G_∞-formality theorem in terms of graphs and associated Chevalley-Eilenberg-Harrison cohomology

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Abstract

In 1997, M. Kontsevich proved the L_∞-formality conjecture (which implies existence of star-products for any Poisson manifold) using graphs. A year later, D. Tamarkin gave another proof of a more general conjecture (for G_∞-structures) using operads and cohomological methods. In this paper, we show how Tamarkin's construction can be written using graphs. For that, we introduce a generalization of Kontsevich graphs on which we define a "Chevalley-Eilenberg-Harrison" complex. We show that this complex on graphs is related to the "Chevalley-Eilenberg-Harrison" complex for maps on polyvector fields, which is is trivial and gave Tamarkin's formality theorem as a consequence. This formality reduces to a L_∞-formality.

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Let $M = \mathbb{R}^n$ and $A = C^\infty(M)$ the commutative algebra of smooth functions on M. Let $C^k(A, A)$ be the space of k-linear differential operators from A to A and $\mathfrak{g}_2 = (C^\cdot(A, A), b)$ be the associated Hochschild cochain complex. The classical Hochschild-Kostant-Rosenberg theorem states that the cohomology of $\mathfrak{g}_2$ is the graded Lie algebra $\mathfrak{g}_1 = \Gamma(M, \Lambda^\cdot TM)$ of multivector fields on M: a k-vector field on M ($\in \Gamma(M, \Lambda^k TM)$) has degree $k-1$ in $\mathfrak{g}_1$ and $k-2$ in $\mathfrak{g}_1[1]$.

The Lie bracket $[-,-]_S$ on $\mathfrak{g}_1$ is the natural extension of the Lie bracket of vector fields and is called the Schouten bracket (see [Kos]).

There is also a graded Lie algebra structure on $\mathfrak{g}_2$ given by the Gerstenhaber bracket (see [GV]). In particular $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are also Lie algebras up to homotopy (L_∞-algebra for short, see Section 1 for definitions). In ([Kon], 1997), using graphs, Kontsevich has proved the formality theorem, *i.e.*, the existence of Lie homomorphisms "up to homotopy" (L_∞-morphisms) from $\mathfrak{g}_1$ to $\mathfrak{g}_2$ (see again Section 1 for definitions). This was a key tool to prove existence of star-products (see [BFFLS]).

In 1998, using cohomological methods and operads, Tamarkin ([Tam]) has proved a more general formality theorem: he constructed a Gerstenhaber algebra up to homotopy homomorphism (G_∞-morphism for short) from $\mathfrak{g}_1$ to $\mathfrak{g}_2$ that restricts to L_∞-morphism between $\mathfrak{g}_1$ and $\mathfrak{g}_2$ (see again Section 1).

In [AM], Arnal and Masmoudi present a graph-cohomological approach of Kontsevich's construction: putting L_∞-maps in a graph framework, they show that those maps can be constructed by induction provided one can go through obstructions that lie in the Chevalley-Eilenberg cohomology group for graphs. In the general case, those cohomology groups are nontrivial.

In this paper, we present a similar approach following Tamarkin's construction: suppose we are given a G_∞-structure on $\mathfrak{g}_2$, we construct a G_∞-morphism between $\mathfrak{g}_1$ and $\mathfrak{g}_2$ by induction. More precisely, thanks to Hochschild-Kostant-Rosenberg theorem, one gets a G_∞-structure on $\mathfrak{g}_1$, G_∞-isomorphic to $\mathfrak{g}_2$. To prove the existence of a G_∞-morphism between the two G_∞-structures on $\mathfrak{g}_1$, we will use a generalization of Kontsevich's graphs. In Section 3, we define those graphs in the framework of which we rewrite this G_∞-morphism. Morphisms corresponding to such graphs (*i.e.* which are graph-mappings according to Definition 3.2) will be called "graph-compatible" . Note that the Hochschild-Kostant-Rosenberg projection can be viewed as a linearization of morphisms (or graphs), that's why the graphs we will construct are linear ones, whereas in [AM], as construction of formality morphism is done directly, non-linear graphs were also considered.

In Section 4, we construct a coboundary operator on those generalized Kontsevich graphs. In Theorem 4.2 we prove that the coboundary acts only on the aerial part of the graph and that it corresponds to the Chevalley-Eilenberg-Harrison differential. We then recall (Theorem 4.3) that the cohomology of this complex $(\mathrm{Hom}(\Lambda^{\cdot}\underline{\mathfrak{g}_1^{\otimes \cdot}}), \partial)$ is acyclic, giving acyclicity of the graph complex. Note here that considering linear or non-linear graphs will not change anything as the coboundary only acts on the aerial part of the graphs.

In Section 1, we recall the definitions of L_∞ and G_∞-structures and in Section 2, we recall the general obstruction theory (as presented in Tamarkin's approach) of G_∞-formality.

1 G_∞-algebras and G_∞-morphisms

For any graded space $\mathfrak{g}$, we denote by $\Lambda^{\cdot}\mathfrak{g}$ the cofree cocommutative coalgebra on $\mathfrak{g}$ and by $\Lambda^{\cdot}\underline{\mathfrak{g}^{\otimes}}$ the graded space

$$\bigoplus_{m\geq 1, r_1+\cdots+r_n=m} \underline{\mathfrak{g}^{\otimes r_1}}\Lambda\cdots\Lambda\underline{\mathfrak{g}^{\otimes r_n}}$$

where $\underline{\mathfrak{g}^{\otimes l}}$ is the quotient of $\mathfrak{g}^{\otimes l}$ by the image of the shuffles of order l. Recall that we denote by $\mathrm{sh}(p,q)$ the set of shuffles of type (p,q), that is the set of permutations σ of S_{p+q} (the symmetric group of $p+q$ elements) such that

$$\sigma(k) < \sigma(k+1) \quad \text{for all} \quad 1 \leq k < p-1,\ p+1 \leq k < p+q-1.$$

(for more details on shuffles see [Lo] or [Sta] for instance). We use the following grading on $\Lambda^{\cdot}\underline{\mathfrak{g}^{\otimes}}$: for homogeneous elements $x_1^1, \ldots, x_n^{r_n}$ in $\mathfrak{g}$ with degree respectively $|x_1^1|, \ldots, |x_n^{r_n}|$, the degree of $x := (x_1^1 \otimes \cdots \otimes x_1^{r_1})\Lambda\cdots\Lambda(x_n^1 \otimes \cdots \otimes x_n^{r_n})$ is

$$|x| = \sum_{i=1}^{r_1} |x_1^i| + \cdots + \sum_{i=1}^{r_n} |x_n^i| - n.$$

It is known that $\Lambda^{\cdot}\underline{\mathfrak{g}^{\otimes}}$ is a cofree coalgebra.

