

Lifts of C_∞ and L_∞ -morphisms to G_∞ -morphisms

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September 30, 2004

Abstract

Let \mathfrak{g}_2 be the Hochschild complex of cochains on $C^\infty(\mathbb{R}^n)$ and \mathfrak{g}_1 be the space of multivector fields on \mathbb{R}^n . In this paper we prove that given any G_∞ -structure (i.e. Gerstenhaber algebra up to homotopy structure) on \mathfrak{g}_2 , and any C_∞ -morphism φ (i.e. morphism of commutative, associative algebra up to homotopy) between \mathfrak{g}_1 and \mathfrak{g}_2 , there exists a G_∞ -morphism Φ between \mathfrak{g}_1 and \mathfrak{g}_2 that restricts to φ . We also show that any L_∞ -morphism (i.e. morphism of Lie algebra up to homotopy), in particular the one constructed by Kontsevich, can be deformed into a G_∞ -morphism, using Tamarkin's method for any G_∞ -structure on \mathfrak{g}_2 . We also show that any two of such G_∞ -morphisms are homotopic.

Keywords : Deformation quantization, star-product, homotopy formulas, homological methods

AMS Classification : Primary 16E40, 53D55, Secondary 18D50, 16S80

Let M be a differential manifold and $\mathfrak{g}_2 = (C^\bullet(A, A), b)$ be the Hochschild cochain complex on $A = C^\infty(M)$. The classical Hochschild-Kostant-Rosenberg theorem states that the cohomology of \mathfrak{g}_2 is the graded Lie algebra $\mathfrak{g}_1 = \Gamma(M, \wedge^\bullet TM)$ of multivector fields on M . There is also a graded Lie algebra structure on \mathfrak{g}_2 given by the Gerstenhaber bracket. In particular \mathfrak{g}_1 and \mathfrak{g}_2 are also Lie algebras up to homotopy (L_∞ -algebra for short). In the case $M = \mathbb{R}^n$, using different methods, Kontsevich ([Ko1] and [Ko2]) and Tamarkin ([Ta]) have proved the existence of Lie homomorphisms “up to homotopy” (L_∞ -morphisms) from \mathfrak{g}_1 to \mathfrak{g}_2 . Kontsevich's proof uses graph complex and is related to multizeta functions whereas Tamarkin's construction uses the existence of Drinfeld's associators. In fact Tamarkin's L_∞ -morphism comes from the restriction of

a Gerstenhaber algebra up to homotopy homomorphism (G_∞ -morphism) from \mathfrak{g}_1 to \mathfrak{g}_2 . The G_∞ -algebra structure on \mathfrak{g}_1 is induced by its classical Gerstenhaber algebra structure and a far less trivial G_∞ -structure on \mathfrak{g}_2 was proved to exist by Tamarkin [Ta] and relies on a Drinfeld's associator. Tamarkin's G_∞ -morphism also restricts into a commutative, associative up to homotopy morphism (C_∞ -morphism for short). The C_∞ -structure on \mathfrak{g}_2 (given by restriction of the G_∞ -one) highly depends on Drinfeld's associator, and any two choices of a Drinfeld associator yields *a priori* different C_∞ -structures. When M is a Poisson manifold, Kontsevich and Tamarkin homomorphisms imply the existence of a star-product (see [BFFLS1] and [BFFLS2] for a definition). A connection between the two approaches has been given in [KS] but the morphisms given by Kontsevich and Tamarkin are not the same. The aim of this paper is to show that, given any G_∞ -structure on \mathfrak{g}_2 and any C_∞ -morphism φ between \mathfrak{g}_1 and \mathfrak{g}_2 , there exists a G_∞ -morphism Φ between \mathfrak{g}_1 and \mathfrak{g}_2 that restricts to φ . We also show that any L_∞ -morphism can be deformed into a G_∞ -one.

In the first section, we fix notation and recall the definitions of L_∞ and G_∞ -structures. In the second section we state and prove the main theorem. In the last section we show that any two G_∞ -morphisms given by Tamarkin's method are homotopic.

Remark : In the sequel, unless otherwise is stated, the manifold M is supposed to be \mathbb{R}^n for some $n \geq 1$. Most results could be generalized to other manifolds using techniques of Kontsevich [Ko1] (also see [TS], [CFT]).

1 C_∞ , L_∞ and G_∞ -structures

For any graded vector space \mathfrak{g} , we choose the following degree on $\wedge^\bullet \mathfrak{g}$: if X_1, \dots, X_k are homogeneous elements of respective degree $|X_1|, \dots, |X_k|$, then

$$|X_1 \wedge \dots \wedge X_k| = |X_1| + \dots + |X_k| - k.$$

In particular the component $\mathfrak{g} = \wedge^1 \mathfrak{g} \subset \wedge^\bullet \mathfrak{g}$ is the same as the space \mathfrak{g} with degree shifted by one. The space $\wedge^\bullet \mathfrak{g}$ with the deconcatenation cobracket is the cofree cocommutative coalgebra on \mathfrak{g} with degree shifted by one (see [LS], Section 2). Any degree one map $d^k : \wedge^k \mathfrak{g} \rightarrow \mathfrak{g}$ ($k \geq 1$) extends into a derivation $d^k : \wedge^\bullet \mathfrak{g} \rightarrow \wedge^\bullet \mathfrak{g}$ of the coalgebra $\wedge^\bullet \mathfrak{g}$ by cofreeness property.

