. Our examples suggest that the natural extension of Duflo's isomorphism does not hold in the case of quantization of Lie Poisson structures on orbifolds. We plan to study this interesting phenomena in future publications.

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