Poisson algebras associated to quasi-Hopf algebras

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Abstract

We define admissible quasi-Hopf quantized universal enveloping (QHQUE) algebras by $\hbar$-adic valuation conditions. We show that any QHQUE algebra is twist-equivalent to an admissible one. We prove a related statement: any associator is twist-equivalent to a Lie associator. We attach a quantized formal series algebra to each admissible QHQUE algebra and study the resulting Poisson algebras.

§ 0 Introduction

In [WX], Weinstein and Xu introduced a geometric counterpart of quasitriangular quantum groups: they proved that if $(\mathfrak{g}, r)$ is a finite dimensional quasi-triangular Lie bialgebra, then the dual group G^* is equipped with a braiding $\mathscr{R}_{\rm WX}$ with properties analogous to those of quantum R-matrices (in particular, it is a set-theoretic solution of the quantum Yang-Baxter Equation). An explicit relation to the theory of quantum groups was later given in [GH, EH, EGH]: to a quasi-triangular QUE algebra $(U_\hbar(\mathfrak{g}), m, R)$ quantizing $(\mathfrak{g}, r)$, one associates its quantized formal series algebra (QFSA) $U_\hbar(\mathfrak{g})' \subset U_\hbar(\mathfrak{g})$; $U_\hbar(\mathfrak{g})'$ is a flat deformation of the Hopf-co-Poisson algebra $\mathscr{O}_{G^*} = (U(\mathfrak{g}^*))^*$ of formal functions of G^*. Then one proves that $\mathrm{Ad}(R)$ preserves $U_\hbar(\mathfrak{g})'^{\bar\otimes 2}$, and $\mathrm{Ad}(R)|_{\hbar=0}$ coincides with the automorphism $\mathscr{R}_{\rm WX}$ of $\mathscr{O}_{G^*}^{\bar\otimes 2}$; moreover, $\rho = \hbar \log(R)|_{\hbar=0}$ is a function of $\mathscr{O}_{G^*}^{\bar\otimes 2}$, independent on a quantization of $\mathfrak{g}^*$, which may be expressed universally in terms of r, and $\mathscr{R}_{\rm WX}$ coincides with the "time one automorphism" of the Hamiltonian vector field generated by ρ.

In this paper, we study the analogous problem in the case of quasi-quantum groups (quasi-Hopf QUE algebras). The classical limit of a QHQUE algebra is a Lie quasi-bialgebra (LQBA). V. Drinfeld proposed to attach Poisson-Lie "quasi-groups" to each LQBA ([Dr4]). Axioms for Poisson-Lie quasi-groups are the quasi-Hopf analogues of the Weinstein-Xu axioms.

A *Poisson-Lie quasi-group* is a Poisson manifold X, together with a "product" Poisson map $X^2 \xrightarrow{m_X} X$, a unit for this product $e \in X$, and Poisson automorphisms $\Phi_X \in \mathrm{Aut}(X^3)$,

$\Phi_X^{12,3,4}$, $\Phi_X^{1,23,4}$ and $\Phi_X^{1,2,34} \in \mathrm{Aut}(X^4)$, such that

$$\begin{aligned} m_X \circ (\mathrm{id} \times m_X) &= m_X \circ (m_X \times \mathrm{id}) \circ \Phi_X, \\ (m_X \times \mathrm{id} \times \mathrm{id}) \circ \Phi_X^{12,3,4} &= \Phi_X \circ (m_X \times \mathrm{id} \times \mathrm{id}), \\ (\mathrm{id} \times m_X \times \mathrm{id}) \circ \Phi_X^{1,23,4} &= \Phi_X \circ (\mathrm{id} \times m_X \times \mathrm{id}), \text{ etc.} \\ \text{and } \Phi_X^{1,2,34} \circ \Phi_X^{12,3,4} &= (\mathrm{id} \times \Phi_X) \circ \Phi_X^{1,23,4} \circ (\Phi_X \times \mathrm{id}). \end{aligned}$$

A *twistor* for the quasi-group (X, m_X, Φ_X) is a collection of Poisson automorphisms $F_X \in \mathrm{Aut}(X^2)$, $F_X^{12,3}$, $F_X^{1,23} \in \mathrm{Aut}(X^3)$, $F_X^{(12)3,4}$, $F_X^{1(23),4}$, $F_X^{12,34}$, $F_X^{1(23),4}$, $F_X^{1,(23)4} \in \mathrm{Aut}(X^4)$ such that

$$\begin{aligned} (m_X \times \mathrm{id}) \circ F_X^{12,3} &= F_X \circ (m_X \times \mathrm{id}), \\ \big((m_X \circ (\mathrm{id} \times m_X)) \times \mathrm{id}\big) \circ F_X^{1(23),4} &= F_X \circ \big((m_X \circ (\mathrm{id} \times m_X)) \times \mathrm{id}\big), \\ F_X^{(12)3,4} &= (\Phi_X \times \mathrm{id}) \circ F_X^{1(23),4} \circ (\Phi_X \times \mathrm{id})^{-1}, \text{ etc.} \end{aligned}$$

A twistor replaces the quasi-group (X, m_X, Φ_X) by (X, m'_X, Φ'_X) with $m'_X = m_X \circ F_X$ and $\Phi'_X = (F_X^{1,23})^{-1} \circ (F_X \times \mathrm{id})^{-1} \circ \Phi_X \circ F_X^{1,23} \circ (\mathrm{id} \times F_X)$.

(Other axioms for Poisson-Lie quasi-groups were proposed in a differential-geometric language in [Ban, KS].)

We do not know a "geometric" construction of a twist-equivalence class of (X, m_X, Φ_X) associated to each Lie quasi-bialgebra, in the spirit of [WX]. Instead we generalize the "construction of a QFS algebra and passage to Poisson geometry" part of the above discussion, and we derive from there a construction of triples (X, m_X, Φ_X), in the case of Lie quasi-bialgebras arising from *metrized* Lie algebras.

Let us describe the generalization of the "construction of a QFS algebra" part (precise statements are in Section 1). We introduce the notion of an *admissible* quasi-Hopf QUE algebra, and we associate a QFSA to such a QHQUE algebra. Each QHQUE algebra can be made admissible after a suitable twist.

We generalize the "passage to Poisson geometry" part as follows. The reduction modulo $\hbar$ of the obtained QFS algebra is a quintuple $(A, m, P, \Delta, \widetilde{\varphi})$ satisfying certain axioms; in particular $\exp(V_{\widetilde{\varphi}})$ is an automorphism of $A^{\widehat{\otimes} 3}$, and $(A, m, \exp(V_{\widetilde{\varphi}}))$ satisfies the axioms dual to those of (X, m_X, Φ_X).

When the Lie quasi-bialgebra arises from a metrized Lie algebra, admissible QHQUE algebras quantizing it are given by Lie associators, and we obtain a quasi-group (X, m_X, Φ_X) using our construction. We also prove that its twist-equivalence class does not depend on the choice of an associator.

Finally, we prove a related result: any associator is twist-equivalent to a unique Lie associator.

§ 1 Outline of results

Let $\mathbb{K}$ be a field of characteristic 0. Let (U, m) be a topologically free $\mathbb{K}[[\hbar]]$-algebra equipped with algebra morphisms

$$\Delta : U \to U \widehat{\otimes} U, \text{ and } \varepsilon : U \to \mathbb{K}[[\hbar]]$$

$$\text{with } (\varepsilon \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id}$$

such that the reduction of (U,m,Δ) modulo $\hbar$ is a universal enveloping algebra. Set

$$U' = \{x \in U | \text{ for any tree } P,\ \delta^{(P)}(x) \in \hbar^{|P|} U^{\otimes |P|}\}$$

(see the definitions of a tree, $\delta^{(P)}$, and $|P|$ in Section 2). We prove:

Theorem 1.1. *U' is a topologically free $\mathbb{K}[[\hbar]]$-algebra. It is equipped with a complete decreasing algebra filtration*

$$(U')^{(n)} = \{x \in U | \textit{ for any tree } P,\ \delta^{(P)}(x) \in \hbar^n U^{\otimes |P|}\}.$$

U' is stable under the multiplication m and the map $\Delta : U \to U^{\widehat{\otimes} 2}$ induces a continuous algebra morphism

$$\Delta_{U'} \,:\, U' \to U'^{\bar{\otimes} 2} = \varprojlim_n \left(U'^{\widehat{\otimes} 2} / \sum_{p,q | p+q=n} U'^{(p)} \otimes U'^{(q)} \right).$$

Set $\mathscr{O} := U'/\hbar U'$. Then $\mathscr{O}$ is a complete commutative local ring and the reduction modulo $\hbar$ of $\Delta_{U'}$ is a continuous ring morphism

$$\Delta_{\mathscr{O}} \,:\, \mathscr{O} \to \mathscr{O}^{\bar{\otimes} 2} = \varprojlim_n \left(\mathscr{O}^{\otimes 2} / \sum_{p,q | p+q=n} \mathscr{O}^{(p)} \otimes \mathscr{O}^{(q)} \right),$$

where $\mathscr{O}^{(p)} = U'^{(p)} / (\hbar U \cap U'^{(p)})$.

Theorem 1.2. *Let (U,m,Δ,Φ) be a quasi-Hopf QUE algebra. Assume that*

$$\hbar \log(\Phi) \in (U')^{\bar{\otimes} 3}. \tag{1.1}$$

Then there is a noncanonical isomorphism of filtered algebras $U'/\hbar U' \to \widehat{S}^{\cdot}(\mathfrak{g})$, where $\widehat{S}^{\cdot}(\mathfrak{g})$ is the formal series completion of the symmetric algebra $S^{\cdot}(\mathfrak{g})$.

When (U,m,Δ,Φ) satisfies the hypothesis (1.1), we say that it is *admissible*. In that case, we say that U' is the quantized formal series algebra (QFSA) corresponding to (U,m,Δ,Φ). Let us recall the notion of a *twist* of a quasi-Hopf QUE algebra (U,m,Δ,Φ). This is an element $F \in \left(U^{\widehat{\otimes} 2}\right)^{\times}$, such that $(\varepsilon \otimes \mathrm{id})(F) = (\mathrm{id} \otimes \varepsilon)(F) = 1$. It transforms (U,m,Δ,Φ) into the quasi-Hopf algebra $(U,m,{}^F\Delta,{}^F\Phi)$, where

$${}^F\Delta = \mathrm{Ad}(F) \circ \Delta, \text{ and } {}^F\Phi = (1 \otimes F)(\mathrm{id} \otimes \Delta)(F)\Phi(\Delta \otimes \mathrm{id})(F)^{-1}(F \otimes 1)^{-1}.$$

Theorem 1.3.

1) *Let (U,m,Δ,Φ) be an admissible quasi-Hopf QUE algebra. Let us say that a twist F of U is admissible if $\hbar \log(F) \in U'^{\bar{\otimes} 2}$. Then the twisted quasi-Hopf algebra $(U,m,{}^F\Delta,{}^F\Phi)$ is also admissible, and its QFSA coincides with U'.*

2) *Let (U,m,Δ,Φ) be an arbitrary quasi-Hopf QUE algebra. There exists a twist F_0 of U such that the twisted quasi-Hopf algebra $(U,m,{}^{F_0}\Delta,{}^{F_0}\Phi)$ is admissible.*

Theorem 1.3 can be interpreted as follows. Let (U,m) be a formal deformation of a universal enveloping algebra. The set of twists of U is a subgroup $\mathscr{T}$ of $(U^{\widehat{\otimes} 2})^{\times}$. Denote by $\mathscr{Q}$ the set of all quasi-Hopf structures on (U,m), and by $\mathscr{Q}_{\mathrm{adm}}$ the subset of admissible structures. If $\mathscr{Q}$ is nonempty, then $\mathscr{Q}_{\mathrm{adm}}$ is also nonempty, and all its elements give rise to the same subalgebra $U' \subset U$ (Theorem 1.3, 1)). Using U', we then define the subgroup $\mathscr{T}_{\mathrm{adm}} \subset \mathscr{T}$ of admissible twists. We have a natural action of $\mathscr{T}$ on $\mathscr{Q}$, which restricts to an action of $\mathscr{T}_{\mathrm{adm}}$ on $\mathscr{Q}_{\mathrm{adm}}$. Theorem 1.3 2) says that the natural map

$$\mathscr{Q}_{\mathrm{adm}}/\mathscr{T}_{\mathrm{adm}} \to \mathscr{Q}/\mathscr{T}$$

is surjective. Let us explain why it is not injective in general. Any QUE Hopf algebra (U,m,Δ) is admissible as a quasi-Hopf algebra. If $u \in U^\times$ and $F = (u\otimes u)\Delta(u)^{-1}$, then $(U,m,{}^F\Delta)$ is a Hopf algebra. So (U,m,Δ) and $(U,m,{}^F\Delta)$ are in the same class of $\mathscr{Q}/\mathscr{T}$. These are also two elements of $\mathscr{Q}_{\mathrm{adm}}$; the corresponding QFS algebras are U' and $\mathrm{Ad}(u)(U')$. In general, these algebras do not coincide, so (U,m,Δ) and $(U,m,{}^F\Delta)$ are not in the same class of $\mathscr{Q}_{\mathrm{adm}}/\mathscr{T}_{\mathrm{adm}}$.

Let us define a Drinfeld algebra as follows:

Definition 1.4. *A Drinfeld algebra is a quintuple $(A,m_0,P,\Delta,\widetilde{\varphi})$, where*

- *(A,m_0) is a formal series algebra,*
- *P is a Poisson strcture on A "vanishing at the origin" (i.e., such that $\mathrm{Im}(P) \subset \mathfrak{m}_A$, where $\mathfrak{m}_A$ is the maximal ideal of A),*
- *$\Delta : A \to A\widehat{\otimes}A$ is a continuous Poisson algebra morphism, such that $(\varepsilon\otimes\mathrm{id})\circ\Delta = (\mathrm{id}\otimes\varepsilon)\circ\Delta = \mathrm{id}$, where $\varepsilon : A \to A/\mathfrak{m}_A = \mathbb{K}$ is the natural projection,*
- *$\widetilde{\varphi} \in (\mathfrak{m}_A)^{\widehat{\otimes}3}$ satisfies*

$$(\mathrm{id}\otimes\Delta)(\Delta(a)) = \widetilde{\varphi}\star(\Delta\otimes\mathrm{id})(\Delta(a))\star(-\widetilde{\varphi}),\ a \in A,$$
$$\widetilde{\varphi}^{1,2,34}\star\widetilde{\varphi}^{12,3,4} = \widetilde{\varphi}^{2,3,4}\star\widetilde{\varphi}^{1,23,4}\star\widetilde{\varphi}^{1,2,3},$$

where we set $f\star g = f+g+\frac{1}{2}P(f,g)+\cdots$, the Cambell-Baker-Hausdorff (CBH) series of the Lie algebra (A,P).

If $\widetilde{f} \in \mathfrak{m}_A^{\widehat{\otimes}2}$, we define the twist of the Drinfeld algebra $(A,m_0,P,\Delta,\widetilde{\varphi})$ by $\widetilde{f}$ as the algebra $(A,m_0,P,{}^{\widetilde{f}}\Delta,{}^{\widetilde{f}}\widetilde{\varphi})$, where

$$^{\widetilde{f}}\Delta(a) = \widetilde{f}\star\Delta(a)\star(-\widetilde{f}),\ \text{and}$$
$$^{\widetilde{f}}\widetilde{\varphi} = \widetilde{f}^{2,3}\star\widetilde{f}^{1,23}\star\widetilde{\varphi}\star(-\widetilde{f}^{12,3})\star(-\widetilde{f}^{1,2});$$

then $(A,m_0,P,{}^{\widetilde{f}}\Delta,{}^{\widetilde{f}}\widetilde{\varphi})$ is again a Drinfeld algebra.

