A $\hbar$-adic valuation property of universal R-matrices

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Abstract

We prove that if $U_\hbar(\mathfrak{g})$ is a quasitriangular QUE algebra with universal R-matrix R, and $\mathcal{O}_\hbar(G^*)$ is the quantized function algebra sitting inside $U_\hbar(\mathfrak{g})$, then $\hbar\log(R)$ belongs to the tensor square $\mathcal{O}_\hbar(G^*)\bar{\otimes}\mathcal{O}_\hbar(G^*)$. This gives another proof of the results of Gavarini and Halbout, saying that R normalizes $\mathcal{O}_\hbar(G^*)\bar{\otimes}\mathcal{O}_\hbar(G^*)$ and therefore induces a braiding of the formal group G^* (in the sense of Weinstein and Xu, or Reshetikhin).

§ 0 Introduction

Let $(\mathfrak{g},r)$ be a finite-dimensional quasitriangular Lie bialgebra over a field k of characteristic 0, and let $(U_\hbar(\mathfrak{g}),R)$ be a quasitriangular quantization of $(\mathfrak{g},r)$. Recall that this means that (see [Dr]):
(1) $(\mathfrak{g},[-,-],\delta)$ is a Lie bialgebra, $r\in\mathfrak{g}\otimes\mathfrak{g}$ is a solution of the classical Yang-Baxter equation, and the cobracket $\delta:\mathfrak{g}\to\wedge^2(\mathfrak{g})$ of $\mathfrak{g}$ is given by $\delta(x)=[x\otimes 1+1\otimes x,r]$ for $x\in\mathfrak{g}$;
(2) $(U_\hbar(\mathfrak{g}),m,\Delta)$ is a quantized universal enveloping (QUE) algebra (m is the product of $U_\hbar(\mathfrak{g})$, Δ is its coproduct), $R\in U_\hbar(\mathfrak{g})^{\hat{\otimes}2}$ satisfies the quasitriangular identities:

$$(\Delta\otimes\mathrm{id})(R)=R^{13}R^{23},\qquad (\mathrm{id}\otimes\Delta)(R)=R^{13}R^{12},$$

$$(\mathrm{id}\otimes\varepsilon)(R)=(\varepsilon\otimes\mathrm{id})(R)=1,\qquad \Delta^{\mathrm{op}}=\mathrm{Ad}(R)\circ\Delta,$$

where $\mathrm{Ad}(R)(x) = RxR^{-1}$ for $x \in U_{\hbar}(\mathfrak{g})^{\widehat{\otimes}2}$, and

$$\left(\frac{1}{\hbar}(R-1) \bmod \hbar\right) = r\,; \tag{0.1}$$

here $\widehat{\otimes}$ denotes the $\hbar$-adically completed tensor product, the map $x \mapsto (x \bmod \hbar)$ is the canonical projection

$$U_{\hbar}(\mathfrak{g})^{\widehat{\otimes}2} \to U_{\hbar}(\mathfrak{g})^{\widehat{\otimes}2} \underset{k[[\hbar]]}{\otimes} k = U(\mathfrak{g})^{\otimes 2},$$

and r is viewed as an element of $U(\mathfrak{g})^{\otimes 2}$. For $n \geq 0$, let us denote by $\delta_n : U_{\hbar}(\mathfrak{g}) \to U_{\hbar}(\mathfrak{g})^{\widehat{\otimes}n}$ the map

$$\delta_n = (\mathrm{id} - \eta \circ \varepsilon)^{\otimes n} \circ \Delta^{(n)},$$

where $\Delta^{(n)} : U_{\hbar}(\mathfrak{g}) \to U_{\hbar}(\mathfrak{g})^{\widehat{\otimes}n}$ is the n-th fold coproduct and η, ε are the unit and counit maps of $U_{\hbar}(\mathfrak{g})$. Set

$$\mathscr{O}_{\hbar}(G^*) = \{x \in U_{\hbar}(\mathfrak{g}) \mid \forall n \geq 0,\ \delta_n(x) \in \hbar^n\, U_{\hbar}(\mathfrak{g})^{\widehat{\otimes}n}\}.$$

Then a classical result (see [Dr, Ga]) says that $\mathscr{O}_{\hbar}(G^*)$ is a quantization of the Hopf-Poisson algebra $\mathscr{O}(G^*)$ of functions on the formal group corresponding to $\mathfrak{g}^*$ (so $\mathscr{O}(G^*) = U(\mathfrak{g}^*)^*$).

According to (0.1), we have $R \in 1 + \hbar U_{\hbar}(\mathfrak{g})^{\widehat{\otimes}2}$. So $\log(R) = \sum_{n\geq 1} (-1)^n \frac{(R-1)^n}{n}$ is a well-defined element of $\hbar U_{\hbar}(\mathfrak{g})^{\widehat{\otimes}2}$.

Theorem 0.1. *Set $\rho = \frac{1}{\hbar}\log(R)$. Then ρ belongs to $\mathscr{O}_{\hbar}(G^*)^{\bar{\otimes}2}$ (this is a subalgebra of $U_{\hbar}(\mathfrak{g})^{\widehat{\otimes}2}$). If $\mathfrak{m}_{\hbar}$ is the augmentation ideal of $\mathscr{O}_{\hbar}(G^*)$, we even have $\rho \in (\mathfrak{m}_{\hbar})^{\bar{\otimes}2}$.*

Here $\bar{\otimes}$ denotes the completed tensor product of formal series algebras. As a corollary, we obtain a result of [GH].

Corollary 0.2. *([GH]) The R-matrix R normalizes $\mathscr{O}_{\hbar}(G^*)^{\bar{\otimes}2}$, in other words, $\mathrm{Ad}(R) : U_{\hbar}(\mathfrak{g})^{\widehat{\otimes}2} \to U_{\hbar}(\mathfrak{g})^{\widehat{\otimes}2}$ restricts to an automorphism of $\mathscr{O}_{\hbar}(G^*)^{\bar{\otimes}2}$.*

Gavarini and one of us (see [GH]) derive from this result that $\mathrm{Ad}(R)_{|\hbar=0}$, the reduction $\bmod\,\hbar$ of $\mathrm{Ad}(R)$ is an automorphism of $\mathscr{O}_{\hbar}(G^*)\bar{\otimes}\mathscr{O}_{\hbar}(G^*)$, satisfying the braiding identities of [WX] or [Re]. In a forthcoming paper, we plan to prove that this braiding coincides with the braiding defined by Weinstein and Xu in [WX].

