

QUANTIZATION OF Γ -LIE BIALGEBRAS

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ABSTRACT. We introduce the notion of Γ -Lie bialgebras, where Γ is a group. These objects give rise to cocommutative co-Poisson bialgebras, for which we construct quantization functors. This enlarges the class of co-Poisson algebras for which a quantization is known. Our result relies on our earlier work, where we showed that twists of Lie bialgebras can be quantized; we complement this work by studying the behavior of this quantization under compositions of twists.

We work over a field \mathbf{k} of characteristic 0.

1. INTRODUCTION

Recall that a co-Poisson bialgebra is a quadruple (U, m, Δ_0, δ) , where (U, m, Δ_0) is a cocommutative bialgebra and $\delta : U \rightarrow \wedge^2(U)$ is a derivation (for m), a coderivation (for Δ_0) and satisfies the co-Jacobi identity. A quantization of (U, m, Δ_0, δ) is a bialgebra $(U_{\hbar}, m_{\hbar}, \Delta_{\hbar})$ such that $U_{\hbar} \simeq U[[\hbar]]$, $m_{\hbar} = m + O(\hbar)$, $\Delta_{\hbar} = \Delta + O(\hbar)$, $\Delta_{\hbar}(a) - \Delta_{\hbar}^{\text{op}}(a) = \hbar\delta(a) + O(\hbar^2)$ ($\Delta_{\hbar}^{\text{op}}$ is the opposite coproduct).

If $(\mathfrak{a}, \mu_{\mathfrak{a}})$ is a Lie algebra ($\mu_{\mathfrak{a}} : \wedge^2(\mathfrak{a}) \rightarrow \mathfrak{a}$ is the Lie bracket), the co-Poisson bialgebra structures on $U(\mathfrak{a})$ correspond bijectively to the maps $\delta_{\mathfrak{a}} : \mathfrak{a} \rightarrow \wedge^2(\mathfrak{a})$ such that $(\mathfrak{a}, \mu_{\mathfrak{a}}, \delta_{\mathfrak{a}})$ is a Lie bialgebra. The quantization of these co-Poisson bialgebras was obtained in [EK].

To a triple $(\Gamma, \mathfrak{a}, \theta_{\mathfrak{a}})$, where Γ is a group, \mathfrak{a} is a Lie algebra and $\theta_{\mathfrak{a}} : \Gamma \rightarrow \text{Aut}(\mathfrak{a}, \mu_{\mathfrak{a}})$ is an action of Γ on \mathfrak{a} , one associates the Γ -graded cocommutative bialgebra $U(\mathfrak{a}) \rtimes \Gamma$. The Γ -graded co-Poisson bialgebra structures on $U(\mathfrak{a}) \rtimes \Gamma$ correspond bijectively to pairs $(\delta_{\mathfrak{a}}, f)$, where $\delta_{\mathfrak{a}} : \mathfrak{a} \rightarrow \wedge^2(\mathfrak{a})$ is such that $(\mathfrak{a}, \mu_{\mathfrak{a}}, \delta_{\mathfrak{a}})$ is a Lie bialgebra, and $f : \Gamma \rightarrow \wedge^2(\mathfrak{a})$ satisfies some conditions (see Section 2); in particular, $f(\gamma)$ is a twist of $(\mathfrak{a}, \mu_{\mathfrak{a}}, \delta_{\mathfrak{a}})$ for any $\gamma \in \Gamma$. We call the resulting 5-uple $(\mathfrak{a}, \mu_{\mathfrak{a}}, \delta_{\mathfrak{a}}, \theta_{\mathfrak{a}}, f)$ a Γ -Lie bialgebra. The main result of this paper is the quantization of the corresponding co-Poisson bialgebra structures.

Examples of Γ -Lie bialgebras arise from the following situation: G is a Poisson-Lie group with Lie bialgebra $(\mathfrak{a}, \mu_{\mathfrak{a}}, \delta_{\mathfrak{a}})$, and $\Gamma \subset G$ is a discrete subgroup. Another example is when \mathfrak{a} is a Kac-Moody Lie algebra \mathfrak{a} , and Γ is the extended Weyl group of \mathfrak{a} . In the latter case, a quantization is known ([MS]).

To achieve our goal, we complement a result obtained in [EnH], namely the compatibility of Etingof-Kazhdan (EK) quantization functors with twists of Lie bialgebras; this result is based on an alternative construction of these quantization functors ([En]). We describe the behavior of this quantization under composition of twists (Section 4).

To give an idea of the result of [EnH], we formulate its main consequence: let $Q : \{\text{Lie bialgebras}\} \rightarrow \{\text{quantized universal enveloping (QUE) algebras}\}$, $(\mathfrak{a}, \mu_{\mathfrak{a}}, \delta_{\mathfrak{a}}) = \mathfrak{a} \mapsto Q(\mathfrak{a}) = (Q(\mathfrak{a}), m(\mathfrak{a}), \Delta(\mathfrak{a}), 1_{\mathfrak{a}}, \varepsilon_{\mathfrak{a}})$ be a quantization functor; to each classical twist $f_{\mathfrak{a}}$ of \mathfrak{a} (i.e., $f_{\mathfrak{a}} \in \wedge^2(\mathfrak{a})$ and $(\delta_{\mathfrak{a}} \otimes \text{id}_{\mathfrak{a}})(f_{\mathfrak{a}}) + [f_{\mathfrak{a}}^{13}, f_{\mathfrak{a}}^{23}] + \text{cyclic permutations} = 0$), one associates $F(\mathfrak{a}, f_{\mathfrak{a}}) \in Q(\mathfrak{a})^{\otimes 2}$, with the following properties: (a) it is a cocycle for $Q(\mathfrak{a})$, i.e.,¹ $(F(\mathfrak{a}, f_{\mathfrak{a}}) \otimes 1_{\mathfrak{a}}) * (\Delta(\mathfrak{a}) \otimes \text{id})(F(\mathfrak{a}, f_{\mathfrak{a}})) = (1_{\mathfrak{a}} \otimes F(\mathfrak{a}, f_{\mathfrak{a}})) * (\text{id} \otimes \Delta(\mathfrak{a}))(F(\mathfrak{a}, f_{\mathfrak{a}}))$, $(\varepsilon_{\mathfrak{a}} \otimes \text{id})(F(\mathfrak{a}, f_{\mathfrak{a}})) = (\text{id} \otimes \varepsilon_{\mathfrak{a}})(F(\mathfrak{a}, f_{\mathfrak{a}})) =$

¹We denote by $*$ the product in $Q(\mathfrak{a})^{\otimes k}$ for $k \geq 1$

$1_{\mathfrak{a}}$; this implies that² $F(\mathfrak{a}, \delta_{\mathfrak{a}})Q(\mathfrak{a}) := (Q(\mathfrak{a}), m(\mathfrak{a}), \text{Ad}(F(\mathfrak{a}, f_{\mathfrak{a}})) \circ \Delta(\mathfrak{a}), 1_{\mathfrak{a}}, \varepsilon_{\mathfrak{a}})$ is a QUE algebra; (b) we have an isomorphism $i(\mathfrak{a}, f_{\mathfrak{a}}) : F(\mathfrak{a}, f_{\mathfrak{a}})Q(\mathfrak{a}) \rightarrow Q(\mathfrak{a}_{f_{\mathfrak{a}}})$ of QUE algebras (here $\mathfrak{a}_{f_{\mathfrak{a}}} = (\mathfrak{a}, \mu_{\mathfrak{a}}, \delta_{\mathfrak{a}} + \text{ad}(f_{\mathfrak{a}}))$, where $\text{ad}(f_{\mathfrak{a}}) : \mathfrak{a} \rightarrow \wedge^2(\mathfrak{a})$ is $x \mapsto [f_{\mathfrak{a}}, x \otimes 1 + 1 \otimes x]$).

In Section 4, we study the behavior of the assignment $(\mathfrak{a}, f_{\mathfrak{a}}) \mapsto F(\mathfrak{a}, f_{\mathfrak{a}})$ under the composition of twists. A composition of twists is a pair $(f_{\mathfrak{a}}, f'_{\mathfrak{a}})$ such that $f_{\mathfrak{a}}$ is a twist of \mathfrak{a} , and $f'_{\mathfrak{a}}$ is a twist of $\mathfrak{a}_{f_{\mathfrak{a}}}$. We formulate the main consequence of our results: (a) there exists an invertible $v(\mathfrak{a}, f_{\mathfrak{a}}, f'_{\mathfrak{a}}) \in Q(\mathfrak{a})$, such that $F(f_{\mathfrak{a}} + f'_{\mathfrak{a}}) = v(\mathfrak{a}, f_{\mathfrak{a}}, f'_{\mathfrak{a}})^{\otimes 2} * i(\mathfrak{a}, f_{\mathfrak{a}})^{-1}(F(\mathfrak{a}_{f_{\mathfrak{a}}}, f'_{\mathfrak{a}})) * F(\mathfrak{a}, f_{\mathfrak{a}}) * \Delta(\mathfrak{a})(v(\mathfrak{a}, f_{\mathfrak{a}}, f'_{\mathfrak{a}}))^{-1}$ (Theorem 4.4); and (b) if $(f_{\mathfrak{a}}, f'_{\mathfrak{a}}, f''_{\mathfrak{a}})$ are such that $f'_{\mathfrak{a}}$ is a twist of $\mathfrak{a}_{f_{\mathfrak{a}}}$ and $f''_{\mathfrak{a}}$ is a twist of $\mathfrak{a}_{f_{\mathfrak{a}}}$, then $f_{\mathfrak{a}} + f'_{\mathfrak{a}}$ is a twist of \mathfrak{a} , and $v(\mathfrak{a}, f_{\mathfrak{a}} + f'_{\mathfrak{a}}, f''_{\mathfrak{a}}) * v(\mathfrak{a}, f_{\mathfrak{a}}, f'_{\mathfrak{a}}) = v(\mathfrak{a}, f_{\mathfrak{a}}, f'_{\mathfrak{a}} + f''_{\mathfrak{a}}) * i(\mathfrak{a}, f_{\mathfrak{a}})^{-1}(v(\mathfrak{a}_{f_{\mathfrak{a}}}, f'_{\mathfrak{a}}, f''_{\mathfrak{a}}))$ (Theorem 4.4).

We use these results in Section 5 to construct a quantization of the Γ -graded co-Poisson bialgebras $U(\mathfrak{a}) \rtimes \Gamma$ as Γ -graded bialgebras. This quantization is based on the facts that that for $\gamma \in \Gamma$, $f(\gamma)$ is a twist of \mathfrak{a} , and for any $\gamma, \gamma' \in \Gamma$, $(f(\gamma), \wedge^2(\theta_{\mathfrak{a}}(\gamma))(f(\gamma')))$ is a composition of twists for \mathfrak{a} ; we then use the results of [EnH] on quantization of twists and those of Section 4 on their composition.

This paper is organized as follows. In Section 2, we define Γ -Lie bialgebras, the corresponding co-Poisson cocommutative bialgebras, and the problem of their quantization. In Section 3, we recall the formalism of (quasi-multi-bi)props, which is the natural framework of the approach of [En] to quantization functors and of the results of [EnH] on quantization of twists. In Section 4, we describe the behavior of composition of twists under quantization (Theorem 4.4 and Theorem 4.6). In Section 5, we apply these results to the construction of quantizations of Γ -Lie bialgebras.

2. Γ -LIE BIALGEBRAS

2.1. Γ -Lie algebras and equivalent categories. Define a group Lie algebra as a triple $(\Gamma, \mathfrak{a}, \theta_{\mathfrak{a}})$, where Γ is a group, \mathfrak{a} is a Lie algebra and $\theta_{\mathfrak{a}} : \Gamma \rightarrow \text{Aut}(\mathfrak{a})$ is a group morphism. Group Lie algebras form a category, where a morphism $(\Gamma, \mathfrak{a}, \theta_{\mathfrak{a}}) \rightarrow (\Gamma', \mathfrak{a}', \theta_{\mathfrak{a}'})$ is the data of a group morphism $i_{\Gamma\Gamma'} : \Gamma \rightarrow \Gamma'$ and a Lie algebra morphism $i_{\mathfrak{a}\mathfrak{a}'} : \mathfrak{a} \rightarrow \mathfrak{a}'$, such that $i_{\mathfrak{a}\mathfrak{a}'}(\theta_{\mathfrak{a}, \gamma}(x)) = \theta_{\mathfrak{a}, i_{\Gamma\Gamma'}(\gamma)}(i_{\mathfrak{a}\mathfrak{a}'}(x))$.

If Γ is a group, a Γ -Lie algebra is a pair $(\mathfrak{a}, \theta_{\mathfrak{a}})$, such that $(\Gamma, \mathfrak{a}, \theta_{\mathfrak{a}})$ is a group Lie algebra. Γ -Lie algebras form a subcategory of group Lie algebras, where the morphisms are restricted by the condition $i_{\Gamma\Gamma} = \text{id}_{\Gamma}$.

Define a group cocommutative bialgebra as a triple (Γ, A, i) , where Γ is a group, A is a cocommutative bialgebra, $A = \bigoplus_{\gamma \in \Gamma} A_{\gamma}$ is a decomposition of A , and $i : \mathbf{k}\Gamma \rightarrow A$ is a bialgebra morphism, such that $A_{\gamma}A_{\gamma'} \subset A_{\gamma\gamma'}$, $\Delta_A(A_{\gamma}) \subset A_{\gamma}^{\otimes 2}$, and i is compatible with the Γ -grading. A morphism $(\Gamma, A, i) \rightarrow (\Gamma', A', i')$ is the data of a group morphism $i_{\Gamma\Gamma'} : \Gamma \rightarrow \Gamma'$ and a bialgebra morphism $i_{AA'} : A \rightarrow A'$, such that $i_{AA'}(A_{\gamma}) \subset A'_{i_{\Gamma\Gamma'}(\gamma)}$, and $i_{AA'} \circ i = i' \circ i_{\mathbf{k}\Gamma, \mathbf{k}\Gamma'}$ (where $i_{\mathbf{k}\Gamma, \mathbf{k}\Gamma'} : \mathbf{k}\Gamma \rightarrow \mathbf{k}\Gamma'$ is the morphism induced by $i_{\Gamma\Gamma'}$).

We then define a Γ -cocommutative bialgebra as a pair (A, i) , such that (Γ, A, i) is a group cocommutative bialgebra. Γ -cocommutative bialgebras form a category, where as before $i_{\Gamma\Gamma} = \text{id}_{\Gamma}$.

