QUANTIZATION OF $\Gamma$-LIE BIALGEBRAS

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Abstract. We introduce the notion of $\Gamma$-Lie bialgebras, where $\Gamma$ is a group. These objects give rise to cocommutative co-Poisson bialgebras, for which we construct quantization functors. This enlarges the class of co-Poisson algebras for which a quantization is known.

Our result relies on our earlier work, where we showed that twists of Lie bialgebras can be quantized; we complement this work by studying the behavior of this quantization under compositions of twists.

We work over a field $k$ of characteristic 0.

1. Introduction

Recall that a co-Poisson bialgebra is a quadruple $(U,m,\Delta,\delta)$, where $(U,m,\Delta_0)$ is a cocommutative bialgebra and $\delta : U \to \wedge^2(U)$ is a derivation (for $m$), a coderivation (for $\Delta_0$) and satisfies the co-Jacobi identity. A quantization of $(U,m,\id)(F(\structures on the opposite coproduct).

As an example, the co-Poisson bialgebra structures on $U(a)$ correspond bijectively to the maps $\delta_a : a \to \wedge^2(a)$ such that $(a,\mu_a,\delta_a)$ is a Lie bialgebra. The quantization of these co-Poisson bialgebras was obtained in [EK].

To a triple $(\Gamma,a,\theta_a)$, where $\Gamma$ is a group, $a$ is a Lie algebra and $\theta_a : \Gamma \to \text{Aut}(a,\mu_a)$ is an action of $\Gamma$ on $a$, one associates the $\Gamma$-graded cocommutative bialgebra $U(a) \times \Gamma$. The $\Gamma$-graded co-Poisson bialgebra structures on $U(a) \rtimes \Gamma$ correspond bijectively to pairs $(\delta_a,f)$, where $\delta_a : a \to \wedge^2(a)$ is such that $(a,\mu_a,\delta_a)$ is a Lie bialgebra, and $f : \Gamma \to \wedge^2(a)$ satisfies some conditions (see Section 2); in particular, $f(\gamma)$ is a twist of $(a,\mu_a,\delta_a)$ for any $\gamma \in \Gamma$. We call the resulting 5-uple $(a,\mu_a,\delta_a,\theta_a,f)$ a $\Gamma$-Lie bialgebra. The main result of this paper is the quantization of the corresponding co-Poisson bialgebra structures.

Examples of $\Gamma$-Lie bialgebras arise from the following situation: $G$ is a Poisson-Lie group with Lie bialgebra $(a,\mu_a,\delta_a)$, and $\Gamma \subset G$ is a discrete subgroup. Another example is when $a$ is a Kac-Moody Lie algebra $a$, and $\Gamma$ is the extended Weyl group of $a$. In the latter case, a quantization in known ([MS]).

To achieve our goal, we complement a result obtained in [EnH], namely the compatibility of Etingof-Kazhdan (EK) quantization functors with twists of Lie bialgebras; this result is based on an alternative construction of these quantization functors ([En]). We describe the behavior of this quantization under composition of twists (Section 4).

To give an idea of the result of [EnH], we formulate its main consequence: let $Q : \{\text{Lie bialgebras}\} \to \{\text{quantized universal enveloping (QUE) algebras}\}$, $(a,\mu_a,\delta_a) = a \mapsto Q(a) = (Q(a),m(a),\Delta(a),1_a,\varepsilon_a)$ be a quantization functor; to each classical twist $f_a$ of $a$ (i.e., $f_a \in \wedge^2(a)$ and $(\delta_a \otimes \id_a)(f_a) + [f_a^{13}, f_{a}^{23}] + \text{cyclic permutations} = 0$), one associates $F(a,f_a) = Q(a)\otimes k$ for $k \geq 1$
1; this implies that \( F(a, \delta a) Q(a) := (Q(a), m(a), \text{Ad}(F(a, f_a)) \circ \Delta(a), 1_a, \varepsilon_a) \) is a QUE algebra; (b) we have an isomorphism \( i(a, f_a) : F(a, f_a) Q(a) \rightarrow Q(a_f) \) of QUE algebras (here \( a_f = (a, \mu_a, \delta_a + \text{ad}(f_a)) \), where \( \text{ad}(f_a) : a \rightarrow \wedge^2(a) \) is \( x \mapsto [f_a, x \otimes 1 + 1 \otimes x] \)).

In Section 4, we study the behavior of the assignment \((a, f_a) \mapsto F(a, f_a)\) under the composition of twists. A composition of twists is a pair \((f_a, f'_a)\) such that \( f_a \) is a twist of \( a \), and \( f'_a \) is a twist of \( a_{f_a} \). We formulate the main consequence of our results: (a) there exists an invertible \( v(a, f_a, f'_a) \in Q(a) \), such that \( F(a, f_a + f'_a) = v(a, f_a, f'_a)^2 \ast i(a, f_a)^{-1}(F(a, f_a) \ast F(a, f'_a) \ast \Delta(a)(v(a, f_a, f'_a))^{-1}) \) (Theorem 4.4); and (b) if \((f_a, f'_a, f''_a)\) are such that \( f'_a \) is a twist of \( a_{f_a} \), then \( f_a + f'_a \) is a twist of \( a \), and \( v(a, f_a + f'_a, f''_a) \ast v(a, f_a, f'_a) = v(a, f_a, f'_a + f''_a) \ast i(a, f_a)^{-1}(v(a, f_a, f'_a, f''_a)) \) (Theorem 4.4).

We use these results in Section 5 to construct a quantization of the \( \Gamma \)-graded co-Poisson bialgebras \( U(a) \times \Gamma \) as \( \Gamma \)-graded bialgebras. This quantization is based on the facts that that for \( \gamma \in \Gamma \), \( f(\gamma) \) is a twist of \( a \), and for any \( \gamma, \gamma' \in \Gamma \), \((f(\gamma), \wedge^2(\theta_a(\gamma))(f(\gamma')))) \) is a composition of twists for \( a \); we then use the results of [EnH] on quantization of twists and those of Section 4 on their composition.