Remark 1.5. If Λ is any Artinian local $\mathbb{K}$-ring with residue field $\mathbb{K}$, set $X = \mathrm{Hom}_{\mathbb{K}}(A,\Lambda)$. Then X is the "Poisson-Lie quasi-group", in the sense of the Introduction. Namely, Δ_0 induces a product $m_X : X\times X \to X$, and $\exp(V_{\widetilde{\varphi}})$, $\exp(V_{\widetilde{\varphi}^{12,3,4}})$, etc., induce automorphisms Φ_X, $\Phi_X^{12,3,4}$, etc., of X, that satisfy the quasi-group axioms (we denote by V_f the Hamiltonian derivation of $A^{\widehat{\otimes}k}$ induced by $f \in A^{\widehat{\otimes}k}$). Moreover, if $\widetilde{f}$ is a twist of A, then $\exp(V_{\widetilde{f}})$, $\exp(V_{\widetilde{f}^{12,3}})$, $\exp(V_{\widetilde{f}^{(12)3,4}})$, etc., define a twistor $(F_X,F_X^{12,3},F_X^{(12)3,4},\dots)$ of (X,m_X,Φ_X). Twisting A by $\widetilde{f}$ corresponds to twisting (X,m_X,Φ_X) by $(F_X,F_X^{12,3},\dots)$.

Lemma 1.6. *If $(A,m_0,P,\Delta,\widetilde{\varphi})$ is a Drinfeld algebra, set $\mathfrak{g} = \mathfrak{m}_A/(\mathfrak{m}_A)^2$; then P induces a Lie bracket μ on $\mathfrak{g}$, $\Delta-\Delta^{1,2}$ induces a linear map $\delta : \mathfrak{g}\to\Lambda^2(\mathfrak{g})$, and the reduction of $\mathrm{Alt}(\widetilde{\varphi})$ is an element φ of $\Lambda^3(\mathfrak{g})$. Then $(\mathfrak{g},\mu,\delta,\varphi)$ is a Lie quasi-bialgebra. Moreover, twisting $(A,m_0,P,\Delta,\widetilde{\varphi})$ by $\widetilde{f}$ corresponds to twisting $(\mathfrak{g},\mu,\delta,\varphi)$ by*

$$f := \big(\mathrm{Alt}(\widetilde{f})\,\mathrm{mod}\ (\mathfrak{m}_A)^2\otimes\mathfrak{m}_A+\mathfrak{m}_A\otimes(\mathfrak{m}_A)^2\big) \in \Lambda^2(\mathfrak{g}).$$

Taking the reduction modulo $\hbar$ induces a natural map

$$\mathcal{Q}_{\text{adm}}/\mathcal{T}_{\text{adm}} \to \{\text{Drinfeld algebra structures on } \widehat{S}^{\cdot}(\mathfrak{g})\}/\text{twists}.$$

To summarize, we have a diagram

$$\begin{array}{ccccc} \mathcal{Q}/\mathcal{T} & \leftarrow & \mathcal{Q}_{\text{adm}}/\mathcal{T}_{\text{adm}} & \rightarrow & \left\{\begin{array}{c}\text{Drinfeld algebra structures}\\ \text{on } \widehat{S}^{\cdot}(\mathfrak{g})\end{array}\right\}/\text{twists} \\ \scriptstyle{class}\downarrow & & & & \downarrow\scriptstyle{red} \\ & & \{\text{Lie quasi-bialgebra structures on } (\mathfrak{g},\mu)\}/\text{twists}, & & \end{array}$$

where *class* is the classical limit map described in [Dr2], and *red* is the map described in Lemma 1.6. It is easy to see that this diagram commutes.

When U is a Hopf QUE algebra, the corresponding Drinfeld algebra is the Hopf-Poisson structure on $\mathcal{O}_{G^*} = (U(\mathfrak{g}^*))^*$, and $\widetilde{\varphi} = 0$.

Let $(\mathfrak{g},\mu,\delta,\varphi)$ be a Lie quasi-bialgebra. A *lift of* $(\mathfrak{g},\mu,\delta,\varphi)$ is a Drinfeld algebra, whose reduction is $(\mathfrak{g},\mu,\delta,\varphi)$. A general problem is to construct a lift for any Lie quasi-bialgebra. We will not solve this problem, but we will give partial existence and unicity results. Recall that a *metrized Lie algebra* is a pair $(\mathfrak{g},t_{\mathfrak{g}})$ of a Lie algebra $\mathfrak{g}$ and $t_{\mathfrak{g}} \in S^2(\mathfrak{g})^{\mathfrak{g}}$. It gives rise to the Lie quasi-bialgebra $(\mathfrak{g},\delta=0,\varphi=[t_{\mathfrak{g}}^{1,2},t_{\mathfrak{g}}^{2,3}])$. Recall that a *Lie associator* is a noncommutative formal series $\Phi(A,B)$, such that $\log\Phi(A,B)$ is a Lie series $[A,B]+$higher degrees terms, satisfying the pentagon and hexagon identities (see [Dr3]).

Theorem 1.7.

1) *If Φ is Lie associator, then the Drinfeld algebra*

$$(\widehat{S}^{\cdot}(\mathfrak{g}),m_0,P_{\mathfrak{g}^*},\Delta_0,\log(\Phi)(\overline{t}_{\mathfrak{g}}^{1,2},\overline{t}_{\mathfrak{g}}^{2,3})) \tag{1.2}$$

is a lift of $(\mathfrak{g},\delta=0,\varphi=[t_{\mathfrak{g}}^{1,2},t_{\mathfrak{g}}^{2,3}])$, where $P_{\mathfrak{g}^*}$ is the Kostant-Kirillov Poisson structure on $\mathfrak{g}^*$, Δ_0 is the coproduct for which the elements of $\mathfrak{g}$ are primitive, $\overline{t}_{\mathfrak{g}}^{1,2}$ is the image of $t_{\mathfrak{g}}^{1,2}$ in $\widehat{S}^{\cdot}(\mathfrak{g})^{\widehat{\otimes}2}$, and we use the Poisson bracket of $\widehat{S}^{\cdot}(\mathfrak{g})^{\widehat{\otimes}2}$ in the expression of $\log(\Phi)(\overline{t}_{\mathfrak{g}}^{1,2},\overline{t}_{\mathfrak{g}}^{2,3})$.

2) *Any two lifts of $(\mathfrak{g},\delta=0,\varphi=[t_{\mathfrak{g}}^{1,2},t_{\mathfrak{g}}^{2,3}])$ of the form $(\widehat{S}^{\cdot}(\mathfrak{g}),m_0,P_{\mathfrak{g}_0},\Delta_0,\widetilde{\varphi})$ are related by a $\mathfrak{g}$-invariant twist. In particular, any two Drinfeld algebra structures of the type (1.2) are related by a $\mathfrak{g}$-invariant twist.*

We prove this theorem in Section 6. If now Φ is a general (non-Lie) associator, $(U(\mathfrak{g})[[\hbar]], m_0,\Delta_0,\Phi(\hbar t_{\mathfrak{g}}^{1,2},\hbar t_{\mathfrak{g}}^{2,3}))$ is a quasi-Hopf QUE algebra, but it is admissible only when Φ is Lie (for general $\mathfrak{g}$). According to Theorem 1.3 2), it is twist-equivalent to an admissible quasi-Hopf QUE algebra. We prove

Theorem 1.8. *Any (non-Lie) associator is twist-equivalent to a unique Lie associator.*

So the "concrete" version of the twist of Theorem 1.8 is an example of the twist F of Theorem 1.3 2).

§ 2 Definition and properties of U'

In this section, we prove Theorem 1.1. We first introduce the material for the definition of U': trees (a); the map $\delta^{(P)}$ (b); then we prove Theorem 1.1 in (c) and (d).

- a - Binary complete planar rooted trees

Definition 2.1. *A n-binary complete planar rooted tree (n-tree for short) is a set of vertices and oriented edges satisfying the following conditions:*

- *each edge carries one of the labels $\{l, r\}$.*
- *if we set:*

$$\textit{valency of a vertex} = (\mathrm{card}(\textit{incoming edges}), \mathrm{card}(\textit{outgoing edges})),$$

  we have

  - *there exists exactly one vertex with valency $(0,2)$ (the root)*
  - *there exists exactly n vertices with valency $(1,0)$ (the leaves)*
  - *all other vertices have valency $(1,2)$*
  - *if a vertex has valency $(x,2)$, then one of its outgoing edges has label l and the other has label r.*
- *the set of leaves has cardinal n.*

Let us denote, for $n \geq 2$,

$$\mathrm{Tree}_n = \{n\text{-binary complete planar rooted trees}\}.$$

By definition, Tree_1 consists of one element (the tree with a root and one nonmarked edge) and Tree_0 consists of one element (the tree with a root and no edge). We will write $|P| = n$ if P is a tree in Tree_n.

Definition 2.2. *(Extracted trees) Let P be a binary complete planar rooted tree. Let L be the set of its leaves and let L' be a subset of L. We define the extracted subtree $P_{L'}$ as follows:*

(1) *$\widetilde{P}_{L'}$ is the set of all edges connecting the root with an element of L',*

(2) *the vertices of $\widetilde{P}_{L'}$ all have valency $(0,2)$, $(1,0)$, $(1,2)$ or $(1,1)$;*

(3) *$P_{L'}$ is obtained from $\widetilde{P}_{L'}$ by replacing each maximal sequence of edges related by a $(1,1)$ vertex, by a single edge whose label is the label of the first edge of the sequence.*

Then $P_{L'}$ is a $|L'|$-binary complete planar rooted tree.

Definition 2.3. *(Descendants of a tree) If we cut the tree P by removing its root and the related vertices, we get two trees P' and P'', its* left and right descendants.

In the same way, we define the left and right descendants of a vertex of P.

If P is a n-tree, there exists a unique bijection of the set of leaves with $\{1, \ldots, n\}$, such that for each vertex, the number attached to any leaf of its left descendant is smaller than the number attached to any leaf of its right descendant.

- b - Definition of $\delta^{(P)} : U \to U^{\widehat{\otimes} n}$

Let us place ourselves in the hypothesis of Theorem 1.1. Let us define $\delta^{(2)} : U \to U^{\widehat{\otimes} 2}$,

$$\delta^{(2)}(x) = \Delta(x) - x \otimes 1 - 1 \otimes x + \varepsilon(x) 1 \otimes 1.$$

For P_2 the only tree of Tree_2, we set

$$\delta^{(P_2)} = \delta^{(2)} = \delta.$$

For P_1, the only tree of Tree_1, we set

$$\Delta^{(P_1)}(x) = \delta^{(1)}(x) = x - \varepsilon(x)1.$$

For P_0 the only tree of Tree_0, we set

$$\delta^{(P_0)}(x) = \delta^{(0)}(x) = \varepsilon(x).$$

When P is a n-tree with descendants P' and P'', we set

$$\delta^{(P)} = (\delta^{(P')} \otimes \delta^{(P'')}) \circ \delta,$$

so $\delta^{(P)}$ is a linear map $U \to U^{\widehat{\otimes} n}$.

- c - Behavior of $\delta^{(P)}$ with respect to multiplication

If $\Sigma = \{i_1, \ldots, i_k\}$ is a subset of $\{1, \ldots, n\}$, where $i_1 < i_2 < \cdots < i_k$, the map $x \mapsto x^\Sigma$ is the linear map $U^{\widehat{\otimes} k} \to U^{\widehat{\otimes} n}$, defined by

$$x_1 \otimes \cdots \otimes x_k \mapsto 1^{\otimes i_1 - 1} \otimes x_1 \otimes 1^{\otimes i_2 - i_1 - 1} \otimes x_2 \otimes \cdots \otimes 1^{\otimes i_k - i_{k-1} - 1} \otimes x_k \otimes 1^{\otimes n - i_k - 1}.$$

If $\Sigma = \emptyset$, $x \mapsto x^\Sigma$ is the map $\mathbb{K} \to U^{\widehat{\otimes} n}$, $1 \mapsto 1^{\otimes n}$.

Proposition 2.4. *For $P \in \mathrm{Tree}_n$, we have the identity*

$$\delta^{(P)}(xy) = \sum_{\substack{\Sigma', \Sigma'' \subset \{1, \ldots, n\} | \\ \Sigma' \cup \Sigma'' = \{1, \ldots, n\}}} (\delta^{(\Sigma')}(x))^{\Sigma'} (\delta^{(\Sigma'')}(y))^{\Sigma''},$$

for any $x, y \in U$.

This proposition is proved in Section 5.

- d - Construction of U'

Let us set

$$U' = \{x \in U | \text{ for any tree } P,\ \delta^{(P)}(x) \in \hbar^{|P|} U^{\widehat{\otimes} |P|}\}.$$

Then U' is a topologically free $\mathbb{K}[[\hbar]]$-submodule of U. Moreover, if $x, y \in U'$, and P is a tree, then

$$\delta^{(P)}([x,y]) = \sum_{\substack{\Sigma, \Sigma' \subset \{1, \ldots, |P|\} \\ \Sigma \cup \Sigma' = \{1, \ldots, |P|\}}} \left[\delta^{(P_\Sigma)}(x)^\Sigma, \delta^{(P_{\Sigma'})}(y)^{\Sigma'} \right];$$

the summand corresponding to a pair (Σ, Σ') with $\Sigma \cap \Sigma' = \emptyset$ is zero, and the $\hbar$-adic valuation of the other summands is $\leq |\Sigma| + |\Sigma'| \leq |P| + 1$; so $\delta^{(P)}([x,y]) \in \hbar^{|P|+1} U^{\widehat{\otimes} |P|}$. On the other hand, there exists $z \in U$ such that $[x,y] = \hbar z$, so $\delta^{(P)}(z) \in \hbar^{|P|} U^{\widehat{\otimes}|P|}$; so $z \in U'$ and we get $[x,y] \in \hbar U'$. It follows that $U'/\hbar U'$ is commutative. Let us set

$$U'^{(n)} = U' \cap \hbar^n U. \tag{2.3}$$

We have a decreasing filtration

$$U' = U'^{(0)} \supset U'^{(1)} \supset U'^{(2)} \supset \cdots;$$

we have $U'^{(n)} \subset \hbar^n U$, so U' is complete for the topology induced by this filtration. This is an algebra filtration, i.e., $U'^{(i)} U'^{(j)} \subset U'^{(i+j)}$. It induces an algebra filtration on $U'/\hbar U'$,

$$U'/\hbar U' \supset \cdots \supset U'^{(i)} / \big(U'^{(i)} \cap \hbar U'\big) \supset \cdots,$$

for which $U'/\hbar U'$ is complete. Moreover, the completed tensor product

$$U' \bar{\otimes} U' = \lim_{\substack{\longleftarrow \\ n}} \big(U' \widehat{\otimes} U' / \sum_{p,q | p+q=n} U'^{(p)} \widehat{\otimes} U'^{(q)}\big)$$

identifies with

$$\lim_{\substack{\longleftarrow \\ n}} \big(\{x \in U \widehat{\otimes} U | \forall P, Q, (\delta^{(P)} \otimes \delta^{(Q)})(x) \in \hbar^{|P|+|Q|} U^{\widehat{\otimes} 2}\} / \\ \{x \in U \widehat{\otimes} U | \forall P, Q,\ (\delta^{(P)} \otimes \delta^{(Q)})(x) \in \hbar^{\max(n,|P|+|Q|)} U^{\widehat{\otimes} 2}\}\big).$$

If $x \in U'$, and P, Q are trees, with $|P|, |Q| \neq 0$, then since $\delta^{(P)}(1) = \delta^{(Q)}(1) = 0$, we have

$$(\delta^{(P)} \otimes \delta^{(Q)})(\Delta(x)) = (\delta^{(P)} \otimes \delta^{(Q)})(\delta(x)) = \delta^{(R)}(x) \in \hbar^{|R|} U^{\widehat{\otimes}|R|} \\ = \hbar^{|P|+|Q|} U^{\widehat{\otimes}|P|+|Q|},$$

where R is the tree whose left and right descendants are P and Q; so $|R| = |P| + |Q|$. On the other hand,

$$(\delta^{(P)} \otimes \varepsilon)(\Delta(x)) = \delta^{(P)}(x) \otimes 1 \in \hbar^{|P|} U^{\otimes|P|}$$
$$(\varepsilon \otimes \delta^{(P)})(\Delta(x)) = 1 \otimes \delta^{(P)}(x) \in \hbar^{|P|} U^{\otimes|P|},$$

so $\Delta(x)$ satisfies $(\delta^{(P)} \otimes \delta^{(Q)})(\Delta(x)) \in \hbar^{|P|+|Q|} U^{\bar{\otimes}|P|+|Q|}$ for any pair of trees (P, Q). $\Delta : U \to U \widehat{\otimes} U$ therefore induces an algebra morphism $\Delta_{U'} : U' \to U'^{\bar{\otimes} 2}$, whose reduction modulo $\hbar$ is a morphism of complete local rings

$$\mathscr{O} \to \mathscr{O}^{\bar{\otimes} 2} = \lim_{\substack{\longleftarrow \\ n}} \left(\mathscr{O}^{\otimes 2} / \sum_{p,q | p+q=n} \mathscr{O}_p \otimes \mathscr{O}_q\right),$$

where $\mathscr{O} = U'/\hbar U'$ and $\mathscr{O}_p = U'^{(p)} / (U'^{(p)} \cap \hbar U')$.