The category of group (resp., Γ -) cocommutative bialgebras contains as a full subcategory the category of group (resp., Γ -) universal enveloping algebras, where (A, Γ, i) satisfies the additional requirement that A_e is a universal enveloping algebra.

Define a group commutative bialgebra (in a symmetric monoidal category \mathcal{S}) as a triple (Γ, \mathcal{O}, j) , where Γ is a group, \mathcal{O} is a commutative algebra (in \mathcal{S}) with a decomposition $\mathcal{O} = \bigoplus_{\gamma \in \Gamma} \mathcal{O}_{\gamma}$, such that $\mathcal{O}_{\gamma}\mathcal{O}_{\gamma'} = 0$ for $\gamma \neq \gamma'$, algebra morphisms $\Delta_{\gamma'\gamma''} : \mathcal{O}_{\gamma'\gamma''} \rightarrow \mathcal{O}_{\gamma'} \otimes \mathcal{O}_{\gamma''}$,

²If A is an algebra and $u \in A$ is invertible, $\text{Ad}(u) : A \rightarrow A$ is $x \mapsto uxu^{-1}$

$\eta : \mathbf{k} \rightarrow \mathcal{O}_e$ and $\varepsilon : \mathcal{O}_e \rightarrow \mathbf{k}$, satisfying axioms such that when Γ is finite, these morphisms add up to a bialgebra structure on \mathcal{O} ; and $j : \mathcal{O} \rightarrow \mathbf{k}^\Gamma$ is a morphism of commutative algebras, compatible with the Γ -gradings and the maps $\Delta_{\gamma'\gamma''}$ on both sides. We define Γ -commutative bialgebras as above.

We define the category of group (resp., Γ -) formal series Hopf (FSH) algebras as a full subcategory of the category of group (resp., Γ -) commutative bialgebras in $\mathcal{S} = \{\text{pro-vector spaces}\}$ by the condition the \mathcal{O}_e (or equivalently, each \mathcal{O}_γ) is a formal series algebra.

Proposition 2.1. *1) We have (anti)equivalences of categories $\{\text{group Lie algebras}\} \leftrightarrow \{\text{group universal enveloping algebras}\} \leftrightarrow \{\text{group FHS algebras}\}$ (the last map is an antiequivalence).*

2) If Γ is a group, these (anti)equivalences restrict to $\{\Gamma\text{-Lie algebras}\} \leftrightarrow \{\Gamma\text{-universal enveloping algebras}\} \leftrightarrow \{\Gamma\text{-FHS algebras}\}$.

Proof. We denote the Γ -universal enveloping algebra corresponding to a Γ -Lie algebra $(\Gamma, \mathfrak{a}, \theta_\mathfrak{a})$ as $U(\mathfrak{a}) \rtimes \Gamma$. It is isomorphic to $U(\mathfrak{a}) \otimes \mathbf{k}\Gamma$ as a vector space; if we denote by $x \mapsto [x]$, $\gamma \mapsto [\gamma]$ the natural maps $\mathfrak{a} \rightarrow U(\mathfrak{a}) \rtimes \Gamma$, $\Gamma \rightarrow U(\mathfrak{a}) \rtimes \Gamma$, then the bialgebra structure of $U(\mathfrak{a}) \rtimes \Gamma$ is given by $[\gamma][x][\gamma^{-1}] = [\theta_\gamma(x)]$, $[\gamma][\gamma'] = [\gamma\gamma']$, $[e] = 1$, $[x][x'] - [x'][x] = [[x, x']]$, $\Delta([x]) = [x] \otimes 1 + 1 \otimes [x]$, $\Delta([\gamma]) = [\gamma] \otimes [\gamma]$.

When Γ is finite, the corresponding Γ -FSH algebra is then $(U(\mathfrak{a}) \rtimes \mathbf{k}\Gamma)^*$, and in general, this is $\bigoplus_{\gamma \in \Gamma} (U(\mathfrak{a}) \otimes \mathbf{k}\gamma)^*$. One checks that these are (anti)equivalences of categories. For example, if A is a group universal enveloping algebra, then one recovers Γ as $\{\text{group-like elements of } A\}$ and \mathfrak{a} as $\{\text{primitive elements of } A\}$. \square

2.2. Γ -Lie bialgebras and equivalent categories. A group Lie bialgebra is a 5-uple $(\Gamma, \mathfrak{a}, \theta_\mathfrak{a}, \delta_\mathfrak{a}, f)$ where $(\Gamma, \mathfrak{a}, \theta_\mathfrak{a})$ is a group Lie algebra, $\delta_\mathfrak{a} : \mathfrak{a} \rightarrow \wedge^2(\mathfrak{a})$ is³ such that $(\mathfrak{a}, \delta_\mathfrak{a})$ is a Lie bialgebra, and $f : \Gamma \rightarrow \wedge^2(\mathfrak{a})$ is a map $\gamma \mapsto f_\gamma$, such that: (a) $\wedge^2(\theta_\gamma) \circ \delta \circ \theta_\gamma^{-1}(x) = \delta(x) + [f_\gamma, x \otimes 1 + 1 \otimes x]$ for any $x \in \mathfrak{a}$, (b) $f_{\gamma\gamma'} = f_\gamma + \wedge^2(\theta_\gamma)(f_{\gamma'})$, and (c) $(\delta \circ \text{id})(f_\gamma) + [f_\gamma^{1,3}, f_\gamma^{2,3}] + \text{cyclic permutations} = 0$.

Group Lie bialgebras form a category, where a morphism $(\Gamma, \mathfrak{a}, \theta_\mathfrak{a}, \delta_\mathfrak{a}, f) \rightarrow (\Gamma', \mathfrak{a}', \theta_{\mathfrak{a}'}, \delta_{\mathfrak{a}'}, f')$ is a group Lie algebra morphism $(\Gamma, \mathfrak{a}, \theta_\mathfrak{a}) \rightarrow (\Gamma', \mathfrak{a}', \theta_{\mathfrak{a}'})$, such that $i_{\mathfrak{a}\mathfrak{a}'} : \mathfrak{a} \rightarrow \mathfrak{a}'$ is a Lie bialgebra morphism and $\wedge^2(i_{\mathfrak{a}\mathfrak{a}'})f_\gamma = f'_{i_{\Gamma\Gamma'}(\gamma)}$. When Γ is fixed, one defines the category of Γ -Lie bialgebras as above.

A co-Poisson structure on a group cocommutative bialgebra (Γ, A, i) is a co-Poisson structure $\delta_A : A \rightarrow \wedge^2(A)$, such that $\delta_A(A_\gamma) \subset \wedge^2(A_\gamma)$. Co-Poisson group cocommutative bialgebras form a category, where a morphism $(\Gamma, A, i, \delta_A) \rightarrow (\Gamma', A', i', \delta_{A'})$ is a morphism $(\Gamma, A, i) \rightarrow (\Gamma', A', i')$ of group cocommutative bialgebras, compatible with the co-Poisson structures. Co-Poisson group universal enveloping algebras form a full subcategory of the latter category. One defines the full subcategories of co-Poisson Γ -cocommutative bialgebras and co-Poisson Γ -enveloping algebras as above.

A Poisson structure on a group commutative bialgebra (Γ, \mathcal{O}, j) is a Poisson bialgebra structure $\{-, -\} : \wedge^2(\mathcal{O}) \rightarrow \mathcal{O}$, such that $\{\mathcal{O}_\gamma, \mathcal{O}_\gamma\} \subset \mathcal{O}_\gamma$ and $\{\mathcal{O}_\gamma, \mathcal{O}_{\gamma'}\} = 0$ if $\gamma \neq \gamma'$. Poisson group bialgebras form a category, and Poisson group FSH algebras form a full subcategory when $\mathcal{S} = \{\text{pro-vector spaces}\}$. One defines the full subcategories of Poisson Γ -bialgebras and Poisson Γ -FSH algebras as above.

Example. Let G be a Poisson-Lie (e.g., algebraic) group, let $\Gamma \subset G$ be a subgroup (which we view as an abstract group). We define $\theta_\gamma := \text{Ad}(\gamma)$, where $\text{Ad} : G \rightarrow \text{Aut}_{\text{Lie}}(\mathfrak{a})$ is the adjoint action. If $P : G \rightarrow \wedge^2(\mathfrak{a})$ is the Poisson bivector, satisfying $P(gg') = P(g) + \wedge^2(\text{Ad}(g))(P(g'))$, then we set $f_\gamma := -P(\gamma)$. Then $(\mathfrak{a}, \Gamma, f)$ is a Γ -Lie bialgebra.

³We view $\wedge^2(V)$ as a subspace of $V^{\otimes 2}$.

Example. Assume that $(\mathfrak{a}, r_{\mathfrak{a}})$ is a quasitriangular Lie bialgebra and $\theta : \Gamma \rightarrow \text{Aut}(\mathfrak{a}, t_{\mathfrak{a}})$ is an action of Γ on \mathfrak{a} by Lie algebra automorphisms preserving $t_{\mathfrak{a}} := r_{\mathfrak{a}} + r_{\mathfrak{a}}^{2,1}$. If we set $f_{\gamma} := \theta_{\gamma}^{\otimes 2}(r) - r$, then $(\mathfrak{a}, \theta, f)$ is a Γ -Lie bialgebra (we call this a quasitriangular Γ -Lie bialgebra). For example, \mathfrak{a} is a Kac-Moody Lie algebra, and $\Gamma = \tilde{W}$ is the extended Weyl group of \mathfrak{a} .

Proposition 2.2. 1) We have category (anti)equivalences $\{\text{group bialgebras}\} \leftrightarrow \{\text{co-Poisson group universal enveloping algebras}\} \leftrightarrow \{\text{Poisson group FSH algebras}\}$.

2) These restrict to category (anti)equivalences $\{\Gamma\text{-bialgebras}\} \leftrightarrow \{\text{co-Poisson } \Gamma\text{-universal enveloping algebras}\} \leftrightarrow \{\text{Poisson } \Gamma\text{-FSH algebras}\}$.

Proof. If $(\mathfrak{a}, \theta_{\mathfrak{a}}, \delta_{\mathfrak{a}})$ is a Γ -Lie bialgebra, then the co-Poisson structure on $A := U(\mathfrak{a}) \rtimes \Gamma$ is given by $\delta_A([x]) = [\delta_{\mathfrak{a}}(x)]$, and $\delta_A([\gamma]) = -[f_{\gamma}][[\gamma] \otimes [\gamma]]$. (Here we also denote by $x \mapsto [x]$ the natural map $\wedge^2(\mathfrak{a}) \rightarrow \wedge^2(U(\mathfrak{a}) \rtimes \Gamma)$.) One checks that this establishes the desired (anti)equivalences. \square

2.3. The problem of quantization of Γ -Lie bialgebras. Define a Γ -graded bialgebra (in a symmetric monoidal category \mathcal{S}) as a bialgebra A (in \mathcal{S}), equipped with a grading $A = \bigoplus_{\gamma \in \Gamma} A_{\gamma}$, such that $A_{\gamma} A_{\gamma'} \subset A_{\gamma\gamma'}$ and $\Delta_A(A_{\gamma}) \subset A_{\gamma}^{\otimes 2}$.

Assume that A is a Γ -graded bialgebra in the category of topologically free $\mathbf{k}[[\hbar]]$ -modules, quasicocommutative (in the sense that $A_0 := A/\hbar A$ is cocommutative). Then we get a co-Poisson structure on A_0 . It is Γ -graded, in the sense that $\delta_{A_0}((A_0)_{\gamma}) \subset \wedge^2((A_0)_{\gamma})$. We therefore get a classical limit functor class : $\{\Gamma\text{-graded quasicocommutative bialgebras}\} \rightarrow \{\Gamma\text{-graded co-Poisson bialgebras}\}$.

Definition 2.3. A quantization functor for Γ -Lie bialgebras is a functor $\{\text{co-Poisson } \Gamma\text{-universal enveloping algebras}\} \rightarrow \{\Gamma\text{-graded quasicocommutative bialgebras}\}$, right inverse to class.

We define the category of group-graded bialgebras as follows: objects are pairs (Γ, A) , where Γ is a group and A is a Γ -graded bialgebra. A morphism $(\Gamma, A) \rightarrow (\Gamma', A')$ is the pairs of a group morphism $i_{\Gamma\Gamma'} : \Gamma \rightarrow \Gamma'$ and a bialgebra morphism $i_{AA'} : A \rightarrow A'$, compatible with the gradings.

One defines similarly the category of group-graded co-Poisson bialgebras and quantization functors for group Lie bialgebras.

2.4. Relation with quantization of co-Poisson bialgebras. We have inclusions of full subcategories $\{\text{co-Poisson universal enveloping algebras}\} \subset \{\text{co-Poisson group universal enveloping algebras}\} \subset \{\text{co-Poisson bialgebras}\}$.

The classical limit functor is class : $\{\text{quasicocommutative bialgebras}\} \rightarrow \{\text{co-Poisson bialgebras}\}$.

A quantization functor of Lie bialgebras is a functor $\{\text{co-Poisson universal enveloping algebras}\} \rightarrow \{\text{quasicocommutative bialgebras}\}$, left inverse fo class. A quantization functor for group Lie bialgebras may then be viewed as a left inverse to class with a wider domain.

3. THE FORMALISM OF PROPS

We recall material from [EnH]. Polynomial Schur functors form a symmetric monoidal abelian category Sch , equipped with an involution. A prop P is an additive symmetric monoidal category, equipped with a tensor functor $\text{Sch} \rightarrow P$, which induces a bijection $\text{Ob}(\text{Sch}) \simeq \text{Ob}(P)$ ([McL]). A prop morphism $P \rightarrow Q$ is a tensor functor, such that the composition $\text{Sch} \rightarrow P \rightarrow Q$ coincides with $\text{Sch} \rightarrow Q$. A topological prop is defined in the same way, with Sch replaced by the category of ‘‘formal series’’ Schur functors (i.e., infinite sums of homogenous Schur functors). If F is a (formal series) Schur functor and P is a (topological) prop, then $F(P)$ is a prop defined by $(F(P))(F_1, F_2) = P(F_1 \circ F, F_2 \circ F)$.

Props may be defined by generators and relations. We will need the props \mathbf{Bialg} of bialgebras, \mathbf{LBA} of Lie bialgebras, \mathbf{LBA}_f of Lie bialgebras with a twist. Generators of \mathbf{Bialg} are $m, \Delta, \varepsilon, \eta$ (the universal analogues of the product, coproduct, counit, unit of a bialgebra); generators of \mathbf{LBA} are μ, δ (universal analogues of the Lie bracket and cobracket); \mathbf{LBA}_f has the additional generator f (universal twist element). \mathbf{LBA} and \mathbf{LBA}_f are graded (μ has degree 0 and δ, f have degree 1) and can be completed into topological props $\mathbf{LBA}, \mathbf{LBA}_f$.