Definition 1.1. A vector space \mathfrak{g} is endowed with a L_∞ -algebra (Lie algebras "up to homotopy") structure if there are degree one linear maps $m^{1, \dots, 1}$, with k ones : $\wedge^k \mathfrak{g} \rightarrow \mathfrak{g}$ such that if we extend them to maps $\wedge^\bullet \mathfrak{g} \rightarrow \wedge^\bullet \mathfrak{g}$, then $d \circ d = 0$ where d is the derivation

$$d = m^1 + m^{1,1} + \dots + m^{1, \dots, 1} + \dots .$$

For more details on L_∞ -structures, see [LS]. It follows from the definition that a L_∞ -algebra structure induces a differential coalgebra structure on $\wedge^\bullet \mathfrak{g}$ and that the map $m^1 : \mathfrak{g} \rightarrow \mathfrak{g}$ is a differential. If $m^{1, \dots, 1} : \wedge^k \mathfrak{g} \rightarrow \mathfrak{g}$ are 0 for $k \geq 3$, we get the usual definition of (differential if $m^1 \neq 0$) graded Lie algebras.

For any graded vector space \mathfrak{g} , we denote $\underline{\mathfrak{g}}^{\otimes n}$ the quotient of $\mathfrak{g}^{\otimes n}$ by the image of all shuffles of length n (see [GK] or [GH] for details). The graded vector space $\bigoplus_{n \geq 0} \underline{\mathfrak{g}}^{\otimes n}$ is a quotient coalgebra of the tensor coalgebra $\bigoplus_{n \geq 0} \mathfrak{g}^{\otimes n}$. It is well known that this coalgebra $\bigoplus_{n \geq 0} \underline{\mathfrak{g}}^{\otimes n}$ is the cofree Lie coalgebra on the vector space \mathfrak{g} (with degree shifted by minus one).

Definition 1.2. A C_∞ -algebra (commutative and associative “up to homotopy” algebra) structure on a vector space \mathfrak{g} is given by a collection of degree one linear maps $m^k : \underline{\mathfrak{g}}^{\otimes k} \rightarrow \underline{\mathfrak{g}}$ such that if we extend them to maps $\underline{\mathfrak{g}}^{\otimes \bullet} \rightarrow \underline{\mathfrak{g}}^{\otimes \bullet}$, then $d \circ d = 0$ where d is the derivation

$$d = m^1 + m^2 + m^3 + \dots.$$

In particular a C_∞ -algebra is an A_∞ -algebra.

For any space \mathfrak{g} , we denote $\wedge^\bullet \underline{\mathfrak{g}}^{\otimes \bullet}$ the graded space

$$\wedge^\bullet \underline{\mathfrak{g}}^{\otimes \bullet} = \bigoplus_{m \geq 1, p_1 + \dots + p_n = m} \underline{\mathfrak{g}}^{\otimes p_1} \wedge \dots \wedge \underline{\mathfrak{g}}^{\otimes p_n}.$$

We use the following grading on $\wedge^\bullet \underline{\mathfrak{g}}^{\otimes \bullet}$: for $x_1^{p_1}, \dots, x_n^{p_n} \in \underline{\mathfrak{g}}$, we define

$$|\underline{x_1^{p_1} \otimes \dots \otimes x_1^{p_1}} \wedge \dots \wedge \underline{x_n^{p_n} \otimes \dots \otimes x_n^{p_n}}| = \sum_{i_1}^{p_1} |x_1^{i_1}| + \dots + \sum_{i_n}^{p_n} |x_n^{i_n}| - n.$$

Notice that the induced grading on $\wedge^\bullet \underline{\mathfrak{g}} \subset \wedge^\bullet \underline{\mathfrak{g}}^{\otimes \bullet}$ is the same than the one introduced above. The cobracket on $\underline{\mathfrak{g}}^{\otimes \bullet}$ and the coproduct on $\wedge^\bullet \underline{\mathfrak{g}}$ extend to a cobracket and a coproduct on $\wedge^\bullet \underline{\mathfrak{g}}^{\otimes \bullet}$ which yield a Gerstenhaber coalgebra structure on $\wedge^\bullet \underline{\mathfrak{g}}^{\otimes \bullet}$. It is well known that this coalgebra structure is cofree (see [Gi], Section 3 for example).

Definition 1.3. A G_∞ -algebra (Gerstenhaber algebra “up to homotopy”) structure on a graded vector space \mathfrak{g} is given by a collection of degree one maps

$$m^{p_1, \dots, p_n} : \underline{\mathfrak{g}}^{\otimes p_1} \wedge \dots \wedge \underline{\mathfrak{g}}^{\otimes p_n} \rightarrow \underline{\mathfrak{g}}$$

indexed by $p_1, \dots, p_n \geq 1$ such that their canonical extension: $\wedge^\bullet \underline{\mathfrak{g}}^{\otimes \bullet} \rightarrow \wedge^\bullet \underline{\mathfrak{g}}^{\otimes \bullet}$ satisfies $d \circ d = 0$ where

$$d = \sum_{m \geq 1, p_1 + \dots + p_n = m} m^{p_1, \dots, p_n}.$$

Again, as the coalgebra structure of $\wedge^\bullet \underline{\mathfrak{g}}^{\otimes \bullet}$ is cofree, the map d makes $\wedge^\bullet \underline{\mathfrak{g}}^{\otimes \bullet}$ into a differential coalgebra. If the maps m^{p_1, \dots, p_n} are 0 for $(p_1, p_2, \dots) \neq (1, 0, \dots), (1, 1, 0, \dots)$ or $(2, 0, \dots)$, we get the usual definition of (differential if $m^1 \neq 0$) Gerstenhaber algebra.