This paper is organized as follows. In Section 2, we define \( \Gamma \)-Lie bialgebras, the corresponding co-Poisson comcomutative bialgebras, and the problem of their quantization. In Section 3, we recall the formalism of (quasi-multi-bi)props, which is the natural framework of the approach of [En] to quantization functors and of the results of [EnH] on quantization of twists. In Section 4, we describe the behavior of composition of twists under quantization (Theorem 4.4 and Theorem 4.6). In Section 5, we apply these results to the construction of quantizations of \( \Gamma \)-Lie bialgebras.

## 2. \( \Gamma \)-Lie Bialgebras

### 2.1. \( \Gamma \)-Lie Algebras and Equivalent Categories

Define a group Lie algebra as a triple \((\Gamma, A, \theta_a)\), where \( \Gamma \) is a group, \( A \) is a Lie algebra and \( \theta_a : \Gamma \rightarrow \text{Aut}(a) \) is a group morphism. Group Lie algebras form a category, where a morphism \((\Gamma, A, \theta_a) \rightarrow (\Gamma', A', \theta'_a)\) is the data of a group morphism \( i_{\Gamma \Gamma'} : \Gamma \rightarrow \Gamma' \) and a Lie algebra morphism \( i_{aa'} : a \rightarrow a' \), such that \( i_{aa'}(\theta_a(\gamma)(x)) = \theta_{a'}(i_{\Gamma \Gamma'}(\gamma))i_{aa'}(x) \).

If \( \Gamma \) is a group, a \( \Gamma \)-Lie algebra is a pair \((a, \theta_a)\), such that \((\Gamma, a, \theta_a)\) is a group Lie algebra. \( \Gamma \)-Lie algebras form a subcategory of group Lie algebras, where the morphisms are restricted by the condition \( i_{\Gamma \Gamma'} = \text{id}_\Gamma \).

Define a group cocommutative bialgebra as a triple \((\Gamma, A, i)\), where \( \Gamma \) is a group, \( A \) is a cocommutative bialgebra, \( A = \oplus_{\gamma \in \Gamma} A_{\gamma} \) is a decomposition of \( A \), and \(i : \Gamma \rightarrow A \) is a bialgebra morphism, such that \( A_{\gamma} A_{\gamma'} \subset A_{\gamma \gamma'} \), \( \Delta A(a_{\gamma}) \subset A_{\gamma}^{\otimes 2} \), and \( i \) is compatible with the \( \Gamma \)-grading. A morphism \((\Gamma, A, i) \rightarrow (\Gamma', A', i')\) is the data of a group morphism \( i_{\Gamma \Gamma'} : \Gamma \rightarrow \Gamma' \) and a bialgebra morphism \( i_{aa'} : A \rightarrow A' \), such that \( i_{aa'}(A_{\gamma}) \subset A'_{i_{\Gamma \Gamma'}(\gamma)} \) and \( i_{aa'} \circ i = i' \circ i_{\Gamma \Gamma} \text{id}_{A'} \) (where \( i_{\Gamma \Gamma} \text{id}_{A'} \) is the morphism induced by \( i_{\Gamma \Gamma'} \)).

We then define a \( \Gamma \)-co-commutative bialgebra as a pair \((A, i)\), such that \((\Gamma, A, i)\) is a group cocommutative bialgebra. \( \Gamma \)-co-commutative bialgebras form a category, where as before \( i_{\Gamma \Gamma'} = \text{id}_\Gamma \).

The category of group (resp., \( \Gamma \)-)cocommutative bialgebras contains as a full subcategory the category of group (resp., \( \Gamma \)-)universal enveloping algebras, where \((A, \Gamma, i)\) satisfies the additional requirement that \( A_{\gamma} \) is a universal enveloping algebra.

Define a group commutative bialgebra (in a symmetric monoidal category \( S \)) as a triple \((\Gamma, O, j)\), where \( \Gamma \) is a group, \( O \) is a commutative algebra (in \( S \)) with a decomposition \( O = \oplus_{\gamma \in \Gamma} O_{\gamma} \), such that \( O_{\gamma} O_{\gamma'} = 0 \) for \( \gamma \neq \gamma' \), algebra morphisms \( \Delta_{\gamma \gamma'} : O_{\gamma} \rightarrow O_{\gamma} \otimes O_{\gamma'} \),

\[2\]If \( A \) is an algebra and \( u \in A \) is invertible, \( \text{Ad}(u) : A \rightarrow A \) is \( x \mapsto uxu^{-1} \)
\(\eta : k \to O_c\) and \(e : O_c \to k\), satisfying axioms such that when \(\Gamma\) is finite, these morphisms add up to a bialgebra structure on \(O\); and \(j : O \to k^\Gamma\) is a morphism of commutative algebras, compatible with the \(\Gamma\)-gradings and the maps \(\Delta_{\gamma^\prime}\) on both sides. We define \(\Gamma\)-commutative bialgebras as above.

We define the category of group (resp., \(\Gamma\)-) formal series Hopf (FSH) algebras as a full subcategory of the category of group (resp., \(\Gamma\)-) commutative bialgebras in \(S = \{\text{pro-vector spaces}\}\) by the condition the \(O_c\) (or equivalently, each \(O_r\)) is a formal series algebra.

**Proposition 2.1.** 1) We have (anti)equivalences of categories \{group Lie algebras\} \(\leftrightarrow\) \{group universal enveloping algebras\} (the last map is an anti-equivalence).

2) If \(\Gamma\) is a group, these (anti)equivalences restrict to \{\(\Gamma\)-Lie algebras\} \(\leftrightarrow\) \{\(\Gamma\)-universal enveloping algebras\}.

**Proof.** We denote the \(\Gamma\)-universal enveloping algebra corresponding to a \(\Gamma\)-Lie algebra \((\Gamma, a, \theta_a)\) as \(U(a) \rtimes \Gamma\). It is isomorphic to \(U(a) \otimes k^\Gamma\) as a vector space; if we denote by \(x \mapsto [x]\), \(\gamma \mapsto [\gamma]\) the natural maps \(a \mapsto U(a) \rtimes \Gamma\), \(\Gamma \mapsto U(a) \rtimes \Gamma\), then the bialgebra structure of \(U(a) \rtimes \Gamma\) is given by \([\gamma][x][\gamma^\prime] = [\theta_a(x)]\), \([\gamma][\gamma^\prime] = [\gamma\gamma^\prime]\), \([e] = 1\), \([x][x'] - [x'][x] = [[x, x']\], \(\Delta([x]) = [x] \otimes 1 + 1 \otimes [x]\), \(\Delta([\gamma]) = [\gamma] \otimes [\gamma]\).