This paper is organized as follows: we prove Theorem 0.1 in Section 2.b. This proof uses a combinatorial result on universal Lie algebras $F_{(n,p)}$, which is stated and proved in Section 1. In Section 3, we prove Corollary 0.2 using Theorem 0.1.

§ 1 A theorem on the universal Lie algebra $F_{(n,p)}$

In this section, we introduce a Lie algebra $F_{(n,p)}$ (Section 1.a). This Lie algebra is universal for the following situation: A is an algebra, $\rho \in A^{\otimes 2}$, and we consider elements $\rho^{i,j} \in A^{\otimes(n+p)}$, $i \in \{1,\dots,n\}$, $j \in \{n+1,\dots,n+p\}$. We construct elements $\delta_{(n,p)}$ in a completion of $F_{(n,p)}$, using the Campbell-Baker-Hausdorff series (Section 1.b). The main result is Theorem 1.2 on the valuation (for the total degree) of $\delta_{(n,p)}$. It will be used in Section 2 to prove Theorem 0.1 on the $\hbar$-adic valuation of universal R-matrices.

Let n, p be integers ≥ 1.

- a - The Lie algebra $F_{(n,p)}$

Let us denote by $\mathrm{Free}_{(n,p)}$ the free Lie algebra with generators $\widetilde{x}_{i,j}$, where $(i,j) \in \{1,\dots,n\} \times \{1,\dots,p\}$. Let us denote by $F_{(n,p)}$ the quotient of $\mathrm{Free}_{(n,p)}$ by the relations

$$[\widetilde{x}_{i,j}, \widetilde{x}_{i',j'}] = 0 \text{ when } i \neq i' \text{ and } j \neq j'.$$

We denote by $x_{i,j}$ the image of $\widetilde{x}_{i,j}$ in $\mathrm{Free}_{(n,p)}$. The Lie algebra $\mathrm{Free}_{(n,p)}$ is graded by $\bigoplus_{(i,j)\in\{1,\dots,n\}\times\{1,\dots,p\}} \mathbb{N}\varepsilon_{(i,j)}$ where $\deg(\widetilde{x}_{i,j}) = \varepsilon_{(i,j)}$. We call this grading the *fine degree*. $\mathrm{Free}_{(n,p)}$ has also a $\mathbb{N}$-grading which we call the *total degree*; the total degree of $\widetilde{x}_{i,j}$ is 1. These gradings induce gradings on $F_{(n,p)}$.
For $A = \mathrm{Free}_{(n,p)}$ or $F_{(n,p)}$ and $\underline{k} \in \oplus_{(i,j)} \mathbb{N}\varepsilon_{(i,j)}$ (resp., $k \in \mathbb{N}$), we denote by $A_{\underline{k}}$ (resp., A_k) the fine degree $\underline{k}$ part (resp., total degree k part) of A. We denote by $\widehat{A}$ the completion of A with respect to the total degree. So

$$\widehat{A} = \prod_{k\geq 0} A_k.$$

- b - The Campbell-Baker-Hausdorff series

If N is an integer, let us denote by Free_N the free Lie algebra with generators $\widetilde{x}_1, \dots, \widetilde{x}_N$ and by $\widehat{\mathrm{Free}}_N$ its completion with respect to the total degree. There exists a series

$$\widetilde{x}_1 \star \cdots \star \widetilde{x}_N \in \widehat{\mathrm{Free}}_N,$$

such that the identity

$$\exp(\widetilde{x}_1 \star \cdots \star \widetilde{x}_N) = \exp(\widetilde{x}_1) \cdots \exp(\widetilde{x}_N)$$

holds in the completion $\widehat{\mathrm{Freealg}}_N$ of the free associative algebra $\mathrm{Freealg}_N$ with generators $\widetilde{x}_1, \dots, \widetilde{x}_N$, with respect to the total degree (see [Bo]).

Let $\mathfrak{g}$ be a $\mathbb{N}$-graded Lie algebra, so $\mathfrak{g} = \bigoplus_{n\geq 0} \mathfrak{g}_n$ and let $\widehat{\mathfrak{g}}$ be its completion; so $\widehat{\mathfrak{g}} = \prod_{n\geq 0} \mathfrak{g}_n$. If $x_1, \dots, x_n$ are elements of $\mathfrak{g}$, with valuation > 0, then there is a unique continuous Lie algebra morphism $\pi : \widehat{\mathrm{Free}}_N \to \widehat{\mathfrak{g}}$, taking each $\widetilde{x}_i$ to x_i. We then define

$$x_1 \star \cdots \star x_N := \pi(\widetilde{x}_1 \star \cdots \star \widetilde{x}_N).$$

- c - The elements $\delta_{(n,p)}$ of $\widehat{F}_{(n,p)}$

Definition 1.1. We set

$$\delta_{(n,p)} = \sum_{k=0}^{n} \sum_{l=0}^{p} \sum_{\substack{1 \le i_1 < \cdots < i_k \le n, \\ 1 \le j_1 < \cdots < j_k \le p}} (-1)^{n+p-k-l}$$

$$(x_{i_1,j_1} \star x_{i_1,j_2} \star \cdots \star x_{i_1,j_l}) \star (x_{i_2,j_l} \star \cdots \star x_{i_2,j_l}) \star \cdots \star (x_{i_k,j_1} \star \cdots \star x_{i_k,j_l}).$$

According to Section 1-b, the element $\delta_{(n,p)}$ belongs to $\widehat{F}_{(n,p)}$.