The space of multivector fields \mathfrak{g}_1 is endowed with a graded Lie bracket $[-, -]_S$ called the Schouten bracket (see [Kos]). This Lie algebra can be extended into a Gerstenhaber algebra, with commutative structure given by the exterior product: $(\alpha, \beta) \mapsto \alpha \wedge \beta$

Setting $d_1 = m_1^{1,1} + m_1^2$, where $m_1^{1,1} : \wedge^2 \mathfrak{g}_1 \rightarrow \mathfrak{g}_1$, and $m_1^2 : \underline{\mathfrak{g}}_1^{\otimes 2} \rightarrow \mathfrak{g}_1$ are the extension of the Schouten bracket and the exterior product, we find that (\mathfrak{g}_1, d_1) is a G_∞ -algebra.

In the same way, one can define a differential Lie algebra structure on the vector space $\mathfrak{g}_2 = C(A, A) = \bigoplus_{k \geq 0} C^k(A, A)$, the space of Hochschild cochains (generated by differential k -linear maps from A^k to A), where $A = C^\infty(M)$ is the algebra of smooth differential functions over M . Its bracket $[-, -]_G$, called the Gerstenhaber bracket, is defined, for $D, E \in \mathfrak{g}_2$, by

$$[D, E]_G = \{D|E\} - (-1)^{|E||D|} \{E|D\},$$

where

$$\{D|E\}(x_1, \dots, x_{d+e-1}) = \sum_{i \geq 0} (-1)^{|E|^i} D(x_1, \dots, x_i, E(x_{i+1}, \dots, x_{i+e}), \dots).$$

The space \mathfrak{g}_2 has a grading defined by $|D|=k \Leftrightarrow D \in C^{k+1}(A, A)$ and its differential is $b = [m, -]_G$, where $m \in C^2(A, A)$ is the commutative multiplication on A .

Tamarkin (see [Ta] or also [GH]) stated the existence of a G_∞ -structure on \mathfrak{g}_2 (depending on a choice of a Drinfeld associator) given by a differential $d_2 = m_2^1 + m_2^{1,1} + m_2^2 + \dots + m_2^{p_1, \dots, p_n} + \dots$, on $\wedge^\bullet \mathfrak{g}_2^{\otimes \bullet}$ satisfying $d_2 \circ d_2 = 0$. Although this structure is non-explicit, it satisfies the following three properties :

- (a) m_2^1 is the extension of the differential b
- (b) $m_2^{1,1}$ is the extension of the Gerstenhaber bracket $[-, -]_G$ and $m_2^{1,1, \dots, 1} = 0$
- (c) m_2^2 induces the exterior product in cohomology and the collection of the $(m^k)_{k \geq 1}$ defines a C_∞ -structure on \mathfrak{g}_2 . (1.1)

Definition 1.4. A L_∞ -morphism between two L_∞ -algebras $(\mathfrak{g}_1, d_1 = m_1^1 + \dots)$ and $(\mathfrak{g}_2, d_2 = m_2^1 + \dots)$ is a morphism of differential coalgebras

$$\varphi : (\wedge^\bullet \mathfrak{g}_1, d_1) \rightarrow (\wedge^\bullet \mathfrak{g}_2, d_2). \quad (1.2)$$

Such a map φ is uniquely determined by a collection of maps $\varphi^n : \wedge^n \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ (again by cofreeness properties). In the case \mathfrak{g}_1 and \mathfrak{g}_2 are respectively the graded Lie algebra $(\Gamma(M, \wedge TM), [-, -]_S)$ and the differential graded Lie algebra $(C(A, A), [-, -]_G)$, the formality theorems of Kontsevich and Tamarkin state the existence of a L_∞ -morphism between \mathfrak{g}_1 and \mathfrak{g}_2 such that φ^1 is the Hochschild-Kostant-Rosenberg quasi-isomorphism.

Definition 1.5. A morphism of C_∞ -algebras between two C_∞ -algebras (\mathfrak{g}_1, d_1) and (\mathfrak{g}_2, d_2) is a map $\phi : (\oplus \mathfrak{g}_1^{\otimes \bullet}, d_1) \rightarrow (\oplus \mathfrak{g}_2^{\otimes \bullet}, d_2)$ of codifferential coalgebras.

A C_∞ -morphism is in particular a morphism of A_∞ -algebras and is uniquely determined by maps $\partial^k : \mathfrak{g}^{\otimes k} \rightarrow \mathfrak{g}$.

Definition 1.6. A morphism of G_∞ -algebras between two G_∞ -algebras (\mathfrak{g}_1, d_1) and (\mathfrak{g}_2, d_2) is a map $\phi : (\wedge^\bullet \mathfrak{g}_1^{\otimes \bullet}, d_1) \rightarrow (\wedge^\bullet \mathfrak{g}_2^{\otimes \bullet}, d_2)$ of codifferential coalgebras.