We define tensor categories $\mathbf{Sch}_{(1)}$ and $\mathbf{Sch}_{(1+1)}$ by $\mathbf{Ob}(\mathbf{Sch}_{(1)}) = \prod'_{n \geq 0} \mathbf{Ob}(\mathbf{Sch}_n)$ and $\mathbf{Sch}_{(1+1)} = \prod'_{p, q \geq 0} \mathbf{Ob}(\mathbf{Sch}_{p+q})$, where $\mathbf{Ob}(\mathbf{Sch}_n)$ is the set of polynomial Schur multifunctors $\mathbf{Vect}^n \rightarrow \mathbf{Vect}$; the tensor product in these categories is denoted \boxtimes . The bifunctor $\mathbf{Sch}_{(1)}^2 \rightarrow \mathbf{Sch}_{(1+1)}$ is denoted $(F, G) \mapsto F \boxtimes G$. A multi(bi)prop is an additive symmetric monoidal category \tilde{P} with a tensor functor $\mathbf{Sch}_{(1)} \rightarrow \tilde{P}$ (resp., $\mathbf{Sch}_{(1+1)} \rightarrow \tilde{P}$), inducing a bijection on the sets of objects. A prop P give rises to a multi-prop \tilde{P} via $\tilde{P}(F, G) := P(c(F), c(G))$, where $c : \mathbf{Ob}(\mathbf{Sch}_n) \rightarrow \mathbf{Ob}(\mathbf{Sch})$ is induced by the diagonal embedding $\mathbf{Vect} \rightarrow \mathbf{Vect}^n$. We introduce the notions of a trace on a symmetric monoidal category, of a quasi-category, we show that a symmetric monoidal category with a trace and an involution gives rise to a symmetric monoidal quasi-category (i.e., the compositions are not always defined). In particular, a trace on a multi-prop gives rise to a quasi-multi-bi-prop (i.e., an additive symmetric monoidal quasi-category with a morphism from $\mathbf{Sch}_{(1+1)}$ inducing a bijection on objects). We define traces on the multi-props arising from \mathbf{LBA} and \mathbf{LBA}_f ; this gives rise to quasi-multi-bi-props $\mathbf{\Pi}, \mathbf{\Pi}_f$ with $\mathbf{\Pi}(F \boxtimes G, F' \boxtimes G') = \mathbf{LBA}(c(F) \otimes c(G')^*, c(F') \otimes c(G)^*)$; the morphisms in $\mathbf{\Pi}_f$ are defined by a similar formula. We also define topological completions $\mathbf{\Pi}, \mathbf{\Pi}_f$. When F, \dots, G' are tensor products (in $\mathbf{Sch}_{(1)}$) of irreducible Schur functors, $\mathbf{\Pi}(F \boxtimes G, F' \boxtimes G')$ is graded by a set of oriented graphs; the composition of two (or several) morphisms is defined if the composition of their diagrams is acyclic. For general $F \in \mathbf{Ob}(\mathbf{Sch}_n), \dots, G' \in \mathbf{Ob}(\mathbf{Sch}_{p'})$, one can define the support of a given element of $\mathbf{\Pi}(F \boxtimes G, F' \boxtimes G')$ (again an oriented graph), and acyclicity is a sufficient condition for the composition of two (or many) morphisms to be defined. Using this criterion, one checks that the compositions involved in the future computations all make sense.

The motivation for working with such structures is that when F, \dots, G' are tensor products (in $\mathbf{Sch}_{(1)}$) of tensor Schur functors (i.e., objects of \mathbf{Sch}_1 of the form $V \mapsto V^{\otimes n}$), $\mathbf{\Pi}(F \boxtimes G, F' \boxtimes G')$ may be viewed as a space of acyclic oriented diagrams; composition is then defined by connecting diagrams, and is of course only defined under acyclicity assumptions. When F, \dots, G' are tensor products (in $\mathbf{Sch}_{(1)}$) of simple Schur functors, the morphisms are obtained from the case of tensor products of tensor functors by applying projectors in the group algebras of products of symmetric groups, preserving a partition of the vertices.

4. COMPOSITIONS OF TWISTS

A quantization functor is a prop morphism $Q : \mathbf{Bialg} \rightarrow S(\mathbf{LBA})$ with certain classical limit properties.

Let Q be an Etingof-Kazhdan (EK) quantization functor. It is constructed as follows. We define elements $m_{\mathbf{\Pi}} \in \mathbf{\Pi}((S \boxtimes S)^{\boxtimes 2}, S \boxtimes S)$, $\Delta_0 \in \mathbf{\Pi}(S \boxtimes S, (S \boxtimes S)^{\boxtimes 2})$, $J \in \mathbf{\Pi}(1 \boxtimes 1, (S \boxtimes S)^{\boxtimes 2})$, $R_+ \in \mathbf{\Pi}(S \boxtimes 1, S \boxtimes S)$, $m_a \in \mathbf{\Pi}((S \boxtimes 1)^{\boxtimes 2}, S \boxtimes 1)$, $\Delta_a \in \mathbf{\Pi}(S \boxtimes 1, (S \boxtimes 1)^{\boxtimes 2})$.

We define $m_{\mathbf{\Pi}}^{(i,j)} \in \mathbf{\Pi}(((S \boxtimes S)^{\boxtimes j})^{\boxtimes i}, (S \boxtimes S)^{\boxtimes j})$ as the j th tensor power of the i fold iterate of $m_{\mathbf{\Pi}}$.

We have

$$m_{\mathbf{\Pi}} \circ R_+^{\boxtimes 2} = R_+ \circ m_a, \quad m_{\mathbf{\Pi}}^{(2,2)} \circ (J \boxtimes (\Delta_0 \circ R_+)) = m_{\mathbf{\Pi}}^{(2,2)} \circ ((R_+^{\boxtimes 2} \circ \Delta_a) \boxtimes J).$$

Then m_a, Δ_a satisfy the bialgebra relations. The functor Q is defined by $m \mapsto m_a, \Delta \mapsto \Delta_a$.

We now recall the results from [EnH] on the quantization of twists. We define prop morphisms $\kappa_i : \text{LBA} \rightarrow \text{LBA}_f$ ($i = 1, 2$) by $\kappa_1 : (\mu, \delta) \mapsto (\mu, \delta)$ and $\kappa_2 : (\mu, \delta) \mapsto (\mu, \delta + \text{ad}(f))$.

Define $\Xi_f \in \mathbf{\Pi}_f(S \boxtimes S, S \boxtimes S)^\times$ as the universal version of the sequence of maps $S(\mathfrak{a}) \otimes S(\mathfrak{a}^*) \simeq U(D(\mathfrak{a})) \simeq U(D(\mathfrak{a}_f)) \simeq S(\mathfrak{a}) \otimes S(\mathfrak{a}^*)$, based on the Lie algebra isomorphism $D(\mathfrak{a}) \simeq D(\mathfrak{a}_f)$. Then

$$\kappa_2^\Pi(m_\Pi) = \Xi_f \circ \kappa_1^\Pi(m_\Pi) \circ (\Xi_f^{-1})^{\boxtimes 2}, \quad \kappa_2^\Pi(\Delta_0) = \Xi_f^{\boxtimes 2} \circ \kappa_1^\Pi(\Delta_0) \circ \Xi_f^{-1}, \quad (1)$$

Proposition 4.1. (see [EnH]) *There exists (F, v, i) with $F \in \mathbf{\Pi}_f(\mathbf{1} \boxtimes \mathbf{1}, (S \boxtimes \mathbf{1})^{\boxtimes 2})$, $v \in \mathbf{\Pi}_f(\mathbf{1} \boxtimes \mathbf{1}, S \boxtimes S)$, and $i \in \mathbf{\Pi}_f(S \boxtimes \mathbf{1}, S \boxtimes \mathbf{1})^\times$, such that $F = 1 + \text{degree} > 0$, $v = 1 + \text{degree} > 0$, $i = \text{id}_{S \boxtimes \mathbf{1}} + \text{degree} > 0$,*

$$\kappa_1^\Pi(m_\Pi^{(2,2)}) \circ \left(((\Xi_f^{-1})^{\boxtimes 2} \circ \kappa_2^\Pi(J)) \boxtimes (\Delta_0 \circ v) \right) = \kappa_1^\Pi(m_\Pi^{(3,2)}) \circ \left((v \boxtimes v) \boxtimes (\kappa_1^\Pi(R_+)^{\boxtimes 2} \circ F) \boxtimes (\kappa_1^\Pi(J)) \right) \quad (2)$$

and

$$\kappa_1^\Pi(m_\Pi) \circ \left((\Xi_f^{-1} \circ \kappa_2^\Pi(R_+) \circ i) \boxtimes v \right) = \kappa_1^\Pi(m_\Pi) \circ \left(v \boxtimes \kappa_1^\Pi(R_+) \right). \quad (3)$$

The set of triples (F', v', i') satisfying these relations is given by $v' = \kappa_1^\Pi(m_\Pi) \circ \left(v \boxtimes (\kappa_1^\Pi(R_+) \circ u) \right)$, $m_a^{(2,2)} \circ \left(F' \boxtimes (\Delta \circ u) \right) = m_a^{(2,2)} \circ \left(u^{\boxtimes 2} \boxtimes F \right)$, $v' = \kappa_1^\Pi \left(v \boxtimes (\kappa_1^\Pi(R_+) \circ u) \right)$, $i' = i \circ \underline{\text{Ad}}(u)$, where $u \in \mathbf{\Pi}_f(\mathbf{1} \boxtimes \mathbf{1}, S \boxtimes \mathbf{1})$ has the form $u = 1 + \text{degree} > 0$, and $\underline{\text{Ad}}(u) \in \mathbf{\Pi}_f(S \boxtimes \mathbf{1}, S \boxtimes \mathbf{1})^\times$ is such that $m_a \circ (\underline{\text{Ad}}(u) \boxtimes u) = m_a \circ (u \boxtimes \text{id}_{S \boxtimes \mathbf{1}})$.

In [EnH], we prove that this proposition has the following consequence:

Theorem 4.2. (Compatibility of quantization functors with twists) *We have*

$$\kappa_2^\Pi(m_a) = i \circ \kappa_1^\Pi(m_a) \circ (i^{-1})^{\boxtimes 2}, \quad \kappa_1^\Pi(m_a^{(2,2)}) \circ \left(((i^{-1})^{\boxtimes 2} \circ \kappa_2^\Pi(\Delta_a) \circ i) \boxtimes F \right) = \kappa_1^\Pi(m_a^{(2,2)}) \circ \left(F \boxtimes \kappa_1^\Pi(\Delta_a) \right),$$

$$\kappa_1^\Pi(m_a^{(3,2)}) \circ \left((F \boxtimes \mathbf{1}) \boxtimes ((\Delta_a \boxtimes \text{id}_{S \boxtimes \mathbf{1}}) \circ F) \right) = \kappa_1^\Pi(m_a^{(3,2)}) \circ \left((\mathbf{1} \boxtimes F) \boxtimes ((\text{id}_{S \boxtimes \mathbf{1}} \boxtimes \Delta_a) \circ F) \right).$$

As before, $m_a^{(i,j)}$ is the j th tensor power of the i fold iterate to m_a .

We will now study the behavior of the composition of twists under quantization.

Define a prop $\text{LBA}_{f,f'}$ by generators $\mu \in \text{LBA}_{f,f'}(\wedge^2, \mathbf{id})$, $\delta \in \text{LBA}_{f,f'}(\mathbf{id}, \wedge^2)$, $f, f' \in \text{LBA}_{f,f'}(\mathbf{1}, \wedge^2)$ and relations: μ, δ, f satisfy the relations of LBA_f , and f' is such that

$$((123) + (231) + (312)) \circ \left((\delta \boxtimes \text{id}_{\mathbf{id}}) \circ f' + (\mu \boxtimes \text{id}_{T_2}) \circ ((1234) + (1324)) \circ (f \boxtimes f') \right) = 0. \quad (4)$$

Define prop morphisms $\kappa_{ij} : \text{LBA}_f \rightarrow \text{LBA}_{f,f'}$, by $\kappa_{12} : (\mu, \delta, f) \mapsto (\mu, \delta, f)$, $\kappa_{23} : (\mu, \delta, f) \mapsto (\mu, \delta + \text{ad}(f), f')$, $\kappa_{13} : (\mu, \delta, f) \mapsto (\mu, \delta, f + f')$.

Define prop morphisms $\bar{\kappa}_1, \bar{\kappa}_2, \bar{\kappa}_3 : \text{LBA} \rightarrow \text{LBA}_{f,f'}$, $(\mu, \delta) \mapsto (\mu, \delta)$, $(\mu, \delta + \text{ad}(f))$, $(\mu, \delta + \text{ad}(f + f'))$.

Then we have $\kappa_{1i} \circ \kappa_1 = \bar{\kappa}_1$, $\kappa_{i3} \circ \kappa_2 = \bar{\kappa}_3$, $\kappa_{23} \circ \kappa_1 = \kappa_{12} \circ \kappa_2 = \bar{\kappa}_2$.