Theorem 1.2. *The valuation of $\delta_{(n,p)}$ for the total degree of $\widehat{F}_{(n,p)}$ is $\geq n+p-1$.*

PROOF. For $\underline{k} \in \bigoplus_{(i,j) \in \{1,\dots,n\} \times \{1,\dots,p\}} \mathbb{N}\varepsilon_{(i,j)}$, we denote by $\delta_{(n,p),\underline{k}}$ the fine degree $\underline{k}$ component of $\delta_{(n,p)}$. Define the support of $\underline{k}$, $\mathrm{supp}(\underline{k})$ as the set of all pairs (i,j) such that $(i,j) \neq 0$. If S is a subset of $\{1,\dots,n\} \times \{1,\dots,p\}$, we call the *i-th column of S* the intersection of S with the i-th column $\{i\} \times \{1,\dots,p\}$, and the *j-th line of S* its intersection with the j-th line $\{1,\dots,n\} \times \{j\}$. If S_1 is a subset of $\{1,\dots,n\}$, we denote by $\overline{S_1}$ its complement $\{1,\dots,n\} \setminus S_1$; if S_2 is a subset of $\{1,\dots,p\}$, we denote by $\overline{S_2}$ its complement $\{1,\dots,p\} \setminus S_2$. We will show:

Proposition 1.3. *For each $\underline{k} \in \oplus_{(i,j)} \mathbb{N}\varepsilon_{(i,j)}$, we have $\delta_{(n,p),\underline{k}} = 0$ unless $S = \mathrm{supp}(\underline{k})$ satisfies the following conditions:*
(1) each column of S is nonempty;
(2) each line of S is nonempty;
(3) if $S_1 \subset \{1,\dots,n\}$ and $S_2 \subset \{1,\dots,p\}$ are proper subsets (i.e., they are nonempty as well as their complements) then $S \not\subset (S_1 \times S_2) \cup (\overline{S_1} \times \overline{S_2})$.

Proposition 1.4. *If S is a subset of $\{1,\dots,n\} \times \{1,\dots,p\}$ satisfying conditions* (1), (2) *and* (3) *of Proposition 1.3, then* $\mathrm{card}(S) \geq n+p-1$.

Theorem 1.2 now follows from these propositions and the fact that the total degree of an element of $\mathrm{Free}_{(n,p)}$ of degree $\underline{k}$ is $\geq \mathrm{card}(\mathrm{supp}(\underline{k}))$. □

- d - PROOF OF PROPOSITION 1.3.

For $i \in \{1,\dots,n\}$, there is a unique continuous Lie algebra morphism

$$\lambda_i : \widehat{F}_{n,p} \to \widehat{F}_{n-1,p}$$

$$x_{i',j'} \mapsto \begin{cases} x_{i',j'} & \text{if } i' < i \\ 0 & \text{if } i' = i \\ x_{i'-1,j'} & \text{if } i' > i. \end{cases}$$

Similarly, for $j \in \{1,\ldots,p\}$, there is a unique continuous Lie algebra morphism

$$\rho_j : \widehat{F}_{n,p} \to \widehat{F}_{n,p-1}$$

$$x_{i',j'} \mapsto \begin{cases} x_{i',j'} & \text{if } j' < j \\ 0 & \text{if } j' = j \\ x_{i',j'-1} & \text{if } j' > j. \end{cases}$$

The proof will essentially consist in the following Lemmas 1.5, 1.6, and 1.7.

Lemma 1.5. *For each $i \in \{1,\ldots,n\}$, we have $\lambda_i(\delta_{(n,p)}) = 0$.*
For each $j \in \{1,\ldots,p\}$, we have $\rho_j(\delta_{(n,p)}) = 0$.

Lemma 1.6. *If $\alpha \in \widehat{F}_{(n,p)}$ is such that $\lambda_i(\alpha) = 0$, then the homogeneous component $\alpha_{\underline{k}}$ of α satisfies $\alpha_{\underline{k}} = 0$, unless the i-th line of $\mathrm{supp}(\underline{k})$ is nonempty.*
In the same way, if $\rho_j(\alpha) = 0$, then $\alpha_{\underline{k}} = 0$ unless the j-th line of $\mathrm{supp}(\underline{k})$ is nonempty.

After Lemmas 1.5 and 1.6 are proved, we know that $\delta_{(n,p),\underline{k}} = 0$ unless each line and each column of $S = \mathrm{supp}(\underline{k})$ is nonempty. This proves that S should satisfy conditions (1) and (2). The fact that S satisfies condition (3) will follow from:

Lemma 1.7. *Let $\alpha \in \widehat{F}_{(n,p)}$ be homogeneous of degree $\underline{k}$. Assume that each line and each column of $\mathrm{supp}(\underline{k})$ is nonempty. Then $\alpha = 0$ unless $\underline{k}$ satisfies conditions (3) of Proposition 1.3.*

So Lemmas 1.5, 1.6, and 1.7 imply Proposition 1.3. □

We now prove Lemmas 1.5, 1.6, and 1.7:

PROOF OF LEMMA 1.5. If $I \subset \{1,\ldots,n\}$ and $J \subset \{1,\ldots,p\}$, let us set $k = \mathrm{card}(I)$, $l = \mathrm{card}(J)$, and $I = \{i_1,\ldots,i_k\}$, $J = \{j_1,\ldots,j_l\}$ where $1 \le i_1 < \cdots < i_k \le n$ and $1 \le j_1 < \cdots < j_l \le p$. Let us then set

$$\delta^{I,J}_{(n,p)} = (-1)^{n+p-k-l} x_{i_1,j_1} \star x_{i_1,j_2} \star \cdots \star x_{i_k,j_l}.$$

Then we have:

$$\delta_{(n,p)} = \sum_{\substack{I \subset \{1,\ldots,n\},\\ J \subset \{1,\ldots,p\}}} \delta^{I,J}_{(n,p)}.$$

Let $i \in \{1, \dots, n\}$ and let us prove that $\lambda_i(\delta_{(n,p)}) = 0$: we write

$$\delta_{(n,p)} = \sum_{\substack{I' \subset \{1,\dots,n\} \setminus \{i\}, \\ J \subset \{1,\dots,p\}}} \delta_{(n,p)}^{I',J} + \delta_{(n,p)}^{I' \cup \{i\}, J},$$

and we now show that for any (I', J),

$$\lambda_i \left(\delta_{(n,p)}^{I',J} + \delta_{(n,p)}^{I' \cup \{i\}, J} \right) = 0.$$

This follows from the following facts:

- if $\varphi : \widehat{\mathfrak{g}} \to \widehat{\mathfrak{g}}'$ is a morphism of complete graded Lie algebras, and $x_1, \dots, x_N$ are elements of $\widehat{\mathfrak{g}}'$ of positive valuation, then

$$\varphi(x_1) \star \cdots \star \varphi(x_N) = \varphi(x_1 \star \cdots \star x_n);$$

- we have the identity $x_1 \star \cdots \star x_k \star 0 \star x_{k+1} \star \cdots \star x_N = x_1 \star \cdots \star x_N$.