There are coalgebras inclusions $\wedge^\bullet \mathfrak{g} \rightarrow \wedge^\bullet \mathfrak{g}^{\otimes \bullet}$, $\oplus \mathfrak{g}^{\otimes \bullet} \rightarrow \wedge^\bullet \mathfrak{g}^{\otimes \bullet}$ and it is easy to check that any G_∞ -morphism between two G_∞ -algebras $(\mathfrak{g}, \Sigma m^{p_1, \dots, p_n})$, $(\mathfrak{g}', \Sigma m'^{p_1, \dots, p_n})$ restricts to a L_∞ -morphism $(\wedge^\bullet \mathfrak{g}, \Sigma m^{1, \dots, 1}) \rightarrow (\wedge^\bullet \mathfrak{g}', \Sigma m'^{1, \dots, 1})$ and a C_∞ -morphism $(\oplus \mathfrak{g}^{\otimes \bullet}, \Sigma m^k) \rightarrow (\oplus \mathfrak{g}'^{\otimes \bullet}, \Sigma m'^k)$. In the case \mathfrak{g}_1 and \mathfrak{g}_2 are as above, Tamarkin's theorem states that there exists a G_∞ -morphism between the two G_∞ algebras \mathfrak{g}_1 and \mathfrak{g}_2 (with the G_∞ structure he built) that restricts to a C_∞ and a L_∞ -morphism.

2 Main theorem

We keep the notations of the previous section, in particular \mathfrak{g}_2 is the Hochschild complex of cochains on $C^\infty(M)$ and \mathfrak{g}_1 its cohomology. Here is our main theorem.

Theorem 2.1. *Given any G_∞ -structure d_2 on \mathfrak{g}_2 satisfying the three properties of (1.1), and any C_∞ -morphism φ between \mathfrak{g}_1 and \mathfrak{g}_2 such that φ^1 is the Hochschild-Kostant-Rosenberg map, there exists a G_∞ -morphism $\Phi : (\mathfrak{g}_1, d_1) \rightarrow (\mathfrak{g}_2, d_2)$ that restricts to φ .*

Also, given any L_∞ -morphism γ between \mathfrak{g}_1 and \mathfrak{g}_2 such that γ^1 is the Hochschild-Kostant-Rosenberg map, there exists a G_∞ -structure (\mathfrak{g}_1, d'_1) on \mathfrak{g}_1 and G_∞ -morphism $\Gamma : (\mathfrak{g}_1, d'_1) \rightarrow (\mathfrak{g}_2, d_2)$ that restricts to γ . Moreover there exists a G_∞ -morphism $\Gamma' : (\mathfrak{g}_1, d_1) \rightarrow (\mathfrak{g}_1, d'_1)$.

In particular, Theorem 2.1 applies to the formality map of Kontsevich and also to any C_∞ -map derived (see [Ta], [GH]) from any B_∞ -structure on \mathfrak{g}_2 lifting the Gerstenhaber structure of \mathfrak{g}_1 .

Let us first recall the proof of Tamarkin's formality theorem (see [GH] for more details):

1. First one proves there exists a G_∞ -structure on \mathfrak{g}_2 , given by a differential d_2 , as in (1.1).
2. Then, one constructs a G_∞ -structure on \mathfrak{g}_1 given by a differential d'_1 together with a G_∞ -morphism Φ between (\mathfrak{g}_1, d'_1) and (\mathfrak{g}_2, d_2) .
3. Finally, one constructs a G_∞ -morphism Φ' between (\mathfrak{g}_1, d_1) and (\mathfrak{g}_1, d'_1) .

The composition $\Phi \circ \Phi'$ is then a G_∞ -morphism between (\mathfrak{g}_1, d_1) and (\mathfrak{g}_2, d_2) , thus restricts to a L_∞ -morphism between the differential graded Lie algebras \mathfrak{g}_1 and \mathfrak{g}_2 .

We suppose now that, in the first step, we take any G_∞ -structure on \mathfrak{g}_2 given by a differential d_2 and we suppose we are given a C_∞ -morphism φ and a L_∞ -morphism γ between \mathfrak{g}_1 and \mathfrak{g}_2 satisfying $\gamma^1 = \varphi^1 = \varphi_{\text{HKR}}$ the Hochschild-Kostant-Rosenberg quasi-isomorphism.

Proof of Theorem 2.1:

The Theorem will follow if we prove that steps 2 and 3 of Tamarkin's construction are still true with the extra conditions that the restriction of the G_∞ -morphism Φ (resp. Φ') on the C_∞ -structures is the C_∞ -morphism $\varphi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ (resp. id).