Definition 1.1.

- *An homotopy Gerstenhaber algebra structure (a G_∞-algebra structure in short) on a vector space $\mathfrak{g}$ is given by a collection of degree one maps:*

  $$m^{r_1,\ldots,r_n} : \underline{\mathfrak{g}^{\otimes r_1}}\Lambda\cdots\Lambda\underline{\mathfrak{g}^{\otimes r_n}} \to \mathfrak{g}$$

  such that their canonical extension to $\Lambda^{\cdot}\underline{\mathfrak{g}^{\otimes}}$ satisfies

  $$d \circ d = 0$$

  where

  $$d = \sum_{m\geq 1} \sum_{r_1+\cdots+r_n=m} m^{r_1,\ldots,r_n}.$$

- *An homotopy Lie algebra structure (a L_∞-algebra structure in short) on a vector space $\mathfrak{g}$ is a G_∞-algebra where the maps $m^{r_1,\ldots,r_n}$: $\underline{\mathfrak{g}^{\otimes r_1}}\Lambda\cdots\Lambda\underline{\mathfrak{g}^{\otimes r_n}} \to \mathfrak{g}$ are all zero for $r_i > 1$.*

Consider the space $\mathfrak{g}_1 = \Gamma(M, \Lambda^{\cdot}TM)$. The graded Lie algebra structure of $\mathfrak{g}_1$ given by the Schouten bracket $[-,-]_S$ can be translated by a map

$$m_1^{1,1} : \Lambda^2\mathfrak{g}_1 \to \mathfrak{g}_1$$

and the commutative graded algebra structure, given by the exterior product $\wedge$, by a map

$$m_1^2 : \underline{\mathfrak{g}_1^{\otimes 2}} \to \mathfrak{g}_1.$$

These maps can be naturally extended to maps, still noted $m_1^{1,1}$ and m_1^2, on $\Lambda^{\cdot}\underline{\mathfrak{g}^{\otimes}}$ (we will see this in more details in Section 2). It is then easy to check that $m_1^{1,1}$ and $d_1 := m_1^{1,1} + m_1^2$ satisfy

$$m_1^{1,1} \circ m_1^{1,1} = 0 \text{ and } d_1 \circ d_1 = 0$$

so that $(\mathfrak{g}_1, m_1^{1,1})$ (resp. $(\mathfrak{g}_1, d_1)$) can be viewed as a L_∞-algebra (resp. G_∞-algebra). More generally, any Gerstenhaber algebra $(G, \mu, [\,,\,])$ has a canonical G_∞-algebra structure given by $m^{1,1} = [\,,\,]$, $m^2 = \mu$, the other maps being 0.

Now, let us consider the Hochschild complex $\mathfrak{g}_2 = C^{\cdot}(A, A)$ where A is $C^\infty(M)$. We shall look at it as a graded vector space: $\mathfrak{g}_2 = \bigoplus_k C^k(A, A)$ and an element of $C^k(A, A)$ is of degree $k-1$. Equipped with the Gerstenhaber bracket $[-,-]_G$ and the Hochschild differential b, it is a graded Lie differential algebra (and then a L_∞-algebra with $m_2^1 = b$ and $m_2^{1,1} = [-,-]_G$). It was conjectured by Deligne [Del] that this space $\mathfrak{g}_2$ can also be endowed with a structure of G_∞-algebra with m_2^2 corresponding to the usual commutative product of cochains. The proof of this conjecture has been obtained by Tamarkin [Tam], by using Etingof's quantization functor [EK].

Definition 1.2. *A L_∞ (resp. G_∞)-morphism between two L_∞ (G_∞)-algebras $(\mathfrak{g}_1, d_1)$ and $(\mathfrak{g}_2, d_2)$ is a map $\psi : (\Lambda^{\cdot}\underline{\mathfrak{g}_1^{\otimes}}, d_1) \to (\Lambda^{\cdot}\underline{\mathfrak{g}_2^{\otimes}}, d_2)$ of codifferential coalgebras.*

A G_∞-morphism ψ between two G_∞- algebras $(\mathfrak{g}_1, d_1)$ and $(\mathfrak{g}_2, d_2)$ is given by a collection of maps $\psi^{[n]}$ where

$$\psi^{[n]} = \sum_{r_1+\cdots+r_k=n} \psi^{r_1,\ldots,r_k}$$

with maps $\psi^{r_1,\ldots,r_k} : \underline{\mathfrak{g}_1^{\otimes r_1}}\Lambda\cdots\Lambda\underline{\mathfrak{g}_1^{\otimes r_k}} \to \mathfrak{g}_2$ satisfying for all n the following equation

$$\psi^{[\le n]} \circ d_1^{[\le n]} = d_2^{[\le n]} \circ \psi^{[\le n]}.$$

Here we have noted

$$\psi^{[\le n]} = \sum_{r_1+\cdots+r_k\le n} \psi^{r_1,\ldots,r_k},$$

$$d_i^{[\le n]} = \sum_{r_1+\cdots+r_k\le n} d_i^{r_1,\ldots,r_k} \quad (i = 1, 2)$$

where $d_i^{r_1,\ldots,r_k}$ are the components of the codifferential d_i $(i = 1, 2)$.