The proof of $\rho_j(\delta_{(n,p)}) = 0$ is similar: we write

$$\delta_{(n,p)} = \sum_{\substack{I \subset \{1,\dots,n\}, \\ J' \subset \{1,\dots,p\} \setminus \{j\}}} \delta_{(n,p)}^{I,J' \cup \{j\}} + \delta_{(n,p)}^{I,J'},$$

and then show that

$$\rho_j \left(\delta_{(n,p)}^{I,J' \cup \{j\}} + \delta_{(n,p)}^{I,J'} \right) = 0$$

for any pair (I, J') using the same arguments. □

PROOF OF LEMMA 1.6. It will be enough to prove Lemma 1.6 when $i = n$. Let α be an element of $F_{n,p}$. Set

$$\alpha = \sum_{\underline{k} \in \bigoplus_{(i,j)} \mathbb{N}\varepsilon_{(i,j)}} \alpha_{\underline{k}},$$

where $\alpha_{\underline{k}}$ has degree $\underline{k}$. Assume that $\lambda_n(\alpha) = 0$, we want to show that $\alpha_{\underline{k}} = 0$ unless the n-th column of $\mathrm{supp}(\underline{k})$ is nonempty. We will now write:

$$\alpha = \sum_{\underline{k} \,\mid\, n\text{-th column}(\mathrm{supp}(\underline{k})) \neq \emptyset} \alpha_{\underline{k}} + \sum_{\underline{k} \,\mid\, n\text{-th column}(\mathrm{supp}(\underline{k})) = \emptyset} \alpha_{\underline{k}}.$$

Now $\lambda_n(\alpha_{\underline{k}}) = 0$ as soon as the n-th column of $\mathrm{supp}(\underline{k})$ is nonempty, so we get

$$\sum_{\underline{k} \,\mid\, n\text{-th column}(\mathrm{supp}(\underline{k})) = \emptyset} \lambda_n(\alpha_{\underline{k}}) = 0 \quad (\text{identity in } F_{(n-1,p)}). \tag{1.2}$$

There is a unique Lie algebra morphism $\iota : F_{n-1,p} \to F_{n,p}$, taking each $x_{i,j}$ to $x_{i,j}$. We have

$$\lambda_n \circ \iota = \mathrm{id}_{F_{n-1,p}},$$

therefore ι is injective. Moreover, let $\mathscr{F}$ be the Lie subalgebra of $F_{n,p}$ generated by the $x_{i,j}$, $i \in \{1,\dots,n\}$, $j \in \{1,\dots,p\}$. $\mathscr{F}$ is the image of the Lie subalgebra $\widetilde{\mathscr{F}} \subset \mathrm{Free}_{(n,p)}$ generated by the $\widetilde{x}_{i,j}$, $i \in \{1,\dots,n-1\}$, $j \in \{1,\dots,p\}$ under the canonical projection $\mathrm{Free}_{(n,p)} \to F_{(n,p)}$. Denote by $\widetilde{\iota} : \mathrm{Free}_{(n-1,p)} \to \mathrm{Free}_{(n,p)}$ the morphism taking each $\widetilde{x}_{i,j}$ to $\widetilde{x}_{i,j}$, then the diagram

$$\begin{array}{ccccc} \mathrm{Free}_{(n-1,p)} & \xrightarrow{\widetilde{\iota}} & \widetilde{\mathscr{F}} & \hookrightarrow & \mathrm{Free}_{(n,p)} \\ \downarrow & & \downarrow & & \downarrow \\ F_{(n-1,p)} & \xrightarrow{\iota} & \mathscr{F} & \hookrightarrow & F_{(n,p)} \end{array}$$

commutes. Since the vertical arrows are onto and $\widetilde{\iota} : \mathrm{Free}_{(n-1,p)} \to \widetilde{\mathscr{F}}$ is onto, $\iota : \mathrm{Free}_{(n-1,p)} \to \mathscr{F}$ is onto. So $\iota : \mathrm{Free}_{(n-1,p)} \to \mathscr{F}$ is an isomorphism. Now if the n-th column of $\mathrm{supp}(\underline{k})$ is empty, we have $\alpha_{\underline{k}} \in \mathscr{F}$. Apply ι to identity (1.2), we get

$$\sum_{\underline{k} \mid n\text{-th column}(\mathrm{supp}(\underline{k}))=\emptyset} \alpha_{\underline{k}} = 0 \quad (\text{identity in } F_{(n-1,p)}).$$

Separating homogeneous components, we get $\alpha_{\underline{k}} = 0$ for each $\underline{k}$, such that n-th $\mathrm{column}(\mathrm{supp}(\underline{k})) = \emptyset$. □

PROOF OF LEMMA 1.7. Let $\alpha \in F_{(n,p)}$ be of fine degree $\underline{k}$, and assume that $\underline{k}$ satisfies (1), (2) but not (3). So we have proper subsets $S_1 \subset \{1,\dots,n\}$ and $S_2 \subset \{1,\dots,p\}$ such that

$$\mathrm{supp}(\underline{k}) \subset (S_1 \times S_2) \cup (\overline{S}_1 \times \overline{S}_2).$$

The rectangles $R = S_1 \times S_2$ and $R' = \overline{S}_1 \times \overline{S}_2$ are disjoint. Moreover, condition (1) (or condition (2)) implies that $\mathrm{supp}(\underline{k}) \cap R$ and $\mathrm{supp}(\underline{k}) \cap R'$ are both nonempty. Let $(k_{(i,j)})_{(i,j)}$ be the component of $\underline{k}$ in the basis $(\varepsilon_{(i,j)})_{(i,j)\in\{1,\dots,n\}\times\{1,\dots,p\}}$. Then $k_{(i,j)} \neq 0$ if and only if $(i,j) \in \mathrm{supp}(\underline{k})$, so

$$\underline{k} = \sum_{(i,j)\in\mathrm{supp}(\underline{k})} k_{(i,j)} \varepsilon_{(i,j)}.$$