Let us recall (see [GH]) that the constructions of Φ and d'_1 can be made by induction. For $i = 1, 2$ and $n \geq 0$, let us set

$$V_i^{[n]} = \bigoplus_{p_1 + \dots + p_k = n} \underline{\mathfrak{g}_i^{\otimes p_1}} \wedge \dots \wedge \underline{\mathfrak{g}_i^{\otimes p_k}}$$

and $V_i^{[\leq n]} = \sum_{k \leq n} V_i^{[k]}$. Let $d_2^{[n]}$ and $d_2^{[\leq n]}$ be the sums

$$d_2^{[n]} = \sum_{p_1 + \dots + p_k = n} d_2^{p_1, \dots, p_k} \quad \text{and} \quad d_2^{[\leq n]} = \sum_{p \leq n} d_2^{[p]}.$$

Clearly, $d_2 = \sum_{n \geq 1} d_2^{[n]}$. In the same way, we denote $d'_1 = \sum_{n \geq 1} d'_1^{[n]}$ with

$$d'_1^{[n]} = \sum_{p_1 + \dots + p_k = n} d'_1{}^{p_1, \dots, p_k} \quad \text{and} \quad d'_1^{[\leq n]} = \sum_{1 \leq k \leq n} d'_1{}^{[k]}.$$

We know from Section 1 that a morphism $\Phi : (\wedge^\bullet \underline{\mathfrak{g}}_1^{\otimes \bullet}, d'_1) \rightarrow (\wedge^\bullet \underline{\mathfrak{g}}_2^{\otimes \bullet}, d_2)$ is uniquely determined by its components $\Phi^{p_1, \dots, p_k} : \underline{\mathfrak{g}}_1^{\otimes p_1} \wedge \dots \wedge \underline{\mathfrak{g}}_1^{\otimes p_k} \rightarrow \underline{\mathfrak{g}}_2$. Again, we have $\Phi = \sum_{n \geq 1} \Phi^{[n]}$ with

$$\Phi^{[n]} = \sum_{p_1 + \dots + p_k = n} \Phi^{p_1, \dots, p_k} \quad \text{and} \quad \Phi^{[\leq n]} = \sum_{1 \leq k \leq n} \Phi^{[k]}.$$

We want to construct the maps $d'_1{}^{[n]}$ and $\Phi^{[n]}$ by induction with the initial condition

$$d'_1{}^{[1]} = 0 \quad \text{and} \quad \Phi^{[1]} = \varphi_{\text{HKR}},$$

where $\varphi_{\text{HKR}} : (\mathfrak{g}_1, 0) \rightarrow (\mathfrak{g}_2, b)$ is the Hochschild-Kostant-Rosenberg quasi-isomorphism (see [HKR]) defined, for $\alpha \in \mathfrak{g}_1$, $f_1, \dots, f_n \in A$, by

$$\varphi_{\text{HKR}} : \alpha \mapsto ((f_1, \dots, f_n) \mapsto \langle \alpha, df_1 \wedge \dots \wedge df_n \rangle).$$

Moreover, we want the following extra conditions to be true:

$$\Phi^{k \geq 2} = \varphi^k, \quad d'_1{}^2 = d_1^2, \quad d'_1{}^{k \geq 3} = 0. \quad (2.3)$$

Now suppose the construction is done for $n-1$ ($n \geq 2$), i.e., we have built maps $(d'_1{}^{[i]})_{i \leq n-1}$ and $(\Phi^{[i]})_{i \leq n-1}$ satisfying conditions (2.3) and

$$\Phi^{[\leq n-1]} \circ d'_1{}^{[\leq n-1]} = d_2^{[\leq n-1]} \circ \Phi^{[\leq n-1]} \quad \text{on } V_1^{[\leq n-1]} \quad \text{and} \quad d'_1{}^{[\leq n-1]} \circ d'_1{}^{[\leq n-1]} = 0 \quad \text{on } V_1^{[\leq n]}. \quad (2.4)$$

In [GH], we prove that for any such $(d'_1{}^{[i]})_{i \leq n-1}$ and $(\Phi^{[i]})_{i \leq n-1}$, one can construct $d'^{[n]}$ and $\Phi^{[n]}$ such that condition (2.4) is true for n instead of $n-1$. To complete the proof of Theorem 2.1 (step 2), we have to show that $d'^{[n]}$ and $\Phi^{[n]}$ can be chosen to satisfy conditions (2.3). In the equation 2.4, the terms d'^k and Φ^k only act on V_1^k . So one can replace Φ^n with φ^n , d'^2 with d_2^2 (or d_1^2 , $i \geq 3$ with 0) provided conditions (2.4) are still satisfied on V_1^n . The other terms acting on V_1^n in the equation (2.4) only involve terms $\Phi^m = \varphi^m$ and d'^m . Then conditions (2.4) on $V_1^{1, \dots, 1}$ are the equations that should be satisfied by a C_∞ -morphism between the C_∞ -algebras $(\mathfrak{g}_1, d'^{1,1} = d_1^{1,1})$ and $(\mathfrak{g}_2, \sum_{k \geq 1} d_2^k)$ restricted to V_1^n . Hence by hypothesis on φ the conditions hold.

Similarly the construction of Φ' can be made by induction. Let us recall the proof given in [GH]. Again a morphism $\Phi' : (\wedge^\bullet \underline{\mathfrak{g}}_1^{\otimes \bullet}, d_1) \rightarrow (\wedge^\bullet \underline{\mathfrak{g}}_2^{\otimes \bullet}, d'_1)$ is uniquely determined by its components $\Phi'^{p_1, \dots, p_k} : \underline{\mathfrak{g}}_1^{\otimes p_1} \wedge \dots \wedge \underline{\mathfrak{g}}_1^{\otimes p_k} \rightarrow \underline{\mathfrak{g}}_1$. We write $\Phi' = \sum_{n \geq 1} \Phi'^{[n]}$ with

$$\Phi'^{[n]} = \sum_{p_1 + \dots + p_k = n} \Phi'^{p_1, \dots, p_k} \quad \text{and} \quad \Phi'^{[\leq n]} = \sum_{1 \leq k \leq n} \Phi'^{[k]}.$$