Consider the G_∞-structure on $\mathfrak{g}_1 = \Gamma(M, \Lambda^{\cdot}TM)$ given by the codifferential $d_1 = m_1^{1,1} + m_1^2$ and denote by $m_2^{r_1,\ldots,r_k}$ the components of the codifferential d_2 defining the G_∞-structure on $\mathfrak{g}_2 = C^{\cdot}(A, A)$ (as defined before). The G_∞-formality theorem (proved by Tamarkin) states the existence of a G_∞-morphism between $(\mathfrak{g}_1, d_1)$ and $(\mathfrak{g}_2, d_2)$. Equations satisfied by such a morphism can be written as

$$\begin{aligned}
&m_2^1 \circ \psi^{[\leq n]}(\alpha_1 \Lambda \cdots \Lambda \alpha_k) = \\
&\qquad \psi^{[\leq n-1]} \circ m_1^{1,1}(\alpha_1 \Lambda \cdots \Lambda \alpha_k) + \psi^{[\leq n-1]} \circ m_1^2(\alpha_1 \Lambda \cdots \Lambda \alpha_k) \\
&\qquad + m_2^{1,1} \circ \psi^{[\leq n-1]}(\alpha_1 \Lambda \cdots \Lambda \alpha_k) + m_2^2 \circ \psi^{[\leq n-1]}(\alpha_1 \Lambda \cdots \Lambda \alpha_k) \\
&\qquad + \sum_{2 \leq j \leq n-2} d_2^{[\leq j]} \circ \psi^{[\leq j]}(\alpha_1 \Lambda \cdots \Lambda \alpha_k),
\end{aligned}$$

where, for all $1 \leq i \leq k$, $\alpha_i \in \underline{\mathfrak{g}_1^{\otimes r_i}}$ and $d_2^{[\leq j]} = \sum_{r_1 + \cdots + r_k \leq j} m_2^{r_1, \ldots, r_k}$.

2 Obstruction theory of G_∞-formalities

2.1 Recallings

In [Tam], after having constructed a G_∞-structure on $\mathfrak{g}_2$ (given by a differential d_2 on $\Lambda^\cdot \underline{\mathfrak{g}_1^{\otimes}}$), Tamarkin gave a cohomological construction of the maps $\psi^{[n]}$, components of a G_∞-morphism between $\mathfrak{g}_1$ and $\mathfrak{g}_2$. In [GH], it is shown that this construction can be made in two steps: first, thanks to the Hochschild-Kostant-Rosenberg theorem, by projection on the cohomology, one can prove the existence of a G_∞-structure on $\mathfrak{g}_1$ (given by a differential d_1') together with a G_∞-morphism ψ' between $(\mathfrak{g}_1, d_1')$ and $(\mathfrak{g}_2, d_2)$. The second step consists in proving that one can construct a G_∞-morphism ψ'' between $(\mathfrak{g}_1, d_1)$ and $(\mathfrak{g}_1, d_1')$. Obstructions of such a construction can be described as follows:

Proposition 2.1. *Suppose we want to construct by induction a G_∞-formality between $(\mathfrak{g}_1, d_1 = m_1^{1,1} + m_1^2)$ and $(\mathfrak{g}_1, d_1')$ (starting by the identity). Assume the components $\psi^{[k]}$ are constructed up to the order $n-1$. Then the obstruction to the construction of $\psi^{[n]}$ is a cocycle in $(\mathrm{Hom}(\Lambda^\cdot \underline{\mathfrak{g}_1^{\otimes \cdot}}, \partial)$ where $\partial = [m_1^{1,1} + m_1^2, .]$, here $[\,,\,]$ denotes the graded commutator of morphisms.*

The complex $(\mathrm{Hom}(\Lambda^\cdot \underline{\mathfrak{g}_1^{\otimes \cdot}}), \partial)$ is in fact acyclic ([Tam], [GH]) and then the existence of ψ'' is proved and so is the existence of a G_∞-formality.

In this paper, we will show how the construction of the G_∞-morphism ψ'' can be made in terms of graphs. Here, we are interested in restricting the cochains to maps which can be written explicitly as linear combinations of Kontsevich's graphs (see Definition 3.2). In the following sections, we will also show that the coboundary operator ∂ can be nicely reduced on graphs to an operator acting only on the aerial part of the graphs. We will construct a morphism between the complex $(\mathrm{Hom}(\Lambda^\cdot \underline{\mathfrak{g}_1^{\otimes \cdot}}), \partial)$ and the corresponding Chevalley-Eilenberg-Harrison complex on generalized Kontsevich graphs. Before doing that, in the next subsections, we will rewrite the boundary map ∂ in a more explicit way.

2.2 Extensions of $m_1^{1,1}$ and m_1^2 to $\Lambda^{\cdot}\underline{\mathfrak{g}_1^{\otimes}}$

The map m_1^2 is extended to $\Lambda^{\cdot}\underline{\mathfrak{g}_1^{\otimes}}$ as follows. For any

$$\alpha = \alpha_1 \Lambda \cdots \Lambda \alpha_k \in \underline{\mathfrak{g}_1^{\otimes r_1}} \Lambda \cdots \Lambda \underline{\mathfrak{g}_1^{\otimes r_k}} \ (\alpha_i \in \underline{\mathfrak{g}_1^{\otimes r_i}}),$$

we have

$$m_1^2(\alpha) = \sum_{\tau \in S_k} \varepsilon(\tau) m_1^2(\alpha_{\tau(1)}) \Lambda \alpha_{\tau(2)} \Lambda \cdots \Lambda \widehat{\alpha_{\tau(1)}} \Lambda \cdots \Lambda \alpha_{\tau(k)}$$

where, if $\alpha = \alpha^1 \otimes \cdots \otimes \alpha^l \in \underline{\mathfrak{g}_1^{\otimes l}}$,

$$m_1^2(\alpha) = \sum_{1 \leq i \leq l-1} \varepsilon_i \alpha^1 \otimes \cdots \otimes m_1^2(\alpha^i \otimes \alpha^{i+1}) \otimes \cdots \otimes \alpha^l.$$