Now there exists $\beta \in \mathrm{Free}_{(n,p)}$ of fine degree $\underline{k}$, whose image under $\mathrm{Free}_{(n,p)} \to F_{(n,p)}$ is α. Moreover, standard results on free Lie algebra (see [Bo]) imply that β has the following form: set $|\underline{k}| = \sum_{(i,j)\in\{1,\dots,n\}\times\{1,\dots,p\}} k_{(i,j)}$, and say that a map $\mu : \{1,\dots,|\underline{k}|\} \to \{1,\dots,n\} \times \{1,\dots,p\}$ is a $\underline{k}$-map if for any $(i,j) \in \{1,\dots,n\} \times \{1,\dots,p\}$, $\mathrm{card}(\mu^{-1}(i,j)) = k_{(i,j)}$. If μ is a $\underline{k}$-map, let us define $\widetilde{x}_\mu$ as the iterated Lie bracket

$$\widetilde{x}_\mu = \left[\dots\left[\left[\widetilde{x}_{\mu(1)}, \widetilde{x}_{\mu(2)}\right], \widetilde{x}_{\mu(3)}\right], \dots, \widetilde{x}_{\mu(|\underline{k}|)}\right].$$

Then β is a linear combination

$$\beta = \sum_{\mu \in \{\underline{k}\text{-maps}\}} \beta_\mu \widetilde{x}_\mu,$$

where the β_μ are scalars. Define x_μ as the image of $\widetilde{x}_\mu$ under the map $\text{Free}_{(n,p)} \to F_{(n,p)}$. Then $\alpha = \sum_{\mu \in \{\underline{k}\text{-maps}\}} \beta_\mu x_\mu$. On the other hand, $x_\mu = 0$ if μ is any $\underline{k}$-map. Indeed

$$x_\mu = \left[\dots\left[\left[x_{\mu(1)}, x_{\mu(2)}\right], x_{\mu(3)}\right], \dots, x_{\mu(|\underline{k}|)}\right].$$

Then, since $R \cap R' = \emptyset$, $\text{supp}(\underline{k}) \cap R \neq \emptyset$ and $\text{supp}(\underline{k}) \cap R' \neq \emptyset$, there exists an integer $\nu \in \{1, \dots, |\underline{k}| - 1\}$, such that one of the following possibilities occurs:

- either $\mu(1), \dots, \mu(\nu) \in R$ and $\mu(\nu+1) \in R'$
- or $\mu(1), \dots, \mu(\nu) \in R'$ and $\mu(\nu+1) \in R$.

Now the bracket $[x_\alpha, x_\beta]$ vanishes when $\alpha \in R$ and $\beta \in R'$. It follows that in both cases, the bracket

$$\left[\dots\left[\left[x_{\mu(1)}, x_{\mu(2)}\right], x_{\mu(3)}\right], \dots, x_{\mu(\nu+1)}\right]$$

vanishes. Therefore $x_\mu = 0$, which implies $\alpha = 0$. □

- e - PROOF OF PROPOSITION 1.4.

The argument will be an induction over $n+p$. Assume that the proposition is proved when $n+p \leq N$ and let us prove it when $n+p = N+1$. Let (n,p) be such that $n+p = N+1$ and let S be as in the proposition. We may assume $n \geq p$. If each column of S has ≥ 2 elements, then

$$\text{card}(S) \geq n.2 \geq n+p-1.$$

Each column of S has ≥ 1 element by assumption (1), so we can assume that S has a column with exactly 1 element. We may assume that this element is (n,p). Let us now set $S' = S \cap (\{1, \dots, n\} \times \{1, \dots, p\})$ and prove that S' satisfies the analogues (1'), (2') and (3') of the assumptions of the proposition, where (n,p) is replaced by $(n-1,p)$:

(1') for $i = 1, \dots, n-1$, the i-th lines of S and of S' coincide so the i-th line of S' is nonempty;

(2') for $j = 1, \dots, p-1$, the j-th lines of S and of S' coincide, so the j-th line of S' is nonempty. Let us prove that the p-th line of S' is also nonempty. If not, we would have $(n\text{-th column of } S) \cup (p\text{-th line of } S) = \{(n,p)\}$, contradicting assumption (3) on S with $S_1 = \{1, \dots, n-1\}$ and $S_2 = \{1, \dots, p-1\}$. Therefore all the lines of S' are nonempty;

(3') assume that there exist proper subsets $S_1' \subset \{1, \dots, n-1\}$, $S_2 \subset \{1, \dots, p\}$, (i.e., nonempty and with nonempty complement), such that

$$S' \subset (S_1' \times S_2) \cup \left((\{1, \dots, n-1\} \setminus S_1') \times (\{1, \dots, p\} \setminus S_2)\right).$$

Then if $p \in S_2$, let us set $S_1 = S_1 \cup \{n\}$, and if $p \in \{1, \dots, p\} \setminus S_2$, let us set $S_1 = S_1'$. Then in each case, since $S = S' \cup \{(n,p)\}$, we get

$$S \subset (S_1 \times S_2) \cup ((\{1, \dots, n\} \setminus S_1) \times (\{1, \dots, p\} \setminus S_2)),$$

thus contradiction the fact that S satisfies assumption (3). Therefore S' satisfies assumption (3').

Now S' satisfies the assumptions of the proposition, with (n,p) replaced with $(n-1,p)$. Since $(n-1)+p = N$, we may apply the induction hypothesis. So $\mathrm{card}(S') \geq (n-1)+p-1$. Now $\mathrm{card}(S) = \mathrm{card}(S')+1$, so $\mathrm{card}(S) \geq n+p-1$. This proves the induction step. □

§ 2 $\hbar$-adic valuation properties of universal R-matrices

In this section, we first identify a tensor product of QFSH (quantized formal series Hopf) algebras as a subspace of the tensor product of the corresponding QUE algebras (Proposition 2.1).

- a - Tensor product of QFSH algebras

Let H be a QUE algebra. The corresponding QFSH algebra is:

$$H' = \{x \in H |\ \forall n \in \mathbb{N},\ \delta_H^{(n)}(x) \in \hbar^n H^{\widehat{\otimes} n}\}$$

(where $\delta_H^{(n)} = (\mathrm{id} - \eta \circ \varepsilon)^{\otimes n} \circ \Delta_H^{(n)}$). We express this condition as follows: set

$$\delta_H^{(\mathrm{tot})} = \prod_{n \geq 0} \delta_H^{(n)}\ :\ H \to \prod_{n \geq 0} H^{\widehat{\otimes} n}.$$

Then

$$H' = \left(\delta_H^{(\mathrm{tot})}\right)^{-1} \left(\prod_{n \geq 0} \hbar^n H^{\widehat{\otimes} n} \right).$$

In the same way, if K is another QUE algebra, we define

$$K' = \left(\delta_K^{(\mathrm{tot})}\right)^{-1} \left(\prod_{n \geq 0} \hbar^n K^{\widehat{\otimes} n} \right).$$

Then $\delta_H^{(\mathrm{tot})} \otimes \delta_K^{(\mathrm{tot})}$ is a linear map $H \widehat{\otimes} K \to \prod_{n,p \geq 0} H^{\widehat{\otimes} n} \widehat{\otimes} K^{\widehat{\otimes} p}$.

