Globalization of Tamarkin's formality theorem

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Abstract

Construction of formality theorem by Kontsevich ([Ko]) and Tamarkin ([Ta]) are first made locally. In [Ko] and [Do], sufficient conditions are given to globalize the fomality maps. Kontsevich formality maps satisfy those conditions. In this letter, we show that Tamarkin's maps can also be constructed to satify those conditions and so can be globalized. This letter is extracted from a general lecture given in Dijon 8 - 12 March 2004 on Tamarkin's works.

Let M be a differential manifold and $\mathfrak{g}_2 = (C^{\cdot}(A,A), b)$ be the Hochschild cochain complex on $A = C^{\infty}(M)$. The classical Hochschild-Kostant-Rosenberg theorem states that the cohomology of $\mathfrak{g}_2$ is the graded Lie algebra $\mathfrak{g}_1 = \Gamma^{\cdot}(M, \wedge^{\cdot} TM)$ of multivector fields on M. There is also a graded Lie algebra structure on $\mathfrak{g}_2$ given by the Gerstenhaber bracket. In the case $M = \mathbb{R}^n$, using different methods, Kontsevich ([Ko]) and Tamarkin ([Ta]) have proved the existence of Lie homomorphisms "up to homotopy" (L_∞-morphisms) form $\mathfrak{g}_1$ to $\mathfrak{g}_2$. Kontsevich's proof uses graphs and maps are related to polyzeta functions whereas Tamarkin's construction uses existence of associators. In fact Tamarkin's L_∞-morphism comes from the restriction of a "up to homotopy" Gerstenhaber algebra homomorphism (G_∞-morphism) from $\mathfrak{g}_1$ to $\mathfrak{g}_2$ (the G_∞-algebra structure on $\mathfrak{g}_1$ is induced by its classical Gerstenhaber algebra structure and a far less trivial G_∞-structure on $\mathfrak{g}_2$ was proved to exist by Tamarkin [Ta] and relies on Drinfeld's associator). When M is a Poisson manifold, Kontsevich and Tamarkin homomorphisms imply the existence of a star-product (see [BFFLS1] and [BFFLS2] for a definition). Connection between the two approaches has been shown in [KS]. The purpose of this letter is to prove that the (local) G_∞-morphism of Tamarkin can be built so that its L_∞ part satisfies globalization hypothesis described by Kontsevich ([Ko]) and Dolgushev ([Do]) and so can be globalized, as well as Tamarkin's L_∞ maps.

In the first section, we fix notations and recall the definitions of L_∞ and G_∞-structures. In the second section we state and prove the main Theorem.

Remark : in the sequel, unless otherwise is stated, the manifold M is supposed to be $\mathbb{R}^n$ for some $n \geq 1$.

1 L_∞ and G_∞-structures

For any graded vector space $\mathfrak{g}$, we choose the following degree on the space $\wedge^{\cdot}\mathfrak{g}$: if $X_1,\dots,X_k$ are homogeneous elements of respective degree $|X_1|,\dots|X_k|$, then

$$|X_1 \wedge \cdots \wedge X_k| = |X_1| + \cdots + |X_k| - k.$$

In particular the component $\mathfrak{g} = \wedge^1\mathfrak{g} \subset \wedge^{\cdot}\mathfrak{g}$ is the same as the space $\mathfrak{g}$ with degree shifted by one. The space $\wedge^{\cdot}\mathfrak{g}$ with the deconcatenation cobracket is the cofree cocommutative coalgebra on $\mathfrak{g}$ with degree shifted by one. Any degree one map $d^k : \wedge^k\mathfrak{g} \to \mathfrak{g}$ ($k \geq 1$) extends into a derivation $d^k : \wedge^{\cdot}\mathfrak{g} \to \wedge^{\cdot}\mathfrak{g}$ of the coalgebra $\wedge^{\cdot}\mathfrak{g}$ (by cofreeness property).