The signs ε_i and $\varepsilon(\tau)$ occuring in these expressions can be explicitly computed (*cf.* [Gi]) applying the Koszul-Quillen rule where m^2 is of degree 0 (for any tensor fields α_1, α_2 with degree $|\alpha_1|$, $|\alpha_2|$ in $\mathfrak{g}_1$, we have $\alpha_1 \Lambda \alpha_2 = (-1)^{(|\alpha_1|-1)(|\alpha_2|-1)} \alpha_2 \Lambda \alpha_1$).
Similarly, the map $m_1^{1,1}$ is extended to $\Lambda^{\cdot}\underline{\mathfrak{g}_1^{\otimes}}$ as follows. For any

$$\alpha = \alpha_1 \Lambda \cdots \Lambda \alpha_k \in \underline{\mathfrak{g}_1^{\otimes r_1}} \Lambda \cdots \Lambda \underline{\mathfrak{g}_1^{\otimes r_k}},$$

we have

$$m_1^{1,1}(\alpha) = \sum_{\sigma \in \mathrm{sh}(2,k-2)} \varepsilon(\sigma) m_1^{1,1}(\alpha_{\sigma(1)} \Lambda \alpha_{\sigma(2)}) \Lambda \cdots \Lambda \alpha_{\sigma(k)}$$

where if $\alpha_1 = \alpha^1 \otimes \cdots \otimes \alpha^l \in \underline{\mathfrak{g}_1^{\otimes l}}$ and $\alpha_2 = \alpha^{l+1} \otimes \cdots \otimes \alpha^{l+m} \in \underline{\mathfrak{g}_1^{\otimes m}}$:

$$m_1^{1,1}(\alpha_1 \Lambda \alpha_2) = \sum_{\gamma \in \mathrm{sh}(l,m)} \sum_{1 \leq i \leq l+m-1} \varepsilon(\gamma) \varepsilon_i \alpha^{\gamma(1)} \otimes \cdots \otimes [\alpha^{\gamma(i)} \alpha^{\gamma(i+1)}] \otimes \cdots \alpha^{\gamma(l+m)},$$

where $[\alpha^{\gamma(i)} \alpha^{\gamma(i+1)}] = m^{1,1}(\alpha^{\gamma(i)} \Lambda \alpha^{\gamma(i+1)})$ if $\gamma(i) \leq l$ and $\gamma(i+1) \geq l+1$ and is $= 0$ if not, and $\varepsilon(\gamma)$ and ε_i are again given by Koszul-Quillen rule where $m^{1,1}$ is of degree -1.

2.3 Boundary operator ∂ on maps $\psi^{[n]}$

Let us consider a mapping ψ from $\Lambda^{\cdot}\underline{\mathfrak{g}_1^{\otimes}}$ to $\mathfrak{g}_1$ whose components $\psi^{r_1,\ldots,r_k}$: $\underline{\mathfrak{g}_1^{\otimes r_1}} \wedge \cdots \underline{\mathfrak{g}_1^{\otimes r_k}} \to \mathfrak{g}_1$ are homogeneous of degree $|\psi|$.

$$\psi = \psi^{[n]} = \sum_k \sum_{r_1 + \cdots + r_k = n} \psi^{r_1,\ldots,r_k}.$$

By definition:

$$\partial\psi = (m_1^{1,1} + m_1^2) \circ \psi - (-1)^{|\psi|} \psi \circ (m_1^{1,1} + m_1^2).$$

We can write $\partial\psi = \sum\limits_{l} \sum\limits_{s_1+\cdots+s_l=n+1} (\partial\psi)^{s_1,\dots,s_l}$, and for all $\alpha = \alpha_1\Lambda\cdots\Lambda\alpha_l$, we have

$$\partial\psi(\alpha) = (1) + (2) + (3) + (4)$$

where

$$(1) = (-1)^{|\psi|+1} \sum_{\tau\in S_l} \varepsilon(\tau) \sum_{1\le i\le l-1} \varepsilon_i \psi(\alpha^1 \otimes \cdots \otimes m_1^2(\alpha^i \otimes \alpha^{i+1}) \otimes \cdots \otimes \alpha^l)$$

$$(2) = \sum_{\tau\in S_l} \varepsilon(\tau) \left(\pm m_1^2(\alpha^1_{\tau(1)} \otimes \psi(\alpha^2_{\tau(1)} \cdots)) \pm m_1^2(\psi(\alpha^1_{\tau(1)} \cdots) \otimes \alpha^{s_{\tau(1)}}_{\tau(1)}) \right)$$

$$(3) = (-1)^{|\psi|+1} \sum_{\sigma\in \mathrm{sh}(2,l-2)} \varepsilon(\sigma) \sum_{\gamma\in \mathrm{sh}(l,m)} \sum_{1\le i\le l+m-1}$$
$$\varepsilon(\gamma)\varepsilon_i \psi(\alpha^{\gamma(1)} \otimes \cdots \otimes [\alpha^{\gamma(i)}\alpha^{\gamma(i+1)}] \otimes \cdots \alpha^{\gamma(l+m)})$$

$$(4) = \sum_{\tau\in S_l, s_{\tau(1)}=1} \varepsilon(\tau)(-1)^{|\psi||\alpha^1_{\tau(1)}|} \varepsilon_\alpha m_1^{1,1}(\alpha^1_{\tau(1)} \Lambda \psi(\cdots \widehat{\alpha^1_{\tau(1)}} \cdots)).$$

3 Generalization of Kontsevich's graphs

3.1 Graphs with "packs"

In this section, we will extract a natural class of cochains associated with finite graphs. We need to consider what we call graphs 'with packs'. These graphs generalize Kontsevich's ones. More precisely, a graph Γ with packs has:

(i) aerial labelled packs $P_1, \dots, P_k$; and each pack P_i contains aerial labelled vertices $p_i^1, \dots, p_i^{r_i}$;

(ii) terrestrial labelled vertices $q_1, \dots, q_m$;

(iii) edges which are arrows starting from an aerial vertex, ending to a terrestrial or aerial vertex. There are no multiple arrows.