Proposition 2.1. *The tensor product of QFSH algebras H' and K' is*

$$H' \bar{\otimes}\, K' = \left(\delta_H^{(\mathrm{tot})} \otimes \delta_K^{(\mathrm{tot})}\right)^{-1} \left(\prod_{n,p \geq 0} \hbar^{n+p} H^{\widehat{\otimes} n} \widehat{\otimes} K^{\widehat{\otimes} p} \right).$$

PROOF. We will need the following lemma:

Lemma 2.2. *Let V_1, V_2, T_1, T_2 be vector spaces and let $\delta_1 : V_1 \hookrightarrow T_1$, $\delta_2 : V_2 \hookrightarrow T_2$ be injections. Let S_1, S_2 be vector subspaces of T_1, T_2, and set $U_i = \delta_i^{-1}(S_i)$. Then*

$$U_1 \otimes U_2 = (\delta_1 \otimes \delta_2)^{-1}(S_1 \otimes S_2).$$

The proposition then follows because the maps $\delta_K^{(\mathrm{tot})}$ and $\delta_H^{(\mathrm{tot})}$ are injective. Indeed, x can be obtained from $\delta_H^{(\mathrm{tot})}(x)$ by the formula:

$$x = \left(\delta_H^{(\mathrm{tot})}(x)\right)_1 + \eta\left(\left(\delta_H^{(\mathrm{tot})}(x)\right)_0\right).$$

□

PROOF OF LEMMA 2.2. Let $W_1 = \mathrm{Im}(\delta_1)$, $W_2 = \mathrm{Im}(\delta_2)$. Then $W_1 \otimes W_2 = \mathrm{Im}(\delta_1 \otimes \delta_2)$. So

$$W_1 \subset T_1, \quad W_2 \subset T_2, \quad W_1 \otimes W_2 \subset T_1 \otimes T_2.$$

We should prove that if $W_1 \subset T_1$, $W_2 \subset T_2$, then the subspace $(W_1 \cap S_1) \otimes (W_2 \cap S_2) \subset W_1 \otimes W_2$ is equal to $(W_1 \otimes W_2) \cap (S_1 \otimes S_2)$. Identifying the spaces V_i with their images $W_i = \mathrm{Im}(\delta_i)$ through the maps δ_i, we should prove the following statement:

Lemma 2.3. *Let T_1, T_2 be vector spaces, and let for each $i = 1, 2$, W_i and S_i be vector subspaces of T_i. Then*

$$(W_1 \cap S_1) \otimes (W_2 \cap S_2) = (W_1 \otimes W_2) \cap (S_1 \otimes S_2).$$

Indeed, $U_1 \otimes U_2$ can be identified with $(W_1 \cap S_1) \otimes (W_2 \cap S_2)$ through $\delta_1 \otimes \delta_2$, whereas $(\delta_1 \otimes \delta_2)^{-1}(S_1 \otimes S_2)$ can be identified with $(W_1 \otimes W_2) \cap (S_1 \otimes S_2)$ through the same map. □

PROOF OF LEMMA 2.3. Introducing supplementary vector spaces $\widetilde{W}_i$ of $W_i \cap S_i$ in W_i and $\widetilde{S}_i$ of $W_i \cap S_i$ in S_i, we get

$$\begin{cases} W_i = & (W_i \cap S_i) \oplus \widetilde{W}_i \\ S_i = & (W_1 \cap S_i) \oplus \widetilde{S}_i \end{cases}$$

$W_i + S_i = (W_i \cap S_i) \oplus \widetilde{W}_i \oplus \widetilde{S}_1$ so $(W_1 + S_1) \otimes (W_2 + S_2)$ is the direct sum of nine summands. The subspaces $W_1 \otimes W_2$ and $S_1 \otimes S_2$ are the sums

$$\begin{cases} W_1 \otimes W_2 = & \left((W_1 \cap S_1) \otimes (W_2 \cap S_2)\right) \oplus \left((W_1 \cap S_1) \otimes \widetilde{W}_2\right) \\ & \oplus \left(\widetilde{W}_1 \otimes (W_2 \cap S_2)\right) \oplus \left(\widetilde{W}_1 \otimes \widetilde{W}_2\right) \\ S_1 \otimes S_2 = & \left((W_1 \cap S_1) \otimes (W_2 \cap S_2)\right) \oplus \left((W_1 \cap S_1) \otimes \widetilde{S}_2\right) \\ & \oplus \left(\widetilde{S}_1 \otimes (W_2 \cap S_2)\right) \oplus \left(\widetilde{S}_1 \otimes \widetilde{S}_2\right), \end{cases}$$

the intersection of which is $(W_1 \cap S_1) \otimes (W_2 \cap S_2)$. □

- b - PROOF OF THEOREM 0.1.

Let $(U_\hbar(\mathfrak{g}), R)$ be a quasitriangular QUE algebra. Set $\rho = \hbar \log(R)$. Then $\rho \in \hbar^2 U_\hbar(\mathfrak{g})^{\widehat{\otimes} 2}$. For $x,\ y \in U_\hbar(\mathfrak{g})^{\widehat{\otimes} n}[\hbar^{-1}]$, let us set

$$\{x,y\}_\hbar = \frac{1}{\hbar}[x,y].$$

Then $\{-,-\}$ restricts to a Lie bracket

$$\{-,-\} \ : \ \wedge^2 \left(\hbar^2 U_\hbar(\mathfrak{g})^{\widehat{\otimes} k}\right) \to \hbar^2 U_\hbar(\mathfrak{g})^{\widehat{\otimes} k}.$$