Definition 1.1. *A vector space $\mathfrak{g}$ is endowed with a L_∞-algebra (Lie algebra "up to homotopy") structure if there are degree one linear maps $m^{1,\dots,1} : \wedge^k\mathfrak{g} \to \mathfrak{g}$ such that if we extend them to maps $\wedge^{\cdot}\mathfrak{g} \to \wedge^{\cdot}\mathfrak{g}$, then $d \circ d = 0$ where d is the derivation*

$$d = m^1 + m^{1,1} + \cdots + m^{1,\dots,1} + \cdots .$$

For more details on L_∞-structures, see [LS]. It follows from the definition that a L_∞-algebra structure induces a differential coalgebra structure on $\wedge^{\cdot}\mathfrak{g}$ and that the map $m^1 : \mathfrak{g} \to \mathfrak{g}$ is a differential. If $m^{1,\dots,1} : \wedge^k\mathfrak{g} \to \mathfrak{g}$ are 0 for $k \geq 3$, we get the usual definition of (differential if $m^1 \neq 0$) graded Lie algebras.

For any graded vector space $\mathfrak{g}$, we denote $\underline{\mathfrak{g}^{\otimes n}}$ the quotient of $\mathfrak{g}^{\otimes n}$ by the image of all the shuffles of length n (see [GK] or [GH] for details). The graded vector space $\oplus_{n\geq 0}\underline{\mathfrak{g}^{\otimes n}}$ is a quotient coalgebra of the tensor coalgebra $\oplus_{n\geq 0}\mathfrak{g}^{\otimes n}$. It is well known that this coalgebra $\oplus_{n\geq 0}\underline{\mathfrak{g}^{\otimes n}}$ is the cofree Lie coalgebra on the vector space $\mathfrak{g}$ (with degree shifted by minus one). For any space $\mathfrak{g}$, we denote $\wedge^{\cdot}\underline{\mathfrak{g}^{\otimes \cdot}}$ the graded space $\bigoplus_{m\geq 1,\ p_1+\cdots+p_n=m} \underline{\mathfrak{g}^{\otimes p_1}} \wedge \cdots \wedge \underline{\mathfrak{g}^{\otimes p_n}}$. We use the following grading on $\wedge^{\cdot}\underline{\mathfrak{g}^{\otimes \cdot}}$ defined as follows: for $x_1^1, \cdots, x_n^{p_n} \in \mathfrak{g}$,

$$|\underline{x_1^1 \otimes \cdots \otimes x_1^{p_1}} \wedge \cdots \wedge \underline{x_n^1 \otimes \cdots \otimes x_n^{p_n}}| = \sum_{i_1}^{p_1} |x_1^{i_1}| + \cdots + \sum_{i_n}^{p_1} |x_n^{i_n}| - n.$$

Notice that the induced grading on $\wedge^{\cdot}\mathfrak{g} \subset \wedge^{\cdot}\underline{\mathfrak{g}^{\otimes \cdot}}$ is the same as the one introduced above. The cobracket on $\oplus\underline{\mathfrak{g}^{\otimes \cdot}}$ and the coproduct on $\wedge^{\cdot}\mathfrak{g}$ extend to a cobracket and a coproduct on $\wedge^{\cdot}\underline{\mathfrak{g}^{\otimes \cdot}}$ which yield a Gerstenhaber coalgebra structure on $\wedge^{\cdot}\underline{\mathfrak{g}^{\otimes \cdot}}$. It is well known that this coalgebra structure is cofree (see [Gi],Section 3 for example).

Definition 1.2. *A G_∞-algebra (Gerstenhaber algebra "up to homotopy") structure on a graded vector space $\mathfrak{g}$ is given by a collection of degree one maps*

$$m^{p_1,\dots,p_n} \ : \ \underline{\mathfrak{g}^{\otimes p_1}} \wedge \cdots \wedge \underline{\mathfrak{g}^{\otimes p_n}} \to \mathfrak{g}$$

indexed by $p_1, \dots p_n \geq 1$ *such that their canonical extension:* $\wedge^{\cdot}\underline{\mathfrak{g}^{\otimes \cdot}} \to \wedge^{\cdot}\underline{\mathfrak{g}^{\otimes \cdot}}$ *satisfies* $d \circ d = 0$ *where*

$$d = \sum_{m \geq 1,\ p_1 + \cdots p_n = m} m^{p_1, \dots, p_n}.$$

Again, as the coalgebra structure of $\wedge^{\cdot}\underline{\mathfrak{g}^{\otimes \cdot}}$ is cofree, the map d makes $\wedge^{\cdot}\underline{\mathfrak{g}^{\otimes \cdot}}$ a differential coalgebra. If the maps $m^{p_1, \dots, p_n}$ are 0 for $(p_1, p_2, \dots) \neq (1, 0, \dots)$, $(1, 1, 0, \dots)$ or $(2, 0, \dots)$, we get the usual definition of (differential if $m^1 \neq 0$) Gerstenhaber algebra.