If we fix a total ordering O on the edges of Γ, we get an oriented graph (Γ, O). We say that O is compatible, if for all i and all j, the arrows starting from p_i^j are before those starting from p_i^{j+1}, and before those starting from p_{i+1}^l for all l. We say that a graph is purely aerial, if it has no terrestrial vertices q_j (and thus no legs *i.e.* no arrows of the form $\overrightarrow{p_i^j q_r}$). Finally, we denote by $GO_{n,m,k}$ (resp. $GO^{(0)}_{n,k}$) the set of oriented compatible graphs with k packs, n aerial vertices and m terrestrial vertices (resp. oriented aerial compatible graphs with k packs and n vertices); and by $GO^{(1)}_{n,m,k}$ the subset of graphs in $GO_{n,m,k}$, having exactly one leg for each foot. For oriented non compatible graphs, we will use the notation $GO'_{n,m,k}$, $GO'^{(1)}_{n,m,k}$ and $GO'^{(0)}_{n,k}$.

Now let (Γ, O) be a graph in $GO_{n,m,k}$. We denote by r_i the number of aerial vertices of the pack P_i, by $p_i^1, \dots, p_i^{r_i}$ the aerial vertices contained in P_i and by

k_i^j the number of edges starting from the vertex p_i^j. As in [Kon], with (Γ, O) we associate an operator $B_{(\Gamma,O)}$ assigning to an element

$$\alpha = \alpha_1 \Lambda \cdots \Lambda \alpha_k \in \underline{\mathfrak{g}_1^{\otimes r_1}} \Lambda \cdots \Lambda \underline{\mathfrak{g}_1^{\otimes r_k}}$$

a m-differential operator $B_{(\Gamma,O)}(\alpha)$, which is zero excepted when for all i,

$$\alpha_i = \alpha_i^1 \otimes \cdots \otimes \alpha_i^{r_i} \text{ with } \alpha_i^l \in \Gamma(M, \Lambda^{k_i^l} TM).$$

For example, consider the graph (Γ_o, O) given in Figure 1 below. Then, the associated operator $B_{(\Gamma_o,O)}$ is defined as follows. If

$$\alpha = \alpha_1^1 \Lambda (\alpha_2^1 \otimes \alpha_2^2) \Lambda (\alpha_3^1 \otimes \alpha_3^2 \otimes \alpha_3^3) \Lambda \alpha_4^1$$

where α_1^1, α_4^1 are 2-tensor fields, α_2^1 is a 4-tensor fields, $\alpha_2^2, \alpha_3^1, \alpha_3^2$ are vector fields and α_3^3 a function on $\mathbb{R}^d$, then $B_{(\Gamma_o,O)}(\alpha)$ is the 7-differential operator:

$$\begin{aligned} B_{(\Gamma_o,O)}(\alpha) = \sum_{1 \le i_1, \ldots, i_{11} \le d} & \partial_{i_4} \alpha_1^{1,i_1 i_2} \alpha_2^{1,i_3 i_4 i_5 i_6} \alpha_2^{2,i_7} \partial_{i_7} \alpha_3^{1,i_8} \alpha_3^{2,i_9} \partial_{i_8} \alpha_3^3 \partial_{i_9} \\ & \alpha_4^{1,i_{10} i_{11}} \partial_{i_3} \otimes \partial_{i_1} \otimes \partial_{i_2} \otimes \partial_{i_5} \otimes \partial_{i_6} \otimes \partial_{i_{10}} \otimes \partial_{i_{11}}. \end{aligned}$$

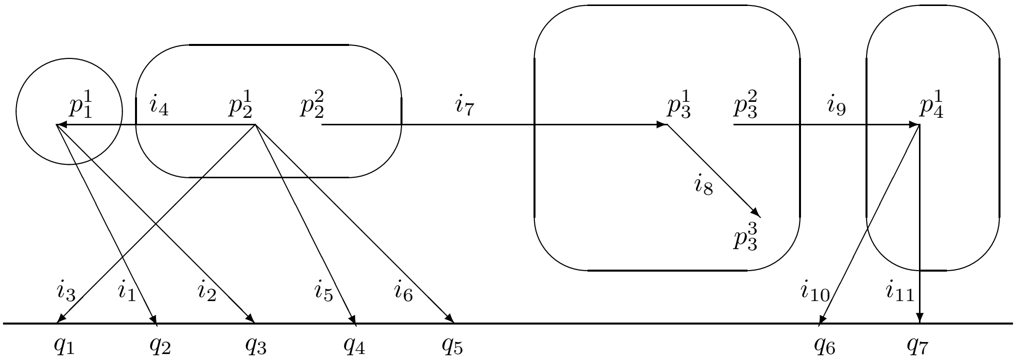

Figure 1

Now suppose that (Γ, O) is a graph in $GO_{n,m,k}^{(1)}$. Then $B_{(\Gamma,O)}(\alpha)$ can be considered as an element of $\mathfrak{g}_1$. Starting from (Γ, O), it will be helpful to put the legs at the end in the order of the feet. This can be done through two permutations of the edges. More precisely, we denote by $E(\Gamma)$ the ordered set of edges of Γ. We can write

$$E(\Gamma)=((p_1^1))\cup\cdots\cup((p_1^{r_1}))\cup((p_2^1))\cup\cdots\cup((p_2^{r_2}))\cup\cdots\cup((p_k^{r_k})),$$

if we denote by $((p_i^j))$ the set of edges starting from the vertex p_i^j of Γ. To explicit the signs interfering in our expressions, we will need to replace $E(\Gamma)$ by a new set of edges, say $\widetilde{E(\Gamma)}$, obtained by adding some extra aerial edges. In fact, for each aerial vertex p_i^j, we will add a fictional aerial edge $\overrightarrow{e_i^j}$, which will be regarded as the last edge starting from p_i^j. Then, for any pack P_t, we will add a fictional aerial edge noted $\overrightarrow{e_{P_t}}$, which will be put between $\overrightarrow{e_t^{r_t}}$ and the first edge starting from p_{t+1}^1. We can write

$$\begin{aligned}\widetilde{E(\Gamma)}=((p_1^1))\cup\{\overrightarrow{e_1^1}\}\cup\cdots\cup((p_1^{r_1}))\cup\{\overrightarrow{e_1^{r_1}}\}\cup\{\overrightarrow{e_{P_1}}\}\cup\\ \cup((p_2^1))\cup\cdots\cdots\cup((p_k^{r_k}))\cup\{\overrightarrow{e_k^{r_k}}\}\cup\{\overrightarrow{e_{P_k}}\}.\end{aligned}$$