Its image is actually contained in $\hbar^3 U_\hbar(\mathfrak{g})^{\widehat{\otimes} k}$. According to Proposition 2.1, the statement $\rho \in \mathscr{O}_\hbar(G^*)^{\bar{\otimes} 2}$ is equivalent to

$$\left(\delta^{(n)}_{U_\hbar(\mathfrak{g})} \otimes \delta^{(p)}_{U_\hbar(\mathfrak{g})}\right)(\rho) \in \hbar^{n+p} U_\hbar(\mathfrak{g})^{\widehat{\otimes}(n+p)} \tag{2.3}$$

for any $n,\ p \geq 0$. Let us therefore prove this statement: for $a, b \in \hbar^2 U_\hbar(\mathfrak{g})^{\widehat{\otimes} k}$, let us set

$$a \star_k b = a + b + \frac{1}{2}\{a,b\}_\hbar + \cdots ; \tag{2.4}$$

the series (2.4) makes sense because of the following fact: if $m \in \mathrm{Free}_2$ is a Lie monomial of degree p in two free variables A, B, and $m\left(\{-,-\}_\hbar, a, b\right)$ is the image of m by the Lie algebra morphism $\mathrm{Free}_2 \to \hbar^2 U_\hbar(\mathfrak{g})^{\widehat{\otimes} k}$ defined by $A \mapsto a$, $B \mapsto b$, then

$$m\left(\{-,-\}_\hbar, a, b\right) \in \hbar^{p+1} U_\hbar(\mathfrak{g})^{\widehat{\otimes} k}.$$

Recall that $\rho \in \hbar^2 U_\hbar(\mathfrak{g})^{\widehat{\otimes} 3}$. The quasitriangular identities then imply that ρ satisfies

$$\left(\Delta_{U_\hbar(\mathfrak{g})} \otimes \mathrm{id}\right)(\rho) = \rho^{1,3} \star_3 \rho^{2,3}, \ \left(\mathrm{id} \otimes \Delta_{U_\hbar(\mathfrak{g})}\right)(\rho) = \rho^{1,3} \star_3 \rho^{1,2}. \tag{2.5}$$

There is a unique Lie algebra morphism

$$\phi_{(n,p)} \ : \ \widehat{F}_{(n,p)} \to \left(\hbar^2 U_\hbar(\mathfrak{g})^{\widehat{\otimes}(n+p)}, \{-,-\}_\hbar\right),$$

$$x_{i,j} \mapsto \rho^{i,n+j}$$

for each $(i,j) \in \{1,\dots,n\} \times \{1,\dots,p\}$ (here $\rho^{i,n+j}$ is the image of ρ by the map

$$U_\hbar(\mathfrak{g})^{\widehat{\otimes} 2} \to U_\hbar(\mathfrak{g})^{\widehat{\otimes}(n+p)},$$

taking the first component of $U_\hbar(\mathfrak{g})^{\widehat{\otimes} 2}$ to the i-th component, and the second component to the $(n+j)$-th component). Then the identities (2.5) imply

$$\left(\delta^{(n)}_{U_\hbar(\mathfrak{g})} \otimes \delta^{(p)}_{U_\hbar(\mathfrak{g})}\right)(\rho) = \phi_{(n,p)}\left(\delta_{(n,p)}\right). \tag{2.6}$$

Let $\alpha \in \mathrm{Free}_{(n,p)}$ be an element of total degree d. Then we have seen that $\phi_{(n,p)}(\alpha) \in \hbar^{d+1}U_\hbar(\mathfrak{g})^{\widehat{\otimes}(n+p)}$. Since $\delta_{(n,p)}$ has valuation $\geq n+p-1$, we get $\phi_{(n,p)}(\delta_{(n,p)}) \in \hbar^{n+p}U_\hbar(\mathfrak{g})^{\widehat{\otimes}(n+p)}$. According to (2.6), this implies identity (2.3). So we get $\rho \in \mathscr{O}_\hbar(G^*)^{\bar{\otimes}2}$. Since we have $(\varepsilon \otimes \mathrm{id})(R) = (\mathrm{id} \otimes \varepsilon)(R) = 1$, we get $(\varepsilon \otimes \mathrm{id})(\rho) = (\mathrm{id} \otimes \varepsilon)(\rho) = 0$, therefore $\rho \in (\mathfrak{m}_\hbar)^{\bar{\otimes}2}$. □

§ 3 Proof of Corollary 0.2.

The space $U_\hbar(\mathfrak{g})^{\widehat{\otimes}2}$ is a quantization of $\mathfrak{g} \times \mathfrak{g}$ and $\mathscr{O}_\hbar(G^*)^{\bar{\otimes}2}$ is the corresponding QFSH algebra. Set $\mathfrak{d} = \mathfrak{g} \times \mathfrak{g}$, let D^* be the corresponding formal group, and set $U_\hbar(\mathfrak{d}) = U_\hbar(\mathfrak{g})^{\widehat{\otimes}2}$, $\mathscr{O}_\hbar(D^*) = \mathscr{O}_\hbar(G^*)^{\bar{\otimes}2}$. Denote by $\mathfrak{m}_\hbar(D^*)$ the augmentation ideal of $\mathscr{O}_\hbar(D^*)$. Then $\mathfrak{m}_\hbar \bar{\otimes} 1$ and $1 \bar{\otimes} \mathfrak{m}_\hbar$ are subspaces of $\mathfrak{m}_\hbar(D^*)$, so $\rho \in \mathfrak{m}_\hbar(D^*)^2$. Corollary 0.2 will follow from the following proposition:

Proposition 3.1. *Let $\mathfrak{d}$ be an arbitrary finite-dimensional Lie bialgebra, let $U_\hbar(\mathfrak{d})$ be a quantization of $\mathfrak{d}$, let $\mathscr{O}_\hbar(D^*)$ be the QFSH subalgebra of $U_\hbar(\mathfrak{d})$, and let $\mathfrak{m}_\hbar(D^*)$ be the augmentation ideal of $\mathscr{O}_\hbar(D^*)$. Let ρ be an arbitrary element of $\mathfrak{m}_\hbar(D^*)^2$.*

(1) the operation $\{-,-\}_\hbar : \wedge^2 \left(U_\hbar(\mathfrak{d})[\hbar^{-1}]\right) \to U_\hbar(\mathfrak{d})[\hbar^{-1}]$ restricts to

$$\{-,-\}_\hbar \; : \; \wedge^2 \left(\mathscr{O}_\hbar(D^*)\right) \to \mathscr{O}_\hbar(D^*);$$

(2) for any k, $l \geq 0$, we have

$$\left\{\mathfrak{m}_\hbar(D^*)^k, \mathfrak{m}_\hbar(D^*)^l\right\}_\hbar \subset \mathfrak{m}_\hbar(D^*)^{k+l-1};$$