§ 3 Classical limit of U'

We will prove Theorem 1.2 as follows. We first compare the various $\delta^{(P)}$, where P is a n-tree (Proposition 3.1). Relations found between the $\delta^{(P)}$ imply that they have $\hbar$-adic valuation properties close to those of the Hopf case (Proposition 3.2). We then prove Theorem 1.2.

- a - Comparison of the various $\delta^{(P)}$

Let P and P_0 be n-trees. There exists an element $\Phi^{P,P_0} \in U^{\widehat{\otimes} n}$, such that $\Delta^{(P)} = \mathrm{Ad}(\Phi^{P,P_0}) \circ \Delta^{(P_0)}$. The element Φ^{P,P_0} is a product of images of Φ and Φ^{-1} by the various maps $U^{\widehat{\otimes} 3} \to U^{\widehat{\otimes} n}$ obtained by iteration of Δ. We have

$$\Phi^{P',P_0} = \Phi^{P',P}\Phi^{P,P_0} \tag{3.4}$$

for any n-trees P_0, P, P'. For example,

$$(\mathrm{id} \otimes \Delta) \circ \Delta = \mathrm{Ad}(\Phi) \circ ((\Delta \otimes \mathrm{id}) \circ \Delta),$$
$$(\Delta \otimes \Delta) \circ \Delta = \mathrm{Ad}(\Phi^{12,3,4}) \circ ((\Delta \otimes \mathrm{id}^{\otimes 2}) \circ (\Delta \otimes \mathrm{id}) \circ \Delta), \text{ etc.}$$

Proposition 3.1. *Assume that* $\hbar \log(\Phi) \in (U')^{\bar{\otimes} 3}$. *Then there exists a sequence of elements*

$$F^{PP_0R\Sigma\nu} = \sum_\alpha F^{PP_0R\Sigma\nu}_{1,\alpha} \otimes \cdots \otimes F^{PP_0R\Sigma\nu}_{\nu,\alpha} \in (U'^{\bar{\otimes} n})^{\bar{\otimes} \nu},$$

indexed by the triples (R, Σ, ν), *where* R *is a tree such that* $|R| < n$, Σ *is a subset of* $\{1, \ldots, n\}$ *with* $\mathrm{card}(\Sigma) = |R|$, *and* ν *is an integer* ≥ 1, *such that the equality*

$$\delta^{(P)} = \mathrm{Ad}(\Phi^{P,P_0}) \circ \delta^{(P_0)} + \sum_{k | k < n} \sum_{R \text{ a } k\text{-tree}} \sum_{\substack{\Sigma \subset \{1,\ldots,n\}, \\ \mathrm{card}(\Sigma) = k}} \sum_{\nu \geq 1} \sum_\alpha \mathrm{ad}_\hbar(F^{PP_0R\Sigma\nu}_{1,\alpha}) \circ \cdots \circ \mathrm{ad}_\hbar(F^{PP_0R\Sigma\nu}_{\nu,\alpha}) \circ (\delta^{(R)})^\Sigma \tag{3.5}$$

holds. Here $\mathrm{ad}_\hbar(x)(y) = \frac{1}{\hbar}[x,y]$.

PROOF. Let us prove this statement by induction on n. When $n = 3$, we find

$$\delta^{(1(23))} = \mathrm{Ad}(\Phi)\delta^{((12)3)} + (\mathrm{Ad}(\Phi) - 1)(\delta^{1,2} + \delta^{1,3} + \delta^{2,3} + \delta^{(1)1} + \delta^{(1)2} + \delta^{(1)3}),$$

so the identity holds with $F^{PP_0R\Sigma\nu} = \frac{1}{\nu!}(\hbar \log \Phi)^{\bar{\otimes} \nu}$ for all choices of (R, Σ, ν), except when $|R| = 0$, in which case $F^{PP_0R\Sigma\nu} = 0$. Assume that the statement holds for any pair of k-trees, $k \leq n$, and let us prove it for a pair (P, P_0) of $(n+1)$-trees. For k any integer, let $P_{\mathrm{left}}(k)$ be the k-tree corresponding to

$$\delta^{(P_{\mathrm{left}}(k))} = (\delta \otimes \mathrm{id}^{\otimes k-2}) \circ \cdots \circ \delta.$$

Thanks to (3.4), we may assume that $P_0 = P_{\mathrm{left}}(n+1)$ and P is arbitrary. Let P' and P'' be the subtrees of P, such that $|P'| + |P''| = n+1$, and $\delta^{(P)} = (\delta^{(P')} \otimes \delta^{(P'')}) \circ \delta$. Let P_1 and P_2 the n-trees such that

$$\delta^{(P_1)} = (\delta^{(P_{\mathrm{left}}(k'))} \otimes \delta^{(P'')}) \circ \delta \text{ and } \delta^{(P_2)} = (\delta^{(P_{\mathrm{left}}(k'))} \otimes \delta^{(P_{\mathrm{left}}(k''))}) \circ \delta$$

Assume that $|P_1| \neq 1$. Using (3.4), we reduce the proof of (3.5) to the case of the pairs $(P, P_1), (P_1, P_2)$ and (P_2, P_0). Then the induction hypothesis applied to the pair $(P', P_{\mathrm{left}}(k'))$, together with $\Phi^{P,P_1} = \Phi^{P',P_{\rm left}(k')} \otimes 1^{\otimes k''}$, implies

$$\delta^{(P)} = \mathrm{Ad}(\Phi^{P,P_1}) \circ \delta^{(P_1)} + \sum_{k|k<k'} \sum_{R \text{ a } k\text{-tree}} \sum_{\substack{\Sigma\subset\{1,\ldots,k'\},\\ \mathrm{card}(\Sigma)=k}}$$

$$\sum_{\nu\geq 1} \sum_{\alpha} \mathrm{Ad}(\Phi^{P,P_1}) \circ \mathrm{ad}_{\hbar}(F_{1,\alpha}^{P' P_{\rm left}(k')\Sigma\nu} \otimes 1^{\otimes k''}) \cdots \mathrm{ad}_{\hbar}(F_{\nu,\alpha}^{P' P_{\rm left}(k')\Sigma\nu} \otimes 1^{\otimes k''})$$
$$\circ ((\delta^{(R)} \otimes \delta^{(P'')}) \circ \delta)^{\Sigma,k'+1,\ldots,n+1},$$

which is (3.5) for (P,P_1). In the same way, one proves a similar identity relating P_1 and P_2. Let us now prove the identity relating P_2 and P_0. We have $\delta^{(P_2)} = (\delta \otimes \mathrm{id}^{\otimes n-1}) \circ \delta^{(P'_2)}$ and $\delta^{(P_0)} = (\delta \otimes \mathrm{id}^{\otimes n-1}) \circ \delta^{(P'_0)}$, where P'_2 and P'_0 are n-trees. We have

$$\Phi^{P_2,P_0} = (\Delta \otimes \mathrm{id}^{\otimes n-2}) \circ \Phi^{P'_2,P'_0}$$

so we get

$$\begin{aligned}
\delta^{(P_2)} &= \mathrm{Ad}(\Phi^{P_2,P_0}) \circ \delta^{(P_0)} \\
&+ \big(\mathrm{Ad}(\Phi^{P_2,P_0}) - \mathrm{Ad}((\Phi^{P'_2,P'_0})^{1,3,\ldots,n+1})\big) \circ (\delta^{(P'_0)})^{1,3,\ldots,n+1} \\
&+ \big(\mathrm{Ad}(\Phi^{P_2,P_0}) - \mathrm{Ad}((\Phi^{P'_2,P'_0})^{2,3,\ldots,n+1})\big) \circ (\delta^{(P'_0)})^{2,3,\ldots,n+1} \\
&+ (\delta \otimes \mathrm{id}^{\otimes n-1})\big(\sum_{k\leq n} \sum_{R \text{ a } k\text{-tree}} \sum_{\substack{\Sigma\subset\{1,\ldots,n\},\\ \mathrm{card}(\Sigma)=k}} \\
&\sum_{\nu\geq 1}\sum_{\alpha} \mathrm{ad}_{\hbar}(F_{1,\alpha}^{P'_2 P'_0 \Sigma\nu}) \cdots \mathrm{ad}_{\hbar}(F_{\nu,\alpha}^{P'_2 P'_0 \Sigma\nu}) \circ (\delta^{(R)})^{\Sigma}\big).
\end{aligned}$$

We have $\hbar \log \Phi^{P_2,P_0} \in U'^{\bar{\otimes} n+1}$ and $\hbar \log \Phi^{P'_2,P'_0} \in U'^{\bar{\otimes} n}$; this fact and the relations

$$\begin{aligned}
&(\delta \otimes \mathrm{id}^{\otimes n-1})(\mathrm{ad}_{\hbar}(x_1) \cdots \mathrm{ad}_{\hbar}(x_\nu) \circ (\delta^{(R)})^{\Sigma}) = \\
&\quad \big(\mathrm{ad}_{\hbar}(x_1^{12,\ldots,n+1}) \circ \cdots \circ \mathrm{ad}_{\hbar}(x_\nu^{12,\ldots,n+1}) - \mathrm{ad}_{\hbar}(x_1^{1,3,\ldots,n+1}) \circ \cdots \circ \mathrm{ad}_{\hbar}(x_\nu^{1,3,\ldots,n+1}) \\
&\qquad - \mathrm{ad}_{\hbar}(x_1^{2,3,\ldots,n+1}) \circ \cdots \circ \mathrm{ad}_{\hbar}(x_\nu^{2,3,\ldots,n+1})\big) \circ (\delta^{(R)})^{\Sigma+1}
\end{aligned}$$

if $1 \notin \Sigma$, and

$$\begin{aligned}
&(\delta \otimes \mathrm{id}^{\otimes n-1})(\mathrm{ad}_{\hbar}(x_1) \cdots \mathrm{ad}_{\hbar}(x_\nu) \circ (\delta^{(R)})^{\Sigma}) = \\
&\quad \mathrm{ad}_{\hbar}(x_1^{12,\ldots,n+1}) \circ \cdots \circ \mathrm{ad}_{\hbar}(x_\nu^{12,\ldots,n+1}) \circ ((\delta \otimes \mathrm{id}^{\otimes n-1}) \circ \delta^{(R)})^{1,2,\Sigma'+1} \\
&\quad + \big(\mathrm{ad}_{\hbar}(x_1^{12,\ldots,n+1}) \circ \cdots \circ \mathrm{ad}_{\hbar}(x_\nu^{12,\ldots,n+1}) - \mathrm{ad}_{\hbar}(x_1^{1,3,\ldots,n+1}) \circ \cdots \circ \mathrm{ad}_{\hbar}(x_\nu^{1,3,\ldots,n+1})\big) \\
&\qquad \circ (\delta^{(R)})^{1,\Sigma'+1} \\
&\quad + \big(\mathrm{ad}_{\hbar}(x_1^{12,\ldots,n+1}) \circ \cdots \circ \mathrm{ad}_{\hbar}(x_\nu^{12,\ldots,n+1}) - \mathrm{ad}_{\hbar}(x_1^{2,3,\ldots,n+1}) \circ \cdots \circ \mathrm{ad}_{\hbar}(x_\nu^{2,3,\ldots,n+1})\big) \\
&\qquad \circ (\delta^{(R)})^{2,\Sigma'+1}.
\end{aligned}$$

if $\Sigma = \Sigma' \cup \{1\}$, where $1 \notin \Sigma'$, imply that $\delta^{(P_2)} - \mathrm{Ad}(\Phi^{P_2,P_0}) \circ \delta^{(P_0)}$ has the desired form. Let us now treat the case $|P_1| = 1$. For this, we introduce the trees P_3 and P_4, such that:

$$\begin{aligned}
\delta^{(P_3)} &= (\mathrm{id}^{\otimes n-1} \otimes \delta) \circ (\mathrm{id}^{\otimes n-2} \otimes \delta) \circ \cdots \circ \delta, \\
\delta^{(P_4)} &= (\mathrm{id}^{\otimes n-1} \otimes \delta) \circ (\delta \otimes \mathrm{id}^{\otimes n-2}) \circ (\delta \otimes \mathrm{id}^{\otimes n-3}) \circ \cdots \circ (\delta \otimes \mathrm{id}) \circ \delta.
\end{aligned}$$

We then prove the relation for the pair (P,P_3) in the same way as for (P_1,P_2) (only the right branch of the tree is changed); the relation for (P_3,P_4) in the same way as for (P_2,P_3) (instead of composing a known relation by $\delta\otimes \mathrm{id}^{\otimes n-1}$, we compose it with $\mathrm{id}^{\otimes n-1}\otimes\delta$); and using the identity

$$\delta^{(P_4)} = (\delta\otimes \mathrm{id}^{\otimes n-1})\circ(\mathrm{id}^{\otimes n-2}\otimes\delta)\circ(\delta\otimes \mathrm{id}^{\otimes n-3})\circ\cdots\circ\delta,$$

we prove the relation for (P_4,P) in the same way as for (P_2,P_3) (composing a known relation by $\delta\otimes \mathrm{id}^{\otimes n-1}$). $\square$

- b - Properties of $\delta^{(P)}$

Proposition 3.2. *Let n be an integer and $x\in U$.*

1) *Assume that for any tree R, such that $|R|<n$, we have $\delta^{(R)}(x)\in \hbar^{|R|}U^{\widehat{\otimes}|R|}$. Then the conditions*

$$\delta^{(P)}(x)\in \hbar^n U^{\widehat{\otimes}n} \tag{3.6}$$

where P is an n-tree, are all equivalent.

2) *Assume that for any tree R, such that $|R|<n$, we have $\delta^{(R)}(x)\in \hbar^{|R|+1}U^{\widehat{\otimes}|R|}$. Then the elements*

$$\left(\frac{1}{\hbar^n}\delta^{(P)}(x)\,\mathrm{mod}\,\hbar\right)\in U(\mathfrak{g})^{\otimes n},$$

where P is an n-tree, are all equal and belong to $(\mathfrak{g}^{\otimes n})^{\mathfrak{S}_n}=S^n(\mathfrak{g})$.