When \(\Gamma\) is finite, the corresponding \(\Gamma\)-FSH algebra is then \((U(a) \otimes k^\gamma)^*\), and in general, this is \(\oplus_{\gamma \in \Gamma}(U(a) \otimes k_\gamma)^*\). One checks that these are (anti)equivalences of categories. For example, if \(A\) is a group universal enveloping algebra, then one recovers \(\Gamma\) as \{group-like elements of \(A\)\} and \(a\) as \{primitive elements of \(A\)\}. \(\square\)

### 2.2. \(\Gamma\)-Lie bialgebras and equivalent categories.

A group Lie bialgebra is a 5-uple \((\Gamma, A, \theta, \delta, f)\) where \((\Gamma, a, \theta_a)\) is a group Lie algebra, \(\delta : a \to \wedge^2(a)\) is\(^3\) such that \((a, \delta_a)\) is a Lie bialgebra, and \(f : \Gamma \to \wedge^2(a)\) is a morphism \(\gamma \mapsto f_\gamma\), such that: (a) \(\wedge^2(\theta_a) \circ \delta \circ \theta_a^{-1}(x) = \delta(x) + [f_\gamma, x \otimes 1 + 1 \otimes x]\) for any \(x \in a\), (b) \(f_{\gamma \gamma'} + f_{\gamma'} + \wedge^2(\theta_a)(f_\gamma)\), and (c) \((\delta \otimes \text{id})(f_\gamma) + [f_\gamma^1, f_\gamma^2] + \text{cyclic permutations} = 0\).

Group Lie bialgebras form a category, where a morphism \((\Gamma, A, a, \theta, \delta, f) \to (\Gamma', A', a', \theta', \delta', f')\) is a group Lie algebra morphism \((\Gamma, a, \theta_a) \to (\Gamma', a', \theta'_a)\), such that \(i_{a a'} : a \to a'\) is a Lie bialgebra morphism and \(\wedge^2(i_{a a'})(f_\gamma) = f'_{\gamma \gamma'}(\gamma)\). When \(\Gamma\) is fixed, one defines the category of \(\Gamma\)-Lie bialgebras as above.

A co-Poisson structure on a group cocommutative bialgebra \((\Gamma, A, i)\) is a co-Poisson structure \(\delta_A : A \to \wedge^2(A)\), such that \(\delta_A(A_r) \subset \wedge^2(A_r)\). Co-Poisson group cocommutative bialgebras form a category, where a morphism \((\Gamma, A, i, \delta_A) \to (\Gamma', A', i', \delta_A')\) is a morphism \((\Gamma, A, i) \to (\Gamma', A', i')\) of group cocommutative bialgebras, compatible with the co-Poisson structures. Co-Poisson group universal enveloping algebras form a full subcategory of the latter category. One defines the full subcategories of co-Poisson \(\Gamma\)-cocommutative bialgebras and co-Poisson \(\Gamma\)-enveloping algebras as above.

A Poisson structure on a group commutative bialgebra \((\Gamma, O, j)\) is a Poisson bialgebra structure \{-, -\} : \(\wedge^2(O) \to O\), such that \(\{O_r, O_r\} \subset O_r\) and \(\{O_r, O_r'\} = 0\) if \(\gamma \neq \gamma'\). Poisson group bialgebras form a category, and Poisson group FSH algebras form a full subcategory when \(S = \{\text{pro-vector spaces}\}\). One defines the full subcategories of Poisson \(\Gamma\)-bialgebras and Poisson \(\Gamma\)-FSH algebras as above.

**Example.** Let \(G\) be a Poisson-Lie (e.g., algebraic) group, let \(\Gamma \subset G\) be a subgroup (which we view as an abstract group). We define \(\theta_\gamma := \text{Ad}(\gamma)\), where \(\text{Ad} : G \to \text{Aut}_L(a)\) is the adjoint action. If \(P : G \to \wedge^2(a)\) is the Poisson bivector, satisfying \(P(g^-1) = P(g') + \wedge^2(\text{Ad}(g))(P(g'))\), then we set \(f_\gamma := -P(\gamma)\). Then \((a, \Gamma, f)\) is a \(\Gamma\)-Lie bialgebra.

\(^3\)We view \(\wedge^2(V)\) as a subspace of \(V \otimes^2\).
Example. Assume that \((\mathfrak{a}, r_\mathfrak{a})\) is a quasitriangular Lie bialgebra and \(\theta : \Gamma \to \text{Aut}(\mathfrak{a}, t_\mathfrak{a})\) is an action of \(\Gamma\) on \(\mathfrak{a}\) by Lie algebra automorphisms preserving \(t_\mathfrak{a} := r_\mathfrak{a} + r_\mathfrak{a}^{-1}\). If we set \(f_\gamma := \theta_\gamma^{-2}(r) - r\), then \((\mathfrak{a}, \theta, f)\) is a \(\Gamma\)-Lie bialgebra (we call this a quasitriangular \(\Gamma\)-Lie bialgebra).

For example, \(\mathfrak{a}\) is a Kac-Moody Lie algebra, and \(\Gamma = \tilde{W}\) is the extended Weyl group of \(\mathfrak{a}\).

**Proposition 2.2.** 1) We have category (anti)equivalences \(\{\text{group bialgebras}\} \leftrightarrow \{\text{co-Poisson group universal enveloping algebras}\} \leftrightarrow \{\text{Poisson group FSH algebras}\}.

2) These restrict to category (anti)equivalences \(\{\Gamma\text{-bialgebras}\} \leftrightarrow \{\text{co-Poisson } \Gamma\text{-universal enveloping algebras}\} \leftrightarrow \{\text{Poisson } \Gamma\text{-FSH algebras}\}.