The space of polyvector fields $\mathfrak{g}_1$ is endowed with a graded Lie bracket $[-,-]_S$ called the Schouten bracket (see [Kos]). This Lie algebra can be extended into a Gerstenhaber algebra, with commutative structure given by the exterior product: $(\alpha, \beta) \mapsto \alpha \wedge \beta$. Setting $d_1 = m_1^{1,1} + m_1^2$, where $m_1^{1,1} : \wedge^2 \mathfrak{g}_1 \to \mathfrak{g}_1$, and $m_1^2 : \underline{\mathfrak{g}_1^{\otimes 2}} \to \mathfrak{g}_1$ are the extension of the Schouten bracket and the exterior product, we find that $(\mathfrak{g}_1, d_1)$ is a G_∞-algebra.

In the same way, one can define a differential Lie algebra structure on the vector space $\mathfrak{g}_2 = C(A,A) = \bigoplus_{k \geq 0} C^k(A,A)$, the space of Hochschild cochains (generated by differential k-linear maps from A^k to A), where $A = C^\infty(M)$ is the algebra of smooth differential functions over M. Its bracket called Gerstenhaber bracket $[-,-]_G$ is defined, for $D, E \in \mathfrak{g}_2$, by

$$[D,E]_G = \{D|E\} - (-1)^{|E||D|}\{E|D\},$$

where

$$\{D|E\}(x_1, \dots, x_{d+e-1}) = \sum_{i \geq 0} (-1)^{|E| \cdot i} D(x_1, \dots, x_i, E(x_{i+1}, \dots, x_{i+e}), \dots).$$

The space $\mathfrak{g}_2$ has a grading defined by $|D| = k \Leftrightarrow D \in C^{k+1}(A,A)$ and its differential is $b = [m, -]_G$, where $m \in C^2(A,A)$ is the commutative multiplication on A.

Proposition 1.3. *([HKR]) The complex* $(\mathfrak{g}_2, b)$ *is quasi-isomorphic to* $(\mathfrak{g}_1, 0)$.

An explicit quasi-isomorphism is given by the Hochschild-Kostant-Rosenberg map (see [HKR]), $\varphi_{\mathrm{HKR}} : (\mathfrak{g}_1, 0) \to (\mathfrak{g}_2, b)$ defined, for $\alpha \in \mathfrak{g}_1$, $f_1, \cdots, f_n \in A$, by

$$\varphi_{\mathrm{HKR}} \ : \ \alpha \mapsto \big((f_1, \dots, f_n) \mapsto \ \langle \alpha, df_1 \wedge \cdots \wedge df_n \rangle\big).$$

Tamarkin (see [Ta] or also [GH]) stated the existence of a G_∞-structure on $\mathfrak{g}_2$ given by a differential $d_2 = m_2^1 + m_2^{1,1} + m_2^2 + \cdots + m_2^{p_1, p_2} + \cdots$, on $\wedge^{\cdot}\underline{\mathfrak{g}_2^{\otimes \cdot}}$ ($m_2^{p_1, p_2, \dots, p_k} = 0$ for $k \geq 3$) satisfying $d_2 \circ d_2 = 0$. Although this structure is non-explicit, it satisfies the following three properties :

$$\begin{aligned} &(a) \quad m_2^1 \text{ is the extension of the differential } b \\ &(b) \quad m_2^{1,1} \text{ is the extension of the Gerstenhaber bracket } [-,-]_G. \\ &(c) \quad m_2^2 \text{ induces the exterior product in cohomology.} \end{aligned} \tag{1.1}$$

Definition 1.4. *A L_∞-morphism between two L_∞-algebras $(\mathfrak{g}_1, d_1 = m_1^1 + \dots)$ and $(\mathfrak{g}_2, d_2 = m_2^1 + \dots)$ is a morphism of differential coalgebras*

$$\varphi \ : \ (\wedge^{\cdot}\mathfrak{g}_1, d_1) \rightarrow (\wedge^{\cdot}\mathfrak{g}_2, d_2). \tag{1.2}$$

Such a map φ is uniquely determined by a collection of maps $\varphi^n : \wedge^n\mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ (again by cofreeness properties). In the case $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are respectively the graded Lie algebra $(\Gamma(M, \wedge TM),\ [-,-]_S)$ and the differential graded Lie algebra $(C(A,A),\ [-,-]_G)$, formality theorems of Kontsevich and Tamarkin state the existence of a L_∞-morphism between $\mathfrak{g}_1$ and $\mathfrak{g}_2$ such that φ^1 is the Hochschild-Kostant-Rosenberg quasi-isomorphism.