PROOF. Let us prove 1). We have $\delta^{(P)}=(\mathrm{id}-\eta\circ\varepsilon)^{\otimes|P|}\circ\delta^{(P)}$, where $\eta:\mathbb{K}[[\hbar]]\to U$ is the unit map of U, so

$$\delta^{(P)} = \mathrm{Ad}(\Phi^{P,P_0})\circ\delta^{(P_0)} + \sum_{k|k<n}\sum_{R\text{ a }k\text{-tree}}\sum_{\substack{\Sigma\subset\{1,\dots,n\},\\ \mathrm{card}(\Sigma)=k}}\sum_{\nu\geq 1}\sum_{\alpha}$$
$$(\mathrm{id}-\eta\circ\varepsilon)^{\otimes n}\circ \mathrm{ad}_{\hbar}(F_{1,\alpha}^{PP_0R\Sigma\nu})\circ\cdots\circ\mathrm{ad}_{\hbar}(F_{\nu,\alpha}^{PP_0R\Sigma\nu})\circ(\delta^{(R)})^{\Sigma}.$$

Then 1) follows from:

Lemma 3.3. *Let Σ be a subset of $\{1,\dots,n\}$ (we will write $|\Sigma|$ instead of $\mathrm{card}(\Sigma)$) and let U_0 be the kernel of the counit of U. Let $x\in\hbar^{|\Sigma|}(U_0)^{\widehat{\otimes}|\Sigma|}$ and $F_1,\dots,F_\nu$ be elements of $(U')^{\bar{\otimes}n}$. Then*

$$(\mathrm{id}-\eta\circ\varepsilon)^{\otimes n}(\mathrm{ad}_{\hbar}(F_1)\cdots\mathrm{ad}_{\hbar}(F_\nu)(x^{\Sigma}))\in\hbar^n(U_0)^{\widehat{\otimes}n}.$$

PROOF OF LEMMA. Each element $F\in(U')^{\bar{\otimes}n}$ is uniquely expressed as a sum $F=\sum_{\Sigma\in\mathscr{P}(\{1,\dots,n\})}F_\Sigma$, where F_Σ belongs to the image of

$$(U_0')^{\bar{\otimes}|\Sigma|}\to(U')^{\bar{\otimes}n},$$
$$f\mapsto f^{\Sigma},$$

$\mathscr{P}(\{1,\dots,n\})$ is the set of subsets of $\{1,\dots,n\}$, and U_0' is the kernel of the counit of U'. Then

$$(\mathrm{id}-\eta\circ\varepsilon)^{\otimes n}(\mathrm{ad}_{\hbar}(F_1)\cdots\mathrm{ad}_{\hbar}(F_\nu)(x^{\Sigma}))$$
$$=\sum_{\Sigma_1,\dots,\Sigma_\nu\in\mathscr{P}(\{1,\dots,n\})}(\mathrm{id}-\eta\circ\varepsilon)^{\otimes n}\big(\mathrm{ad}_{\hbar}((F_1)_{\Sigma_1})\cdots\mathrm{ad}_{\hbar}((F_\nu)_{\Sigma_\nu})(x^{\Sigma})\big).$$

The summands corresponding to $(\Sigma_1,\dots,\Sigma_\nu)$ such that $\Sigma_1\cup\cdots\Sigma_\nu\cup\Sigma\neq\{1,\dots,n\}$ are all zero. Moreover, each $(F_\alpha)_{\Sigma_\alpha}$ can be expressed as $(f_\alpha)^{\Sigma_\alpha}$, where $f_\alpha\in\hbar^{|\Sigma_\alpha|}(U_0)^{\widehat{\otimes}|\Sigma_\alpha|}$. The lemma then follows from the statement:

Statement 3.4. *If $\Sigma,\Sigma'\subset\{1,\dots,n\}$, $x\in\hbar^{|\Sigma|}(U_0)^{\widehat{\otimes}|\Sigma|}$, $y\in\hbar^{|\Sigma'|}(U_0)^{\widehat{\otimes}|\Sigma'|}$, then $\frac{1}{\hbar}[x,y]$ can be expressed as $z^{\Sigma\cup\Sigma'}$, where $z\in\hbar^{|\Sigma\cup\Sigma'|}(U_0)^{\widehat{\otimes}|\Sigma\cup\Sigma'|}$.*

PROOF. If $\Sigma\cap\Sigma'=\emptyset$, then $[x,y]=0$, so the statement holds. If $\Sigma\cap\Sigma'\neq\emptyset$, then the $\hbar$-adic valuation of $\frac{1}{\hbar}[x,y]$ is $\geq -1+|\Sigma|+|\Sigma'|\geq|\Sigma|+|\Sigma'|-|\Sigma\cap\Sigma'|=|\Sigma\cup\Sigma'|$. □

Let us now prove property 2). The above arguments immediately imply that the $(\frac{1}{\hbar^n}\delta^{(P)}(x)\bmod\hbar)$, $|P|=n$, are all equal. This defines an element $S_n(x)\in U(\mathfrak{g})^{\otimes n}$. If $|P|=n$, we have $(\mathrm{id}^{\otimes k}\otimes\delta\otimes\mathrm{id}^{\otimes n-k-1})\circ\delta^{(P)}(x)\in\hbar^{n+1}U^{\widehat{\otimes}n+1}$, so if $\delta_0: U(\mathfrak{g})\to U(\mathfrak{g})\otimes U(\mathfrak{g})$ is defined by $\delta_0(x)=\Delta_0(x)-x\otimes 1-1\otimes x+\varepsilon(x)1\otimes 1$, Δ_0 being the coproduct of $U(\mathfrak{g})$, then $(\mathrm{id}^{\otimes k}\otimes\delta_0\otimes\mathrm{id}^{\otimes n-k-1})(S_n(x))=0$, so

$$S_n\in\mathfrak{g}^{\otimes n}. \tag{3.7}$$

Let us denote by $\sigma_{i,i+1}$ the permutation of the factors i and $i+1$ in a tensor power. For $i=1,\dots,n-1$, let us compute $(\sigma_{i,i+1}-\mathrm{id})(S_n(x))$. Let P' be a $(n-1)$-tree and let P be the n-tree such that $\delta^{(P)}=(\mathrm{id}^{\otimes i-1}\otimes\delta\otimes\mathrm{id}^{\otimes n-i-1})\circ\delta^{(P')}$. Then

$$(\sigma_{i,i+1}-\mathrm{id})(S_n)=\left[\frac{1}{\hbar}(\mathrm{id}^{\otimes i-1}\otimes(\delta^{2,1}-\delta)\otimes\mathrm{id}^{\otimes n-i-1})\circ\delta^{(P')}(x)\bmod\hbar\right].$$

By assumption, $\delta^{(P')}(x)\in\hbar^n U^{\widehat{\otimes}n-1}$; moreover, $\delta^{2,1}-\delta=\Delta^{2,1}-\Delta$, so $(\delta^{2,1}-\delta)(U)\subset\hbar(U\widehat{\otimes}U)$; therefore

$$(\mathrm{id}^{\otimes i-1}\otimes(\delta^{2,1}-\delta)\otimes\mathrm{id}^{\otimes n-i-1})\circ\delta^{(P')}(x)\in\hbar^{n+1}U^{\widehat{\otimes}n};$$

it follows that $(\sigma_{i,i+1}-\mathrm{id})(S_n(x))=0$, therefore $S_n(x)$ is a symmetric tensor of $U(\mathfrak{g})^{\otimes n}$. Together with (3.7), this gives $S_n(x)\in(\mathfrak{g}^{\otimes n})^{\mathfrak{S}_n}$. This ends the proof of Proposition 3.2. □

- c - Flatness of U' (proof of Theorem 1.2)

Let us set

$$U''^{(n)}=\{x\in U'|\delta^{(P)}(x)\in\hbar^{|P|+1}U^{\widehat{\otimes}|P|}\text{ if }|P|\leq n\}.$$

Then by Proposition 2.4, we have a decreasing algebra filtration

$$U'=U''^{(0)}\supset U''^{(1)}\supset U''^{(2)}\supset\cdots\supset\hbar U'. \tag{3.8}$$

Each $U''^{(n)}$ is divisible in U', i.e., $U''^{(n)}\cap\hbar U'=\hbar U''^{(n)}$. We also have $U''^{(n)}\supset U'^{(n)}+\hbar U'$ (we will see later that this is an equality). We derive from (3.8) a decreasing filtration

$$\mathscr{O}=\mathscr{O}''^{(0)}\supset\mathscr{O}''^{(1)}\supset\mathscr{O}''^{(2)}\supset\cdots,$$

where $\mathscr{O}=U'/\hbar U'$ and $\mathscr{O}''^{(n)}=U''^{(n)}/\hbar U''^{(n)}$. We have clearly

$$\bigcap_{n\geq 0}\mathscr{O}''^{(n)}=\{0\};$$

the fact that $\mathscr{O}$ is complete for this filtration will follow from its identification with the filtration $\mathscr{O}\supset\mathscr{O}'^{(1)}\supset\cdots$ (see Proposition 3.6), where $\mathscr{O}'^{(i)}=U'^{(i)}/\hbar U\cap U'^{(i)}$ and $U'^{(i)}$ is defined in (2.3). We first prove:

Proposition 3.5. *Set* $\widehat{\mathrm{gr}}''(\mathscr{O}) = \widehat{\bigoplus}_{n\geq 0} \mathscr{O}''^{(n)}/\mathscr{O}''^{(n+1)}$*. Then there is a unique linear map* λ_n *:* $\mathrm{gr}''_n(\mathscr{O}) \to S^n(\mathfrak{g})$*, taking the class of x to the common value of all* $\frac{1}{n!}(\frac{1}{\hbar^n}\delta^{(P)}(x) \bmod \hbar)$*, where P is a n-tree. The resulting map* $\lambda : \widehat{\mathrm{gr}}''(\mathscr{O}) \to \widehat{S}^{\cdot}(\mathfrak{g})$ *is an isomorphism of graded complete algebras.*

PROOF. In Proposition 3.2, we constructed a map $U''^{(n)} \to S^n(\mathfrak{g})$, by $x \mapsto$ common value of $\frac{1}{n!}(\frac{1}{\hbar^n}\delta^{(P)}(x) \bmod \hbar)$ for all n-trees P. The subspace $U''^{(n+1)} \subset U''^{(n)}$ is clearly contained in the kernel of this map, so we obtain a map

$$\lambda_n : U''^{(n)}/U''^{(n+1)} = \mathscr{O}''^{(n)}/\mathscr{O}''^{(n+1)} \to S^n(\mathfrak{g}).$$

Let us prove that $\lambda = \widehat{\bigoplus}_{n\geq 1} \lambda_n$ is a morphism of algebras. If $x \in U''^{(n)}$ and $y \in U''^{(m)}$, Proposition 2.4 implies that if R is any $(n+m)$-tree, we have

$$\delta^{(P)}(xy) = \sum_{\substack{\Sigma',\Sigma''\subset\{1,\ldots,n+m\}| \\ \Sigma'\cup\Sigma''=\{1,\ldots,n+m\}}} \delta^{(R_{\Sigma'})}(x)^{\Sigma'}\delta^{(R_{\Sigma''})}(y)^{\Sigma''}.$$

The $\hbar$-adic valuation of the term corresponding to (Σ',Σ'') is $\geq |\Sigma'|+|\Sigma''|$ if $|\Sigma'|\geq n$ and $|\Sigma''|\geq m$, and $\geq |\Sigma'|+|\Sigma''|+1$ otherwise, so the only contributions to $(\frac{1}{\hbar^{n+m}}\delta^{(R)}(xy) \bmod \hbar)$ are those of the pairs (Σ',Σ'') such that $\Sigma'\cap\Sigma''=\emptyset$. Then:

$$\begin{aligned}
&(\frac{1}{\hbar^{n+m}}\delta^{(R)}(xy) \bmod \hbar) \\
&\quad = \sum_{\substack{\Sigma',\Sigma''\subset\{1,\ldots,n+m\}| \\ |\Sigma'|=n, |\Sigma''|=m, \\ \Sigma'\cap\Sigma''=\emptyset}} (\frac{1}{\hbar^n}\delta^{(R_{\Sigma'})}(x) \bmod \hbar)(\frac{1}{\hbar^m}\delta^{(R_{\Sigma''})}(y) \bmod \hbar) \\
&\quad = \sum_{\substack{\Sigma',\Sigma''\subset\{1,\ldots,n+m\}| \\ |\Sigma'|=n, |\Sigma''|=m, \\ \Sigma'\cap\Sigma''=\emptyset}} (n!\lambda_n(x)^{\Sigma'})(m!\lambda_m(y)^{\Sigma''}) \\
&\quad = (n+m)!\lambda_n(x)\lambda_m(y),
\end{aligned}$$

because the map

$$S^{\cdot}(\mathfrak{g}) \to (T(\mathfrak{g}), \text{shuffle product}),$$
$$x_1\cdots x_n \mapsto \sum_{\sigma\in\mathfrak{S}_n} x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(n)}$$

is an algebra morphism. Therefore $\lambda_{n+m}(xy) = \lambda_n(x)\lambda_m(y)$. Let us prove that λ_n is injective. If $x \in U''^{(n)}$ is such that $(\frac{1}{\hbar^n}\delta^{(P)}(x) \bmod \hbar) = 0$ for any n-tree P, then $x \in U''^{(n+1)}$, so its class in $\mathscr{O}''^{(n)}/\mathscr{O}''^{(n+1)} = U''^{(n)}/U''^{(n+1)}$ is zero. So each λ_n is injective, so λ is injective.

To prove that λ is surjective, it suffices to prove that λ_1 is surjective. Let us fix $x \in \mathfrak{g}$. We will construct a sequence $x_n \in U$, $n \geq 0$ such that $\varepsilon(x_n) = 0$, $(\frac{1}{\hbar}x_n \bmod \hbar) = x$, $x_{n+1} \in x_n + \hbar^{n+1}U$ for any $n \geq 1$, and if P is any tree such that $|P| \leq n$, $\delta^{(P)}(x_n) \in \hbar^{|P|}U^{\widehat{\otimes}|P|}$ (this last condition implies that $\delta^{(Q)}(x_n) \in \hbar^n U^{\widehat{\otimes}|Q|}$ for $|Q| \geq n$). Then the limit $\widetilde{x} = \lim_{n\to\infty}(x_n)$ exists, belongs to U', satisfies $\varepsilon(\widetilde{x}) = 0$ and $(\frac{1}{\hbar}\delta_1(\widetilde{x}) \bmod \hbar) = x$, so its class in $U''^{(1)}/U''^{(2)}$ is a preimage of x.

Let us now construct the sequence $(x_n)_{n\geq 0}$. We fix a linear map $\mathfrak{g} \to \{y \in U | \varepsilon(y) = 0\}$, $y \mapsto \overline{y}$, such that for any $y \in \mathfrak{g}$, $(\overline{y} \bmod \hbar) = y$. We set $x_1 = \overline{\hbar x}$. Let us construct x_{n+1} knowing x_n. By Proposition 3.2, if Q is any $(n+1)$-tree, $\delta^{(Q)}(x_n) \in \hbar^n U^{\widehat{\otimes} n+1}$, and $(\frac{1}{\hbar^n}\delta^{(Q)}(x_n) \bmod \hbar)$ is an element of $S^{n+1}(\mathfrak{g})$, independent of Q. Let us write this element as

$$\sum_{\sigma\in\mathfrak{S}_{n+1}} \sum_\alpha y^\alpha_{\sigma(1)} \cdots y^\alpha_{\sigma(n+1)}, \text{ where } \sum_\alpha y^\alpha_1 \otimes \cdots \otimes y^\alpha_{n+1} \in \mathfrak{g}^{\otimes n+1}.$$

Then we set

$$x_{n+1} = x_n - \frac{\hbar^{n+1}}{(n+1)!} \sum_{\sigma\in\mathfrak{S}_{n+1}} \overline{y}^\alpha_{\sigma(1)} \cdots \overline{y}^\alpha_{\sigma(n+1)}.$$ □

We now prove:

Proposition 3.6.