**Proof.** If \((\mathfrak{a}, \theta_\mathfrak{a}, \delta_\mathfrak{a})\) is a \(\Gamma\)-Lie bialgebra, then the co-Poisson structure on \(A := U(\mathfrak{a}) \times \Gamma\) is given by \(\delta_A([x]) = [\delta_\mathfrak{a}(x)]\), and \(\delta_A([\gamma]) = -[f_\gamma([\gamma] \otimes [\gamma])].\) (Here we also denote by \(x \mapsto [x]\) the natural map \(\wedge^2(\mathfrak{a}) \to \wedge^2(U(\mathfrak{a}) \times \Gamma)\).) One checks that this establishes the desired (anti)equivalences. \(\square\)

2.3. The problem of quantization of \(\Gamma\)-Lie bialgebras. Define a \(\Gamma\)-graded bialgebra (in a symmetric monoidal category \(S\)) as a bialgebra \(A\) (in \(S\)), equipped with a grading \(A = \oplus_{\gamma \in \Gamma} A_\gamma\), such that \(A_\gamma A_\nu \subset A_{\gamma \nu}\) and \(\Delta_A(A_\gamma) \subset A_\gamma^\otimes 2\).

Assume that \(A\) is a \(\Gamma\)-graded bialgebra in the category of topologically free \(k[[h]]\)-modules, quasicocommutative (in the sense that \(A_0 := A/hA\) is cocommutative). Then we get a co-Poisson structure on \(A_0\). It is \(\Gamma\)-graded, in the sense that \(\delta_{A_0}((A_0)_\gamma) \subset \wedge^2((A_0)_\gamma)\). We therefore get a classical limit functor class : \(\{\text{\(\Gamma\)-graded quasicocommutative bialgebras}\} \to \{\text{\(\Gamma\)-graded co-Poisson bialgebras}\}.

**Definition 2.3.** A quantization functor for \(\Gamma\)-Lie bialgebras is a functor \(\{\text{co-Poisson } \Gamma\text{-universal enveloping algebras}\} \to \{\text{\(\Gamma\)-graded quasicocommutative bialgebras}\},\) right inverse to class.

We define the category of group-graded bialgebras as follows: objects are pairs \((\Gamma, A)\), where \(\Gamma\) is a group and \(A\) is a \(\Gamma\)-graded bialgebra. A morphism \((\Gamma, A) \to (\Gamma', A')\) is the pairs of a group morphism \(i_{\Gamma'} : \Gamma \to \Gamma'\) and a bialgebra morphism \(i_{A'} : A \to A'\), compatible with the gradings.

One defines similarly the category of group-graded co-Poisson bialgebras and quantization functors for group Lie bialgebras.

2.4. Relation with quantization of co-Poisson bialgebras. We have inclusions of full subcategories \(\{\text{co-Poisson universal enveloping algebras}\} \subset \{\text{co-Poisson group universal enveloping algebras}\} \subset \{\text{co-Poisson bialgebras}\}.

The classical limit functor is class : \(\{\text{quasicocommutative bialgebras}\} \to \{\text{co-Poisson bialgebras}\}.

A quantization functor for Lie bialgebras is a functor \(\{\text{co-Poisson universal enveloping algebras}\} \to \{\text{quasicocommutative bialgebras}\},\) left inverse fo class. A quantization functor for group Lie bialgebras may then be viewed as a left inverse to class with a wider domain.

3. The formalism of props

We recall material from [EnH]. Polynomial Schur functors form a symmetric monoidal abelian category \(\text{Sch}\), equipped with an involution. A prop \(P\) is an additive symmetric monoidal category, equipped with a tensor functor \(\text{Sch} \to P\), which induces a bijection \(\text{Ob}(\text{Sch}) \simeq \text{Ob}(P)\) ([McL]). A prop morphism \(P \to Q\) is a tensor functor, such that the composition \(\text{Sch} \to P \to Q\) coincides with \(\text{Sch} \to Q\). A topological prop is defined in the same way, with \(\text{Sch}\) replaced by the category of "formal series" Schur functors (i.e., infinite sums of homogenous Schur functors). If \(F\) is a (formal series) Schur functor and \(P\) is a (topological) prop, then \(F(P)\) is a prop defined by \((F(P))(F_1, F_2) = P(F_1 \circ F, F_2 \circ F)\).
Props may be defined by generators and relations. We will need the props Bialg of bialgebras, LBA of Lie bialgebras, LBA\(_f\) of Lie bialgebras with a twist. Generators of Bialg are \(m, \Delta, \epsilon, \eta\) (the universal analogues of the product, coproduct, counit, unit of a bialgebra); generators of LBA are \(\mu, \delta\) (universal analogues of the Lie bracket and cobracket); LBA\(_f\) has the additional generator \(f\) (universal twist element). LBA and LBA\(_f\) are graded (\(\mu\) has degree 0 and \(\delta, f\) have degree 1) and can be completed into topological props LBA, LBA\(_f\).

We define tensor categories Sch\(_{(1)}\) and Sch\(_{(1+1)}\) by \(\text{Ob}(\text{Sch\(_{(1)}\)}) = \bigoplus_{n \geq 0} \text{Ob}(\text{Sch}_n)\) and \(\text{Sch}\(_{(1+1)}\) = \bigoplus_{p,q \geq 0} \text{Ob}(\text{Sch}_{p+q})\), where \(\text{Ob}(\text{Sch}_n)\) is the set of polynomial Schur multifunctors \(\text{Vect}^n \rightarrow \text{Vect}\); the tensor product in these categories is denoted \(\otimes\). The bifunctor \(\text{Sch}\(_{(1)}\) \rightarrow \text{Sch}\(_{(1+1)}\)\) is denoted \((F,G) \mapsto F \boxtimes G\). A multi(bi)prop is an additive symmetric monoidal category \(\tilde{P}\) with a tensor functor \(\text{Sch}\(_{(1)}\) \rightarrow \tilde{P}\) (resp., \(\text{Sch}\(_{(1+1)}\) \rightarrow \tilde{P}\)), inducing a bijection on the sets of objects. A prop \(P\) gives rise to a multi-prop \(\tilde{P}\) via \(\tilde{P}(F,G) := P(c(F), c(G))\), where \(c : \text{Ob}(\text{Sch}_n) \rightarrow \text{Ob}(\text{Sch})\) is induced by the diagonal embedding \(\text{Vect} \rightarrow \text{Vect}^n\). We introduce the notions of a trace on a symmetric monoidal category, of a quasi-category, that a symmetric monoidal category with a trace and an involution gives rise to a symmetric monoidal quasi-category (i.e., the compositions are not always defined). In particular, a trace on a multi-prop gives rise to a quasi-multi-bi-prop (i.e., an additive symmetric monoidal quasi-category with a morphism from \(\text{Sch}\(_{(1)}\)\) inducing a bijection on objects). We define traces on the multi-props arising from LBA and LBA\(_f\); this gives rise to quasi-multi-bi-props \(\Pi, \Pi_f\) with \(\Pi(F \boxtimes G, F' \boxtimes G') = \text{LBA}(c(F) \otimes c(G)^*, c(F') \otimes c(G')^*)\); the morphisms in \(\Pi_f\) are defined by a similar formula. We also define topological completions \(\Pi, \Pi_f\). When \(F, \ldots, G'\) are tensor products (in \(\text{Sch}\(_{(1)}\)\) of irreducible Schur functors, \(\Pi(F \boxtimes G, F' \boxtimes G')\) is graded by a set of oriented graphs; the composition of two (or several) morphisms is defined if the composition of their diagrams is acyclic. For general \(F \in \text{Ob}(\text{Sch}_n), \ldots, G' \in \text{Ob}(\text{Sch}_p)\), one can define the support of a given element of \(\Pi(F \boxtimes G, F' \boxtimes G')\) (again an oriented graph), and acyclicity is a sufficient condition for the composition of two (or several) morphisms to be defined. Using this criterion, one checks that the compositions involved in the future computations all make sense.