We construct the maps $\Phi'^{[n]}$ by induction with the initial condition $\Phi'^{[1]} = \text{id}$. Moreover, we want the following extra conditions to be true:

$$\Phi'^m = 0 \quad \text{for } n \geq 2. \quad (2.5)$$

Now suppose the construction is done for $n-1$ ($n \geq 2$), i.e., we have built maps $(\Phi'^{[i]})_{i \leq n-1}$ satisfying conditions (2.5) and

$$\Phi'^{[\leq n-1]} d_1^{[\leq n]} = d'_1{}^{[\leq n]} \Phi'^{[\leq n-1]} \quad \text{on } V_1^{[\leq n]}. \quad (2.6)$$

In [GH], we prove that for any such $(\Phi^{[i]})_{i \leq n-1}$, one can construct $\Phi'^{[n]}$ such that condition (2.6) is true for n instead of $n-1$ in the following way : making the equation $\Phi' d_1 = d'_1 \Phi'$ on $V_1^{[n+1]}$ explicit, we get

$$\Phi'^{[\leq n]} d_1^{[\leq n+1]} = d'_1^{[\leq n+1]} \Phi'^{[\leq n]}. \quad (2.7)$$

If we now take into account that $d_1^{[i]} = 0$ for $i \neq 2$, $d'_1^{[1]} = 0$ and that on $V_1^{[n+1]}$ we have $\Phi'^{[k]} d_1^{[l]} = d'_1^{[\leq k]} \Phi'^{[l]} = 0$ for $k+l > n+2$, the identity (2.7) becomes

$$\Phi'^{[\leq n]} d_1^{[2]} = \sum_{k=2}^{n+1} d'_1^{[k]} \Phi'^{[\leq n-k+2]}.$$

As $d_1^{[2]} = d_1^{[2]}$, (2.7) is equivalent to

$$d_1^{[2]} \Psi'^{[\leq n]} - \Phi'^{[\leq n]} d_1^{[2]} = \left[d_1^{[2]}, \Phi'^{[\leq n]} \right] = - \sum_{k=3}^{n+1} d'_1^{[k]} \Phi'^{[\leq n-k+2]}.$$

Notice that $d_1^{[2]} = m_1^{1,1} + m_1^2$. Then the construction will be possible when the term $\sum_{k=3}^{n+1} d'_1^{[k]} \Psi'^{[\leq n-k+2]}$ is a coboundary in the subcomplex of $(\text{End}(\wedge^\bullet \underline{\mathfrak{g}}_1^{\otimes \bullet}), [d_1^{[2]}, -])$ consisting of maps which restrict to zero on $\oplus_{n \geq 2} \underline{\mathfrak{g}}_1^{\otimes n}$. It is always a cocycle by straightforward computation (see [GH]) and the subcomplex is acyclic because both $(\text{End}(\wedge^\bullet \underline{\mathfrak{g}}_1^{\otimes \bullet}), [d_1^{[2]}, -])$ and the Harrison cohomology of \mathfrak{g}_1 are trivial according to Tamarkin [Ta] (see also [GH] Proposition 5.1 and [Hi] 5.4).

In the case of the L_∞ -morphism γ , the first step is similar: the fact that γ is a L_∞ -map enables us to build a G_∞ -structure (\mathfrak{g}_1, d'_1) on \mathfrak{g}_1 and a G_∞ -morphism $\Gamma : (\mathfrak{g}_1, d'_1) \rightarrow (\mathfrak{g}_2, d_2)$ such that:

$$\Gamma^{1, \dots, 1} = \gamma^{1, \dots, 1}, \quad d_1^{1,1} = d_1^{1,1}, \quad d_1^{1,1, \dots, 1} = 0. \quad (2.8)$$

For the second step, we have to build a map Γ' satisfying the equation

$$d_1^{[2]} \Gamma'^{[\leq n]} - \Gamma'^{[\leq n]} d_1^{[2]} = \left[d_1^{[2]}, \Gamma'^{[\leq n]} \right] = - \sum_{k=3}^{n+1} d'_1^{[k]} \Gamma'^{[\leq n-k+2]}$$

on $V_1^{[n+1]}$ for any $n \geq 1$. Again, because Tamarkin has proved that the complex $(\text{End}(\wedge^\bullet \underline{\mathfrak{g}}_1^{\otimes \bullet}), [d_1^{[2]}, -])$ is acyclic (we are in the case $M = \mathbb{R}^n$), the result follows from the fact that $\sum_{k=3}^{n+1} d'_1^{[k]} \Gamma'^{[\leq n-k+2]}$ is a cocycle. The difference with the C_∞ -case is that the $\Gamma^{1, \dots, 1}$ could be non zero. ■

3 The difference between two G_∞ -maps

In this section we investigate the difference between two different G_∞ -formality maps. We fix once for all a G_∞ -structure on \mathfrak{g}_2 (given by a differential d_2) satisfying the conditions (1.1) and a morphism of G_∞ -algebras $T : (\mathfrak{g}_1, d_1) \rightarrow (\mathfrak{g}_2, d_2)$ such that $T^1 : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ is Φ_{HKR} . Let $K : (\mathfrak{g}_1, d_1) \rightarrow (\mathfrak{g}_2, d_2)$ be any other G_∞ -morphism with $K^1 = \Phi_{\text{HKR}}$ (for example any lift of a Kontsevich formality map or any G_∞ -maps lifting another C_∞ -morphism).