1) *For any $n \geq 0$, $U''^{(n)} = U'^{(n)} + \hbar U'$;*

2) *The filtrations $\mathcal{O} = \mathcal{O}'^{(0)} \supset \mathcal{O}'^{(1)} \supset \cdots$ and $\mathcal{O} = \mathcal{O}''^{(0)} \supset \mathcal{O}''^{(1)} \supset \cdots$ coincide, and $\mathcal{O}$ is complete and separated for this filtration.*

PROOF. Let us prove 1). We have to show that $U''^{(n)} \subset U'^{(n)} + \hbar U'$. Let $x \in U''^{(n)}$. We have $\delta^{(P)}(x) \in \hbar^{|P|+1} U^{\widehat{\otimes}|P|}$ for $|P| \leq n-1$, and for P an n-tree, $(\frac{1}{\hbar^n}\delta^{(P)}(x) \bmod \hbar) \in S^n(\mathfrak{g})$ and is independent on P. Write this element of $S^n(\mathfrak{g})$ as $\sum_{\sigma\in\mathfrak{S}_n}\sum_\alpha y^\alpha_{\sigma(1)} \otimes \cdots y^\alpha_{\sigma(n)}$ and set $f_n = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\sum_\alpha \overline{y}^\alpha_{\sigma(1)} \cdots \overline{y}^\alpha_{\sigma(n)}$. Then each $\overline{y}^\alpha_i$ belongs to $U' \cap \hbar U$, so $f_n \in U' \cap \hbar^n U = U'^{(n)}$. Moreover, $x - f_n$ belongs to $U''^{(n+1)}$. Iterating this procedure, we construct elements $f_{n+1}, f_{n+2}, \ldots$, where each f_k belongs to $U'^{(k)}$. The series $\sum_{k\geq n} f_k$ converges in U'; denote by f its sum, then $x - f$ belongs to $\bigcap_{k\geq n} U''^{(k)} = \hbar U'$. So $U''^{(n)} \subset U'^{(n)} + \hbar U'$. The inverse inclusion is obvious. This proves 1). Then 1) immediately implies that for any n, $\mathcal{O}'^{(n)} = \mathcal{O}''^{(n)}$. We already know $\mathcal{O}$ is complete and separated for $\mathcal{O} = \mathcal{O}'^{(0)} \supset \mathcal{O}'^{(1)} \supset \cdots$, which proves 2). □

END OF PROOF OF THEOREM 1.2. $\mathcal{O}$ is a complete local ring, and we have a ring isomorphism $\widehat{\mathrm{gr}}(\mathcal{O}) \to \widehat{S}^\cdot(\mathfrak{g})$. Then any lift $\mathfrak{g} \to \mathcal{O}'^{(1)}$ of $\mathcal{O}'^{(1)} \to \mathcal{O}'^{(1)}/\mathcal{O}'^{(2)} = \mathfrak{g}$ yields a continuous ring morphism $\mu : \widehat{S}^\cdot(\mathfrak{g}) \to \mathcal{O}$. The associated graded of μ is the identity, so μ is an isomorphism. So $\mathcal{O}$ is noncanonically isomorphic to $\widehat{S}^\cdot(\mathfrak{g})$. □

Remark 3.7. When U is Hopf and $\mathfrak{g}$ is finite-dimensional, $U'/\hbar U'$ identifies canonically with $\mathcal{O}_{G^*} = (U(\mathfrak{g}^*))^*$, where $\mathfrak{g}^*$ is the dual Lie bialgebra of $\mathfrak{g}$ (see [Dr1], [Ga]). The natural projection $T(\mathfrak{g}^*) \to U(\mathfrak{g}^*)$ and the identification $T(\mathfrak{g}^*)^* = \widehat{T}(\mathfrak{g})$ (where $\widehat{T}(\mathfrak{g})$ means the degree completion) induce an injection $U'/\hbar U' = \mathcal{O}_{G^*} = (U(\mathfrak{g}^*))^* \hookrightarrow \widehat{T}(\mathfrak{g})$. The map $U'/\hbar U' \hookrightarrow \widehat{T}(\mathfrak{g})$ can be interpreted simply as follows. For any $x \in U'$, we have $(\frac{1}{\hbar^n}\delta_n(x) \bmod \hbar) \in \mathfrak{g}^{\otimes n}$. Then $U'/\hbar U' \hookrightarrow \widehat{T}(\mathfrak{g})$ takes the class of $x \in U'$ to the sequence $(\frac{1}{\hbar^n}\delta_n(x) \bmod \hbar)_{n\geq 0}$.

In the quasi-Hopf case, we have no canonical embedding $U'/\hbar U' \hookrightarrow \widehat{T}(\mathfrak{g})$ because the various $(\frac{1}{\hbar^n}\delta^{(P)}(x) \bmod \hbar)$ do not necessarily coincide for all the n-trees P. This is related to the fact that one cannot expect a Hopf pairing $U(\mathfrak{g}^*)\otimes(U'/\hbar U') \to \mathbb{K}$ since $\mathfrak{g}^*$ is no longer a Lie algebra, so $U(\mathfrak{g}^*)$ does not make sense.

In the other hand, Theorem 1.2 can be interpreted as follows: in the Hopf case, the exponential induces an isomorphism of formal schemes $\mathfrak{g}^* \to G^*$, so $U'/\hbar U'$ identifies noncanonically with $\mathscr{O}_{\mathfrak{g}^*} = \widehat{S}^{\cdot}(\mathfrak{g})$. In the quasi-Hopf case, although there is no formal group G^*, we still have an isomorphism $U'/\hbar U' \xrightarrow{\sim} \widehat{S}^{\cdot}(\mathfrak{g})$. ☐

§ 4 Twists

- a - Admissible twists

If (U,m,Δ,Φ) is an arbitrary QHQUE algebra, we will call a twist $F \in (U^{\widehat{\otimes}2})^\times$ *admissible* if $\hbar\log(F) \in (U')^{\bar{\otimes}2}$.

Proposition 4.1. *Let (U,m,Δ,Φ) be an admissible quasi-Hopf algebra and F an admissible twist. Then the twisted quasi-Hopf algebra $(U,m,{}^F\Delta,{}^F\Phi)$ is admissible.*

PROOF. Set $f = \hbar\log(F)$. Then we have

$$\hbar\log({}^F\Phi) = f^{1,2} \star f^{12,3} \star (\hbar\log(\Phi)) \star (-f^{1,23}) \star (-f^{2,3}),$$

where $a \star b = a + b + \frac{1}{\hbar}[a,b] + \cdots$ (the CBH series for $U'^{\bar{\otimes}3}$ equipped with the bracket $\frac{1}{\hbar}[-,-]$). Since $U'^{\bar{\otimes}3}$ is stable under $\star$, we have $\hbar\log({}^F\Phi) \in U'^{\bar{\otimes}3}$. So $(U,m,{}^F\Delta,{}^F\Phi)$ is admissible. ☐

Let us now prove

Proposition 4.2. *Under the hypothesis of Proposition 4.1, the QFS algebra U'_F corresponding to $(U,m,{}^F\Delta,{}^F\Phi)$ coincides with the QFS algebra U' corresponding to (U,m,Δ,Φ).*

We will first prove the following lemma:

Lemma 4.3. *Let P be an n-tree. Then*

$$\delta_F^{(P)} = \delta^{(P)} + \sum_{k\le n}\sum_{R \text{ a } k\text{-tree}}\sum_{\substack{\Sigma\subset\{1,\dots,n\}|\\ \mathrm{card}(\Sigma)=k}} \sum_{\nu\ge1}\sum_{\alpha} \mathrm{ad}_{\hbar}(f_{1,\alpha}^{\Sigma,P})\circ\cdots\circ\mathrm{ad}_{\hbar}(f_{\nu,\alpha}^{\Sigma,P})\circ(\delta^{(R)})^{\Sigma}, \quad (4.9)$$

where for each ν, $\sum_\alpha f_{1,\alpha}^{\Sigma,P}\otimes\cdots\otimes f_{\nu,\alpha}^{\Sigma,P} \in (U'^{\bar{\otimes}n})^{\bar{\otimes}\nu}$.

Remark 4.4. One can prove that in the right hand side of (4.9), the contribution of all terms with $k = n$ is $(\mathrm{Ad}(F^{(P)}) - \mathrm{id})\circ\delta^{(P)}$ where $F^{(P)}$ is the product of $F^{I,J}$ (I,J subsets of $\{1,\dots,n\}$, such that $\max(I) < \min(J)$) and their inverses such that

$$\Delta_F^{(P)} = \mathrm{Ad}(F^{(P)})\circ\Delta^{(P)}.$$

PROOF OF THE LEMMA. equation (4.9) may be proved by induction on $|P|$. Let us prove it for the unique tree P such that $|P| = 2$:

$$\delta_F^{(2)} = \delta^{(2)} + \sum_{\nu\geq 1}\frac{1}{\nu!}\,\mathrm{ad}_{\hbar}(f)^{\nu}(\delta^{(2)}(x) + \delta^{(1)}(x)^1 + \delta^{(1)}(x)^2),$$

where (1) and (2) are the 1- and 2-trees. Assume that (4.9) is proved when $|P| = n$. Let P' be an $(n+1)$-tree. Then for some $i \in \{1,\ldots,n\}$, we have

$$\delta_F^{(P)} = (\mathrm{id}^{\otimes i-1} \otimes \delta_F^{(2)} \otimes \mathrm{id}^{\otimes n-i}) \circ \delta_F^{(P')},$$

where $|P'| = n$. Then:

$$\begin{aligned}
\delta_F^{(P)} =& (\mathrm{id}^{\otimes i-1} \otimes \Delta_F \otimes \mathrm{id}^{\otimes n-i}) \circ \delta_F^{(P')} - (\delta_F^{(P')})^{1,\ldots,\widehat{i},\ldots,n+1} - (\delta_F^{(P')})^{1,\ldots,\widehat{i+1},\ldots,n+1}\\
=& (\mathrm{id}^{\otimes i-1} \otimes \Delta_F \otimes \mathrm{id}^{\otimes n-i}) \circ \big(\delta^{(P')} + \sum_{k\leq n}\sum_{R \text{ a } k\text{-tree}}\ \sum_{\substack{\Sigma\subset\{1,\ldots,n\}|\\ \mathrm{card}(\Sigma)=k}}\\
&\qquad \sum_{\nu\geq 1}\sum_{\alpha}\mathrm{ad}_{\hbar}(f_{1,\alpha}^{\Sigma,P'}) \circ \cdots \circ \mathrm{ad}_{\hbar}(f_{\nu,\alpha}^{\Sigma,P'}) \circ (\delta^{(R)})^{\Sigma}\big)\\
&- (\cdots)^{1\ldots\widehat{i},\ldots,n+1} - (\cdots)^{1,\ldots,\widehat{i+1},\ldots,n+1}\\
=& \mathrm{Ad}(F^{i,i+1}) \circ \big(\delta^{(P)} + (\delta^{(P')})^{1,\ldots,\widehat{i},\ldots,n+1} + (\delta^{(P')})^{1,\ldots,\widehat{i+1},\ldots,n+1}\\
&+ \sum_{k\leq n}\sum_{R \text{ a } k\text{-tree}}\ \sum_{\substack{\Sigma\subset\{1,\ldots,n\}|\\ \mathrm{card}(\Sigma)=k}}\ \sum_{\nu\geq 1}\sum_{\alpha}\mathrm{ad}_{\hbar}((f_{1,\alpha}^{\Sigma,P'})^{1,\ldots,\{i,i+1\},\ldots,n+1})\circ\\
&\qquad \circ\,\mathrm{ad}_{\hbar}((f_{\nu,\alpha}^{\Sigma,P'})^{1,\ldots,\{i,i+1\},\ldots,n+1}) \circ (1^{\otimes i-1} \otimes \Delta \otimes 1^{\otimes n-i}) \circ (\delta^{(R)})^{\Sigma}\big)\\
&- (\cdots)^{1,\ldots,\widehat{i},\ldots,n+1} - (\cdots)^{1,\ldots,\widehat{i+1},\ldots,n+1};
\end{aligned}$$

this has the desired form because:

$$\begin{aligned}
&(\mathrm{Ad}(F^{i,i+1}) - 1) \circ \big(\delta^{(P)} + (\delta^{(P')})^{1,\ldots,\widehat{i},\ldots,n+1} + (\delta^{(P')})^{1,\ldots,\widehat{i+1},\ldots,n+1}\big)\\
&\qquad = \sum_{\nu\geq 1}\frac{1}{\nu!}\,\mathrm{ad}_{\hbar}(f^{i,i+1})^{\nu}\big(\delta^{(P)} + (\delta^{(P')})^{1,\ldots,\widehat{i},\ldots,n+1} + (\delta^{(P')})^{1,\ldots,\widehat{i+1},\ldots,n+1}\big).
\end{aligned}$$

This proves (4.9). □

END OF PROOF OF PROPOSITION 4.2. One repeats the proof of Proposition 3.2 to prove that if $x \in U'$, then we have $\delta^{(P)}(x) \in \hbar^{|P|}U^{\widehat{\otimes}|P|}$ for any tree P. So $U' \subset U'_F$. Since (U,m,Δ,Φ) is the twist by F^{-1} of $(U,m,{}^F\Delta,{}^F\Phi)$, and $\hbar\log(F^{-1}) = -\hbar\log(F) \in (U')^{\bar{\otimes}2} \subset (U'_F)^{\otimes 2}$, F^{-1} is admissible for $(U,m,{}^F\Delta,{}^F\Phi)$, so we have also $U'_F \subset U'$, so $U'_F = U'$. □

- b - Twisting any algebra into an admissible algebra

Proposition 4.5. *Let (U,m,Δ,Φ) be a quasi-Hopf algebra. There exists a twist F_0 such that the twisted quasi-Hopf algebra $(U,m,{}^{F_0}\Delta,{}^{F_0}\Phi)$ is admissible.*

PROOF. We construct F_0 as a convergent infinite product $F_0 = \cdots F_n \cdots F_2$, where $F_n \in 1 + \hbar^{n-1} U^{\hat{\otimes} 2}$, and the F_n have the following property: if $\bar{F_n} = F_n F_{n-1} \cdots F_2$, if $\Phi_n = {}^{\bar{F}_n}\Phi$, and $\delta_n^{(P)} : U \to U^{\hat{\otimes}|P|}$ is the map corresponding to a tree P and to $\Delta_n = \mathrm{Ad}(\bar{F_n}) \circ \Delta$, then we have

$$(\delta_n^{(P)} \otimes \delta_n^{(Q)} \otimes \delta_n^{(R)})(\hbar \log(\Phi_n)) \in \hbar^{|P|+|Q|+|R|} U^{\hat{\otimes}|P|+|Q|+|R|}$$

for any trees P, Q, R such that $|P| + |Q| + |R| \leq n$.

Assume that we have constructed $F_1, \dots, F_n$, and let us construct F_{n+1}. The argument of Proposition 3.2 shows that for any integers (n_1, n_2, n_3) such that $n_1 + n_2 + n_3 = n + 1$, and any trees P, Q, R such that $|P| = n_1$, $|Q| = n_2$, $|R| = n_3$,

$$\big(\frac{1}{\hbar^n}(\delta_n^{(P)} \otimes \delta_n^{(Q)} \otimes \delta_n^{(R)})(\hbar \log(\Phi_n)) \bmod \hbar\big) \in S^{n_1}(\mathfrak{g}) \otimes S^{n_2}(\mathfrak{g}) \otimes S^{n_3}(\mathfrak{g}),$$

and is independent of the trees P, Q, R. The direct sum of these elements is an element $\bar{\varphi}_n$ of $S^{\cdot}(\mathfrak{g})^{\otimes 3}$, homogeneous of degree $n+1$. Since Φ_n satisfies the pentagon equation

$$(\mathrm{id} \otimes \mathrm{id} \otimes \Delta_n)(\Phi_n)^{-1}(1 \otimes \Phi_n)(\mathrm{id} \otimes \Delta_n \otimes \mathrm{id})(\Phi_n)(\Phi_n \otimes 1)(\Delta_n \otimes \mathrm{id} \otimes \mathrm{id})(\Phi_n)^{-1} = 1,$$

$\varphi_n^{\hbar} := \hbar \log(\Phi_n)$ satisfies the equation

$$\begin{aligned}\big(-(\mathrm{id} \otimes \mathrm{id} \otimes \Delta_n)(\varphi_n^{\hbar})\big) \star (1 \otimes \varphi_n^{\hbar}) \star \big((\mathrm{id} \otimes \Delta_n \otimes \mathrm{id})(\varphi_n^{\hbar})\big) \star \\ (\varphi_n^{\hbar} \otimes 1) \star \big(-(\Delta_n \otimes \mathrm{id} \otimes \mathrm{id})(\varphi_n^{\hbar})\big) = 0,\end{aligned} \tag{4.10}$$

where we set

$$a \star b = a + b + \frac{1}{2}[a,b]_{\hbar} + \cdots$$

(the CBH series for the Lie bracket $[-,-]_{\hbar}$). Let (n_1, n_2, n_3, n_4) be integers such that $n_1 + \cdots + n_4 = n + 1$. Let P, Q, R, S be trees such that $|P| = n_1, \dots, |S| = n_4$. Let us apply $\delta_n^{(P)} \otimes \cdots \otimes \delta_n^{(S)}$ to (4.10). The left hand side of (4.10) is equal to

$$(-\Delta_n \otimes \mathrm{id} \otimes \mathrm{id} + \mathrm{id} \otimes \Delta_n \otimes \mathrm{id} - \mathrm{id} \otimes \mathrm{id} \otimes \Delta_n)(\varphi_n^{\hbar}) + (1 \otimes \varphi_n^{\hbar}) - (\varphi_n^{\hbar} \otimes 1) + \text{ brackets.}$$