Definition 1.5. *A morphism of G_∞-algebras between two G_∞-algebras $(\mathfrak{g}_1, d_1)$ and $(\mathfrak{g}_2, d_2)$ is a map $\phi : (\wedge^{\cdot}\underline{\mathfrak{g}_1^{\otimes\cdot}}, d_1) \rightarrow (\wedge^{\cdot}\underline{\mathfrak{g}_2^{\otimes\cdot}}, d_2)$ of codifferential coalgebras.*

There is a coalgebra inclusion $\wedge^{\cdot}\mathfrak{g} \rightarrow \wedge^{\cdot}\underline{\mathfrak{g}^{\otimes\cdot}}$, and it is easy to check that any G_∞-morphism between two G_∞-algebras $(\mathfrak{g}, \sum m^{p_1,\dots,p_n})$, $(\mathfrak{g}', \sum m'^{p_1,\dots,p_n})$ restricts into a L_∞-morphism $\left(\wedge^{\cdot}\mathfrak{g}, \sum m^{1,\dots,1}\right) \rightarrow \left(\wedge^{\cdot}\mathfrak{g}', \sum m'^{1,\dots,1}\right)$. In the case $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are as above, Tamarkin's theorem states that there exists a G_∞-morphism between the two G_∞ algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ (with the G_∞ structure he built) that restricts to a L_∞-morphism between the differential graded Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$..

2 Main theorem

We keep the notations of the previous section, in particular $\mathfrak{g}_2$ is the Hochschild complex of cochains on $C^\infty(M)$ and $\mathfrak{g}_1$ its cohomology.

Here is our main theorem.

Theorem 2.1. *Suppose $M = \mathbb{R}^d$ and we are given a G_∞-structure on $\mathfrak{g}_2$ given by a differential d_2 as in 1.1. One can construct, following Tamarkin, a G_∞-morphism $\Phi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ satisfying the extra conditions:*

1. *The G_∞-morphism is local (one can replace $\mathbb{R}^d$ by its formal completion $\mathbb{R}_0^d$ at the origin, or in other words, one can replace the functions with their Taylor expansion) and it can be made equivariant with respect to linear transformations of the coordinates on $\mathbb{R}_0^d$.*
2. *For any set of vector fields $(v_i)_{1 \leq i \leq 2} \in \Gamma(\mathbb{R}_0^d, T\mathbb{R}_0^d)$,*

$$\Phi^{1,1}(v_1 \wedge v_2) = 0. \tag{2.3}$$

3. *If $n \geq 2$ and $v \in \Gamma(\mathbb{R}_0^d, T\mathbb{R}_0^d)$ is linear in the coordinates on $\mathbb{R}_0^d$, then for any set of tensor product of multivector fields $\gamma_i \in \underline{\Gamma(\mathbb{R}_0^d, \wedge T\mathbb{R}_0^d)^{\otimes p_i}}$:*

$$\Phi^{1,p_2,\dots,p_n}(v \wedge \gamma_2 \wedge \dots \wedge \gamma_n) = 0. \tag{2.4}$$

Corollary 2.2. *The restriction φ of Φ as a L_∞-morphism*

$$\varphi\ :(\mathfrak{g}_1,[-,-]_S)\to(\mathfrak{g}_2,[-,-]_G+b)$$

satisfies the conditions:

1. *The L_∞-morphism is local and it can be made equivariant with respect to linear transformations of the coordinates on $\mathbb{R}^d_0$.*
2. *For any set of vector fields $(v_i)_{1\leq i\leq 2}\in\Gamma(\mathbb{R}^d_0,T\mathbb{R}^d_0)$,*
$$\varphi^{1,1}(v_1\wedge v_2)=0. \tag{2.5}$$
3. *If $n\geq 2$ and $v\in\Gamma(\mathbb{R}^d_0,T\mathbb{R}^d_0)$ is linear in the coordinates on $\mathbb{R}^d_0$, then for any set of multivector fields $\gamma_i\in\Gamma(\mathbb{R}^d_0,\wedge T\mathbb{R}^d_0)$:*
$$\varphi^{1,1,\dots,1}(v\wedge\gamma_2\wedge\cdots\wedge\gamma_n)=0. \tag{2.6}$$

Those are exactly the conditions written in [Ko] and [Do] for globalization. So one can build a global L_∞ morphism using Tamarkin's methods.