We denote by U_j^i (resp. V_j^i) the ordered set of legs (resp. aerial edges) starting from p_i^j and by s_O the permutation given by:

$$s_O:\widetilde{E(\Gamma)}\rightarrow\bigcup_{i,j}V_i^j\cup\bigcup_{i,j}U_i^j.$$

We get then a new ordering O' on Γ, no more compatible, such that all the legs are in the end, and we define a permutation τ_O of the legs of (Γ,O'), by putting first the (unique) leg arriving on the first terrestrial vertex q_1, then the (unique) leg arriving on q_2... and finally the one arriving on q_m. We extend τ_O to $\bigcup_{i,j}V_i^j\cup\bigcup_{i,j}U_i^j$ just by putting $\tau_O(v)=v$ for all v in $\bigcup_{i,j}V_i^j$.

Definition 3.1. *For any (Δ,O_Δ) in $GO_{n,k}^{(0)}$, we define an operator*

$$C_{(\Delta,O_\Delta)}:\underline{\mathfrak{g}_1^{\otimes\cdot}}\Lambda\cdots\Lambda\underline{\mathfrak{g}_1^{\otimes\cdot}}\rightarrow\mathfrak{g}_1,$$

by

$$C_{(\Delta,O_\Delta)}=\sum_{m\geq0}\frac{1}{m!}\sum_{(\Gamma,O)\supset(\Delta,O_\Delta);(\Gamma,O)\in GO_{n,m,k}^{(1)}}\varepsilon_\Gamma\varepsilon(s_O)\varepsilon(\tau_O)B_{(\Gamma,O)}$$

where

$$\varepsilon_\Gamma=\frac{l!}{k!}:=\frac{\Pi_{i,j}l_i^j!}{\Pi_{i,j}k_i^j!},$$

if k_i^j (resp. l_i^j) denotes the number of edges (resp. aerial edges) starting from the vertex p_i^j of Γ.

We extend the definition of C_Δ by linearity to all combination of graphs $\Upsilon=\sum c_\Delta C_\Delta$. Remark that this definition is justified by the fact that the explicit formality of Kontsevich can be written in terms of such operators (see [AM]). Remark also that the above definition can be extended naturally to aerial graphs $(\Delta', O_{\Delta'})$ which are not compatible. We will use this fact in the proof of Theorem 4.2.

To simplify the writing, we will note in the sequel Δ and Γ instead of (Δ, O_Δ) and (Γ, O) respectively.

Definition 3.2. *A mapping* $\phi : \underline{\mathfrak{g}_1^{\otimes r_1}}\Lambda\cdots\Lambda\underline{\mathfrak{g}_1^{\otimes r_k}} \to \mathfrak{g}_1$ *will be called a graph-mapping if it can be written as:*

$$\phi = \sum_k \sum_{\Delta\in GO^{(0)}_{n,k}} c_\Delta C_\Delta$$

with real c_Δ*.*

We also define the degree of a graph Δ in $GO^{(0)}_{n,k}$ by

$$|\Delta| := |l| - n - k$$

if $|l| = \sum_{i,k} l_i^k$ denotes the total number of (aerial) edges of Δ, n is the number of vertices and k, the number of packs of Δ. By construction, the operators B_Γ occuring in C_Δ have a degree $|B_\Gamma|$ which depends only on the aerial part of Γ, and it is easy to check that it satisfies:

$$|B_\Gamma| \equiv |\Delta| \mod 2.$$

The aim will is now to write Tamarkin's formality theorem with morphisms beeing graph morphisms.

3.2 Symmetrization properties

Let (Γ, O) be a graph in $GO_{n,m,k}$. We still denote by r_i the number of aerial vertices contained in the pack P_i $(1 \le i \le k)$. There is an obvious action of S_k and of each S_{r_i} on (Γ, O). For any σ in S_k (resp. any τ in S_{r_i}), we denote by σ_Γ (resp. τ_Γ) the permutation induced by σ (resp. τ) on the modified set of edges $\widetilde{E(\Gamma)}$. We also denote by $\varepsilon_\Gamma(\sigma_\Gamma)$ (resp. $\varepsilon_\Gamma(\tau_\Gamma)$) the sign of σ_Γ (resp. τ_Γ). We have:

Proposition 3.3. *If* (Γ, O) *is a graph in* $GO^{(1)}_{n,m,k}$ *and* σ *a permutation in* S_k*, then for all* α_j *in* $\underline{\mathfrak{g}_1{}^{\otimes r_j}}$*, we have:*

$$B_{\sigma(\Gamma)}(\alpha_1\Lambda\cdots\Lambda\alpha_k) = B_\Gamma(\alpha_{\sigma(1)}\Lambda\cdots\Lambda\alpha_{\sigma(k)})$$

Moreover:

Proposition 3.4. *If* (Γ, O) *is a graph in* $GO^{(1)}_{n,m,k}$ *and* τ *a* $(k, r_i - k)$*-shuffle* $(1 \le kr_i - 1)$*, then for all* $\alpha_j = \alpha_j^1 \otimes\cdots\otimes \alpha_j^{r_j}$ *in* $\underline{\mathfrak{g}_1{}^{\otimes r_j}}$*, we have:*

$$B_{\tau(\Gamma)}(\alpha_1\Lambda\cdots\Lambda\alpha_k) = B_\Gamma(\alpha_1\Lambda\cdots\Lambda(\alpha_i^{\tau(1)}\otimes\cdots\otimes\alpha_i^{\tau(r_i)})\Lambda\cdots\Lambda\alpha_k).$$