Now

$$(\delta_n^{(P)} \otimes \delta_n^{(Q)} \otimes \delta_n^{(R)} \otimes \delta_n^{(S)})(\Delta_n \otimes \mathrm{id} \otimes \mathrm{id})(\varphi_n^{\hbar}) = (\delta_n^{(P \cup Q)} \otimes \delta_n^{(R)} \otimes \delta_n^{(S)}).$$

where $P \cup Q$ is the tree with left descendant P and right descendant Q. Therefore

$$\big(\frac{1}{\hbar^n}(\delta_n^{(P)} \otimes \delta_n^{(Q)} \otimes \delta_n^{(R)} \otimes \delta_n^{(S)})(\Delta_n \otimes \mathrm{id} \otimes \mathrm{id})(\varphi_n^{\hbar}) \bmod \hbar\big) = (\Delta_0 \otimes \mathrm{id} \otimes \mathrm{id})(\bar{\varphi}_n)_{n_1, n_2, n_3, n_4}$$

where the index $(n_1, \dots, n_4)$ means the component in $\otimes_{i=1}^4 S^{n_i}(\mathfrak{g})$. On the other hand, if a_1 and $a_2 \in U^{\hat{\otimes} 4}$ are such that

$$(\delta_n^{(P)} \otimes \cdots \otimes \delta_n^{(S)})(a_i) \in \hbar^{\inf(|P|+\cdots+|S|, n)} U^{\hat{\otimes} 4}$$

for any trees $(P, \dots, S)$, then if $(P, \dots, S)$ are such that $|P| + \cdots + |S| = n$, we have

$$(\delta_n^{(P)} \otimes \cdots \otimes \delta_n^{(S)})(\frac{1}{\hbar}[a_1, a_2]) \in \hbar^{n+1} U^{\hat{\otimes} n};$$

one proves this in the same way as the commutativity of $U'/\hbar U'$ (see Theorem 1.1). Then $\frac{1}{\hbar^n}(\delta_n^{(P)}\otimes\cdots\otimes\delta_n^{(S)})(4.10)|_{\hbar=0}$ yields $\mathrm{d}(\bar{\varphi}_n)=0$, where $\mathrm{d}: S^{\cdot}(\mathfrak{g})^{\otimes 2}\to S^{\cdot}(\mathfrak{g})^{\otimes 3}$ is the co-Hochschild cohomology differential. This relation implies that

$$\bar{\varphi}_n=\mathrm{d}(\bar{f}_n)+\lambda_n,$$

where $\bar{f}_n\in S^{\cdot}(\mathfrak{g})^{\otimes 2}$ and $\lambda_n\in\Lambda^3(\mathfrak{g})$. Moreover, f_n and λ_n both have degree $n+1$. This implies that $\lambda_n=0$. Let $f_n\in(U(\mathfrak{g})^{\otimes 2})_{\le n+1}$ be a preimage of $\bar{f}_n$ by the projection

$$(U(\mathfrak{g})^{\otimes 2})_{\le n+1}\to(U(\mathfrak{g})^{\otimes 2})_{\le n+1}/(U(\mathfrak{g})^{\otimes 2})_{\le n}=(S^{\cdot}(\mathfrak{g})^{\otimes 2})_{n+1}$$

(where the indices n and $\le n$ mean "homogeneous part of degree n" and "part of degree $\le n$"). Let $f_n^{\hbar}\in U^{\hat{\otimes}2}$ be a preimage of f_n by the projection $U^{\hat{\otimes}2}\to U^{\hat{\otimes}2}/\hbar U^{\hat{\otimes}2}=U(\mathfrak{g})^{\otimes 2}$. Set $F_{n+1}=\exp(\hbar^{n-1}f_n)$. We may assume that $\hbar^n f_n\in(U(\bar{F}_n)')^{\bar{\otimes}2}$, where $U(\bar{F}_n)'=\{x\in U|\delta_n^{(P)}(x)\in\hbar^{\inf(n,|P|)}U^{\hat{\otimes}|P|}\}$. Then $\Phi_{n+1}={}^{F_{n+1}}\Phi_n$. If P,Q,R are such that $|P|+|Q|+|R|=n+1$, then

$$(\delta_n^{(P)}\otimes\delta_n^{(Q)}\otimes\delta_n^{(R)})(\hbar\log(\Phi_{n+1}))\in\hbar^{n+1}U^{\hat{\otimes}n+1}.$$

Then according to Lemma 4.3,

$$(\delta_{n+1}^{(P)}\otimes\delta_{n+1}^{(Q)}\otimes\delta_{n+1}^{(R)}-\delta_n^{(P)}\otimes\delta_n^{(Q)}\otimes\delta_n^{(R)})(\hbar\log(\Phi_{n+1}))$$

has $\hbar$-adic valuation $>|P|+|Q|+|R|$ when $|P|+|Q|+|R|\le n+1$. So $(\delta_{n+1}^{(P)}\otimes\delta_{n+1}^{(Q)}\otimes\delta_{n+1}^{(R)})(\hbar\log(\Phi_{n+1}))\in\hbar^{|P|+|Q|+|R|}U^{\hat{\otimes}|P|+|Q|+|R|}$ whenever $|P|+|Q|+|R|\le n+1$. □

§ 5 Proof of Proposition 2.4

We work by induction on n. The statement is obvious when $n=0,1$. For $n=2$, we get

$$\begin{aligned}\delta^{(2)}(xy)=&\delta^{(2)}(x)\delta^{(2)}(y)+\delta^{(2)}(x)\big(\delta^{(1)}(y)^1+\delta^{(1)}(y)^2+\delta^{(0)}(y)^{\emptyset}\big)\\&+\big(\delta^{(1)}(x)^1+\delta^{(1)}(y)^1+\delta^{(0)}(y)^{\emptyset}\big)\delta^{(2)}(y)\\&+\delta^{(1)}(x)^1\delta^{(2)}(y)^2+\delta^{(1)}(x)^2\delta^{(2)}(y)^1,\end{aligned}\tag{5.11}$$

so the statement also holds.
Assume that the statement is proved when P is a n-tree. Let $\bar{P}$ be a $(n+1)$-tree. There exists an integer $k\in\{0,\dots,n-1\}$, such that $\bar{P}$ may be viewed as the glueing of the 2-tree on the k-th leaf of a n-tree P. Then we have

$$\delta^{(\bar{P})}=(\mathrm{id}^{\otimes k}\otimes\delta^{(2)}\otimes\mathrm{id}^{\otimes n-k-1})\circ\delta^{(P)}.$$

Let us assume, for instance, that $k=n-1$. If ν is an integer, set

$$S_\nu=\{(\Sigma',\Sigma'')|\Sigma',\Sigma''\subset\{1,\dots,\nu\}\text{ and }\Sigma'\cup\Sigma''=\{1,\dots,\nu\}\}.$$

Then

$$S_n=f_{\{n\},\emptyset}(S_{n-1})\cup f_{\emptyset,\{n\}}(S_{n-1})\cup f_{\{n\},\{n\}}(S_{n-1})\text{ (disjoint union)},$$

where $f_{\alpha,\beta}(\Sigma',\Sigma'')=(\Sigma'\cup\alpha,\Sigma''\cup\beta)$. By hypothesis, we have

$$\delta^{(P)}(xy)=\sum_{(\Sigma_1,\Sigma_2)\in S_n}\delta^{(P_{\Sigma_1})}(x)^{\Sigma_1}\delta^{(P_{\Sigma_2})}(y)^{\Sigma_2},$$

therefore

$$\begin{aligned}\delta^{(P)}(xy) = \sum_{(\Sigma',\Sigma'')\in S_{n-1}} & \delta^{(P_{\Sigma'\cup\{n\}})}(x)^{\Sigma'\cup\{n\}}\delta^{(P_{\Sigma''})}(y)^{\Sigma''} \\ & + \delta^{(P_{\Sigma'})}(x)^{\Sigma'}\delta^{(P_{\Sigma''\cup\{n\}})}(y)^{\Sigma''\cup\{n\}} \\ & + \delta^{(P_{\Sigma'\cup\{n\}})}(x)^{\Sigma'\cup\{n\}}\delta^{(P_{\Sigma''\cup\{n\}})}(y)^{\Sigma''\cup\{n\}}.\end{aligned}$$

Applying $\mathrm{id}^{\otimes n-1}\otimes\delta^{(2)}$ to this identity and using (5.11) and the identities

$$\begin{aligned}(\mathrm{id}^{\otimes k}\otimes\delta^{(1)}\otimes\mathrm{id}^{\otimes|P|-k-1})\circ\delta^{(P)} = \delta^{(P)},\\ (\mathrm{id}^{\otimes k}\otimes\delta^{(0)}\otimes\mathrm{id}^{\otimes|P|-k-1})\circ\delta^{(P)} = 0,\end{aligned}$$

we get $\delta^{(\bar{P})}(xy) =$

$$\begin{aligned}\sum_{(\Sigma',\Sigma'')\in S_{n-1}} \Big(& \left((\mathrm{id}^{\otimes|\Sigma'|}\otimes\delta^{(2)})\circ\delta^{(P_{\Sigma'\cup\{n\}})}\right)(x)^{\Sigma'\cup\{n,n+1\}}\delta^{(P_{\Sigma''})}(y)^{\Sigma''} \\ & + \delta^{(P_{\Sigma'})}(x)^{\Sigma'}\left((\mathrm{id}^{\otimes|\Sigma''|}\otimes\delta^{(2)})\circ\delta^{(P_{\Sigma''\cup\{n\}})}\right)(y)^{\Sigma''\cup\{n,n+1\}} \\ & + \left((\mathrm{id}^{\otimes|\Sigma'|}\otimes\delta^{(2)})\circ\delta^{(P_{\Sigma'\cup\{n\}})}\right)(x)^{\Sigma'\cup\{n,n+1\}}\left((\mathrm{id}^{\otimes|\Sigma''|}\otimes\delta^{(2)})\circ\delta^{(P_{\Sigma''\cup\{n\}})}\right)(y)^{\Sigma''\cup\{n,n+1\}} \\ & + \left((\mathrm{id}^{\otimes|\Sigma'|}\otimes\delta^{(2)})\circ\delta^{(P_{\Sigma'\cup\{n\}})}\right)(x)^{\Sigma'\cup\{n,n+1\}}\left(\delta^{(P_{\Sigma''\cup\{n\}})}(y)^{\Sigma''\cup\{n\}} + \delta^{(P_{\Sigma''\cup\{n\}})}(y)^{\Sigma''\cup\{n+1\}}\right) \\ & + \left(\delta^{(P_{\Sigma'\cup\{n\}})}(x)^{\Sigma'\cup\{n\}} + \delta^{(P_{\Sigma'\cup\{n\}})}(x)^{\Sigma'\cup\{n+1\}}\right)\left((\mathrm{id}^{\otimes|\Sigma''|}\otimes\delta^{(2)})\circ\delta^{(P_{\Sigma''\cup\{n\}})}\right)(y)^{\Sigma''\cup\{n,n+1\}} \\ & + \delta^{(P_{\Sigma'\cup\{n\}})}(x)^{\Sigma'\cup\{n\}}\delta^{(P_{\Sigma''\cup\{n\}})}(y)^{\Sigma''\cup\{n+1\}} + \delta^{(P_{\Sigma'\cup\{n\}})}(x)^{\Sigma'\cup\{n+1\}}\delta^{(P_{\Sigma''\cup\{n\}})}(y)^{\Sigma''\cup\{n\}}\Big).\end{aligned}$$

So we get $\delta^{(\bar{P})}(xy) =$

$$\begin{aligned}\sum_{(\Sigma',\Sigma'')\in S_{n-1}} \Big(& \delta^{(\bar{P}_{\Sigma'\cup\{n,n+1\}})}(x)^{\Sigma'\cup\{n,n+1\}}\delta^{(\bar{P}_{\Sigma''})}(y)^{\Sigma''} \\ & + \delta^{(\bar{P}_{\Sigma'})}(x)^{\Sigma'}\delta^{(\bar{P}_{\Sigma''\cup\{n,n+1\}})}(y)^{\Sigma''\cup\{n,n+1\}} \\ & + \delta^{(\bar{P}_{\Sigma'\cup\{n,n+1\}})}(x)^{\Sigma'\cup\{n,n+1\}}\delta^{(\bar{P}_{\Sigma''\cup\{n,n+1\}})}(y)^{\Sigma''\cup\{n,n+1\}} \\ & + \delta^{(\bar{P}_{\Sigma'\cup\{n,n+1\}})}(x)^{\Sigma'\cup\{n,n+1\}}\left(\delta^{(\bar{P}_{\Sigma''\cup\{n\}})}(y)^{\Sigma''\cup\{n\}} + \delta^{(\bar{P}_{\Sigma''\cup\{n+1\}})}(y)^{\Sigma''\cup\{n+1\}}\right) \\ & + \left(\delta^{(\bar{P}_{\Sigma'\cup\{n\}})}(x)^{\Sigma'\cup\{n\}} + \delta^{(\bar{P}_{\Sigma'\cup\{n+1\}})}(x)^{\Sigma'\cup\{n+1\}}\right)\delta^{(\bar{P}_{\Sigma''\cup\{n,n+1\}})}(y)^{\Sigma''\cup\{n,n+1\}} \\ & + \delta^{(\bar{P}_{\Sigma'\cup\{n\}})}(x)^{\Sigma'\cup\{n\}}\delta^{(\bar{P}_{\Sigma''\cup\{n+1\}})}(y)^{\Sigma''\cup\{n+1\}} + \delta^{(\bar{P}_{\Sigma'\cup\{n+1\}})}(x)^{\Sigma'\cup\{n+1\}}\delta^{(\bar{P}_{\Sigma''\cup\{n\}})}(y)^{\Sigma''\cup\{n\}}\Big).\end{aligned}$$

We have

$$\begin{aligned}S_{n+1} = & f_{\{n,n+1\},\{n,n+1\}}(S_{n-1}) \cup f_{\{n,n+1\},\{n\}}(S_{n-1}) \cup f_{\{n,n+1\},\{n+1\}}(S_{n-1}) \\ & \cup f_{\{n,n+1\},\emptyset}(S_{n-1}) \cup f_{\{n\},\{n,n+1\}}(S_{n-1}) \cup f_{\{n+1\},\{n,n+1\}}(S_{n-1}) \\ & \cup f_{\emptyset,\{n,n+1\}}(S_{n-1}) \cup f_{\{n\},\{n+1\}}(S_{n-1}) \cup f_{\{n+1\},\{n\}}(S_{n-1}) \text{ (disjoint union)},\end{aligned}$$

where we recall that $f_{\alpha,\beta}(\Sigma',\Sigma'') = (\Sigma' \cup \alpha, \Sigma'' \cup \beta)$. So we get

$$\delta^{(\bar{P})}(xy) = \sum_{(\bar{\Sigma}',\bar{\Sigma}'')\in S_{n+1}} \delta^{(P_{\bar{\Sigma}'})}(x)^{|\bar{\Sigma}'|}\delta^{(P_{\bar{\Sigma}''})}(y)^{|\bar{\Sigma}''|}.$$

The proof is the same for a general $k \in \{0,\dots,n-1\}$. This establishes the induction. □

§ 6 Proof of Theorem 1.7

1) According to [Dr3], $(U(\mathfrak{g}), m_0, \Delta_0, e^{\hbar t_{\mathfrak{g}}/2}, \Phi(\hbar t_{\mathfrak{g}}^{1,2}, \hbar t_{\mathfrak{g}}^{2,3}))$ is a quasi-triangular quasi-Hopf algebra. One checks that it is admissible; then the reduction modulo $\hbar$ of the corresponding QFS algebra is the Drinfeld algebra of 1).

2) Let $\widetilde{\varphi}_1$, $\widetilde{\varphi}_2$ be the elements of $\widehat{S}^{\cdot}(\mathfrak{g})^{\widehat{\otimes} 3}$ such that $(\widehat{S}^{\cdot}(\mathfrak{g}), m_0, P_{\mathfrak{g}}, \Delta_0, \widetilde{\varphi}_i)$ are Drinfeld algebras. Let C be the lowest degree component of $\widetilde{\varphi}_1 - \widetilde{\varphi}_2$. Then the degree k of C is ≥ 4. Taking the degree k part of the difference of the pentagon identities for $\widetilde{\varphi}_1$ and $\widetilde{\varphi}_2$, we find $\mathrm{d}(C) = 0$, where $\mathrm{d} : S^{\cdot}(\mathfrak{g})^{\otimes 3} \to S^{\cdot}(\mathfrak{g})^{\otimes 4}$ is the co-Hochschild differential. So $\mathrm{Alt}(C) \in \Lambda^3(\mathfrak{g})$, and since $\mathrm{Alt}(C)$ also has degree ≥ 4, $\mathrm{Alt}(C) = 0$. If C_{p_1,p_2,p_3} is the component of C in $\otimes_{i=1}^3 S^{p_i}(\mathfrak{g})$ then we may define inductively B by $B_{0,k} = B_{1,k-1} = 0$, $B_{2,k-2} = \frac{1}{2}(\mathrm{id}\otimes m)(C_{1,1,k-2})$, and

$$B_{i+1,k-i-1} = \frac{1}{i+1}(\mathrm{id}\otimes m)[C_{i,1,k-i-1} + \big((\mathrm{id}\otimes \mathrm{d})(B_{i,k-i})\big)_{i,1,k-i-1}],$$

where $B_{i,j}$ is the component of B in $S^i(\mathfrak{g}) \otimes S^j(\mathfrak{g})$ and m is the product of $S^{\cdot}(\mathfrak{g})$. So B can be chosen to be $\mathfrak{g}$-invariant. Applying successive twists, we obtain the result. □

Remark 6.1. To prove 2), we cannot use Theorem A of [Dr2] because we do not know that the twist constructed there is admissible.