Proof of Theorem 2.1:

Let us first recall the proof of Tamarkin's formality theorem (see [GH] for more details):

1. Firstly one proves there exists a G_∞-structure on $\mathfrak{g}_2$, given by a differential d_2, as in (1.1), satisying the extra property:
$$d_2=\sum_{p_1,p_2}d_2^{p_1,p_2},$$
and by construction, $d_2^{1,p}(u,v_1\cdots v_p)=0$ for $p>1$ and any vector fields u.
2. Then, one constructs a G_∞-structure on $\mathfrak{g}_1$ given by a differential d_1' together with a G_∞-morphism Φ' between $(\mathfrak{g}_1,d_1')$ and $(\mathfrak{g}_2,d_2)$.
3. Finaly, one constructs a G_∞-morphism Φ'' between $(\mathfrak{g}_1,d_1)$ and $(\mathfrak{g}_1,d_1')$.

The composition $\Phi=\Phi'\circ\Phi''$ is then a G_∞-morphism between $(\mathfrak{g}_1,d_1)$ and $(\mathfrak{g}_2,d_2)$, that rectricts to a L_∞-morphism between the differential graded Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$.

The theorem will follow if we prove that points 2 and 3 of Tamarkin's construction are still true with Φ' and Φ'' satisfying the extra conditions of Theorem 2.1 and d_1' satisfying conditions (2.4) for $n\geq 3$.

- Point 2: let us recall (see [GH]) that the constructions of Φ' and d_1' can be made by induction. For $i=1,2$ and $n\geq 0$, let us set

$$V_i^{[n]}=\bigoplus_{p_1+\cdots+p_k=n}\underline{\mathfrak{g}_i^{\otimes p_1}}\wedge\cdots\wedge\underline{\mathfrak{g}_i^{\otimes p_k}}$$

and $V_i^{[\leq n]} = \sum_{k\leq n} V_i^{[k]}$. Let $d_2^{[n]}$ and $d_2^{[\leq n]}$ be the sums

$$d_2^{[n]} = \sum_{p_1+\cdots+p_k=n} d_2^{p_1,\dots,p_k} \qquad \text{and} \qquad d_2^{[\leq n]} = \sum_{p\leq n} d_2^{[p]}.$$

Clearly, $d_2 = \sum_{n\geq 1} d_2^{[n]}$. In the same way, we denote $d_1' = \sum_{n\geq 1} {d'}_1^{[n]}$ with

$${d'}_1^{[n]} = \sum_{p_1+\cdots+p_k=n} {d}_1^{\prime\, p_1,\dots,p_k} \qquad \text{and} \qquad {d'}_1^{[\leq n]} = \sum_{1\leq k\leq n} {d'}_1^{[k]}.$$

We know from Section 1 that a morphism $\Phi' : (\wedge^{\cdot}\underline{\mathfrak{g}_1^{\otimes\cdot}}, d_1') \to (\wedge^{\cdot}\underline{\mathfrak{g}_2^{\otimes\cdot}}, d_2)$ is uniquely determined by its components $\Phi'^{p_1,\dots,p_k} : \underline{\mathfrak{g}_1^{\otimes p_1}} \wedge\cdots\wedge \underline{\mathfrak{g}_1^{\otimes p_k}} \to \mathfrak{g}_2$. Again, we have $\Phi' = \sum_{n\geq 1} \Phi'^{[n]}$ with

$$\Phi'^{[n]} = \sum_{p_1+\cdots+p_k=n} \Phi'^{p_1,\dots,p_k} \qquad \text{and} \qquad \Phi'^{[\leq n]} = \sum_{1\leq k\leq n} \Phi'^{[k]}.$$