The motivation for working with such structures is that when \(F, \ldots, G'\) are tensor products (in \(\text{Sch}\(_{(1)}\)\) of tensor Schur functors (i.e., objects of \(\text{Sch}\(_{(1)}\) of the form \(V \mapsto V^\otimes n\)). \(\Pi(F \boxtimes G, F' \boxtimes G')\) may be viewed as a space of acyclic oriented diagrams; composition is then defined by connecting diagrams, and is of course only defined under acyclicity assumptions. When \(F, \ldots, G'\) are tensor products (in \(\text{Sch}\(_{(1)}\)\) of simple Schur functors, the morphisms are obtained from the case of tensor products of tensor functors by applying projectors in the group algebras of products of symmetric groups, preserving a partition of the vertices.

4. Compositions of twists

A quantization functor is a prop morphism \(Q : \text{Bialg} \rightarrow S(\text{LBA})\) with certain classical limit properties.

Let \(Q\) be an Etingof-Kazhdan (EK) quantization functor. It is constructed as follows. We define elements \(m_\Pi \in \Pi((S \boxtimes S)^{\otimes 2}, S \boxtimes S), \Delta_0 \in \Pi(S \boxtimes S, (S \boxtimes S)^{\otimes 2}), J \in \Pi(1\boxtimes 1, (S \boxtimes S)^{\otimes 2}), R_+ \in \Pi(S \boxtimes 1, S \boxtimes S), m_a \in \Pi((S \boxtimes 1)^{\otimes 2}, S \boxtimes 1), \Delta_a \in \Pi(S \boxtimes 1, (S \boxtimes 1)^{\otimes 2})\).

We define \(m_\Pi^{(j)} \in \Pi(((S \boxtimes S)^{\otimes j})^{\otimes 2}, (S \boxtimes S)^{\otimes 2})\) as the \(j\)th tensor power of the \(i\)fold iterate of \(m_\Pi\).

We have

\[
m_\Pi \circ R_+^{\otimes 2} = R_+ \circ m_a, \quad m_\Pi^{(2,2)} \circ \left( J \boxtimes (\Delta_0 \circ R_+) \right) = m_\Pi^{(2,2)} \circ \left( (R_+^{\otimes 2} \circ \Delta_a) \otimes J \right).
\]

Then \(m_a, \Delta_a\) satisfy the bialgebra relations. The functor \(Q\) is defined by \(m \mapsto m_a, \Delta \mapsto \Delta_a\).
Theorem 4.2. (Compatibility of quantization functors with twists) We have

\[ \kappa_2^p(m_{11}) = \Xi_f \circ \kappa_1^p(m_{11}) \circ (\Xi_f^{-1})^{\otimes 2}, \quad \kappa_2^{\Pi}(\Delta_0) = \Xi_f^{\otimes 2} \circ \kappa_1^\Pi(\Delta_0) \circ \Xi_f^{-1}, \] (1)

Proposition 4.1. (see [EnH]) There exists \(F, v, i\) with \(F \in \Pi_f(1, S(2), S(1)), v \in \Pi_f(1, S(2), S(1)),\) and \(i \in \Pi_f(S(2), S(1))\), such that \(F = 1 + \text{degree} > 0, v = 1 + \text{degree} > 0, i = \text{id}_{S(2)}\), is such that \(m_0 \circ (\text{Ad}(u) \circ u) = m_0 \circ (u \circ \text{id}_{S(2)})\).

In [EnH], we prove that this proposition has the following consequence:

Theorem 4.2. (Compatibility of quantization functors with twists) We have

\[ \kappa_2^{\Pi}(m_{11}) = i \circ \kappa_1^\Pi(m_{11}) \circ (i^{-1})^{\otimes 2}, \quad \kappa_1^\Pi(m_{11}^{2,2}) \circ (\Xi_f^{\otimes 2} \circ \kappa_2^\Pi(\Delta_0) \circ i) \circ F) = \kappa_1^\Pi(m_{11}^{2,2}) \circ (F \otimes \kappa_1^\Pi(\Delta_0),\) \]

As before, \(m_0^{(i,j)}\) is the \(j\)th tensor power of the \(i\) fold iterate to \(m_0\).

We will now study the behavior of the composition of twists under quantization.