Lemma 4.3. *We have*

$$\kappa_{23}^\Pi(\Xi_f) \circ \kappa_{12}^\Pi(\Xi_f) = \kappa_{13}^\Pi(\Xi_f). \quad (5)$$

Proof. This follows from the fact that if f_a is a twist for \mathfrak{a} and f'_a is a twist for \mathfrak{a}_{f_a} , then $f_a + f'_a$ is a twist for \mathfrak{a} , and $(\mathfrak{a}_{f_a})_{f'_a} \simeq (\mathfrak{a}_{f_a})_{f'_a}$. \square

Theorem 4.4. *There exists $v \in \Pi_{f,f'}(\mathbf{1} \boxtimes \mathbf{1}, S \boxtimes \mathbf{1})$, such that $v = 1 + \text{degree} > 0$,*

$$\bar{\kappa}_1^\Pi(m_a^{(2,2)}) \circ \left(\kappa_{13}^\Pi(F) \boxtimes (\bar{\kappa}_1^\Pi(\Delta_a) \circ v) \right) = \bar{\kappa}_1^\Pi(m_a^{(3,2)}) \circ \left(v^{\boxtimes 2} \boxtimes (\kappa_{12}^\Pi(i^{-1})^{\boxtimes 2} \circ \kappa_{23}^\Pi(F)) \boxtimes \kappa_{12}^\Pi(F) \right), \quad (6)$$

and

$$\bar{\kappa}_1^\Pi(m_a) \circ \left(v \boxtimes (\kappa_{12}^\Pi(i^{-1}) \circ \kappa_{23}^\Pi(i^{-1})) \right) = \bar{\kappa}_1^\Pi(m_a) \circ \left(\kappa_{13}^\Pi(i^{-1}) \boxtimes v \right), \quad (7)$$

Proof. Let us prove (6). Applying κ_{23}^Π to (2), we get

$$\begin{aligned} & \bar{\kappa}_2^\Pi(m_\Pi^{(2,2)}) \circ \left((\kappa_{23}^\Pi(\Xi_f^{-1})^{\boxtimes 2} \circ \bar{\kappa}_3^\Pi(J)) \boxtimes (\Delta_0 \circ \kappa_{23}^\Pi(v)) \right) \\ &= \bar{\kappa}_2^\Pi(m_\Pi^{(3,2)}) \circ \left((\kappa_{23}^\Pi(v)^{\boxtimes 2}) \boxtimes (\bar{\kappa}_2^\Pi(R_+)^{\boxtimes 2} \circ \kappa_{23}^\Pi(F)) \boxtimes (\bar{\kappa}_2^\Pi(J)) \right). \end{aligned} \quad (8)$$

(1) implies the identities

$$\bar{\kappa}_2^\Pi(m_\Pi) = \kappa_{12}^\Pi(\Xi_f) \circ \bar{\kappa}_1^\Pi(m_\Pi) \circ \kappa_{12}^\Pi(\Xi_f^{-1})^{\boxtimes 2}, \quad \bar{\kappa}_2^\Pi(\Delta_0) = \kappa_{12}^\Pi(\Xi_f)^{\boxtimes 2} \circ \bar{\kappa}_1^\Pi(\Delta_0) \circ \kappa_{12}^\Pi(\Xi_f^{-1}). \quad (9)$$

Left composing (8) with $\kappa_{12}^\Pi(\Xi_f^{-1})^{\boxtimes 2}$ and using these identities, we get

$$\begin{aligned} & \bar{\kappa}_1^\Pi(m_\Pi^{(2,2)}) \circ \left(([\kappa_{12}^\Pi(\Xi_f^{-1}) \circ \kappa_{23}^\Pi(\Xi_f^{-1})]^{\boxtimes 2} \circ \bar{\kappa}_3^\Pi(J)) \boxtimes (\Delta_0 \circ \kappa_{12}^\Pi(\Xi_f^{-1}) \circ \kappa_{23}^\Pi(v)) \right) \\ &= \kappa_{12}^\Pi(\Xi_f^{-1})^{\boxtimes 2} \circ \bar{\kappa}_2^\Pi(m_\Pi^{(3,2)}) \circ \left((\kappa_{23}^\Pi(v)^{\boxtimes 2}) \boxtimes (\bar{\kappa}_2^\Pi(R_+)^{\boxtimes 2} \circ \kappa_{23}^\Pi(F)) \boxtimes (\bar{\kappa}_2^\Pi(J)) \right). \end{aligned}$$

Applying κ_{23}^Π to (3), we get

$$\bar{\kappa}_2^\Pi(m_\Pi) \circ \left(\kappa_{23}^\Pi(v) \boxtimes (\bar{\kappa}_2^\Pi(R_+)) \right) = \bar{\kappa}_2^\Pi(m_\Pi) \circ \left((\kappa_{23}^\Pi(\Xi_f^{-1}) \circ \bar{\kappa}_3^\Pi(R_+) \circ \kappa_{23}^\Pi(i)) \boxtimes \kappa_{23}^\Pi(v) \right), \quad (10)$$

and applying κ_{12}^Π to (3), we get

$$\bar{\kappa}_1^\Pi(m_\Pi) \circ \left(\kappa_{12}^\Pi(v) \boxtimes (\bar{\kappa}_1^\Pi(R_+)) \right) = \bar{\kappa}_1^\Pi(m_\Pi) \circ \left((\kappa_{12}^\Pi(\Xi_f^{-1}) \circ \bar{\kappa}_2^\Pi(R_+) \circ \kappa_{12}^\Pi(i)) \boxtimes \kappa_{12}^\Pi(v) \right). \quad (11)$$

(10) then implies

$$\begin{aligned} & \bar{\kappa}_1^\Pi(m_\Pi^{(2,2)}) \circ \left(([\kappa_{12}^\Pi(\Xi_f^{-1}) \circ \kappa_{23}^\Pi(\Xi_f^{-1})]^{\boxtimes 2} \circ \bar{\kappa}_3^\Pi(J)) \boxtimes (\Delta_0 \circ \kappa_{12}^\Pi(\Xi_f^{-1}) \circ \kappa_{23}^\Pi(v)) \right) \\ &= \kappa_{12}^\Pi(\Xi_f^{-1})^{\boxtimes 2} \circ \bar{\kappa}_2^\Pi(m_\Pi^{(3,2)}) \circ \left((\kappa_{23}^\Pi(\Xi_f^{-1})^{\boxtimes 2} \circ \bar{\kappa}_3^\Pi(R_+)^{\boxtimes 2} \circ \kappa_{23}^\Pi(i)^{\boxtimes 2} \circ \kappa_{23}^\Pi(F)) \boxtimes (\kappa_{23}^\Pi(v)^{\boxtimes 2}) \boxtimes \bar{\kappa}_2^\Pi(J) \right) \\ &= \bar{\kappa}_1^\Pi(m_\Pi^{(3,2)}) \circ \left((\kappa_{12}^\Pi(\Xi_f^{-1}) \circ \kappa_{23}^\Pi(\Xi_f^{-1}) \circ \bar{\kappa}_3^\Pi(R_+) \circ \kappa_{23}^\Pi(i))^{\boxtimes 2} \circ \kappa_{23}^\Pi(F) \right) \\ & \boxtimes (\kappa_{12}^\Pi(\Xi_f^{-1}) \circ \kappa_{23}^\Pi(v))^{\boxtimes 2} \boxtimes (\kappa_{12}^\Pi(\Xi_f^{-1})^{\boxtimes 2} \circ \bar{\kappa}_2^\Pi(J)). \end{aligned}$$

Applying κ_{12}^Π to (2), we get

$$\begin{aligned} & \bar{\kappa}_1^\Pi(m_\Pi^{(2,2)}) \circ \left((\kappa_{12}^\Pi(\Xi_f^{-1})^{\boxtimes 2} \circ \bar{\kappa}_2^\Pi(J)) \boxtimes \kappa_{12}^\Pi(\Delta_0 \circ v) \right) \\ &= \bar{\kappa}_1^\Pi(m_\Pi^{(3,2)}) \circ \left(\kappa_{12}^\Pi(v)^{\boxtimes 2} \boxtimes (\bar{\kappa}_1^\Pi(R_+)^{\boxtimes 2} \circ \kappa_{12}^\Pi(F)) \boxtimes \bar{\kappa}_1^\Pi(J) \right). \end{aligned} \quad (12)$$

Therefore "right multiplication" (using m_Π) of the previous identity by $\kappa_{12}^\Pi(\Delta_0 \circ v)$ yields

$$\begin{aligned}
& \bar{\kappa}_1^\Pi(m_\Pi^{(3,2)}) \circ \left(([\kappa_{12}^\Pi(\Xi_f^{-1}) \circ \kappa_{23}^\Pi(\Xi_f^{-1})]^{\boxtimes 2} \circ \bar{\kappa}_3^\Pi(\mathbf{J})) \boxtimes (\Delta_0 \circ \kappa_{12}^\Pi(\Xi_f^{-1}) \circ \kappa_{23}^\Pi(v)) \boxtimes \kappa_{12}^\Pi(\Delta_0 \circ v) \right) \\
&= \bar{\kappa}_1^\Pi(m_\Pi^{(4,2)}) \circ \left(\left((\kappa_{12}^\Pi(\Xi_f^{-1}) \circ \kappa_{23}^\Pi(\Xi_f^{-1}) \circ \bar{\kappa}_3^\Pi(\mathbf{R}_+) \circ \kappa_{23}^\Pi(\mathbf{i}))^{\boxtimes 2} \circ \kappa_{23}^\Pi(\mathbf{F}) \right) \right. \\
&\quad \left. \boxtimes (\kappa_{12}^\Pi(\Xi_f^{-1}) \circ \kappa_{23}^\Pi(v))^{\boxtimes 2} \boxtimes (\kappa_{12}^\Pi(\Xi_f^{-1})^{\boxtimes 2} \circ \bar{\kappa}_2^\Pi(\mathbf{J})) \boxtimes \kappa_{12}^\Pi(\Delta_0 \circ v) \right) \\
&= \bar{\kappa}_1^\Pi(m_\Pi^{(5,2)}) \circ \left(\left((\kappa_{12}^\Pi(\Xi_f^{-1}) \circ \kappa_{23}^\Pi(\Xi_f^{-1}) \circ \bar{\kappa}_3^\Pi(\mathbf{R}_+) \circ \kappa_{23}^\Pi(\mathbf{i}))^{\boxtimes 2} \circ \kappa_{23}^\Pi(\mathbf{F}) \right) \right. \\
&\quad \left. \boxtimes (\kappa_{12}^\Pi(\Xi_f^{-1}) \circ \kappa_{23}^\Pi(v))^{\boxtimes 2} \boxtimes \kappa_{12}^\Pi(v)^{\boxtimes 2} \boxtimes (\bar{\kappa}_1^\Pi(\mathbf{R}_+)^{\boxtimes 2} \circ \kappa_{12}^\Pi(\mathbf{F})) \boxtimes \bar{\kappa}_1^\Pi(\mathbf{J}) \right),
\end{aligned}$$

where the last equality follows from (12).

According to (9), the last term is equal to

$$\begin{aligned}
& \kappa_{12}^\Pi(\Xi_f^{-1}) \circ \bar{\kappa}_2^\Pi(m_\Pi^{(5,2)}) \circ \left(\left((\kappa_{23}^\Pi(\Xi_f^{-1}) \circ \bar{\kappa}_3^\Pi(\mathbf{R}_+) \circ \kappa_{23}^\Pi(\mathbf{i}))^{\boxtimes 2} \circ \kappa_{23}^\Pi(\mathbf{F}) \right) \right. \\
&\quad \left. \boxtimes (\kappa_{23}^\Pi(v))^{\boxtimes 2} \boxtimes (\kappa_{12}^\Pi(\Xi_f) \circ \kappa_{12}^\Pi(v))^{\boxtimes 2} \boxtimes (\kappa_{12}^\Pi(\Xi_f)^{\boxtimes 2} \circ \bar{\kappa}_1^\Pi(\mathbf{R}_+)^{\boxtimes 2} \circ \kappa_{12}^\Pi(\mathbf{F})) \boxtimes \kappa_{12}^\Pi(\Xi_f)^{\boxtimes 2} \circ \bar{\kappa}_1^\Pi(\mathbf{J}) \right),
\end{aligned}$$

which according to (10) is equal to

$$\begin{aligned}
& \kappa_{12}^\Pi(\Xi_f^{-1}) \circ \bar{\kappa}_2^\Pi(m_\Pi^{(5,2)}) \circ \left((\kappa_{23}^\Pi(v))^{\boxtimes 2} \boxtimes (\bar{\kappa}_2^\Pi(\mathbf{R}_+)^{\boxtimes 2} \circ \kappa_{23}^\Pi(\mathbf{F})) \right. \\
&\quad \left. \boxtimes (\kappa_{12}^\Pi(\Xi_f) \circ \kappa_{12}^\Pi(v))^{\boxtimes 2} \boxtimes (\kappa_{12}^\Pi(\Xi_f)^{\boxtimes 2} \circ \bar{\kappa}_1^\Pi(\mathbf{R}_+)^{\boxtimes 2} \circ \kappa_{12}^\Pi(\mathbf{F})) \boxtimes \kappa_{12}^\Pi(\Xi_f)^{\boxtimes 2} \circ \bar{\kappa}_1^\Pi(\mathbf{J}) \right),
\end{aligned}$$

which we rewrite as

$$\begin{aligned}
& \bar{\kappa}_1^\Pi(m_\Pi^{(5,2)}) \circ \left((\kappa_{12}^\Pi(\Xi_f^{-1})^{\boxtimes 2} \circ \kappa_{23}^\Pi(v)^{\boxtimes 2}) \boxtimes (\kappa_{12}^\Pi(\Xi_f^{-1})^{\boxtimes 2} \circ \bar{\kappa}_2^\Pi(\mathbf{R}_+)^{\boxtimes 2} \circ \kappa_{23}^\Pi(\mathbf{F})) \right. \\
&\quad \left. \boxtimes (\kappa_{12}^\Pi(v))^{\boxtimes 2} \boxtimes (\bar{\kappa}_1^\Pi(\mathbf{R}_+)^{\boxtimes 2} \circ \kappa_{12}^\Pi(\mathbf{F})) \boxtimes \bar{\kappa}_1^\Pi(\mathbf{J}) \right).
\end{aligned}$$

(11) allows then to rewrite this as

$$\begin{aligned}
& \bar{\kappa}_1^\Pi(m_\Pi^{(5,2)}) \circ \left((\kappa_{12}^\Pi(\Xi_f^{-1})^{\boxtimes 2} \circ \kappa_{23}^\Pi(v)^{\boxtimes 2}) \boxtimes \kappa_{12}^\Pi(v)^{\boxtimes 2} \right. \\
&\quad \left. \boxtimes (\bar{\kappa}_1^\Pi(\mathbf{R}_+)^{\boxtimes 2} \circ \kappa_{12}^\Pi(\mathbf{i}^{-1})^{\boxtimes 2} \circ \kappa_{23}^\Pi(\mathbf{F})) \boxtimes (\bar{\kappa}_1^\Pi(\mathbf{R}_+)^{\boxtimes 2} \circ \kappa_{12}^\Pi(\mathbf{F})) \boxtimes \bar{\kappa}_1^\Pi(\mathbf{J}) \right).
\end{aligned}$$