We want to construct the maps ${d'}_1^{[n]}$ and $\Phi'^{[n]}$ by induction with the initial condition

$${d'}_1^{[1]} = 0 \qquad \text{and} \qquad \Phi'^{[1]} = \varphi_{\mathrm{HKR}},$$

where $\varphi_{\mathrm{HKR}} : (\mathfrak{g}_1, 0) \to (\mathfrak{g}_2, b)$ is the Hochschild-Kostant-Rosenberg quasi-isomorphism (defined after Proposition 1.3). Note that this map satisfies the first conditions of Theorem 2.1.
Now suppose the construction is done for $n-1$ ($n \geq 2$), i. e., we have built maps $({d'}_1^{[i]})_{i\leq n-1}$ and $(\Phi'^{[i]})_{i\leq n-1}$ satisfying the extra conditions of Theorem 2.1 and

$$\Phi'^{[\leq n-1]} \circ {d'}_1^{[\leq n-1]} = d_2^{[\leq n-1]} \circ \Phi'^{[\leq n-1]} \text{ on } V_1^{[\leq n-1]} \text{ and } {d'}_1^{[\leq n-1]} \circ {d'}_1^{[\leq n-1]} = 0 \text{ on } V_1^{[\leq n]}. \tag{2.7}$$

In [GH], we proved that for any such $({d'}_1^{[i]})_{i\leq n-1}$ and $(\Phi'^{[i]})_{i\leq n-1}$, one can construct ${d'}_1^{[n]}$ and $\Phi'^{[n]}$ such that condition (2.7) is true for n instead of $n-1$. as this last statement is equivalent to

$$\varphi_{\mathrm{HKR}} d_1'^{[n]} = b\Phi'^{[n]} + A \tag{2.8}$$

where A is always a Hochschild cocycle (existence of $d_1'^{[n]}$ and $\Phi'^{[n]}$ are then consequences of Proposition 1.3).
- It is obvious (use homotopy formulas of [Ha] to solve (2.8)) that the first condition in Theorem 2.1 can then be satisfied for those maps $d_1'^{[n]}$ and $\Phi^{[n]}$.
- Using Equation (2.7), condition (2.3) is equivalent to:

$$\Phi([\alpha,\beta]_S)\ (= \varphi_{\mathrm{HKR}}([\alpha,\beta]_S)) = [\Phi(\alpha), \Phi(\beta)]_G\ (= [\varphi_{\mathrm{HKR}}(\alpha), \varphi_{\mathrm{HKR}}(\beta)]_G),$$

for any set of vector fields $\alpha, \beta \in \Gamma(\mathbb{R}_0^d, T\mathbb{R}_0^d)$, which is true.
- Let us check conditions (2.4) for $d_1^{[n]}$ and $\Phi'^{[n]}$ when they are supposed to be true by induction for $k \leq n-1$. Using the induction hypothesis in Equation (2.7) and the fact

that $d_2^{p_1,\dots,p_n} = 0$ for $n > 2$ and $d_2^{1,p}(u, v_1 \cdots v_p) = 0$ for $p > 1$ and any vector fields u, one can see that those conditions are equivalent to

$$[X, \Phi^{[n-1]}(\cdots \wedge \underline{x_i^1 \otimes \cdots \otimes x_i^{p_i}} \wedge \cdots)]_G = \sum \Phi^{[n-1]}(\cdots \wedge \underline{\cdots \otimes [X, x_i^{n_{ij}}]_S \otimes \cdots} \wedge \cdots), \tag{2.9}$$

where X is a linerar vector fields and $x_i^{n_{ij}}$ are tensor fields, which is exactely the equivariance with respect to linear transformations of the coordinates on $\mathbb{R}_0^d$ and whas already proved.
So one can construct $d_1^{[n]}$ and $\Phi'^{[n]}$ satisfying the conditions of Theorem 2.1.

- Point 2: let us now recall that the construction of Φ'' can also be made by induction. Again, we will use the same notations for $V_1^{[n]}$, $V_1^{[\leq n]}$, $d_1'^{[n]}$, $d_1'^{[\leq n]}$ $\Phi'^{[n]}$, $\Phi'^{[\leq n]}$ and we will also write

$$d_1 = \sum d_1^{[n]}, \qquad d_1'^{[\leq n]} = \sum_{1 \leq k \leq n} d_1^{[n]}$$

and

$$\Phi'' = \sum \Phi''^{[n]}, \qquad \Phi''^{[\leq n]} = \sum_{1 \leq k \leq n} \Phi''^{[n]}.$$