Define a prop \(\text{LBA}_{f,f'}\) by generators \(\mu \in \text{LBA}_{f,f'}(\Delta^2, \text{id}), \delta \in \text{LBA}_{f,f'}(\text{id}, \Delta^2),\) and \(f, f' \in \text{LBA}_{f,f'}(1, \Delta^2)\) and relations: \(\mu, \delta, f\) satisfy the relations of \(\text{LBA}_f\), and \(f'\) is such that

\[ ((123) + (231) + (312)) \circ (\delta \otimes \text{id}_{T_2}) \circ f' + (\mu \otimes \text{id}_{T_2}) \circ ((1234) + (1324)) \circ (f \otimes f') = 0. \] (4)

Define prop morphisms \(\kappa_{i,j} : \text{LBA}_f \to \text{LBA}_{f,f'},\) by \(\kappa_{12} : (\mu, \delta, f) \mapsto (\mu, \delta, f), \kappa_{23} : (\mu, \delta, f) \mapsto (\mu, \delta + \text{ad}(f), f'), \kappa_{13} : (\mu, \delta, f) \mapsto (\mu, \delta, f + f').\)

Define prop morphisms \(\tilde{k}_{1}, \tilde{k}_{2}, \tilde{k}_{3} : \text{LBA} \to \text{LBA}_{f,f'}, (\mu, \delta) \mapsto (\mu, \delta), (\mu, \delta + \text{ad}(f)), (\mu, \delta + \text{ad}(f + f')).\)

Then we have \(\kappa_{14} \circ \kappa_1 = \tilde{k}_1, \kappa_{13} \circ \kappa_2 = \tilde{k}_3, \kappa_{12} \circ \kappa_1 = \kappa_{12} \circ \kappa_2 = \tilde{k}_2.\)

Lemma 4.3. We have

\[ \kappa_{12}^\Pi(\Xi_f) \circ \kappa_{12}^\Pi(\Xi_f) = \kappa_{13}^\Pi(\Xi_f). \] (5)

Proof. This follows from the fact that if \(f_a\) is a twist for \(a\) and \(f'_a\) is a twist for \(a_{f_a}\), then \(f_a + f'_a\) is a twist for \(a\), and \((a_{f_a})_{f'_a} \simeq (a_{f_a})_{f_a}.\)
Theorem 4.4. There exists $\nu \in \Pi_{f,f}((1,\bar{2})1, S\bar{(2)}1)$, such that $\nu = 1 + \text{degree} > 0$,
\[
\kappa_1^\Pi(m_{(2,2)}) \circ \left( \kappa_{12}^\Pi(F) \boxtimes (\kappa_1^\Pi(\Delta_a) \circ v) \right) = \kappa_1^\Pi(m_{(3,2)}) \circ \left( \nu \boxtimes (\kappa_{12}^\Pi(\Xi_2^1) \boxtimes \kappa_{23}^\Pi(F)) \boxtimes \kappa_{12}^\Pi(F) \right),
\]
(6)

and
\[
\kappa_1^\Pi(m_a) \circ \left( \nu \boxtimes (\kappa_{12}^\Pi(i_1) \circ \kappa_{23}^\Pi(i_1)) \right) = \kappa_1^\Pi(m_a) \circ \left( \kappa_{13}^\Pi(i_1) \boxtimes v \right),
\]
(7)

Proof. Let us prove (6). Applying $\kappa_1^\Pi$ to (2), we get
\[
\kappa_1^\Pi(m_{(2,2)}) \circ \left( (\kappa_{12}^\Pi(\Xi_f^1) \boxtimes \kappa_3^\Pi(J)) \boxtimes (\Delta_0 \circ \kappa_{23}^\Pi(v)) \right)
\]
(8)

Left composing (8) with $\kappa_{12}^\Pi(\Xi_f^1) \boxtimes$ and using these identities, we get
\[
kappa_1^\Pi(m_{(2,2)}) \circ \left( (\kappa_{12}^\Pi(\Xi_f^1) \circ \kappa_{23}^\Pi(I)) \boxtimes \kappa_{23}^\Pi(J) \right)
\]
(9)

Applying $\kappa_1^\Pi$ to (3), we get
\[
\kappa_1^\Pi(m_{(2,2)}) \circ \left( \kappa_{23}^\Pi(v) \boxtimes (\kappa_{23}^\Pi(R_+)) \right) = \kappa_1^\Pi(m_{(2,2)}) \circ \left( \kappa_{23}^\Pi(v) \boxtimes (\kappa_{23}^\Pi(R_+) \circ \kappa_{23}^\Pi(1)) \boxtimes \kappa_{23}^\Pi(v) \right),
\]
(10)

and applying $\kappa_1^\Pi$ to (3), we get
\[
\kappa_1^\Pi(m_{(2,2)}) \circ \left( \kappa_{23}^\Pi(v) \boxtimes (\kappa_{23}^\Pi(R_+)) \right) = \kappa_1^\Pi(m_{(2,2)}) \circ \left( \kappa_{23}^\Pi(v) \boxtimes (\kappa_{23}^\Pi(R_+) \circ \kappa_{23}^\Pi(1)) \boxtimes \kappa_{23}^\Pi(v) \right),
\]
(11)

(10) then implies
\[
\kappa_1^\Pi(m_{(2,2)}) \circ \left( (\kappa_{12}^\Pi(\Xi_f^1) \circ \kappa_{23}^\Pi(I)) \boxtimes (\Delta_0 \circ \kappa_{23}^\Pi(v)) \right)
\]
(12)