We therefore get:

$$\bar{\kappa}_1^\Pi(m_\Pi^{(2,2)}) \circ \left(((\kappa_{12}^\Pi(\Xi_f^{-1}) \circ \kappa_{23}^\Pi(\Xi_f^{-1}))^{\boxtimes 2} \circ \bar{\kappa}_3^\Pi(\mathbf{J})) \boxtimes (\Delta_0 \circ v_1) \right) = \bar{\kappa}_1^\Pi(m_\Pi^{(3,2)}) \circ \left(v_1^{\boxtimes 2} \boxtimes F_1 \boxtimes \bar{\kappa}_1^\Pi(\mathbf{J}) \right), \tag{13}$$

where

$$\begin{aligned}
v_1 &= \bar{\kappa}_1^\Pi(m_\Pi) \circ \left((\kappa_{12}^\Pi(\Xi_f^{-1}) \circ \kappa_{23}^\Pi(v)) \boxtimes \kappa_{12}^\Pi(v) \right) \\
F_1 &= \bar{\kappa}_1^\Pi(m_\Pi^{(2,2)}) \circ \left((\bar{\kappa}_1^\Pi(\mathbf{R}_+)^{\boxtimes 2} \circ \kappa_{12}^\Pi(\mathbf{i}^{-1})^{\boxtimes 2} \circ \kappa_{23}^\Pi(\mathbf{F})) \boxtimes (\bar{\kappa}_1^\Pi(\mathbf{R}_+)^{\boxtimes 2} \circ \kappa_{12}^\Pi(\mathbf{F})) \right) \\
&= \bar{\kappa}_1^\Pi(\mathbf{R}_+)^{\boxtimes 2} \circ \bar{\kappa}_1^\Pi(m_a^{(2,2)}) \circ \left((\kappa_{12}^\Pi(\mathbf{i}^{-1})^{\boxtimes 2} \circ \kappa_{23}^\Pi(\mathbf{F})) \boxtimes \kappa_{12}^\Pi(\mathbf{F}) \right) = \bar{\kappa}_1^\Pi(\mathbf{R}_+)^{\boxtimes 2} \circ F_1,
\end{aligned}$$

where

$$F_1 = \bar{\kappa}_1^\Pi(m_a^{(2,2)}) \circ \left((\kappa_{12}^\Pi(\mathbf{i}^{-1})^{\boxtimes 2} \circ \kappa_{23}^\Pi(\mathbf{F})) \boxtimes \kappa_{12}^\Pi(\mathbf{F}) \right).$$

(5) implies that (13) is rewritten as

$$\bar{\kappa}_1^\Pi(m_\Pi^{(2,2)}) \circ \left((\kappa_{13}^\Pi(\Xi_f^{-1})^{\boxtimes 2} \circ \bar{\kappa}_3^\Pi(\mathbf{J})) \boxtimes (\Delta_0 \circ v_1) \right) = \bar{\kappa}_1^\Pi(m_\Pi^{(3,2)}) \circ \left(v_1^{\boxtimes 2} \boxtimes (\bar{\kappa}_1^\Pi(\mathbf{R}_+)^{\boxtimes 2} \circ F_1) \boxtimes \bar{\kappa}_1^\Pi(\mathbf{J}) \right). \tag{14}$$

On the other hand, applying κ_{13}^Π to (2), we get

$$\bar{\kappa}_1^\Pi(m_\Pi^{(2,2)}) \circ \left((\kappa_{13}^\Pi(\Xi_f^{-1})^{\boxtimes 2} \circ \bar{\kappa}_3^\Pi(J)) \boxtimes (\Delta_0 \circ v'_1) \right) = \bar{\kappa}_1^\Pi(m_\Pi^{(3,2)}) \circ \left((v'_1)^{\boxtimes 2} \boxtimes (\bar{\kappa}_1^\Pi(\mathbf{R}_+)^{\boxtimes 2} \circ F'_1) \boxtimes \bar{\kappa}_1^\Pi(J) \right), \quad (15)$$

where

$$v'_1 = \kappa_{13}^\Pi(v), \quad F'_1 = \kappa_{13}^\Pi(F),$$

where $F'_1 = \kappa_{13}^\Pi(F)$.

The result of uniqueness (up to gauge) for solutions $(F, v) \in \Pi_f(\mathbf{1} \boxtimes \mathbf{1}, S \boxtimes S)^\times \times \Pi_f(\mathbf{1} \boxtimes \mathbf{1}, S \boxtimes \mathbf{1})^\times$ of equation (2), which was established in [EnH], Lemma 5.3, can be generalized as follows.

Lemma 4.5. *The set of pairs (F'_1, v'_1) satisfying (14), where $F'_1 \in \Pi_{f,f'}(\mathbf{1} \boxtimes \mathbf{1}, S \boxtimes \mathbf{1})$, $v'_1 \in \Pi_{f,f'}(\mathbf{1} \boxtimes \mathbf{1}, S \boxtimes S)$, $F'_1 = 1 + \text{degree} > 1$, $v'_1 = 1 + \text{degree} > 1$, is given by $v_1 = \bar{\kappa}_1^\Pi(m_\Pi) \circ \left(v''_1 \boxtimes (\bar{\kappa}_1^\Pi(\mathbf{R}_+) \circ v'' \right)$, $\bar{\kappa}_1^\Pi(m_a^{(2,2)}) \circ \left(F'_1 \boxtimes (\bar{\kappa}_1^\Pi(\Delta_a) \circ v'' \right) = \bar{\kappa}_1^\Pi(m_a^{(2,2)}) \circ \left(v''^{\boxtimes 2} \boxtimes F_1 \right)$, where $v'' \in \Pi_{f,f'}(\mathbf{1} \boxtimes \mathbf{1}, S \boxtimes \mathbf{1})$, $v'' = 1 + \text{degree} > 0$.*

The proof is parallel to that of [EnH], Lemma 5.3. The computation of the co-Hochschild cohomology of $(\mathbf{U}_{n,f})_{n \geq 0}$ is replaced by that of $(\mathbf{U}_{n,f,f'})_{n \geq 0}$, where $\mathbf{U}_{f,f',n} = \Pi_{f,f'}(\mathbf{1} \boxtimes \mathbf{1}, (S \boxtimes S)^{\boxtimes n})$, and the argument of the vanishing of $\text{LBA}_f(\mathbf{id}, \mathbf{1})$ is replaced by the vanishing of $\text{LBA}_{f,f'}(\mathbf{id}, \mathbf{1})$.

It follows that there exists $v \in \Pi_{f,f'}(\mathbf{1} \boxtimes \mathbf{1}, S \boxtimes \mathbf{1})$, $v = 1 + \text{degree} > 0$, such that

$$v_1 = \bar{\kappa}_1^\Pi(m_\Pi) \circ \left(v'_1 \boxtimes (\bar{\kappa}_1^\Pi(\mathbf{R}_+) \circ v) \right), \quad \bar{\kappa}_1^\Pi(m_a^{(2,2)}) \circ \left(F'_1 \boxtimes (\bar{\kappa}_1^\Pi(\Delta_a) \circ v) \right) = \bar{\kappa}_1^\Pi(m_a^{(2,2)}) \circ \left(v^{\boxtimes 2} \boxtimes F_1 \right). \quad (16)$$

The second of these identities is (6).

Let us now prove (7). Right composing (3) with i^{-1} , applying κ_{23}^Π , left composing with $\kappa_{12}^\Pi(\Xi_f^{-1})$, and right multiplying the resulting identity by $\kappa_{12}^\Pi(v)$ using $\bar{\kappa}_1^\Pi(m_\Pi)$, we get

$$\begin{aligned} & \bar{\kappa}_1^\Pi(m_\Pi^{(3,1)}) \circ \left((\kappa_{13}^\Pi(\Xi_f^{-1}) \circ \bar{\kappa}_3^\Pi(\mathbf{R}_+)) \boxtimes (\kappa_{12}^\Pi(\Xi_f^{-1}) \circ \kappa_{23}^\Pi(v)) \boxtimes \kappa_{12}^\Pi(v) \right) \\ &= \bar{\kappa}_1^\Pi(m_\Pi^{(3,1)}) \circ \left((\kappa_{12}^\Pi(\Xi_f^{-1}) \circ \kappa_{23}^\Pi(v)) \boxtimes (\kappa_{12}^\Pi(\Xi_f^{-1}) \circ \bar{\kappa}_2^\Pi(\mathbf{R}_+) \circ \kappa_{23}^\Pi(i^{-1})) \boxtimes \kappa_{12}^\Pi(v) \right). \end{aligned} \quad (17)$$

Right composing (3) by i^{-1} , applying κ_{12}^Π , right composing with $\kappa_{23}^\Pi(i^{-1})$, and left multiplying by $\kappa_{12}^\Pi(\Xi_f^{-1}) \circ \kappa_{23}^\Pi(v)$ using $\bar{\kappa}_1^\Pi(m_\Pi)$, we get

$$\begin{aligned} & \bar{\kappa}_1^\Pi(m_\Pi^{(3,1)}) \circ \left((\kappa_{12}^\Pi(\Xi_f^{-1}) \circ \kappa_{23}^\Pi(v)) \boxtimes (\kappa_{12}^\Pi(\Xi_f^{-1}) \circ \bar{\kappa}_2^\Pi(\mathbf{R}_+) \circ \kappa_{23}^\Pi(i^{-1})) \boxtimes \kappa_{12}^\Pi(v) \right) \\ &= \bar{\kappa}_1^\Pi(m_\Pi^{(3,1)}) \circ \left((\kappa_{12}^\Pi(\Xi_f^{-1}) \circ \kappa_{23}^\Pi(v)) \boxtimes \kappa_{12}^\Pi(v) \boxtimes (\bar{\kappa}_1^\Pi(\mathbf{R}_+) \circ \kappa_{12}^\Pi(i^{-1}) \circ \kappa_{23}^\Pi(i^{-1})) \right). \end{aligned} \quad (18)$$

The first identity of (16) is rewritten as

$$\bar{\kappa}_1^\Pi(m_\Pi) \circ \left((\kappa_{12}^\Pi(\Xi_f^{-1}) \circ \kappa_{23}^\Pi(v)) \boxtimes \kappa_{12}^\Pi(v) \right) = \bar{\kappa}_1^\Pi(m_\Pi) \circ \left(\kappa_{13}^\Pi(v) \boxtimes (\bar{\kappa}_1^\Pi(\mathbf{R}_+) \circ v) \right), \quad (19)$$

therefore

$$\begin{aligned} & \bar{\kappa}_1^\Pi(m_\Pi^{(3,1)}) \circ \left((\kappa_{12}^\Pi(\Xi_f^{-1}) \circ \kappa_{23}^\Pi(v)) \boxtimes \kappa_{12}^\Pi(v) \boxtimes (\bar{\kappa}_1^\Pi(\mathbf{R}_+) \circ \kappa_{12}^\Pi(i^{-1}) \circ \kappa_{23}^\Pi(i^{-1})) \right) \\ &= \bar{\kappa}_1^\Pi(m_\Pi^{(3,1)}) \circ \left(\kappa_{13}^\Pi(v) \boxtimes (\bar{\kappa}_1^\Pi(\mathbf{R}_+) \circ v) \boxtimes (\bar{\kappa}_1^\Pi(\mathbf{R}_+) \circ \kappa_{12}^\Pi(i^{-1}) \circ \kappa_{23}^\Pi(i^{-1})) \right). \end{aligned} \quad (20)$$

Combining (17), (18) and (20), we get:

$$\begin{aligned} & \bar{\kappa}_1^\Pi(m_\Pi^{(3,1)}) \circ \left((\kappa_{13}^\Pi(\Xi_f^{-1}) \circ \bar{\kappa}_3^\Pi(\mathbf{R}_+)) \boxtimes (\kappa_{12}^\Pi(\Xi_f^{-1}) \circ \kappa_{23}^\Pi(v)) \boxtimes \kappa_{12}^\Pi(v) \right) \\ & = \bar{\kappa}_1^\Pi(m_\Pi) \circ \left(\kappa_{13}^\Pi(v) \boxtimes [\bar{\kappa}_1^\Pi(\mathbf{R}_+) \circ \bar{\kappa}_1^\Pi(m_a) \circ (\mathbf{v} \boxtimes (\kappa_{12}^\Pi(i^{-1}) \circ \kappa_{23}^\Pi(i^{-1})))] \right). \end{aligned} \quad (21)$$

On the other hand, left multiplying (19) by $\kappa_{13}^\Pi(\Xi_f^{-1}) \circ \bar{\kappa}_3^\Pi(\mathbf{R}_+)$ using $\bar{\kappa}_1^\Pi(m_\Pi)$, we get

$$\begin{aligned} & \bar{\kappa}_1^\Pi(m_\Pi^{(3,1)}) \circ \left((\kappa_{13}^\Pi(\Xi_f^{-1}) \circ \bar{\kappa}_3^\Pi(\mathbf{R}_+)) \boxtimes (\kappa_{12}^\Pi(\Xi_f^{-1}) \circ \kappa_{23}^\Pi(v)) \boxtimes \kappa_{12}^\Pi(v) \right) \\ & = \bar{\kappa}_1^\Pi(m_\Pi^{(3,1)}) \circ \left((\kappa_{13}^\Pi(\Xi_f^{-1}) \circ \bar{\kappa}_3^\Pi(\mathbf{R}_+)) \boxtimes \kappa_{13}^\Pi(v) \boxtimes (\bar{\kappa}_1^\Pi(\mathbf{R}_+) \circ \mathbf{v}) \right). \end{aligned} \quad (22)$$

Right composing (3) by i^{-1} , applying κ_{13}^Π and right multiplying by $\bar{\kappa}_1^\Pi(\mathbf{R}_+) \circ \mathbf{v}$ using $\bar{\kappa}_1^\Pi(m_\Pi)$, we get

$$\begin{aligned} & \bar{\kappa}_1^\Pi(m_\Pi^{(3,1)}) \circ \left((\kappa_{13}^\Pi(\Xi_f^{-1}) \circ \bar{\kappa}_3^\Pi(\mathbf{R}_+)) \boxtimes \kappa_{13}^\Pi(v) \boxtimes (\bar{\kappa}_1^\Pi(\mathbf{R}_+) \circ \mathbf{v}) \right) \\ & = \bar{\kappa}_1^\Pi(m_\Pi^{(3,1)}) \circ \left(\kappa_{13}^\Pi(v) \boxtimes (\bar{\kappa}_1^\Pi(\mathbf{R}_+) \circ \kappa_{13}^\Pi(i^{-1})) \boxtimes (\bar{\kappa}_1^\Pi(\mathbf{R}_+) \circ \mathbf{v}) \right) \\ & = \bar{\kappa}_1^\Pi(m_\Pi) \circ \left(\kappa_{13}^\Pi(v) \boxtimes [\bar{\kappa}_1^\Pi(\mathbf{R}_+) \circ \bar{\kappa}_1^\Pi(m_a) \circ (\kappa_{13}^\Pi(i^{-1}) \boxtimes \mathbf{v})] \right). \end{aligned} \quad (23)$$

Combining (21), (22) and (23), we get

$$\begin{aligned} & \bar{\kappa}_1^\Pi(m_\Pi) \circ \left(\kappa_{13}^\Pi(v) \boxtimes [\bar{\kappa}_1^\Pi(\mathbf{R}_+) \circ \bar{\kappa}_1^\Pi(m_a) \circ (\mathbf{v} \boxtimes (\kappa_{12}^\Pi(i^{-1}) \circ \kappa_{23}^\Pi(i^{-1})))] \right) \\ & = \bar{\kappa}_1^\Pi(m_\Pi) \circ \left(\kappa_{13}^\Pi(v) \boxtimes [\bar{\kappa}_1^\Pi(\mathbf{R}_+) \circ \bar{\kappa}_1^\Pi(m_a) \circ (\kappa_{13}^\Pi(i^{-1}) \boxtimes \mathbf{v})] \right). \end{aligned}$$

Since v is invertible for m_Π , and \mathbf{R}_+ is left invertible, this implies

$$\bar{\kappa}_1^\Pi(m_a) \circ (\mathbf{v} \boxtimes (\kappa_{12}^\Pi(i^{-1}) \circ \kappa_{23}^\Pi(i^{-1}))) = \bar{\kappa}_1^\Pi(m_a) \circ (\kappa_{13}^\Pi(i^{-1}) \boxtimes \mathbf{v}),$$

i.e., (7). \square

We define the prop $\text{LBA}_{f,f',f''}$ by generators μ, δ, f, f', f'' , where $\mu \in \text{LBA}_{f,f',f''}(\wedge^2, \mathbf{id})$, $\delta \in \text{LBA}_{f,f',f''}(\mathbf{id}, \wedge^2)$, $f, f', f'' \in \text{LBA}_{f,f',f''}(\mathbf{1}, \wedge^2)$ and relations: μ, δ, f, f' satisfy the relations of $\text{LBA}_{f,f'}$, and (μ, δ, f, f') satisfy the relation (4) satisfied by (μ, δ, f, f') .