With the same notations, we put

Definition 3.5. *Let Υ be a linear combination of aerial graphs:*

$$\Upsilon = \sum_{\Delta \in GO^{(0)}_{n,k}} c_\Delta C_\Delta.$$

We say that Υ has the good symmetries if for all σ in S_k and all τ a $(k, r_i - k)$-shuffle ($1 \leq kr_i - 1$), we have

$$c_{\sigma(\Delta)} = \varepsilon_\Delta(\sigma_\Delta) c_\Delta \quad \textit{and} \quad c_{\tau(\Delta)} = \varepsilon_\Delta(\tau_\Delta) c_\Delta.$$

It is then obvious that the following proposition holds:

Proposition 3.6. *Let Υ be a linear combination of aerial graphs. If Υ has the good symmetries then the operator C_Υ is well defined on $\Lambda^{\cdot}\underline{\mathfrak{g}_1^{\otimes}}$.*

4 The coboundary operator ∂ on graphs

In this section, we want to translate the coboundary operator ∂ on graphs. We need first to introduce more notations. Let us consider an aerial graph Δ in $GO^{(0)}_{n,k}$.

- We say that a graph Δ' in ${GO'}^{(0)}_{n+1,k}$ (non necessarily compatible) reduces to Δ for m_1^2 in the index i and in the permutation τ of S_k (we write : $\Delta' \underset{m_1^2, i\tau}{\longrightarrow} \Delta$) if:
  - **(i)** there is no edge between the vertices $p^i_{\tau(1)}$ and $p^{i+1}_{\tau(1)}$ of Δ
  - **(ii)** when collapsing $p^i_{\tau(1)}$ and $p^!_{\tau(1)} i+1$, with the induced ordering, we get Δ.
- We say that Δ' reduces properly to Δ for m_1^2 in i, τ (and we will write $\Delta' \underset{m_1^2, i\tau, \text{prop}}{\longrightarrow} \Delta$) if for $i = 1$ or $i = r_{\tau(1)}$, there is at least one edge starting or ending to $p^i_{\tau(1)}$.
- We say that Δ' in ${GO'}^{(0)}_{n+1,k+1}$ (non necessarily compatible) reduces to Δ for $m_1^{1,1}$ in the shuffle σ, in the indexes i, j and in the permutation τ of S_{k+1} (we write $\Delta' \underset{m_1^{1,1}, ij\sigma\tau}{\longrightarrow} \Delta$) if:
  - **(i)** there is exactly one edge between the vertices $p^i_{\tau(1)}$ and $p^j_{\tau(2)}$, namely the edge $\overrightarrow{p^i_{\tau(1)} p^j_{\tau(2)}}$

(ii) when we shuffle the elements of the two pakcs using σ, suppress the edge $\overrightarrow{p^i_{\tau(1)}p^j_{\tau(2)}}$, collapse $p^i_{\tau(1)}$ and $p^j_{\tau(2)}$, collapse the corresponding two packs in one pack and consider the induced ordering, we get Δ.

- We say that Δ' reduces properly to Δ for $m_1^{1,1}$ in $i, j, \sigma\tau$ (and we will write $\Delta' \underset{m_1^{1,1},ij\sigma\tau,\mathrm{prop}}{\longrightarrow} \Delta$) if when $r_{\tau(1)} = 1$ (or $r_{\tau(2)} = 1$), there is a least one edge starting or ending at $p^1_{\tau(1)}$ (or $p^1_{\tau(2)}$).

Definition 4.1. *If Δ is a graph in $GO^{(0)}_{n,k}$, we define the coboundary $\partial\Delta$ of Δ by:*

$$\begin{aligned}\partial\Delta =& (-1)^{|\Delta|+1} \sum_{\tau\in S_k} \sum_{i} \sum_{\Delta' \underset{m_1^2,i\tau,\mathrm{prop}}{\longrightarrow} \Delta} \varepsilon_1(\Delta',\Delta)\Delta' \\ &+ (-1)^{\Delta|+1} \sum_{\tau\in S_{k+1}} \sum_{i,j} \sum_{\Delta' \underset{m_1^{1,1},ij\sigma\tau,\mathrm{prop}}{\longrightarrow} \Delta} \varepsilon_2(\Delta',\Delta)\Delta'.\end{aligned}$$

Here $\varepsilon_1(\Delta,\Delta')$ and $\varepsilon_2(\Delta,\Delta')$ are sign obtained usign the corresponding Koszul Quillen rules.

We extend Υ by linearity to all combination of graphs. The restriction to combinations that have the good symmetries yields an operator of cohomology. In fact, we have:

Theorem 4.2. *Let Υ be a combination of graphs. We suppose that Υ has the good symmetries. Then,*

$$\partial C_\Upsilon = C_{\partial\Upsilon}.$$

Proof: Let us denote by P the part of the coboundary operator ∂ corresponding to m_1^2. For any aerial graph Δ in $GO^{(0)}_{n,k}$, we can write:

$$\begin{aligned}[P, C_\Delta] &= m_1^2 \circ C_\Delta - (-1)^{|C_\Delta|} C_\Delta \circ m_1^2 \\ &= m_1^2 \circ C_\Delta - (-1)^{|\Delta|} C_\Delta \circ m_1^2,\end{aligned}$$

since $|C_\Delta| \equiv |\Delta| \bmod 2$. Now, for all $\alpha = \alpha_1\Lambda\cdots\Lambda\alpha_k$, $\alpha_i \in \underline{\mathfrak{g}_1^{\otimes s_i}}$ we have

$$[P, C_\Delta](\alpha) = (i) + (ii)$$

where

$$\begin{aligned}(i) &= (-1)^{|\Delta|+1} \sum_{\tau\in S_k} \varepsilon(\tau) \sum_{1\le i\le l-1} \varepsilon_i C_\Delta(\alpha^1 \otimes \cdots \otimes m_1^2(\alpha^i \otimes \alpha^{i+1}) \otimes \cdots \otimes \alpha^l) \\ (ii) &= \sum_{\tau\in S_k} \varepsilon(\tau) \left(\pm m_1^2(\alpha^1_{\tau(1)} \otimes C_\Delta(\alpha^2_{\tau(1)}\cdots)) \pm m_1^2(C_\Delta(\alpha^1_{\tau(1)}\cdots) \otimes \alpha^{s_{\tau(1)}}_{\tau(1)}) \right).\end{aligned}$$