§ 7 Associators and Lie associators

In this section, we state precisely and prove Theorem 1.8.

- a - Statement of the result

Recall that the algebra $\mathscr{T}_n$, $n \geq 2$, has generators $t^{i,j}$, $1 \leq 1 \neq j \leq n$, and relations $t^{j,i} = t^{i,j}$,

$$[t^{i,j} + t^{i,k}, t^{j,k}] = 0 \text{ when } i,j,k \text{ are all distinct,}$$
$$[t^{i,j}, t^{k,l}] = 0 \text{ when } i,j,k,l \text{ are all distinct.}$$

$\mathfrak{t}_n$ is defined as the Lie algebra with the same generators and relations. Then $\mathscr{T}_n = U(\mathfrak{t}_n)$. When $n \leq m$ and $(I_1,\dots,I_n)$ is a collection of disjoint subsets of $\{1,\dots,m\}$, there is a unique algebra morphism $\mathscr{T}_n \to \mathscr{T}_m$ taking $t^{i,j}$ to $\sum_{\alpha\in I_i,\beta\in I_j} t^{\alpha,\beta}$. We call it an insertion-coproduct morphism and denote it by $x \mapsto x^{I_1,\dots,I_n}$. In particular, we have an action of $\mathfrak{S}_n$ on $\mathscr{T}_n$. Let us attribute degree 1 to each generator $t^{i,j}$; this defines gradings on the algebra

$\mathcal{T}_n$ and on the Lie algebra $\mathfrak{t}_n$. We denote by $\widehat{\mathcal{T}_n}$ and $\widehat{\mathfrak{t}}_n$ their completions for this grading. Then $\widehat{\mathcal{T}_n}$ is the preimage of $\mathbb{K}^\times$ by the natural projection $\widehat{\mathcal{T}_n} \to \mathbb{K}$, and the exponential is a bijection $(\widehat{\mathcal{T}_n})_0 \to 1 + (\widehat{\mathcal{T}_n})_0$ (where $(\widehat{\mathcal{T}_n})_0 = \mathrm{Ker}(\widehat{\mathcal{T}_n} \to \mathbb{K})$). We have an exact sequence

$$1 \to 1 + (\widehat{\mathcal{T}_n})_0 \to (\widehat{\mathcal{T}_n})^\times \to \mathbb{K}^\times \to 1.$$

An *associator* is an element Φ of $1 + (\widehat{\mathcal{T}_n})_0$, satisfying the pentagon equation

$$\Phi^{1,2,34}\Phi^{12,3,4} = \Phi^{2,3,4}\Phi^{1,23,4}\Phi^{1,2,3}, \tag{7.12}$$

the hexagon equations

$$e^{\frac{t^{1,3}+t^{2,3}}{2}} = \Phi^{3,1,2} e^{\frac{t^{1,3}}{2}} \left(\Phi^{1,3,2}\right)^{-1} e^{\frac{t^{2,3}}{2}} \Phi^{1,2,3}$$

and

$$e^{\frac{t^{1,2}+t^{1,3}}{2}} = \left(\Phi^{2,3,1}\right)^{-1} e^{\frac{t^{1,3}}{2}} \Phi^{2,1,3} e^{\frac{t^{1,3}}{2}} \left(\Phi^{1,2,3}\right)^{-1}$$

and $\mathrm{Alt}(\Phi) = \frac{1}{8}[t^{1,2}, t^{2,3}] +$ terms of degree > 2. We denote by $\underline{\mathrm{Assoc}}$ the set of associators. If Φ satisfies the duality condition $\Phi^{3,2,1} = \Phi^{-1}$, then both hexagon equations are equivalent. We denote by $\underline{\mathrm{Assoc}}^0$ the subset of all $\Phi \in \underline{\mathrm{Assoc}}$ satisfying the duality condition. If $F \in 1 + (\widehat{\mathcal{T}_2})_0$ and $\Phi \in 1 + (\widehat{\mathcal{T}_3})_0$, the *twist of* Φ *by* F is

$${}^F\Phi = F^{2,3}F^{1,23}\Phi(F^{1,2}F^{12,3})^{-1}.$$

This defines an action of $1 + (\widehat{\mathcal{T}_2})_0$ on $1 + (\widehat{\mathcal{T}_3})_0$, which preserves $\underline{\mathrm{Pent}} = \{\Phi \in 1 + (\widehat{\mathcal{T}_3})_0 | \Phi \text{ satisfies (7.12)}\}$, $\underline{\mathrm{Assoc}}$ and $\underline{\mathrm{Assoc}}^0$ ($\underline{\mathrm{Pent}}$ and $\underline{\mathrm{Assoc}}$ are preserved because F has the form $f(t^{1,2})$, $f \in 1 + t\mathbb{K}[[t]]$, so the "twisted R-matrix" ${}^FR = F^{2,1}RF^{-1} = f(t^{2,1})e^{t^{1,2}/2} f(t^{1,2})^{-1} = e^{t^{1,2}/2}$. $\underline{\mathrm{Assoc}}^0$ is preserved because each F is such that $F = F^{2,1}$.) We denote by $\underline{\mathrm{Assoc}}^0_{\mathrm{Lie}}$, $\underline{\mathrm{Assoc}}_{\mathrm{Lie}}$ and $\underline{\mathrm{Pent}}_{\mathrm{Lie}}$ the subsets of all Φ in $\underline{\mathrm{Assoc}}$, $\underline{\mathrm{Assoc}}^0$ and $\underline{\mathrm{Pent}}$, such that $\log(\Phi) \in \widehat{\mathfrak{t}}_3$.

Theorem 7.1. *There is exactly one element of* $\underline{\mathrm{Pent}}_{\mathrm{Lie}}$ *resp.,* $\underline{\mathrm{Assoc}}_{\mathrm{Lie}}$, $\underline{\mathrm{Assoc}}^0_{\mathrm{Lie}}$*) in each orbit of the action of* $1 + (\widehat{\mathcal{T}_2})_0$ *on* $\underline{\mathrm{Pent}}$ *(resp.,* $\underline{\mathrm{Assoc}}$, $\underline{\mathrm{Assoc}}^0$*). The isotropy group of each element of* $\underline{\mathrm{Pent}}$ *is* $\{e^{\lambda t^{1,2}} | \lambda \in \mathbb{K}\} \subset 1 + (\widehat{\mathcal{T}_2})_0$.

- b - Proof of Theorem 7.1

The arguments are the same in all three cases, so we treat the case of $\underline{\mathrm{Assoc}}$.

Let Φ belongs to $\underline{\mathrm{Assoc}}$. Set $\Phi = 1 + \sum_{i>0} \Phi_i$, where Φ_i is the degree i component of Φ. Let d be the co-Hochschild differential,

$$\begin{aligned} \mathrm{d} \,:\, & \mathcal{T}_n \to \mathcal{T}_{n+1} \\ & x \mapsto \sum_{i=1}^{n} (-1)^{i+1} x^{1,\dots,\{i,i+1\},\dots,n+1} - x^{2,3,\dots,n+1} + (-1)^n x^{1,2,\dots,n}. \end{aligned}$$

Then $\mathrm{d}(\Phi_2) = 0$, and $\mathrm{Alt}(\Phi_2) = \frac{1}{8}[t^{1,2}, t^{2,3}]$. Computation shows that this implies that for some $\lambda \in \mathbb{K}$, we have $\Phi_2 = \frac{1}{8}[t^{1,2}, t^{2,3}] + \lambda \mathrm{d}((t^{1,2})^2)$. We construct $F \in 1 + (\widehat{\mathcal{T}})_0$, such that ${}^F\Phi \in \underline{\mathrm{Assoc}}_{\mathrm{Lie}}$, as an infinite product $F = \cdots F_n \cdots F_2$, where $F_i \in 1 + (\widehat{\mathcal{T}_2})_{\geq i}$

(the index $\geq i$ means the part of degree $\geq i$). If we set $F_2 = 1 + \lambda(t^{1,2})^2$, then $\log({}^{F_2}\Phi) \in \mathfrak{t}_3 + (\widehat{\mathscr{T}_3})_{\geq 3}$. Assume that we have found $F_3, \ldots, F_{n-1}$, such that $\log({}^{\bar{F}_{n-1}}\Phi) \in \mathfrak{t}_3 + (\widehat{\mathscr{T}_3})_{\geq n}$, where $\bar{F}_{n-1} = F_{n-1} \cdots F_2$. Then $\varphi^{(n-1)} := \log({}^{\bar{F}_{n-1}}\Phi)$ satisfies

$$\left(\varphi^{(n-1)}\right)^{1,2,34} \star \left(\varphi^{(n-1)}\right)^{12,3,4} = \left(\varphi^{(n-1)}\right)^{2,3,4} \star \left(\varphi^{(n-1)}\right)^{1,23,4} \star \left(\varphi^{(n-1)}\right)^{1,2,3},$$

where $\star$ is the CBH product in $(\widehat{\mathscr{T}_3})_0$. Let $\varphi_n^{(n-1)}$ be the degree n part of $\varphi^{(n-1)}$. Then we get $\mathrm{d}(\varphi^{(n-1)}) \in \mathfrak{t}_4$. We now use the following satement, which will be proved in the next subsection.

Proposition 7.2. *If $\gamma \in \mathscr{T}_3$ is such that $\mathrm{d}(\gamma) \in \mathfrak{t}_4$, then there exists $\beta \in \mathscr{T}_2$, such that $\gamma + \mathrm{d}(\beta) \in \mathfrak{t}_3$. If γ has degree n, one can choose β of degree n.*

It follows that there exists $\beta \in \mathscr{T}_2$ of degree n, such that $\varphi_n^{(n-1)} - \mathrm{d}(\beta) \in \mathfrak{t}_3$. Set $F_n = 1 + \beta$, then $\varphi^{(n)} = \log({}^{\bar{F}_n}\Phi)$ is such that $\varphi^{(n)} \in \varphi^{(n-1)} - \mathrm{d}(\beta) + (\widehat{\mathscr{T}_3})_{\geq n+1}$, so $\varphi^{(n)} \in \mathfrak{t}_3 + (\widehat{\mathscr{T}_3})_{\geq n+1}$. Moreover, the product $F = \cdots F_n \cdots F_2$ is convergent, and ${}^F\Phi$ then satisfies $\log({}^F\Phi) \in \hat{\mathfrak{t}}_3$. This proves the existence of F, such that ${}^F\Phi \in \underline{\mathrm{Assoc}}_{\mathrm{Lie}}$.

Let us now prove the unicity of an element of $\underline{\mathrm{Assoc}}_{\mathrm{Lie}}$, twist-equivalent to $\Phi \in \underline{\mathrm{Assoc}}$. This follows from:

Proposition 7.3. *Let Φ' and Φ'' be elements of $\underline{\mathrm{Assoc}}_{\mathrm{Lie}}$, and let F belong to $1 + (\widehat{\mathscr{T}_2})_0$. Then ${}^F\Phi' = \Phi''$ if and only if there exists $\lambda \in \mathbb{K}$ such that $F = e^{\lambda t^{1,2}}$ and $\Phi'' = \Phi'$.*

PROOF OF PROPOSITION 7.3. Since $t^{1,2} + t^{1,3} + t^{2,3}$ is central in $\widehat{\mathscr{T}_3}$, we have ${}^{F_\lambda}\Phi' = \Phi'$ when $F_\lambda = e^{\lambda t}$, for any $\lambda \in \mathbb{K}$. Conversely, let F_i be the degree i part of F. Then for some $\lambda_0 \in \mathbb{K}$, we have $F_1 = \lambda_0 t$. Replacing F by $F' = FF_{-\lambda_0}$, we get ${}^{F'}\Phi' = \Phi''$, and $F' - 1$ has valuation ≥ 2 (for the degree in t). Assume that $F' - 1 \neq 0$ and let ν be its valuation. Let F'_ν be the degree ν part of F'. Then $\mathrm{d}(F'_\nu) \in \mathfrak{t}_3$. On the other hand, $F'_\nu = \mu(t^{1,2})^\nu$, where $\mu \in \mathbb{K} - \{0\}$. Now $\mathrm{d}((t^{1,2})^\nu) \in \mathscr{T}_3 = U(\mathfrak{t}_3)$ has degree $\leq \nu$ for the filtration of $U(\mathfrak{t}_3)$, and its symbol in $S^\nu(\mathfrak{t}_3) = \mathrm{gr}_\nu(U(\mathfrak{t}_3))$ is $\sum_{\nu'=1}^{\nu-1} \binom{\nu}{\nu'} (t^{1,3})^{\nu'}(t^{2,3})^{\nu-\nu'} - \sum_{\nu''=1}^{\nu-1} \binom{\nu}{\nu''} (t^{1,2})^{\nu''}(t^{1,3})^{\nu-\nu''}$: this is the image of a non-zero element in $S^\nu(\mathbb{K}t^{1,2} \oplus \mathbb{K}t^{1,3} \oplus \mathbb{K}t^{2,3})$ under the injection $S^\nu(\bigoplus_{1\leq i<j\leq 3} \mathbb{K}t^{i,j}) \hookrightarrow S^\nu(\mathfrak{t}_3)$, so it is non-zero. So $F' \neq 1$ leads to a contradiction. So $F = F_{\lambda_0}$, therefore $\Phi'' = \Phi'$. □

Note that we have proved the analogue of Proposition 7.2, where the indices of $\mathscr{T}_3, \mathfrak{t}_4$, etc., are shifted by -1.

- c - Decomposition of $\mathfrak{t}_3$ and proof of Proposition 7.2

To end the proof of the first part of Theorem 7.1, it remains to prove Proposition 7.2. For this, we construct a decomposition of $\mathfrak{t}_n$. For $i = 1, \ldots, n$, there is a unique algebra morphism $\varepsilon_i : \mathscr{T}_n \to \mathscr{T}_{n-1}$, taking $t_{i,j}$ to 0 for any $j \neq i$, and taking $t_{j,k}$ to $t_{j-\lambda_i(j),k-\lambda_i(k)}$ if $j, k \neq i$, where $\lambda_i(j) = 0$ if $j < i$ and $= 1$ if $j > i$. Then ε_i induces a Lie algebra morphism $\widetilde{\varepsilon}_i : \mathfrak{t}_n \to \mathfrak{t}_{n-1}$. Set $\widetilde{\mathfrak{t}}_n = \bigcap_{i=1}^n \mathrm{Ker}(\widetilde{\varepsilon}_i)$. Then we have

Lemma 7.4.

$$\mathfrak{t}_n = \bigoplus_{k=0}^{n} \bigoplus_{I \in \mathscr{P}_k(\{1,\ldots,n\})} (\widetilde{\mathfrak{t}}_k)^I,$$

where $\mathscr{P}_k(\{1,\dots,n\})$ is the set of subsets of $\{1,\dots,n\}$ of cardinal k, and $(\widetilde{\mathfrak{t}}_k)^I$ is the image of $\widetilde{\mathfrak{t}}_k$ under $\mathfrak{t}_k \to \mathfrak{t}_n$, $x \mapsto x^{i_1,\dots,i_k}$, where $I = \{i_1,\dots,i_k\}$.