Define prop morphisms $\kappa_{ijk} : \text{LBA}_{f,f'} \rightarrow \text{LBA}_{f,f',f''}$, $\kappa_{123} : (\mu, \delta, f, f') \mapsto (\mu, \delta, f, f')$, $\kappa_{124} : (\mu, \delta, f, f') \mapsto (\mu, \delta, f, f' + f'')$, $\kappa_{134} : (\mu, \delta, f, f') \mapsto (\mu, \delta, f + f', f'')$, $\kappa_{234} : (\mu, \delta, f, f') \mapsto (\mu, \delta + \text{ad}(f), f', f'')$.

Define prop morphisms $\bar{\kappa}_{ij} : \text{LBA}_f \rightarrow \text{LBA}_{f,f',f''}$ by $\bar{\kappa}_{12} : (\mu, \delta, f) \mapsto (\mu, \delta, f)$, $\bar{\kappa}_{13} : (\mu, \delta, f) \mapsto (\mu, \delta, f + f')$, $\bar{\kappa}_{14} : (\mu, \delta, f) \mapsto (\mu, \delta, f + f' + f'')$, $\bar{\kappa}_{23} : (\mu, \delta, f) \mapsto (\mu, \delta + \text{ad}(f), f')$, $\bar{\kappa}_{24} : (\mu, \delta, f) \mapsto (\mu, \delta + \text{ad}(f), f' + f'')$, $\bar{\kappa}_{34} : (\mu, \delta, f) \mapsto (\mu, \delta + \text{ad}(f + f'), f'')$.

Define prop morphisms $\bar{\kappa}_i : \text{LBA} \rightarrow \text{LBA}_{f,f',f''}$ for $i = 1, \dots, 4$, by $\bar{\kappa}_1 : (\mu, \delta) \mapsto (\mu, \delta)$, $\bar{\kappa}_2 : (\mu, \delta) \mapsto (\mu, \delta + \text{ad}(f))$, $\bar{\kappa}_3 : (\mu, \delta) \mapsto (\mu, \delta + \text{ad}(f + f'))$, $\bar{\kappa}_4 : (\mu, \delta) \mapsto (\mu, \delta + \text{ad}(f + f' + f''))$.

Then $\bar{\kappa}_{ij} \circ \kappa_1 = \bar{\kappa}_i$, $\bar{\kappa}_{ij} \circ \kappa_2 = \bar{\kappa}_j$ (where $1 \leq i < j \leq 4$); we also have $\kappa_{1ij} \circ \bar{\kappa}_1 = \bar{\kappa}_1$ (where $2 \leq i < j \leq 4$), $\kappa_{234} \circ \bar{\kappa}_1 = \kappa_{12i} \circ \bar{\kappa}_2 = \bar{\kappa}_2$ (where $i = 3, 4$), $\kappa_{i34} \circ \bar{\kappa}_2 = \kappa_{123} \circ \bar{\kappa}_3 = \bar{\kappa}_3$ (where $i = 1, 2$), $\kappa_{ij4} \circ \bar{\kappa}_3 = \bar{\kappa}_4$ (where $1 \leq i < j \leq 3$); finally $\kappa_{12i} \circ \kappa_{12} = \bar{\kappa}_{12}$ ($i = 3, 4$), $\kappa_{i14} \circ \kappa_{13} = \bar{\kappa}_{14}$ ($i = 2, 3$), $\kappa_{i34} \circ \kappa_{23} = \bar{\kappa}_{34}$ ($i = 1, 2$), $\kappa_{134} \circ \kappa_{12} = \kappa_{123} \circ \kappa_{13} = \bar{\kappa}_{13}$, $\kappa_{234} \circ \kappa_{12} = \kappa_{123} \circ \kappa_{23} = \bar{\kappa}_{23}$, $\kappa_{234} \circ \kappa_{13} = \kappa_{124} \circ \kappa_{23} = \bar{\kappa}_{24}$.

Theorem 4.6.

$$\bar{\kappa}_1(m_a) \circ (\kappa_{134}^\Pi(\mathbf{v}) \boxtimes \kappa_{123}^\Pi(\mathbf{v})) = \bar{\kappa}_1(m_a) \circ (\kappa_{124}^\Pi(\mathbf{v}) \boxtimes (\bar{\kappa}_{12}^\Pi(i^{-1}) \circ \kappa_{234}^\Pi(\mathbf{v}))). \quad (24)$$

Proof. Applying κ_{134}^Π to (6), we get

$$\bar{\kappa}_1^\Pi(m_a^{(2,2)}) \circ (\bar{\kappa}_{14}^\Pi(\mathbb{F}) \boxtimes \kappa_{134}^\Pi(\Delta_a \circ \mathbb{v})) = \bar{\kappa}_1^\Pi(m_a^{(2,2)}) \circ (\kappa_{134}^\Pi(\mathbb{v})^{\boxtimes 2} \boxtimes (\bar{\kappa}_{13}^\Pi(i^{-1})^{\boxtimes 2} \circ \bar{\kappa}_{34}^\Pi(\mathbb{F})) \boxtimes \bar{\kappa}_{13}^\Pi(\mathbb{F})).$$

Using the fact that $\bar{\kappa}_{13}^\Pi(\mathbb{F}) = \kappa_{123}^\Pi \circ \kappa_{13}^\Pi(\mathbb{F})$ and the image of (6) by κ_{123}^Π , we get

$$\begin{aligned} \bar{\kappa}_1(m_a^{(3,2)}) \circ (\bar{\kappa}_{14}^\Pi(\mathbb{F}) \boxtimes (\bar{\kappa}_1^\Pi(\Delta_a) \circ \kappa_{134}^\Pi(\mathbb{v})) \boxtimes (\bar{\kappa}_1^\Pi(\Delta_a) \circ \kappa_{123}^\Pi(\mathbb{v}))) \\ = \bar{\kappa}_1(m_a^{(5,2)}) \circ (\kappa_{134}^\Pi(\mathbb{v})^{\boxtimes 2} \boxtimes (\bar{\kappa}_{13}^\Pi(i^{-1})^{\boxtimes 2} \circ \bar{\kappa}_{34}^\Pi(\mathbb{F})) \boxtimes \kappa_{123}^\Pi(\mathbb{v}^{\boxtimes 2}) \boxtimes (\bar{\kappa}_{12}^\Pi(i^{-1})^{\boxtimes 2} \circ \bar{\kappa}_{23}^\Pi(\mathbb{F})) \boxtimes \bar{\kappa}_{12}^\Pi(\mathbb{F})). \end{aligned}$$

Applying κ_{123}^Π to (7), we get $\bar{\kappa}_1^\Pi(m_a) \circ (\bar{\kappa}_{13}^\Pi(i^{-1}) \boxtimes \kappa_{123}^\Pi(\mathbb{v})) = \bar{\kappa}_1^\Pi(m_a) \circ (\kappa_{123}^\Pi(\mathbb{v}) \boxtimes (\bar{\kappa}_{12}^\Pi(i^{-1}) \circ \bar{\kappa}_{23}^\Pi(i^{-1})))$, which implies that

$$\begin{aligned} \bar{\kappa}_1(m_a^{(2,2)}) \circ (\bar{\kappa}_{14}^\Pi(\mathbb{F}) \boxtimes (\bar{\kappa}_1^\Pi(\Delta_a) \circ \mathbb{v}_1)) \\ = \bar{\kappa}_1(m_a^{(4,2)}) \circ (\mathbb{v}_1^{\boxtimes 2} \boxtimes (\bar{\kappa}_{12}^\Pi(i^{-1})^{\boxtimes 2} \circ \bar{\kappa}_{23}^\Pi(i^{-1})^{\boxtimes 2} \circ \bar{\kappa}_{34}^\Pi(\mathbb{F})) \boxtimes (\bar{\kappa}_{12}^\Pi(i^{-1})^{\boxtimes 2} \circ \bar{\kappa}_{23}^\Pi(\mathbb{F})) \boxtimes \bar{\kappa}_{12}^\Pi(\mathbb{F})), \end{aligned} \quad (25)$$

where $\mathbb{v}_1 = \bar{\kappa}_1(m_a^{(2,1)}) \circ (\kappa_{134}^\Pi(\mathbb{v}) \boxtimes \kappa_{123}^\Pi(\mathbb{v}))$.

Applying κ_{124}^Π to (6), we get

$$\bar{\kappa}_1^\Pi(m_a)^{(2,2)} \circ (\bar{\kappa}_{14}^\Pi(\mathbb{F}) \boxtimes (\bar{\kappa}_1^\Pi(\Delta_a) \circ \kappa_{124}^\Pi(\mathbb{v}))) = \bar{\kappa}_1^\Pi(m_a^{(3,2)}) \circ (\kappa_{124}^\Pi(\mathbb{v}^{\boxtimes 2}) \boxtimes (\bar{\kappa}_{12}^\Pi(i^{-1})^{\boxtimes 2} \circ \bar{\kappa}_{24}^\Pi(\mathbb{F})) \boxtimes \bar{\kappa}_{12}^\Pi(\mathbb{F})), \quad (26)$$

and applying κ_{234}^Π to the same identity, we get

$$\bar{\kappa}_2^\Pi(m_a^{(2,2)}) \circ (\bar{\kappa}_{24}^\Pi(\mathbb{F}) \boxtimes (\bar{\kappa}_2^\Pi(\Delta_a) \circ \kappa_{234}^\Pi(\mathbb{v}))) = \bar{\kappa}_2^\Pi(m_a^{(3,2)}) \circ (\kappa_{234}^\Pi(\mathbb{v}^{\boxtimes 2}) \boxtimes (\bar{\kappa}_{23}^\Pi(i^{-1})^{\boxtimes 2} \circ \bar{\kappa}_{34}^\Pi(\mathbb{F})) \boxtimes \bar{\kappa}_{23}^\Pi(\mathbb{F})).$$

Since $\bar{\kappa}_2(m_a) = \bar{\kappa}_{12}(i) \circ \bar{\kappa}_1(m_a) \circ (\bar{\kappa}_{12}(i)^{\boxtimes 2})^{-1}$, we get

$$\begin{aligned} \bar{\kappa}_1^\Pi(m_a^{(2,2)}) \circ (\bar{\kappa}_{12}^\Pi(i^{-1})^{\boxtimes 2} \circ \bar{\kappa}_{24}^\Pi(\mathbb{F})) \boxtimes (\bar{\kappa}_{12}^\Pi(i^{-1})^{\boxtimes 2} \circ \bar{\kappa}_2^\Pi(\Delta_a) \circ \kappa_{234}^\Pi(\mathbb{v})) \\ = \bar{\kappa}_1^\Pi(m_a^{(3,2)}) \circ ((\bar{\kappa}_{12}^\Pi(i^{-1})^{\boxtimes 2} \circ \kappa_{234}^\Pi(\mathbb{v}^{\boxtimes 2})) \boxtimes (\bar{\kappa}_{12}^\Pi(i^{-1})^{\boxtimes 2} \circ \bar{\kappa}_{23}^\Pi(i^{-1})^{\boxtimes 2} \circ \bar{\kappa}_{34}^\Pi(\mathbb{F})) \boxtimes (\bar{\kappa}_{12}^\Pi(i^{-1})^{\boxtimes 2} \circ \bar{\kappa}_{23}^\Pi(\mathbb{F}))). \end{aligned}$$

Right multiplying this identity by $\bar{\kappa}_{12}^\Pi(\mathbb{F})$ using $\bar{\kappa}_1^\Pi(m_a)$, and using $\bar{\kappa}_1^\Pi(m_a) \circ ([\bar{\kappa}_{12}^\Pi(i^{-1})^{\boxtimes 2} \circ \bar{\kappa}_2^\Pi(\Delta_a)] \boxtimes \bar{\kappa}_{12}^\Pi(\mathbb{F})) = \bar{\kappa}_1^\Pi(m_a) \circ (\bar{\kappa}_{12}^\Pi(\mathbb{F}) \boxtimes [\bar{\kappa}_1^\Pi(\Delta_a) \circ \bar{\kappa}_{12}^\Pi(i^{-1})])$, we get

$$\begin{aligned} \bar{\kappa}_1^\Pi(m_a^{(3,2)}) \circ ((\bar{\kappa}_{12}^\Pi(i^{-1})^{\boxtimes 2} \circ \bar{\kappa}_{24}^\Pi(\mathbb{F})) \boxtimes \bar{\kappa}_{12}^\Pi(\mathbb{F}) \boxtimes (\bar{\kappa}_1^\Pi(\Delta_a) \circ \bar{\kappa}_{12}^\Pi(i^{-1}) \circ \kappa_{234}^\Pi(\mathbb{v}))) \\ = \bar{\kappa}_1^\Pi(m_a^{(4,2)}) \circ ((\bar{\kappa}_{12}^\Pi(i^{-1}) \circ \kappa_{234}^\Pi(\mathbb{v}))^{\boxtimes 2} \boxtimes \\ (\bar{\kappa}_{12}^\Pi(i^{-1})^{\boxtimes 2} \circ \bar{\kappa}_{23}^\Pi(i^{-1})^{\boxtimes 2} \circ \bar{\kappa}_{34}^\Pi(\mathbb{F})) \boxtimes (\bar{\kappa}_{12}^\Pi(i^{-1})^{\boxtimes 2} \circ \bar{\kappa}_{23}^\Pi(\mathbb{F})) \boxtimes \bar{\kappa}_{12}^\Pi(\mathbb{F})). \end{aligned}$$