Suppose the construction is done for $n-1$, *i.e.* we have built maps $(\Phi''^{[i]})_{i \leq n-1}$ satisfying the extra conditions of Theorem 2.1 and

$$\Phi''^{[1]} = \mathrm{id}, \qquad \Phi''^{[\leq n-1]} d_1^{[\leq n]} = d_1'^{[\leq n]} \Phi''^{[\leq n-1]} \tag{2.10}$$

on $V_1^{[\leq n-1]}$. Again, in [GH], we prove that one can construct $\Phi''^{[n]}$ such that condition (2.10) is true for n instead of $n-1$: indeed this last statement is equivalent to

$$[d^{[2]}, \Phi''^{[\leq n]}] = -\sum_{k=3}^{n+1} d_1'^{[k]} \Phi''^{[n-k+2]}$$

where the complex $\left(\mathrm{Hom}(\wedge^{\cdot} \underline{\mathfrak{g}_1^{\otimes \cdot}}, \wedge^{\cdot} \underline{\mathfrak{g}_1^{\otimes \cdot}}), [d^{[2]}, -]\right)$ is acyclic. To prove this last fact (see [Ta] and [GH]), one can see this complex as a bicomplex where

$$[d^{[2]}, -] = [[-,-]_S, -] + [\wedge, -]$$

(the sum of $\mathrm{d_{CE}}$ the Chevalley-Eilenberg differential and $\mathrm{d_H}$ the Harrison differential) and use the following properties :

1. The complex $\left(\mathrm{Hom}(\mathfrak{g}_1^{\otimes \cdot}, \mathfrak{g}_1), \mathrm{d_H}\right)$ is concentrated in degree 0
2. The complex $\left(\mathrm{Hom}_{\mathfrak{g}_1}(\wedge_{\mathfrak{g}_1}^{\cdot} \Omega_{\mathfrak{g}_1}, \mathfrak{g}_1), \mathrm{d_{CE}}\right)$ is concentrated in degree 0 where $\Omega_{\mathfrak{g}_1}$ is the degreee 0 cohomology of the former complex and is the module of Kähler differential one-forms of the algebra $\mathfrak{g}_1$,

which implies the acyclicity of the complex $\left(\mathrm{Hom}(\wedge^{\cdot} \underline{\mathfrak{g}_1^{\otimes \cdot}}, \wedge^{\cdot} \underline{\mathfrak{g}_1^{\otimes \cdot}}), [d^{[2]}, -]\right)$. Those facts can be checked using again homotopy formulas (see [GH]). Using those formulas, one can again easily check that Φ'' can be built to satisfies the first conditions of Theorem 2.1.

To prove the third condition for Φ'' (the second one can be proved in the same way), using the induction hypothesis, one has to check that the two points giving acyclicity of the complex $\left(\mathrm{Hom}(\wedge^{\cdot}\underline{\mathfrak{g}_1^{\otimes\cdot}},\wedge^{\cdot}\underline{\mathfrak{g}_1^{\otimes\cdot}}),[d^{[2]},-]\right)$ are also true when we impose the extra conditions. More precisely, we have to prove :

1. If a map ψ' of degree more that 2 satisfies $\mathrm{d_H}\psi=\psi'$ and the third extra conditions of Theorem 2.1, one can chose the map ψ to satisfy also those conditions. This can be done as the Harrison differential acts independentely on each part of the exterior product.

1'. If ψ satisfies the third extra conditions of Theorem 2.1, $\mathrm{d_{CE}}\psi$ satisfieds also those conditions. This is true as the only equation that should be satisfied is of the Type of Equation (2.9) and is again a consequence of equivariance with respect to linear change of coordinates.

2. The complex $\left(\widetilde{\mathrm{Hom}}_{\mathfrak{g}_1}(\wedge^{\cdot}_{\mathfrak{g}_1}\Omega_{\mathfrak{g}_1},\mathfrak{g}_1),\mathrm{d_{CE}}\right)$ is concentrated in degree 0 where, with $\widetilde{\mathrm{Hom}}$, we restrict our complex to the maps satisfying the third condition. It is true because $\widetilde{\mathrm{Hom}}=\mathrm{Hom}$ since the maps $\wedge^{\cdot}_{\mathfrak{g}_1}\Omega_{\mathfrak{g}_1}\to\mathfrak{g}_1$ are not concerned with the third condition of Theorem 2.1.