Now, by definition of C_Δ, we have:

$$\begin{aligned}&C_\Delta(\alpha^1\otimes\cdots\otimes m_1^2(\alpha^i\otimes\alpha^{i+1})\otimes\cdots\otimes\alpha^l) =\\ &\quad\sum_{m\geq 0}\frac{1}{m!}\sum_{\Gamma\subset\Delta,\Gamma\in GO^{(1)}_{n,m,k}}\varepsilon_\Gamma\varepsilon(s_O)\varepsilon(\tau_O)\sum_i\varepsilon_\alpha(i)\sum_{\Gamma'\xrightarrow[m_1^2,i\tau]{}\Gamma}B_{\Gamma'}(\alpha_1\Lambda\cdots\Lambda\alpha_k).\end{aligned}$$

Let us consider a graph Γ' occuring in this sum. Reordering the edges of Γ', putting the edges in the order induced by τ, and then putting the legs in the order of the feet gives the same sign as putting first the legs at the end in the order of the feet, then the aerial edges with the order induced by τ. So we have:

$$\begin{aligned}(i) &= (-1)^{|\Delta|+1}\sum_{\tau\in S_k}\sum_i\sum_m\frac{1}{m!}\sum_{\Gamma'\xrightarrow[m_1^2,i\tau]{}\Gamma}\varepsilon_{\Gamma'}\varepsilon_\alpha^{\Delta'}(i)\varepsilon^{\Delta'}(\tau)\varepsilon(s'_O)\varepsilon(\tau'_O)B_{\Gamma'}(\alpha)\\ &= (-1)^{|\Delta|+1}\sum_{\tau\in S_k}\varepsilon^{\Delta'}(\tau)\sum_i\sum_{\Delta'\xrightarrow[m_1^2,i\tau]{}\Delta}\varepsilon_\alpha^{\Delta'}(i)C_{\Delta'}(\alpha)\\ &= (-1)^{|\Delta|+1}\sum_{\tau\in S_k}\sum_i\sum_{\Delta'\xrightarrow[m_1^2,i\tau]{}\Delta}\varepsilon_1(\Delta',\Delta)C_{\Delta'}(\alpha),\end{aligned}$$

where we put

$$\varepsilon_1(\Delta',\Delta)=\varepsilon^{\Delta'}(\tau)\varepsilon_\alpha^{\Delta'}(i).$$

Now, it is easy to see that (ii) corresponds to the non proper terms. More precisely, if $\Upsilon=\sum c_\Delta C_\Delta$ is a combination of graphs, with the good symmetries, we can prove that the terms of

$$\sum_\Delta c_\Delta(m_1^2\circ C_\Delta)$$

are exactly the non proper terms of

$$\sum_\Delta(-1)^{|\Delta|}c_\Delta(C_\Delta\circ m_1^2).$$

Finally, the same proof can be repeated for the part Q (where $[Q,.]=[m_1^{1,1},.]$) of the coboundary operator ∂ corresponding to $m_1^{1,1}$. The result follows. ■

We end this paper, recalling (*cf.* [Tam, GH])

Theorem 4.3. *The corresponding Chevalley-Eilenberg-Harrison complex*

$$(\mathrm{Hom}(\Lambda^{\cdot}\underline{\mathfrak{g}_1^{\otimes\cdot}}),\partial)$$

is acyclic.

So we have:

Theorem 4.4. *The Chevalley-Eilenberg-Harrison complex for graphs:* $(GO^{(0)}_{n,k}, \partial)$ *is acyclic*

Proof: Theorem 4.2 says that for any combination of graphs Υ having the good symmetries (*i.e.* the corresponding map $C_\Upsilon : \Lambda^{\cdot}\underline{\mathfrak{g}_1^{\otimes \cdot}} \to \mathfrak{g}_1$ is a graph-mapping with symmetry properties), we have $\partial C_\Upsilon = C_{\partial\Upsilon}$. Moreover, we have a one to one correspondence between graphs Γ and the differential operators B_Γ. Then, to prove the theorem, we shall prove that for any graph-mapping (with good symmetries) ψ satisfying $\partial\psi = 0$, there exists a graph-mapping ψ' (with again good symmetries) so that $\psi = \partial(\psi')$. Let ψ be such a graph-mapping with $\psi = C_\Upsilon$ satisfying $\partial\psi = 0$. Thanks to Theorem 4.3, there exists a map φ such that $\psi = \partial(\varphi)$. One can find a familly of graphs $(\Gamma_i)_{i\in I}$ such that $\varphi = \sum_{i\in I} B_{\Gamma_i}$. Let $\Gamma = \sum_{i\in I} \Gamma_i$. We define the following projection onto graph-mappings: for any graph $\Gamma_i \in GO^{(1)}_{n,m,k}$, we note $\Gamma_i^{(0)}$ the same graph with only aerial vertices. Let us denote $\varphi' = \sum_{i\in I} C_{\Gamma_i}$. As ψ is a graph mapping, one can check that we still have $\psi = \partial(\varphi')$ which gives the result. ■

As a consequence, given any G_∞-structure on $\mathfrak{g}_1$ corresponding to a differential d_1', one can construct a G_∞-morphism by induction between $(\mathfrak{g}_1, d_1)$ and $(\mathfrak{g}_1, d_1')$ such that the maps are graph-mappings as defined in Definition 3.2.

Remark: We did not compute the homology of the corresponding graph (where differentials are working the other way, collapsing the vertices). One can see, for example in [Ko2, Ko2] works on the corresponding Chevalley-Eilenberg complex.

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