Right multiplying (26) by $\bar{\kappa}_1^\Pi(\Delta_a) \circ \bar{\kappa}_{12}^\Pi(i^{-1}) \circ \kappa_{234}^\Pi(\mathbb{v})$ using $\bar{\kappa}_1(m_a)$, we then get

$$\begin{aligned} \bar{\kappa}_1(m_a^{(2,2)}) \circ (\bar{\kappa}_{14}^\Pi(\mathbb{F}) \boxtimes (\bar{\kappa}_1^\Pi(\Delta_a) \circ \mathbb{v}_2)) \\ = \bar{\kappa}_1(m_a^{(4,2)}) \circ (\mathbb{v}_2^{\boxtimes 2} \boxtimes (\bar{\kappa}_{12}^\Pi(i^{-1})^{\boxtimes 2} \circ \bar{\kappa}_{23}^\Pi(i^{-1})^{\boxtimes 2} \circ \bar{\kappa}_{34}^\Pi(\mathbb{F})) \boxtimes (\bar{\kappa}_{12}^\Pi(i^{-1})^{\boxtimes 2} \circ \bar{\kappa}_{23}^\Pi(\mathbb{F})) \boxtimes \bar{\kappa}_{12}^\Pi(\mathbb{F})), \end{aligned} \quad (27)$$

where $\mathbb{v}_2 = \bar{\kappa}_1(m_a^{(2,1)}) \circ (\kappa_{124}^\Pi(\mathbb{v}) \boxtimes (\bar{\kappa}_{12}^\Pi(i^{-1}) \circ \kappa_{234}^\Pi(\mathbb{v})))$.

There exists a unique $w \in \Pi_{f,f',f''}(\mathbf{1} \boxtimes \mathbf{1}, S \boxtimes \mathbf{1})$, of the form $w = 1 + \text{degree} > 0$, such that $\mathbb{v}_1 = \bar{\kappa}_1^\Pi(m_a) \circ (w \boxtimes \mathbb{v}_2)$. Then (25) and (27) imply that

$$\bar{\kappa}_1^\Pi(m_a^{(2,2)}) \circ (\bar{\kappa}_{14}^\Pi(\mathbb{F}) \boxtimes (\bar{\kappa}_1^\Pi(\Delta_a) \circ w)) = \bar{\kappa}_1^\Pi(m_a^{(2,2)}) \circ (w^{\boxtimes 2} \boxtimes \bar{\kappa}_{14}^\Pi(\mathbb{F})),$$

i.e., $w' := \bar{\kappa}_{14}(i) \circ w$ satisfies $(w')^{\boxtimes 2} = \bar{\kappa}_4(\Delta_a) \circ w'$.

Identity (24) now follows from:

Proposition 4.7. *If $x \in \Pi_{f,f',f''}(\mathbf{1} \boxtimes \mathbf{1}, S \boxtimes \mathbf{1})$ is of the form $X = 1 + \text{degree} > 0$ and if $x^{\boxtimes 2} = \bar{\kappa}_4(\Delta_a) \circ x$, then $x = 1$.*

Proof of Proposition. $\text{LBA}_{f,f',f''}$ is equipped with a prop automorphism ι , where $\iota^2 = \text{id}$, uniquely defined by $(\mu, \delta, f, f', f'') \mapsto (\mu, \delta + \text{ad}(f + f' + f''), -f'', -f', -f)$. Then $\iota \circ \bar{\kappa}_4 = \bar{\kappa}_1$. Set $y := \iota^\Pi(x)$, then $\bar{\kappa}_1^\Pi(y) = y^{\boxtimes 2}$.

The prop $\text{LBA}_{f,f',f''}$ is equipped with a degree, such that $\deg(\mu) = 0$ and $\deg(\delta) = \deg(f) = \deg(f') = \deg(f'') = 1$. We then decompose $y = 1 + y_1 + \dots$ for this degree. Assume that we showed $y_1 = \dots = y_{n-1} = 0$. We then get: $y_n \boxtimes 1 + 1 \boxtimes y_n =$ the degree n part of $\bar{\kappa}_1^\Pi(\Delta_a) \circ y_n$, i.e., $= \Delta_0 \circ y_n$. According to the computation of the co-Hochschild cohomology of the complex $S^{\otimes 0} \rightarrow S \rightarrow S^{\otimes 2} \rightarrow \dots$ of Schur functors, we get $y_n \in \Pi_{f,f',f''}(\mathbf{1} \boxtimes \mathbf{1}, \mathbf{id} \boxtimes \mathbf{1}) \subset \Pi_{f,f',f''}(\mathbf{1} \boxtimes \mathbf{1}, S \boxtimes \mathbf{1})$.

The degree $n + 1$ part of the equation $\bar{\kappa}_1^\Pi(\Delta_a) \circ y = y^{\boxtimes 2}$ then yields (degree $n + 1$ part of $\Delta_0 \circ y_{n+1} + \bar{\kappa}_1^\Pi(\Delta_a) \circ y_n = y_{n+1} \boxtimes 1 + 1 \boxtimes y_{n+1}$). Antisymmetrizing, we get $\delta \circ y_n = 0$.

We then show:

Lemma 4.8. *The map $\text{LBA}_{f,f',f''}(\mathbf{1}, \mathbf{id}) \rightarrow \text{LBA}_{f,f',f''}(\mathbf{1}, \wedge^2)$, $y \mapsto \delta \circ y$ is injective.*

Proof of Lemma. As in [EnH], we will construct a retraction of this map. As in [EnH], one shows that $\text{LBA}_{f,f',f''}(F, G)$ is the cokernel of $\text{LBA}(C \otimes D \otimes F, G) \rightarrow \text{LBA}(C \otimes F, G)$, $x \mapsto x \circ ([(\text{id}_C \boxtimes p) \circ \Delta_C] \boxtimes \text{id}_F)$ where $C = S(\wedge^2 \oplus \wedge^2 \oplus \wedge^2)$, $D = \wedge^3 \oplus \wedge^3 \oplus \wedge^3$, $\Delta_C : C \rightarrow C^{\otimes 2}$ is induced by the coalgebra structure of S , and $p \in \text{LBA}(C, D) = \bigoplus_{k \geq 0} \text{LBA}(S^k \circ (\wedge^2 \oplus \wedge^2 \oplus \wedge^2), \wedge^3 \oplus \wedge^3 \oplus \wedge^3)$ has nonzero components for $k = 1, 2$ only; the $k = 1$, this component specializes to $\wedge^3(\mathfrak{a})^{\oplus 3} \rightarrow \wedge^3(\mathfrak{a})^{\oplus 3}$,

$$(f_a, f'_a, f''_a) \mapsto ((\delta_a \otimes \text{id}_a)(f_a) + \text{c.p.}, (\delta_a \otimes \text{id}_a)(f'_a) + \text{c.p.}, (\delta_a \otimes \text{id}_a)(f''_a) + \text{c.p.}),$$

where c.p. means cyclic permutation, and for $k = 2$ is specializes to $S^2(\wedge^3(\mathfrak{a})^{\oplus 3}) \rightarrow \wedge^3(\mathfrak{a})^{\oplus 3}$,

$$(f_a, f'_a, f''_a)^{\otimes 2} \mapsto ([f_a^{12}, f_a^{13}] + \text{c.p.}, [f_a^{12}, f_a^{13} + f_a^{23}] + [f_a^{12}, f_a^{13}] + \text{c.p.}, [f_a^{12} + f_a^{12}, f_a^{13} + f_a^{23}] + [f_a^{12}, f_a^{13}] + \text{c.p.}).$$

Since left and right compositions commute, we have a commutative diagram,

$$\begin{array}{ccc} \text{LBA}(C \otimes D, \mathbf{id}) & \xrightarrow{\delta \circ -} & \text{LBA}(C \otimes D, \wedge^2) \\ \downarrow & & \downarrow \\ \text{LBA}(C, \mathbf{id}) & \xrightarrow{\delta \circ -} & \text{LBA}(C, \wedge^2) \end{array}$$

whose vertical cokernel is the map $\text{LBA}_{f,f',f''}(\mathbf{1}, \mathbf{id}) \rightarrow \text{LBA}_{f,f',f''}(\mathbf{1}, \wedge^2)$, $y \mapsto \delta \circ y$.

For any Schur functor A , we will construct a retraction $r_A : \text{LBA}(A, \wedge^2) \rightarrow \text{LBA}(A, \mathbf{id})$ of the map $\text{LBA}(A, \mathbf{id}) \rightarrow \text{LBA}(A, \wedge^2)$, such that the diagram

$$\begin{array}{ccc} \text{LBA}(C \otimes D, \wedge^2) & \xrightarrow{r_A \circ \delta} & \text{LBA}(C \otimes D, \mathbf{id}) \\ \downarrow & & \downarrow \\ \text{LBA}(C, \wedge^2) & \xrightarrow{r_A} & \text{LBA}(C, \mathbf{id}) \end{array} \quad (28)$$

commutes. The vertical cokernel of this map is then the desired retraction.

We have $\text{LBA}(A, \mathbf{id}) = \bigoplus_{Z \in \text{Irr}(\text{Sch})} \text{LCA}(A, Z) \otimes \text{LA}(Z, \mathbf{id})$. As in [EnH], one shows that $\text{LCA}(Z, \mathbf{id} \otimes Z)$ is 1-dimensional, and one constructs an element $\delta_Z \in \text{LCA}(Z, \mathbf{id} \otimes Z)$, such that the component $(Z', Z'') = (\mathbf{id}, Z)$ of the map $\text{LA}(Z, \mathbf{id}) \xrightarrow{\delta \circ -} \text{LBA}(Z, \wedge^2) \subset \text{LBA}(Z, \mathbf{id}^{\otimes 2}) \simeq \bigoplus_{Z', Z'' \in \text{Irr}(\text{Sch})} \text{LCA}(Z, Z' \otimes Z'') \otimes \text{LA}(Z', \mathbf{id}) \otimes \text{LA}(Z'', \mathbf{id})$ is $\lambda \mapsto \delta_Z \otimes \text{id}_{\mathbf{id}} \otimes \lambda$.

It follows that the component $Z \mapsto (Z', Z'') = (\mathbf{id}, Z)$ of the map $\bigoplus_{Z \in \text{Irr}(\text{Sch})} \text{LCA}(A, Z) \otimes \text{LA}(Z, \mathbf{id}) \simeq \text{LBA}(A, \mathbf{id}) \xrightarrow{\delta \circ -} \text{LBA}(A, \wedge^2) \subset \text{LBA}(A, \mathbf{id}^{\otimes 2}) \simeq \bigoplus_{Z', Z'' \in \text{Irr}(\text{Sch})} \text{LCA}(A, Z' \otimes Z'') \otimes \text{LA}(Z', \mathbf{id}) \otimes \text{LA}(Z'', \mathbf{id})$ is $\kappa \otimes \lambda \mapsto (\lambda \circ \delta_Z) \otimes \text{id}_{\mathbf{id}} \otimes \kappa$.

Dually to [EnH], we construct a retraction of the map $\lambda \mapsto \lambda \circ \delta_Z$; it gives rise to the section r_A . One then proves the commutativity of (28) as in [EnH].

This ends the proof of the lemma, and therefore also of Proposition 4.7 and Theorem 4.6. \square

We now draw the consequences of the results of the previous Subsection for the quantization of twists of Lie bialgebras.

Let $\mathfrak{a} = (\mathfrak{a}, \mu_{\mathfrak{a}}, \delta_{\mathfrak{a}})$ be a Lie bialgebra. Its quantization is $Q(\mathfrak{a}) = (S(\mathfrak{a})[[\hbar]], m(\mathfrak{a}), \Delta(\mathfrak{a}))$, where $m(\mathfrak{a}) := m_{\mathfrak{a}}(\mu_{\mathfrak{a}}, \hbar\delta_{\mathfrak{a}})$, $\Delta(\mathfrak{a}) := \Delta_{\mathfrak{a}}(\mu_{\mathfrak{a}}, \hbar\delta_{\mathfrak{a}})$. We set $F(\mathfrak{a}, f_{\mathfrak{a}}) := F(\mu_{\mathfrak{a}}, \hbar\delta_{\mathfrak{a}}, \hbar f_{\mathfrak{a}})$, $i(\mathfrak{a}, f_{\mathfrak{a}}) := i(\mu_{\mathfrak{a}}, \hbar\delta_{\mathfrak{a}}, \hbar f_{\mathfrak{a}})$.

Then $F(\mathfrak{a}, f_{\mathfrak{a}}) \in Q(\mathfrak{a})^{\otimes 2}$ and $i(\mathfrak{a}, f_{\mathfrak{a}}) : Q(\mathfrak{a}) \rightarrow Q(\mathfrak{a}_{f_{\mathfrak{a}}})$ are such that

$$m(\mathfrak{a}_{f_{\mathfrak{a}}}) = i(\mathfrak{a}, f_{\mathfrak{a}}) \circ m(\mathfrak{a}) \circ (i(\mathfrak{a}, f_{\mathfrak{a}})^{\otimes 2})^{-1}, \quad \Delta(\mathfrak{a}_{f_{\mathfrak{a}}}) = i(\mathfrak{a}, f_{\mathfrak{a}})^{\otimes 2} \circ \text{Ad}(F(\mathfrak{a}, f_{\mathfrak{a}})) \circ \Delta(\mathfrak{a}) \circ i(\mathfrak{a}, f_{\mathfrak{a}})^{-1},$$

$$(F(\mathfrak{a}, f_{\mathfrak{a}}) \otimes 1) * (\Delta(\mathfrak{a}) \otimes \text{id})(F(\mathfrak{a}, f_{\mathfrak{a}})) = (1 \otimes F(\mathfrak{a}, f_{\mathfrak{a}})) * (\text{id} \otimes \Delta(\mathfrak{a}))(F(\mathfrak{a}, f_{\mathfrak{a}}))$$

(where the product $m(\mathfrak{a})$ is denoted $*$).