Theorem 3.1. *There exists a map $h : \wedge^{\bullet} \underline{\mathfrak{g}}_1^{\otimes \bullet} \rightarrow \wedge^{\bullet} \underline{\mathfrak{g}}_2^{\otimes \bullet}$ such that*

$$T - K = h \circ d_1 + d_2 \circ h.$$

In other words the formality G_{∞} -morphisms K and T are homotopic.

The maps T and K are elements of the cochain complex $(\text{Hom}(\wedge^{\bullet} \underline{\mathfrak{g}}_1^{\otimes \bullet}, \wedge^{\bullet} \underline{\mathfrak{g}}_2^{\otimes \bullet}), \delta)$ with differential given, for all $f \in \text{Hom}(\wedge^{\bullet} \underline{\mathfrak{g}}_1^{\otimes \bullet}, \wedge^{\bullet} \underline{\mathfrak{g}}_2^{\otimes \bullet}), |f| = k$, by

$$\delta(f) = d_2 \circ f - (-1)^k f \circ d_1.$$

We first compare this cochain complex with the complexes $(\text{End}(\wedge^{\bullet} \underline{\mathfrak{g}}_1^{\otimes \bullet}), [d_1; -])$ and $(\text{End}(\wedge^{\bullet} \underline{\mathfrak{g}}_2^{\otimes \bullet}), [d_2; -])$ (where $[-; -]$ is the graded commutator of morphisms). There are morphisms

$$T_* : \text{End}(\wedge^{\bullet} \underline{\mathfrak{g}}_1^{\otimes \bullet}) \rightarrow \text{Hom}(\wedge^{\bullet} \underline{\mathfrak{g}}_1^{\otimes \bullet}, \wedge^{\bullet} \underline{\mathfrak{g}}_2^{\otimes \bullet}), \quad T^* : \text{End}(\wedge^{\bullet} \underline{\mathfrak{g}}_2^{\otimes \bullet}) \rightarrow \text{Hom}(\wedge^{\bullet} \underline{\mathfrak{g}}_1^{\otimes \bullet}, \wedge^{\bullet} \underline{\mathfrak{g}}_2^{\otimes \bullet})$$

defined, for $f \in \text{End}(\wedge^{\bullet} \underline{\mathfrak{g}}_2^{\otimes \bullet})$ and $g \in \text{Hom}(\wedge^{\bullet} \underline{\mathfrak{g}}_1^{\otimes \bullet}, \wedge^{\bullet} \underline{\mathfrak{g}}_2^{\otimes \bullet})$, by

$$T_*(f) = T \circ f, \quad T^*(g) = g \circ T.$$

Lemma 3.2. *The morphisms*

$$T_* : (\text{End}(\wedge^{\bullet} \underline{\mathfrak{g}}_1^{\otimes \bullet}), [d_1; -]) \rightarrow (\text{Hom}(\wedge^{\bullet} \underline{\mathfrak{g}}_1^{\otimes \bullet}, \wedge^{\bullet} \underline{\mathfrak{g}}_2^{\otimes \bullet}), \delta) \leftarrow (\text{End}(\wedge^{\bullet} \underline{\mathfrak{g}}_2^{\otimes \bullet}), [d_2; -]) : T^*$$

of cochain complexes are quasi-isomorphisms.

Remark: This lemma holds for every manifold M and any G_{∞} -morphism $T : (\mathfrak{g}_1, d_1) \rightarrow (\mathfrak{g}_2, d_2)$.

Proof: First we show that T_* is a morphism of complexes. Let $f \in \text{End}(\wedge^{\bullet} \underline{\mathfrak{g}}_2^{\otimes \bullet})$ with $|f| = k$, then

$$\begin{aligned} T_*([d_1; f]) &= T \circ d_1 \circ f - (-1)^k T \circ f \circ d_1 \\ &= d_2 \circ (T \circ f) - (-1)^k (T \circ f) \circ d_1 \\ &= \delta(T_*(f)). \end{aligned}$$

Let us prove now that T_* is a quasi-isomorphism. For any graded vector space \mathfrak{g} , the space $\wedge^{\bullet} \underline{\mathfrak{g}}^{\otimes \bullet}$ has the structure of a filtered space where the m -level of the filtration is $F^m(\wedge^{\bullet} \underline{\mathfrak{g}}^{\otimes \bullet}) = \bigoplus_{p_1 + \dots + p_n - 1 \leq m} \underline{\mathfrak{g}}^{\otimes p_1} \wedge \dots \wedge \underline{\mathfrak{g}}^{\otimes p_n}$. Clearly the differential d_1 and d_2 are compatible with the filtrations on $\wedge^{\bullet} \underline{\mathfrak{g}}_1^{\otimes \bullet}$ and $\wedge^{\bullet} \underline{\mathfrak{g}}_2^{\otimes \bullet}$, hence $\text{End}(\wedge^{\bullet} \underline{\mathfrak{g}}_1^{\otimes \bullet})$ and $\text{Hom}(\wedge^{\bullet} \underline{\mathfrak{g}}_1^{\otimes \bullet}, \wedge^{\bullet} \underline{\mathfrak{g}}_2^{\otimes \bullet})$ are filtered cochain complex. This yields two spectral sequences (lying in the first quadrant) $E_{\bullet, \bullet}^*$ and $\tilde{E}_{\bullet, \bullet}^*$ which converge respectively toward the cohomology $H^{\bullet}(\text{End}(\wedge^{\bullet} \underline{\mathfrak{g}}_1^{\otimes \bullet}))$ and $H^{\bullet}(\text{Hom}(\wedge^{\bullet} \underline{\mathfrak{g}}_1^{\otimes \bullet}, \wedge^{\bullet} \underline{\mathfrak{g}}_2^{\otimes \bullet}))$. By standard spectral sequence techniques it is enough to prove that the map $T_*^0 : E_0^{\bullet, \bullet} \rightarrow \tilde{E}_0^{\bullet, \bullet}$ induced by T_* on the associated grading is a quasi-isomorphism.