So the theorem is proved. ■

Remark: Let $\widetilde{\mathrm{Hom}}(\wedge^{\cdot}\underline{\mathfrak{g}_1^{\otimes\cdot}},\wedge^{\cdot}\underline{\mathfrak{g}_1^{\otimes\cdot}})$ be the subspace of $\mathrm{Hom}(\wedge^{\cdot}\underline{\mathfrak{g}_1^{\otimes\cdot}},\wedge^{\cdot}\underline{\mathfrak{g}_1^{\otimes\cdot}})$ consisting of maps satisfying conditions of Theorem 2.1. What we did can be reformulated as follows: $\left(\widetilde{\mathrm{Hom}}(\wedge^{\cdot}\underline{\mathfrak{g}_1^{\otimes\cdot}},\wedge^{\cdot}\underline{\mathfrak{g}_1^{\otimes\cdot}}),[d^{[2]},-]\right)$ is a subcomplex of $\left(\mathrm{Hom}(\wedge^{\cdot}\underline{\mathfrak{g}_1^{\otimes\cdot}},\wedge^{\cdot}\underline{\mathfrak{g}_1^{\otimes\cdot}}),[d^{[2]},-]\right)$ which is acyclic. We proved acyclicity of this complex following the same steps used to prove acyclicity of $\left(\mathrm{Hom}(\wedge^{\cdot}\underline{\mathfrak{g}_1^{\otimes\cdot}},\wedge^{\cdot}\underline{\mathfrak{g}_1^{\otimes\cdot}}),[d^{[2]},-]\right)$ but it can be deduced directly: $\mathrm{Hom}(\wedge^{\cdot}\underline{\mathfrak{g}_1^{\otimes\cdot}},\wedge^{\cdot}\underline{\mathfrak{g}_1^{\otimes\cdot}}))$ can be seen as a subcomplex H of an extended complex $\widehat{H}$ where we do admit 1 on the left hand side. Both $\widehat{H}$ and H are acyclic (elements of H consist of all elements which are given by polydifferiential expressions and whose projection gives a polyvector field whose 0-ary component is a function vanishing at 0). Note now that H is a $\mathfrak{g}l_n[\varepsilon]$-module, where $\mathfrak{g}l_n[\varepsilon]=\mathfrak{g}l_n\oplus\mathfrak{g}l_n\cdot\varepsilon$, $|\varepsilon|=-1$, the differential is $\partial/\partial\varepsilon$ and operations on H are given by maps L_X and i_X, respectively the natural action and the contraction by vector fields $X\in\mathfrak{g}l_n$.

The complex $\widetilde{\mathrm{Hom}}(\wedge^{\cdot}\underline{\mathfrak{g}_1^{\otimes\cdot}},\wedge^{\cdot}\underline{\mathfrak{g}_1^{\otimes\cdot}})$ can be seen as a subcomplex $H'\subset H$ consisting of all $\mathfrak{g}l_n$-equivariant polyvector fields whose 0-ary component vanishes at 0 (and therefore vanishes itself), *i. e.* $U\in H$ is in $H'\Leftrightarrow i_XU=L_XU=0$. It suffices now to show H' is acyclic which is true because so is H and H' is quasi-isomorphic to the relative cochain complex $C^*(\mathfrak{g}l_n[\varepsilon],\mathfrak{g}l_n;H)$.

To prove this quasi-isomorphism, split $\mathfrak{g}l_n$-equivariantly $\mathfrak{g}_1=\mathfrak{g}l_n\oplus h$, this induces an isomorphism of $\mathfrak{g}l_n[\varepsilon]$-modules $H\cong\prod_i\mathrm{hom}(\Lambda^i\mathfrak{g}l_n,H')$. Let us discuss the differential on the right hand side of this formula corresponding to that on H under our identification. Let F be the filtration of H' given by $F^kH'=H'\cap F^kH$, where in turn, F^kH consists of all elements which vanish on $\underline{T^{n_1}\mathfrak{g}_1}\wedge\cdots\wedge\underline{T^{n_i}\mathfrak{g}_1}$ as long as $n_1+\cdots+n_i<k$. The differential is induced by that in $C^*(\mathfrak{g}l_n,H')\cong\prod_i\mathrm{hom}(\Lambda^i\mathfrak{g}l_n,H')$ modulo a term which increases F. An easy spectral sequence argument imply then the statement.

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