Applying $\kappa_{12}^\Pi$ to (2), we get
\[
\kappa_1^\Pi(m_{(3,2)}) \circ \left( \nu \boxtimes (\kappa_{12}^\Pi(i_1) \circ \kappa_{23}^\Pi(i_1)) \right) = \kappa_1^\Pi(m_{(3,2)}) \circ \left( \kappa_{12}^\Pi(v) \boxtimes (\kappa_{23}^\Pi(1)) \boxtimes \kappa_{23}^\Pi(v) \right),
\]
(13)
Therefore "right multiplication" (using $\Pi_{11}$) of the previous identity by $\kappa_{12}^\Pi(\Delta_0 \circ v)$ yields
\[
\kappa_1^\Pi(m_{11}^{(3,2)}) \circ \left( \left( \kappa_{12}^\Pi(\Xi_{f}^{-1}) \circ \kappa_{23}^\Pi(\Xi_{f}^{-1}) \right)^{\otimes 2} \circ \kappa_3^\Pi(J) \right) \otimes \left( \Delta_0 \circ \kappa_{12}^\Pi(\Xi_{f}^{-1}) \circ \kappa_{23}^\Pi(v) \right) \cong \kappa_1^\Pi(\Delta_0 \circ v)
\]
\[
= \kappa_1^\Pi(m_{11}^{(4,2)}) \circ \left( \left( \kappa_{12}^\Pi(\Xi_{f}^{-1}) \circ \kappa_{23}^\Pi(\Xi_{f}^{-1}) \circ \kappa_3^\Pi(R_+) \circ \kappa_{23}^\Pi(i) \right)^{\otimes 2} \circ \kappa_2^\Pi(F) \right)
\]
\[
\otimes \left( \kappa_{12}^\Pi(\Xi_{f}^{-1}) \circ \kappa_{23}^\Pi(v) \right)^{\otimes 2} \cong \left( \kappa_{12}^\Pi(\Xi_{f}^{-1}) \circ \kappa_{23}^\Pi(\Xi_{f}^{-1}) \circ \kappa_3^\Pi(R_+) \circ \kappa_{23}^\Pi(i) \right)^{\otimes 2} \circ \kappa_2^\Pi(F)
\]
\[
= \kappa_1^\Pi(m_{11}^{(5,2)}) \circ \left( \left( \kappa_{12}^\Pi(\Xi_{f}^{-1}) \circ \kappa_{23}^\Pi(\Xi_{f}^{-1}) \circ \kappa_3^\Pi(R_+) \circ \kappa_{23}^\Pi(i) \right)^{\otimes 2} \circ \kappa_2^\Pi(F) \right)
\]
\[
\otimes \left( \kappa_{12}^\Pi(\Xi_{f}^{-1}) \circ \kappa_{23}^\Pi(v) \right)^{\otimes 2} \cong \left( \kappa_{12}^\Pi(\Xi_{f}^{-1}) \circ \kappa_{23}^\Pi(\Xi_{f}^{-1}) \circ \kappa_3^\Pi(R_+) \circ \kappa_{23}^\Pi(i) \right)^{\otimes 2} \circ \kappa_2^\Pi(F)
\]
where the last equality follows from (12).
According to (9), the last term is equal to
\[
\kappa_{12}^\Pi(\Xi_{f}^{-1}) \circ \kappa_{23}^\Pi(m_{11}^{(5,2)}) \circ \left( \left( \kappa_{12}^\Pi(\Xi_{f}^{-1}) \circ \kappa_{23}^\Pi(R_+) \circ \kappa_{23}^\Pi(i) \right)^{\otimes 2} \circ \kappa_2^\Pi(F) \right)
\]
\[
\otimes \left( \kappa_{12}^\Pi(v) \right)^{\otimes 2} \cong \left( \kappa_{12}^\Pi(\Xi_{f}^{-1}) \circ \kappa_{23}^\Pi(\Xi_{f}^{-1}) \circ \kappa_3^\Pi(R_+) \circ \kappa_{23}^\Pi(i) \right)^{\otimes 2} \circ \kappa_2^\Pi(F)
\]
which according to (10) is equal to
\[
\kappa_{12}^\Pi(\Xi_{f}^{-1}) \circ \kappa_{23}^\Pi(m_{11}^{(5,2)}) \circ \left( \left( \kappa_{12}^\Pi(\Xi_{f}^{-1}) \circ \kappa_{23}^\Pi(R_+) \circ \kappa_{23}^\Pi(i) \right)^{\otimes 2} \circ \kappa_2^\Pi(F) \right)
\]
\[
\otimes \left( \kappa_{12}^\Pi(v) \right)^{\otimes 2} \cong \left( \kappa_{12}^\Pi(\Xi_{f}^{-1}) \circ \kappa_{23}^\Pi(\Xi_{f}^{-1}) \circ \kappa_3^\Pi(R_+) \circ \kappa_{23}^\Pi(i) \right)^{\otimes 2} \circ \kappa_2^\Pi(F)
\]
which we rewrite as
\[
\kappa_1^\Pi(m_{11}^{(5,2)}) \circ \left( \left( \kappa_{12}^\Pi(\Xi_{f}^{-1}) \circ \kappa_{23}^\Pi(\Xi_{f}^{-1}) \right)^{\otimes 2} \circ \kappa_3^\Pi(J) \right) \otimes \left( \kappa_{12}^\Pi(\Xi_{f}^{-1}) \circ \kappa_{23}^\Pi(R_+) \circ \kappa_{23}^\Pi(i) \right)^{\otimes 2} \circ \kappa_2^\Pi(F)
\]
\[
\otimes \left( \kappa_{12}^\Pi(v) \right)^{\otimes 2} \cong \left( \kappa_{12}^\Pi(\Xi_{f}^{-1}) \circ \kappa_{23}^\Pi(\Xi_{f}^{-1}) \circ \kappa_3^\Pi(R_+) \circ \kappa_{23}^\Pi(i) \right)^{\otimes 2} \circ \kappa_2^\Pi(F)
\]
(11) allows then to rewrite this as
\[
\kappa_1^\Pi(m_{11}^{(5,2)}) \circ \left( \left( \kappa_{12}^\Pi(\Xi_{f}^{-1}) \circ \kappa_{23}^\Pi(\Xi_{f}^{-1}) \right)^{\otimes 2} \circ \kappa_3^\Pi(J) \right) \otimes \left( \kappa_{12}^\Pi(\Xi_{f}^{-1}) \circ \kappa_{23}^\Pi(R_+) \circ \kappa_{23}^\Pi(i) \right)^{\otimes 2} \circ \kappa_2^\Pi(F)
\]
\[
\cong \left( \kappa_{12}^\Pi(\Xi_{f}^{-1}) \circ \kappa_{23}^\Pi(R_+) \circ \kappa_{12}^\Pi(\Xi_{f}^{-1}) \circ \kappa_{23}^\Pi(R_+) \circ \kappa_{23}^\Pi(i) \right)^{\otimes 2} \circ \kappa_2^\Pi(F)
\]
We therefore get:
\[
\kappa_1^\Pi(m_{11}^{(5,2)}) \circ \left( \left( \kappa_{12}^\Pi(\Xi_{f}^{-1}) \circ \kappa_{23}^\Pi(\Xi_{f}^{-1}) \right)^{\otimes 2} \circ \kappa_3^\Pi(J) \right) \otimes \left( \kappa_{12}^\Pi(\Xi_{f}^{-1}) \circ \kappa_{23}^\Pi(R_+) \circ \kappa_{23}^\Pi(i) \right)^{\otimes 2} \circ \kappa_2^\Pi(F)
\]
\[
= \kappa_1^\Pi(m_{11}^{(3,2)}) \circ \left( v_1^{\otimes 2} \otimes F_1 \right) \circ \kappa_1^\Pi(J) \ ,
\]
where
\[
v_1 = \kappa_1^\Pi(m_{11}) \circ \left( \kappa_{12}^\Pi(\Xi_{f}^{-1}) \circ \kappa_{23}^\Pi(v) \right)^{\otimes 2}
\]
\[
F_1 = \kappa_1^\Pi(m_{11}^{(2,2)}) \circ \left( \left( \kappa_{12}^\Pi(R_+) \circ \kappa_{12}^\Pi(i^{-1}) \circ \kappa_3^\Pi(J) \right)^{\otimes 2} \circ \kappa_{12}^\Pi(F) \right)
\]
\[
= \kappa_1^\Pi(R_+) \circ \kappa_1^\Pi(m_{11}^{(2,2)}) \circ \left( \left( \kappa_{12}^\Pi(i^{-1}) \circ \kappa_3^\Pi(J) \right)^{\otimes 2} \circ \kappa_{12}^\Pi(F) \right) = \kappa_1^\Pi(R_+) \circ \kappa_1^\Pi(F)
\]
(5) implies that (13) is rewritten as
\[
\kappa_1^\Pi(m_{11}^{(2,2)}) \circ \left( \left( \kappa_{12}^\Pi(\Xi_{f}^{-1}) \circ \kappa_{23}^\Pi(J) \right) \otimes \left( \kappa_{12}^\Pi(\Xi_{f}^{-1}) \circ \kappa_{23}^\Pi(R_+) \circ \kappa_{23}^\Pi(i) \right)^{\otimes 2} \circ \kappa_2^\Pi(F) \right) \circ \kappa_1^\Pi(J)
\]
\[
= \kappa_1^\Pi(m_{11}^{(3,2)}) \circ \left( v_1^{\otimes 2} \otimes \left( \kappa_{12}^\Pi(R_+) \circ \kappa_{12}^\Pi(i^{-1}) \circ \kappa_3^\Pi(J) \right)^{\otimes 2} \circ \kappa_2^\Pi(F) \right) \circ \kappa_1^\Pi(J) \ ,
\]
(14)
On the other hand, applying $\kappa_H^1$ to (2), we get
\[ \kappa_H^1(m_{(2,2)}^{13}) \circ \left( (\kappa_H^1(\Xi_f^{-1}) \circ \kappa_H^1(J)) \otimes (\Delta_0 \circ v_1') \right) = \kappa_H^1(m_{(3,2)}^{13}) \circ \left( (v_1')^{22} \otimes (\kappa_H^1(R_+)^{22} \circ F_1') \right) \otimes \kappa_H^1(J), \]
where
\[ v_1' = \kappa_H^1(v), \quad F_1' = \kappa_H^1(F), \]
where $F_1' = \kappa_H^1(F)$.