PROOF OF LEMMA. Let $\mathfrak{F}$ be the free Lie algebra with generators $\widetilde{t}_{i,j}$, where $1 \le i < j \le n$. It is graded by $\Gamma := \mathbb{N}^{\{(i,j)|1\le i<j\le n\}}$: the degree of $\widetilde{t}_{i,j}$ is the vector $\mathbf{d}_{i,j}$, whose (i',j') coordinate is $\delta_{(i,j),(i',j')}$. For $\underline{k} \in \Gamma$, we denote by $\mathfrak{F}_{\underline{k}}$ the part of $\mathfrak{F}$ of degree $\underline{k}$. Let $\pi : \mathfrak{F} \to \mathfrak{t}_n$ be the canonical projection. Since the defining ideal of $\mathfrak{t}_n$ is graded, we have

$$\mathfrak{t}_n = \bigoplus_{\underline{k}\in\Gamma} \pi(\mathfrak{F}_{\underline{k}}). \tag{7.13}$$

On the other hand, one checks that $\widetilde{\mathfrak{t}}_n = \bigoplus_{\underline{k}\in\widetilde{\Gamma}} \pi(\mathfrak{F}_{\underline{k}})$, where $\widetilde{\Gamma}$ is the set of maps $k : \{(i,j)|1 \le i < j \le n\} \to \mathbb{N}$, such that for each i, $\sum_{j|j>i} k(i,j) + \sum_{j|j<i} k(j,i) \neq 0$. Define a map $\lambda : \Gamma \to \mathscr{P}(\{1,\dots,n\})$ as follows ($\mathscr{P}(\{1,\dots,n\})$ is the set of subsets of $\{1,\dots,n\}$): λ takes the map $k : \{(i,j)|1 \le i < j \le n\} \to \mathbb{N}$ to $\{i| \sum_{j|j>i} k(i,j) + \sum_{j|j<i} k(j,i) \neq 0\}$. Then for each $I \in \mathscr{P}(\{1,\dots,n\})$, $\big(\widetilde{\mathfrak{t}}_{|I|}\big)^I$ identifies with $\bigoplus_{\underline{k}\in\lambda^{-1}(I)} \pi(\mathfrak{F}_{\underline{k}})$. Comparing with (7.13), we get

$$\mathfrak{t}_n = \bigoplus_{I\in\mathscr{P}(\{1,\dots,n\})} \big(\widetilde{\mathfrak{t}}_{|I|}\big)^I. \qquad \square$$

When $n = 3$, we get $\mathfrak{t}_3 = \mathbb{K}t^{1,2} \oplus \mathbb{K}t^{1,3} \oplus \mathbb{K}t^{2,3} \oplus \widetilde{\mathfrak{t}}_3$. On the other hand, the fact that the insertion-coproduct maps take $\mathfrak{t}_n$ to $\mathfrak{t}_m$ implies that $\mathrm{d} : \mathscr{T}_n \to \mathscr{T}_{n+1}$ is compatible with the filtrations induced by the identification $\mathscr{T}_n = U(\mathfrak{t}_n)$, $\mathscr{T}_{n+1} = U(\mathfrak{t}_{n+1})$. The associated graded map is

$$\mathrm{gr}^{\cdot}(\mathrm{d}) \ : \ S^{\cdot}(\mathfrak{t}_n) \to S^{\cdot}(\mathfrak{t}_{n+1}).$$

Proposition 7.2 now follows from:

Lemma 7.5. *When $k \geq 2$, the cohomology of the complex*

$$S^k(\mathfrak{t}_2) \stackrel{\mathrm{gr}^k(\mathrm{d})}{\longrightarrow} S^k(\mathfrak{t}_3) \stackrel{\mathrm{gr}^k(\mathrm{d})}{\longrightarrow} S^k(\mathfrak{t}_4)$$

vanishes.

PROOF OF LEMMA. We have

$$S^k(\mathfrak{t}_3) = \bigoplus_{\alpha=0}^{k} S^{k-\alpha}\Big(\bigoplus_{1\le i<j\le 3} \mathbb{K}t^{i,j}\Big) \otimes S^{\alpha}(\widetilde{\mathfrak{t}}_3). \tag{7.14}$$

Let $x \in S^k(\mathfrak{t}_3)$, and let $(x_\alpha)_{\alpha=0,\dots,k}$ be its components in the decomposition (7.14). We have

$$S^{\cdot}(\mathfrak{t}_4) = S^{\cdot}(\widetilde{\mathfrak{t}}_4) \otimes \bigotimes_{2\le i<j\le 4} S^{\cdot}(\widetilde{\mathfrak{t}}_3^{1,i,j}) \otimes \bigotimes_{i=2}^{4} S^{\cdot}(\widetilde{\mathfrak{t}}_2^{1,i}) \otimes S^{\cdot}(\mathfrak{t}_3^{2,3,4}).$$

We denote by p the projection

$$p \ : \ S^{\cdot}(\mathfrak{t}_4) \to \widetilde{\mathfrak{t}}_3^{1,3,4} \otimes S^{\cdot}(\mathfrak{t}_3^{2,3,4}),$$

which is the tensor product of: the identity on the last factor, the projection to degree 1 on the factor $S^{\cdot}(\widetilde{\mathfrak{t}}_3^{1,3,4})$, and the projection to degree 0 in all other factors. We also denote by $m : \widetilde{\mathfrak{t}}_3^{1,3,4} \otimes S^{\cdot}(\mathfrak{t}_3^{2,3,4}) \to S^{\cdot}(\mathfrak{t}_3)$ the map induced by the identifications $\widetilde{\mathfrak{t}}_3^{1,3,4} \subset \mathfrak{t}_3^{1,3,4} \simeq \mathfrak{t}_3$, $\mathfrak{t}_3^{2,3,4} \simeq \mathfrak{t}_3$ followed by the product map in $S^{\cdot}(\mathfrak{t}_3)$. We denote by $\mathrm{d}_1, \mathrm{d}_2, \mathrm{d}_3$ the maps $\mathscr{T}_3 \to \mathscr{T}_4$ defined by

$$\mathrm{d}_1(x) = x^{12,3,4} - x^{1,3,4} - x^{2,3,4},$$
$$\mathrm{d}_2(x) = x^{1,23,4} - x^{1,2,4} - x^{1,3,4},$$
$$\mathrm{d}_3(x) = x^{1,2,34} - x^{1,2,3} - x^{1,2,4},$$

so $\mathrm{d} = \mathrm{d}_1 - \mathrm{d}_2 + \mathrm{d}_3$. The maps d_i are compatible with the filtrations of $\mathscr{T}_3$ and $\mathscr{T}_4$; we denote by $\mathrm{gr}^k(\mathrm{d}_i)$ the corresponding graded maps, so $\mathrm{gr}^k(\mathrm{d}) = \mathrm{gr}^k(\mathrm{d}_1) - \mathrm{gr}^k(\mathrm{d}_2) + \mathrm{gr}^k(\mathrm{d}_3)$. Then if we set

$$x_1 = \sum_{a,b,c | a+b+c=k-1} (t^{1,2})^a (t^{1,3})^b (t^{2,3})^c \otimes e_{a,b,c},$$

where $e_{a,b,c} \in \widetilde{\mathfrak{t}}_3$, we have

$$m \circ p \circ \mathrm{gr}^k(\mathrm{d}_1)(x) = \big(\sum_{\alpha=0}^{k} \alpha x_\alpha \big) - (t^{2,3})^{k-1} e_{0,0,k-1}.$$

On the other hand, let us define the i-degree of an element of $\big(\widetilde{\mathfrak{t}}_{|I|}\big)^I$ to be 1 if $i \in I$ and 0 if $i \notin I$. Then the i-degree of $\otimes_{I \subset \{1,\dots,n\}} S^{\alpha_I}\big(\big(\widetilde{\mathfrak{t}}_{|I|}\big)^I\big) \subset S^{\cdot}(\mathfrak{t}_n)$ is $\sum_{I | i \in I} \alpha_I$. If x is homogeneous for the 1-degree, then so is $\mathrm{gr}^k(\mathrm{d}_2)(x)$, and 1-degree$(\mathrm{gr}^k(\mathrm{d}_2)(x)) =$ 1-degree(x). On the other hand, the elements of $S^{\cdot}(\mathfrak{t}_4)$ whose 1-degree is $\neq 1$ are in the kernel of p. It follows that

$$m \circ p \circ \mathrm{gr}^k(\mathrm{d}_2)(x_\alpha) = 0 \text{ if } \alpha \neq 1,$$

and $p \circ \mathrm{gr}^k(\mathrm{d}_2)(x_1) = (e_{0,0,k-1})^{1,3,4}\big[(t^{2,4} + t^{3,4})^{k-1} - (t^{3,4})^{k-1}\big]$, so

$$m \circ p \circ \mathrm{gr}^k(\mathrm{d}_2)(x_1) = e_{0,0,k-1}\big[(t^{1,3} + t^{2,3})^{k-1} - (t^{2,3})^{k-1}\big].$$

Finally, $p \circ \mathrm{gr}^k(\mathrm{d}_3)(x) = 0$. If x is such that $\mathrm{gr}^k(\mathrm{d})(x) = 0$, we have $m \circ p \circ \mathrm{gr}^k(\mathrm{d})(x) = 0$, so

$$\sum_{\alpha \geq 0} \alpha x_\alpha = e_{0,0,k-1}(t^{1,3} + t^{2,3})^{k-1}.$$

Looking at degrees in the decomposition (7.14), we get $x_\alpha = 0$ for $\alpha \geq 2$, and $x_1 = e_{0,0,k-1}(t^{1,3} + t^{2,3})^{k-1}$. Using the projection $p' : S^{\cdot}(\mathfrak{t}_4) \to \widetilde{\mathfrak{t}}_3^{1,2,4} \otimes S^{\cdot}(\mathfrak{t}_3^{1,2,3})$, we get in the same way $x_1 = e_{k-1,0,0}(t^{1,2} + t^{1,3})^{k-1}$. Now $e_{k-1,0,0}(t^{1,2} + t^{1,3})^{k-1} = e_{0,0,k-1}(t^{1,3} + t^{2,3})^{k-1}$ implies $e_{k-1,0,0} = e_{0,0,k-1} = 0$ so $x_1 = 0$. Therefore $x \in S^k\big(\oplus_{1 \leq i < j \leq 3} \mathbb{K} t^{i,j}\big)$. Let us set $x = S(t^{1,2}, t^{1,3}, t^{2,3})$, where S is a homogeneous polynomial of degree k of $\mathbb{K}[u,v,w]$. Since $\mathrm{d}(x) = 0$, we have

$$\begin{aligned} &S(t^{1,3} + t^{2,3}, t^{1,4} + t^{2,4}, t^{3,4}) - S(t^{1,2} + t^{1,3}, t^{1,4}, t^{2,4} + t^{3,4}) \\ &\quad + S(t^{1,2}, t^{1,3} + t^{1,4}, t^{2,3} + t^{2,4}) = S(t^{2,3}, t^{2,4}, t^{3,4}) + S(t^{1,2}, t^{1,3}, t^{2,3}) \end{aligned}$$

(equality in $S^{\cdot}\big(\oplus_{1 \leq i < j \leq 4} \mathbb{K} t^{i,j}\big)$).

Applying $\frac{\partial}{\partial t^{1,2}} \circ \frac{\partial}{\partial t^{3,4}}$ to this equality, we get

$$(\partial_u \partial_w S)(t^{1,2} + t^{1,3}, t^{1,4}, t^{2,4} + t^{3,4}) = 0,$$

therefore $\partial_u\partial_w S = 0$. We have therefore

$$S(u,v,w) = P(u,v) + Q(v,w),$$

where P and Q are homogeneous polynomials of degree k. Moreover, $\mathrm{d}(x) = 0$, so

$$\begin{aligned}
&\big[P(t^{1,2},t^{1,3}+t^{1,4}) - P(t^{1,2}+t^{1,3},t^{1,4}) - P(t^{1,2},t^{1,3})\big]\\
&+\big[Q(t^{1,4}+t^{2,4},t^{3,4}) - Q(t^{1,4},t^{2,4}+t^{3,4}) - Q(t^{2,4},t^{3,4})\big]\\
&+\big[P(t^{1,3}+t^{2,3},t^{1,4}+t^{2,4}) + Q(t^{1,3}+t^{1,4},t^{2,3}+t^{2,4}) - P(t^{2,4},t^{2,4}) - Q(t^{1,3},t^{2,3})\big] = 0.
\end{aligned} \tag{7.15}$$

Write this as an identity

$$B(t^{1,2},t^{1,3},t^{1,4}) + C(t^{1,4},t^{2,4},t^{3,4}) + A(t^{2,3},t^{1,4},t^{1,3},t^{2,4}) = 0.$$

Then A (resp., B,C) is independent on $t^{2,3}$ (resp., $t^{1,2}$, $t^{3,4}$). Let us now determine P and Q. Since $B(t^{1,2},t^{1,3},t^{1,4}) = B(0,t^{1,3},t^{1,4})$, we have $P(u,v+w) - P(u+v,w) - P(u,v) = P(0,v+w) - P(v,w) - P(0,v)$. Therefore $(\mathrm{d}\widetilde{P})(u,v,w) = 0$, where $\widetilde{P}(u,v) = P(u,v) - P(0,v)$ and d is the co-Hochschild differential of polynomials in one variable. The corresponding cohomology is zero, so we have a polynomial $\bar{P}$, such that

$$P(u,v) - P(0,v) = \bar{P}(u+v) - \bar{P}(u) - \bar{P}(v).$$

We conclude that $P(u,v)$ has the form

$$P(u,v) = \bar{P}(u+v) - \bar{P}(u) - R(v) \tag{7.16}$$

where $\bar{P}$ and R are polynomials in one variable of degree k; since $P(u,v)$ is homogeneous of degree k, we can assume that $\bar{P}$ and R are monomials of degree k. In the same way, since $C(t^{1,4},t^{2,4},t^{3,4}) = C(t^{1,4},t^{2,4},0)$, we have $Q(u+v,w) - Q(u,v+w) - Q(v,w) = Q(u+v,0) - Q(u,v) - Q(v,0)$, so $(\mathrm{d}\widetilde{Q})(u,v,w) = 0$, where $\widetilde{Q}(u,v) = Q(u,v) - Q(u,0)$. So $Q(u,v)$ has the form

$$Q(u,v) = \bar{Q}(u+v) - \bar{Q}(v) - S(u), \tag{7.17}$$

where $\bar{Q}$ and S are polynomials in one variable of degree k, which can be assumed to be monomials of degree k. We have therefore

$$x = \bar{P}^{1,23} + \bar{Q}^{12,3} - \bar{P}^{1,2} - \bar{Q}^{2,3} - T^{1,3},$$

where $\bar{P} = \bar{P}(t^{1,2})$, $\bar{Q} = \bar{Q}(t^{1,2})$ and $T = (R+S)(t^{1,2})$. So $x = \mathrm{d}(\bar{Q}) + (\bar{P}+\bar{Q})^{1,23} - (\bar{P}+\bar{Q})^{1,2} - T^{1,3}$. Set $a = \bar{P}+\bar{Q}$; we have $\mathrm{d}(y) = 0$, where $y = a^{1,23} - a^{1,2} - T^{1,3}$; applying ε_1 to $\mathrm{d}(y) = 0$, we get $T^{2,3} - T^{2,4} = 0$, so $T = 0$. We then get $a^{12,34} - a^{12,3} - a^{2,34} + a^{2,3} = 0$. Applying $\varepsilon_3 \circ \varepsilon_2$ to this identity, we get $a^{1,4} = 0$. Finally $\bar{P} = -\bar{Q}$, so $x = \mathrm{d}(\bar{Q})$, which proves the lemma. □

- c - Isotropy groups

Proposition 7.3 can be generalized to the case of a pair of elements of $\underline{\mathrm{Pent}}_{\mathrm{Lie}}$, and it implies that the isotropy group of each element of $\underline{\mathrm{Pent}}_{\mathrm{Lie}}$ is the additive group $\{e^{\lambda t^{1,2}}, \lambda \in \mathbb{K}\}$. Let Φ be an element of $\underline{\mathrm{Pent}}$. There exists an element Φ_{Lie} of $\underline{\mathrm{Pent}}_{\mathrm{Lie}}$ in the orbit of Φ. So the isotropy groups of Φ and Φ_{Lie} are conjugated. Since $1 + (\widehat{\mathscr{T}_2})_0$ is commutative, the isotropy group of Φ is $\{e^{\lambda t^{1,2}}, \lambda \in \mathbb{K}\}$. □

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