The induced differentials on $E_0^{\bullet, \bullet}$ and $\tilde{E}_0^{\bullet, \bullet}$ are respectively $[d_1^1, -] = 0$ and $d_2^1 \circ (-) - (-) \circ d_1^1 = b \circ (-)$ where b is the Hochschild coboundary. By cofreeness property we have the following two isomorphisms

$$E_0^{\bullet, \bullet} \cong \text{End}(\mathfrak{g}_1), \quad \tilde{E}_0^{\bullet, \bullet} \cong \text{Hom}(\mathfrak{g}_1, \mathfrak{g}_2).$$

The map $T_*^0 : E_0^{\bullet, \bullet} \rightarrow \tilde{E}^0$ induced by T_* is $\varphi_{\text{HKR}} \circ (-)$. Let $p : \mathfrak{g}_2 \rightarrow \mathfrak{g}_1$ be the projection onto the cohomology, i.e. $p \circ \varphi_{\text{HKR}} = \text{id}$. Let $u : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ be any map satisfying $b(u) = 0$ and set $v = p \circ u \in \text{End}(\mathfrak{g}_1)$. One can choose a map $w : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ which satisfies for any $x \in \mathfrak{g}_1$ the following identity

$$\varphi_{\text{HKR}} \circ p \circ u(x) - u(x) = b \circ w(x).$$

It follows that $\varphi_{\text{HKR}}(v)$ has the same class of homology as u which proves the surjectivity of T_*^0 in cohomology. The identity $p \circ \varphi_{\text{HKR}} = \text{id}$ implies easily that T_*^0 is also injective in cohomology which finish the proof of the lemma for T_* .

The proof that T^* is also a quasi-isomorphism is analogous. ■

Proof of Theorem 3.1:

It is easy to check that $T - K$ is a cocycle in $(\text{Hom}(\wedge^{\bullet} \underline{\mathfrak{g}}_1^{\otimes \bullet}, \wedge^{\bullet} \underline{\mathfrak{g}}_2^{\otimes \bullet}), \delta)$. The complex of cochain $(\text{End}(\wedge^{\bullet} \underline{\mathfrak{g}}_1^{\otimes \bullet}), [d_1, -]) \cong (\text{Hom}(\wedge^{\bullet} \underline{\mathfrak{g}}_1^{\otimes \bullet}, \mathfrak{g}_1), [d_1, -])$ is trigraded with $||_1$ being the degree coming from the graduation of \mathfrak{g}_1 and any element x lying in $\underline{\mathfrak{g}}_1^{\otimes p_1} \wedge \dots \wedge \underline{\mathfrak{g}}_1^{\otimes p_q}$ satisfies $|x|_2 = q - 1$, $|x|_3 = p_1 + \dots + p_q - q$. In the case $M = \mathbb{R}^n$, the cohomology $H^{\bullet}(\text{End}(\wedge^{\bullet} \underline{\mathfrak{g}}_1^{\otimes \bullet}), [d_1, -])$ is concentrated in bidegree $(||_2, ||_3) = (0, 0)$ (see [Ta], [Hi]).

By Lemma 3.2, this is also the case for the cochain complex $(\text{Hom}(\wedge^{\bullet} \underline{\mathfrak{g}}_1^{\otimes \bullet}, \wedge^{\bullet} \underline{\mathfrak{g}}_2^{\otimes \bullet}), \delta)$. Thus, its cohomology classes are determined by complex morphisms $(\mathfrak{g}_1, 0) \rightarrow (\mathfrak{g}_2, d_2^1)$ and it is enough to prove that T and K determine the same complex morphism $(\mathfrak{g}_1, 0) \rightarrow (\mathfrak{g}_2, d_2^1 = b)$ which is clear because T^1 and K^1 are both equal to the Hochschild-Kostant-Rosenberg map. ■

Remark 3.3. It is possible to have an explicit formula for the map h in Theorem 3.1. In fact the quasi-isomorphism coming from Lemma 3.2 can be made explicit using explicit homotopy formulae for the Hochschild-Kostant-Rosenberg map (see [Ha] for example) and deformation retract techniques (instead of spectral sequences) as in [Ka]. The same techniques also apply to give explicit formulae for the quasi-isomorphism giving the acyclicity of $(\text{End}(\wedge^{\bullet} \underline{\mathfrak{g}}_1^{\otimes \bullet}), [d_1; -])$ in the proof of theorem 3.1 (see [GH] for example)

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