(3) $\mathscr{O}_\hbar(D^*) = \varprojlim_k \left(\mathscr{O}_\hbar(D^*)/\left(\mathfrak{m}_\hbar(D^*) + \hbar\mathscr{O}_\hbar(D^*)\right)^k\right)$; let us set

$$\mathrm{ad}_\hbar(\rho)(x) := \{\rho, x\}_\hbar,$$

then the exponential $\exp\left(\mathrm{ad}_\hbar(\rho)\right)$ is a well-defined continuous algebra automorphism of $\mathscr{O}_\hbar(D^*)$;

(4) We have $\rho \in \hbar^2 U_\hbar(\mathfrak{d})$, so $\exp\left(\frac{\rho}{\hbar}\right)$ is a well-defined element of $1 + \hbar U_\hbar(\mathfrak{d})$. Let $\mathrm{Ad}\left(\exp\left(\frac{\rho}{\hbar}\right)\right)$ be the inner automorphism $x \mapsto \exp\left(\frac{\rho}{\hbar}\right) x \exp\left(\frac{\rho}{\hbar}\right)^{-1}$ of $U_\hbar(\mathfrak{d})$. Then $\mathrm{Ad}\left(\exp\left(\frac{\rho}{\hbar}\right)\right)$ restricts to an automorphism of $\mathscr{O}_\hbar(D^*)$, which coincides with $\exp(\mathrm{ad}_\hbar(\rho))$.

PROOF.

(1) $\mathscr{O}_\hbar(D^*)$ is a subalgebra of $U_\hbar(\mathfrak{d})$, and its reduction modulo $\hbar$ is commutative, so for x, $y \in \mathscr{O}_\hbar(D^*)$, $[x,y] \in \hbar\mathscr{O}_\hbar(D^*)$, i.e., $\{x,y\}_\hbar \in \mathscr{O}_\hbar(D^*)$.

(2) Let $\varepsilon : \mathscr{O}_\hbar(D^*) \to k[[\hbar]]$ be the augmentation map. If x, $y \in \mathscr{O}_\hbar(D^*)$, then $\varepsilon([x,y]) = 0$. So $\hbar\varepsilon(\{x,y\}_\hbar) = 0$, i.e., $\varepsilon(\{x,y\}_\hbar) = 0$, so $\{x,y\}_\hbar \in \mathfrak{m}_\hbar(D^*)$. Therefore $\{\mathfrak{m}_\hbar(D^*), \mathfrak{m}_\hbar(D^*)\} \subset \mathfrak{m}_\hbar(D^*)$. Applying the Leibniz rule, we get (2).

(3) The first part is contained in [Ga]. The second part follows from (2): $\rho \in \mathfrak{m}_{\hbar}(D^*)^2$, therefore $\mathrm{ad}_{\hbar}(\rho)\left((\mathfrak{m}_{\hbar}(D^*)^k\right) \subset \mathfrak{m}_{\hbar}(D^*)^{k+1}$ for any k.

(4) If $x \in \mathscr{O}_{\hbar}(D^*)$, then $x - \varepsilon(x) \in \hbar U_{\hbar}(\mathfrak{d})$. If, in addition, $x \in \mathfrak{m}_{\hbar}(D^*)$, then $\varepsilon(x) = 0$ so $x \in \hbar U_{\hbar}(\mathfrak{d})$. So $\mathfrak{m}_{\hbar}(D^*) \subset \hbar U_{\hbar}(\mathfrak{d})$ and $\mathfrak{m}_{\hbar}(D^*)^2 \subset \hbar^2 U_{\hbar}(\mathfrak{d})$. Let us now show that the diagram

$$\begin{array}{ccc} U_{\hbar}(\mathfrak{d}) & \xrightarrow{\mathrm{Ad}\left(\exp\left(\frac{\rho}{\hbar}\right)\right)} & U_{\hbar}(\mathfrak{d}) \\ \uparrow & & \uparrow \\ \mathscr{O}_{\hbar}(D^*) & \xrightarrow{\exp\left(\mathrm{ad}_{\hbar}(\rho)\right)} & \mathscr{O}_{\hbar}(D^*) \end{array} \tag{3.7}$$

commutes. This means that we have the identity (in $U_{\hbar}(\mathfrak{d})$):

$$\exp\left(\mathrm{ad}_{\hbar}(\rho)\right)(x) = \exp\left(\frac{\rho}{\hbar}\right) x \exp\left(\frac{\rho}{\hbar}\right)^{-1},$$

for any $x \in \mathscr{O}_{\hbar}(D^*)$. To show this identity, let us introduce a parameter t and show the same identity, where ρ is replaced by $t\rho$. This means that we have the identity

$$\exp\left(\mathrm{ad}_{\hbar}(t\rho)\right)(x) = \exp\left(\frac{t\rho}{\hbar}\right) x \exp\left(\frac{t\rho}{\hbar}\right)^{-1}.$$

This last identity follows from the fact that its two sides coincide when $t = 0$, and that they both satisfy the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\,t} x(t) = \mathrm{ad}_{\hbar}(\rho)(x(t)).$$

It follows that the diagram (3.7) commutes. Therefore the map $\mathrm{Ad}\left(\exp\left(\frac{\rho}{\hbar}\right)\right)$ restricts to an automorphism of $\mathscr{O}_{\hbar}(D^*)$, which coincides with $\exp\left(\mathrm{ad}_{\hbar}(\rho)\right)$. □

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References

[Bo] Bourbaki N. "Lie groups and Lie algebras. Chapters 1–3." Translated from the French. Reprint of the 1975 edition. Elements of Mathematics. *Springer-Verlag*, Berlin, 1989.

[Dr] Drinfeld V. "Quantum groups", *Proceedings of the International Congress of Mathematicians, Amer. Math. Soc., Providence, RI* **1, 2** (1987), 798–820.

[Ga] Gavarini F. "The quantum duality principle", *Ann. Inst. Fourier.* **52** (2002), no. 3, 1-26.

[GH] Gavarini F., Halbout G. "Tressages des groupes de Poisson formels à dual quasitriangulaire", *J. Pure Appl. Algebra* **161** (2001), no. 3, 295–307.

[Re] Reshetikhin N. "Quasitriangularity of quantum groups at roots of 1", *Commun. Math. Phys.* **170** (1995), 79–99.

[WX] Weinstein A., Xu P. "Classical Solutions of the Quantum Yang-Baxter Equation", *Commun. Math. Phys.* **148** (1992), 309–343.