Assume that $f'_{\mathfrak{a}}$ is a twist of $\mathfrak{a}_{f_{\mathfrak{a}}}$. Then Theorem 4.4 implies that $v(\mathfrak{a}, f_{\mathfrak{a}}, f'_{\mathfrak{a}}) := v(\mu_{\mathfrak{a}}, \hbar\delta_{\mathfrak{a}}, \hbar f_{\mathfrak{a}}, \hbar f'_{\mathfrak{a}})$ satisfies

$$F(\mathfrak{a}, f_{\mathfrak{a}} + f'_{\mathfrak{a}}) = v(\mathfrak{a}, f_{\mathfrak{a}}, f'_{\mathfrak{a}})^{\otimes 2} * (i(\mathfrak{a}, f_{\mathfrak{a}})^{\otimes 2})^{-1} (F(\mathfrak{a}_{f_{\mathfrak{a}}}, f'_{\mathfrak{a}})) * F(\mathfrak{a}, f_{\mathfrak{a}}) * \Delta(\mathfrak{a})(v(\mathfrak{a}, f_{\mathfrak{a}}, f'_{\mathfrak{a}}))^{-1},$$

$$i(\mathfrak{a}, f_{\mathfrak{a}} + f'_{\mathfrak{a}}) = i(\mathfrak{a}_{f_{\mathfrak{a}}}, f'_{\mathfrak{a}}) \circ i(\mathfrak{a}, f_{\mathfrak{a}}) \circ \text{Ad}(v(\mathfrak{a}, f_{\mathfrak{a}}, f'_{\mathfrak{a}}))^{-1}$$

(in both equalities, $m(\mathfrak{a})$ is understood; it is denoted $*$ in the first equality).

Finally, Theorem 4.6 implies that if $f''_{\mathfrak{a}}$ is a twist of $\mathfrak{a}_{f_{\mathfrak{a}} + f'_{\mathfrak{a}}}$, then

$$v(\mathfrak{a}, f_{\mathfrak{a}} + f'_{\mathfrak{a}}, f''_{\mathfrak{a}}) * v(\mathfrak{a}, f_{\mathfrak{a}}, f'_{\mathfrak{a}}) = v(\mathfrak{a}, f_{\mathfrak{a}}, f'_{\mathfrak{a}} + f''_{\mathfrak{a}}) * i(\mathfrak{a}, f_{\mathfrak{a}})^{-1} (v(\mathfrak{a}_{f_{\mathfrak{a}} + f'_{\mathfrak{a}}}, f''_{\mathfrak{a}})).$$

5. QUANTIZATION OF Γ -LIE BIALGEBRAS

5.1. Assume that $(\mathfrak{a}, \theta, f)$ is a Γ -Lie bialgebra. We construct its quantization as follows. Set $A = S(\mathfrak{a}) \otimes \mathbf{k}\Gamma[[\hbar]]$. We set $[x|\gamma] := x \otimes \gamma$, $[x \otimes x'|\gamma, \gamma'] := (x \otimes \gamma) \otimes (x' \otimes \gamma') \in A^{\otimes 2}$.

There are unique linear maps $m : A^{\otimes 2} \rightarrow A$ and $\Delta : A \rightarrow A^{\otimes 2}$, such that

$$m : [x|\gamma][x'|\gamma'] \mapsto [x * i(\mathfrak{a}, f_{\gamma})^{-1}(\theta_{\gamma}(x')) * v(\mathfrak{a}, f_{\gamma}, \wedge^2(\theta_{\gamma})(f_{\gamma'}))^{-1}|\gamma\gamma']$$

$$\Delta : [x|\gamma] \mapsto [\Delta(\mathfrak{a})(x) * F(\mathfrak{a}, f_{\gamma})^{-1}|\gamma, \gamma].$$

The unit for A is $[1|e]$, and the counit is the map $[x|\gamma] \mapsto \delta_{\gamma, e}\varepsilon(x)$ (recall that $*$ denotes the product $m(\mathfrak{a})$ on $S(\mathfrak{a})[[\hbar]]$ or its tensor square).

Proposition 5.1. *This defines a bialgebra structure on A , quantizing the co-Poisson bialgebra structure induced by $(\mathfrak{a}, \theta, f)$.*

Proof. This follows from the above relations on twists. \square

5.2. Propic version. The quantization of Γ -Lie bialgebras has a propic version, which we now describe.

Define LA_{Γ} as the prop with generators $\mu \in \text{LA}(\wedge^2, \mathbf{id})$ and $\theta_{\gamma} \in \text{LA}(\mathbf{id}, \mathbf{id})^{\times}$, and relations: Jacobi identity on μ , $\Gamma \rightarrow \text{LA}_{\Gamma}(\mathbf{id}, \mathbf{id})^{\times}$, $\gamma \mapsto \theta_{\gamma}$ is a group morphism, and $\wedge^2(\theta_{\gamma}) \circ \mu \circ \theta_{\gamma}^{-1} = \mu$.

Define LBA_{Γ} as the prop with generators $\mu \in \text{LBA}_{\Gamma}(\wedge^2, \mathbf{id})$, $\delta \in \text{LBA}_{\Gamma}(\mathbf{id}, \wedge^2)$, $\theta_{\gamma} \in \text{LBA}_{\Gamma}(\mathbf{id}, \mathbf{id})$ and $f_{\gamma} \in \text{LBA}(\mathbf{1}, \wedge^2)$, and relations: (μ, δ) satisfy the relations of the prop LBA , $(\mu, (\theta_{\gamma})_{\gamma}, (f_{\gamma})_{\gamma})$ satisfy the relations of LA_{Γ} ; for each $\gamma \in \Gamma$, $(\mu, \delta, f_{\gamma})$ satisfy the defining relations of LBA_f , as well as $\wedge^2(\theta_{\gamma}) \circ \delta \circ \theta_{\gamma}^{-1} = \delta + \text{ad}(f_{\gamma})$, and for each pair $\gamma, \gamma' \in \Gamma$, $f_{\gamma\gamma'} = f_{\gamma} + \wedge^2(\theta_{\gamma}) \circ f_{\gamma'}$.

Define the prop Bialg_{Γ} of Γ -bialgebras as follows. When Γ is finite, in addition to the generators $m, \Delta, \varepsilon, \eta$ of Bialg , it has generators $e_{\gamma} \in \text{Bialg}_{\Gamma}(\mathbf{id}, \mathbf{id})$, and the additional relations

are $\sum_{\gamma \in \Gamma} e_\gamma = \text{id}_{\mathbf{id}}$, $e_\gamma \circ e_{\gamma'} = \delta_{\gamma\gamma'} e_\gamma$, $m \circ (e_\gamma \boxtimes e_{\gamma'}) = e_{\gamma\gamma'} \circ m$, $\Delta \circ e_\gamma = e_\gamma^{\boxtimes 2} \circ \Delta$, $e_\gamma \circ \eta = \delta_{\gamma e} \eta$, $\varepsilon \circ e_\gamma = \delta_{e\gamma} \varepsilon$.

In general, Bialg_Γ is defined as follows. If S is a set, define Sch_S as the category of polynomial Schur functors $\text{Vect}^S \rightarrow V$ of the form $(V_s)_{s \in S} \rightarrow \bigoplus_{(Z_s)} M_{(Z_s)} \otimes (\bigotimes_{s \in S} Z_s(V_s))$, where Z_s are almost all $\mathbf{1}$ (the unit Schur functor $\mathbf{1}(V) = \mathbf{k}$). Then Sch_S is a symmetric tensor category. We define a S -prop as a symmetric tensor category P together with a natural transformation $\text{Sch}_S \rightarrow P$, which is the identity on objects. A S -prop may be defined by generators and relations. Then Bialg_Γ is the Γ -prop defined by generators $m_{\gamma, \gamma'} \in \text{Bialg}_\Gamma(\mathbf{id}_\gamma \boxtimes \mathbf{id}'_{\gamma'}, \mathbf{id}_{\gamma\gamma'})$, $\Delta_\gamma \in \text{Bialg}_\Gamma(\mathbf{id}_\gamma, \mathbf{id}_\gamma^{\boxtimes 2})$, $\varepsilon \in \text{Bialg}_\Gamma(\mathbf{id}_e, \mathbf{1})$, $\eta \in \text{Bialg}_\Gamma(\mathbf{1}, \mathbf{id}_e)$, and the relations derived from the finite case. The diagonal embedding $\text{Vect} \rightarrow \text{Vect}^\Gamma$ gives rise to a functor $\Delta : \text{Sch} \rightarrow \text{Sch}_S$, and we set $\text{Bialg}_\Gamma(F, G) := \text{Bialg}_\Gamma(\Delta(F), \Delta(G))$.

Then any EK quantization functor gives rise to a prop morphism $\text{Bialg}_\Gamma \rightarrow S(\mathbf{LBA}_\Gamma)^\Gamma$ with suitable classical limit properties. A group morphism $\Gamma \rightarrow \Gamma'$ gives rise to a commutative diagram

$$\begin{array}{ccc} \text{Bialg}_\Gamma & \rightarrow & S(\mathbf{LBA}_\Gamma)^\Gamma \\ \downarrow & & \downarrow \\ \text{Bialg}_{\Gamma'} & \rightarrow & S(\mathbf{LBA}_{\Gamma'})^{\Gamma'} \end{array}$$

We have therefore a quantization functor $\{\text{group Lie bialgebras}\} \rightarrow \{\text{quasicocommutative group bialgebras}\}$ (where both sides are full subcategories of $\{\text{co-Poisson cocomutative bialgebras}\}$ and $\{\text{quasicocommutative bialgebras}\}$).

5.3. Quantization of quasitriangular Γ -Lie bialgebras. We defined a quasitriangular Γ -Lie bialgebra as a triple $(\mathfrak{a}, r_\mathfrak{a}, \theta_\mathfrak{a})$, where $(\mathfrak{a}, r_\mathfrak{a})$ be a quasitriangular Lie bialgebra (i.e., $(r_\mathfrak{a} \in \mathfrak{a}^{\otimes 2})$ satisfies the classical Yang-Baxter identity, and $t_\mathfrak{a} := r_\mathfrak{a} + r_\mathfrak{a}^{21}$ is \mathfrak{a} -invariant), and $\theta_\mathfrak{a} : \Gamma \rightarrow \text{Aut}(\mathfrak{a}, t_\mathfrak{a})$ be an action of Γ by Lie algebra automorphisms of \mathfrak{a} , preserving $t_\mathfrak{a}$. It gives rise to a Γ -Lie bialgebra, with $\delta(x) = [r, x^1 + x^2]$ and $f_\gamma := \theta_\gamma^{\otimes 2}(r_\mathfrak{a}) - r_\mathfrak{a}$.

In that case a quantization can be constructed directly: we set $A = U(\mathfrak{a}) \rtimes \Gamma[[\hbar]]$, the product is undeformed, and the coproduct is $\Delta(x) = J(\hbar r_\mathfrak{a}) \Delta_0(x) J(\hbar r_\mathfrak{a})^{-1}$ (Δ_0 is the standard coproduct).

Denote by qt_Γ the prop of quasitriangular Γ -Lie bialgebras, and by \mathbf{qt}_Γ its completion. We have a natural prop morphism $\text{LBA}_\Gamma \rightarrow \text{qt}_\Gamma$. We claim that the prop morphisms $\text{Bialg}_\Gamma \rightarrow S(\mathbf{qt}_\Gamma)^\Gamma$ (the above direct construction) and the composed morphism $\text{Bialg}_\Gamma \rightarrow S(\mathbf{LBA}_\Gamma)^\Gamma \rightarrow S(\mathbf{qt}_\Gamma)^\Gamma$ are equivalent (i.e., can be obtained from each other using an inner automorphism of $S(\mathbf{qt}_\Gamma)^\Gamma$).

This is a consequence of the following statement on twists. Let $(\mathfrak{a}, r_\mathfrak{a})$ be a quasitriangular Lie bialgebra and let $f_\mathfrak{a} \in \wedge^2(\mathfrak{a})$ be a twist. According to [EK], there exists an invertible $j(\mathfrak{a}, r_\mathfrak{a}) : U(\mathfrak{a})[[\hbar]] \rightarrow S(\mathfrak{a})[[\hbar]]$, such that $m(\mathfrak{a}) = j(\mathfrak{a}, r_\mathfrak{a}) \circ m_0 \circ (j(\mathfrak{a}, r_\mathfrak{a})^{\otimes 2})^{-1}$, $\Delta(\mathfrak{a}) = j(\mathfrak{a}, r_\mathfrak{a})^{\otimes 2} \circ \text{Ad}(J(\hbar r_\mathfrak{a})) \circ \Delta_0 \circ j(\mathfrak{a}, r_\mathfrak{a})^{-1}$. Then one proves that $j(\mathfrak{a}, r_\mathfrak{a} + f_\mathfrak{a}) = i(\mathfrak{a}, f_\mathfrak{a}) \circ j(\mathfrak{a}, r_\mathfrak{a}) \circ \text{Ad}(v(\mathfrak{a}, r_\mathfrak{a})^{-1})$, and $J(\mathfrak{a}, r_\mathfrak{a} + f_\mathfrak{a}) = (v(\mathfrak{a}, r_\mathfrak{a}) \otimes v(\mathfrak{a}, r_\mathfrak{a})) * (j(\mathfrak{a}, r_\mathfrak{a})^{\otimes 2})^{-1} (F(\mathfrak{a}, f_\mathfrak{a})) * J(\mathfrak{a}, r_\mathfrak{a}) * \Delta_0(v(\mathfrak{a}, r_\mathfrak{a}))^{-1}$ (here $*$ is the undeformed product on $U(\mathfrak{a})^{\otimes 2}[[\hbar]]$).

5.4. Open questions. Let \mathfrak{a} be a simple Lie algebra and let \tilde{W} be its extended Weyl group. One expects that the only possible quantization of $U(\mathfrak{a}) \times \tilde{W}$ is the Majid-Soibelman algebra of \mathfrak{a} . When \mathfrak{a} is a Mac-Moody Lie algebra, one expects that if Q is any EK quantization functor, then $Q(\mathfrak{a}, \tilde{W})$ is the Majid-Soibelman algebra of \mathfrak{a} .

Both statements are analogues of well-known results [Dr, EK6].

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