The result of uniqueness (up to gauge) for solutions $(F, v) \in \Pi_f(1 \otimes S, S) \times \Pi_f(1 \otimes S, S)$ of equation (2), which was established in [EnH], Lemma 5.3, can be generalized as follows.

**Lemma 4.5.** The set of pairs $(F''_v, v'')$ satisfying (14), where $F''_v \in \Pi_{f,f'}(1 \otimes S, S)^{2}(\alpha)$, $v''_v \in \Pi_{f,f'}(1 \otimes S, S)^{2}$, $\Pi''_v = 1 + \text{degree} > 1, v''_v = 1 + \text{degree} > 1$, is given by $v_1 = \kappa_H^1(m_{11}) \circ \left( v''_v \otimes (\kappa_H^1(R_+) \circ v'') \right)$.

The proof of that parallel to that of [EnH], Lemma 5.3. The computation of the co-Hochschild cohomology of $(\mathbb{U}_n, f)_{n=0}^n$ is replaced by that of $(\mathbb{U}_{f,n}, f)_{n=0}^n$, where $\mathbb{U}_{f,n} \times \Pi_{f,n}(1 \otimes S, S)^{2}(\alpha)$, and the argument of the vanishing of LBA of $\mathbf{id}, 1$ replaced by the vanishing of LBA of $\mathbf{id}, 1$.

It follows that there exists $v \in \Pi_{f,f'}(1 \otimes S, S)^{2}$, $v = 1 + \text{degree} > 0$, such that
\[ v_1 = \kappa_H^1(m_{11}) \circ \left( v_1' \otimes (\kappa_H^1(R_+) \circ v) \right), \quad \kappa_H^1(m_{a}^{(2,2)}) \circ \left( F_1'' \otimes (\kappa_H^1(\Delta_0) \circ v) \right) = \kappa_H^1(m_{a}^{(3,2)}) \circ \left( v^{22} \otimes F_1 \right). \]

The second of these identities is (6).

Let us now prove (7). Right composing (3) with $i^{-1}$, applying $\kappa_{H2}$, left composing with $\kappa_{H2}(\Xi_1^{-1})$, and right multiplying the resulting identity by $\kappa_{H2}(v)$ using $\kappa_{H1}(m_{11})$, we get
\[ \kappa_{H2}(m_{(3,1)}^{11}) \circ \left( (\kappa_{H2}(\Xi_1^{-1}) \circ \kappa_{H2}(R_+)) \otimes (\kappa_{H2}(v) - \kappa_{H2}^{-1}(v)) \otimes \kappa_{H1}(v) \right) \]
\[ = \kappa_{H2}(m_{(3,1)}^{11}) \circ \left( (\kappa_{H2}(\Xi_1^{-1}) \circ \kappa_{H2}(v)) \otimes (\kappa_{H2}(v) \otimes (\kappa_{H2}(R_+), \kappa_{H2}^{-1}(v))) \otimes \kappa_{H1}(v) \right). \]

Right composing (3) by $i^{-1}$, applying $\kappa_{H2}$, right composing with $\kappa_{H2}^{-1}(i^{-1})$, and left multiplying by $\kappa_{H2}(\Xi_1^{-1}) \circ \kappa_{H2}(v)$ using $\kappa_{H1}(m_{11})$, we get
\[ \kappa_{H2}(m_{(3,1)}^{11}) \circ \left( (\kappa_{H2}(\Xi_1^{-1}) \circ \kappa_{H2}(v)) \otimes (\kappa_{H2}(v) \otimes (\kappa_{H2}(R_+), \kappa_{H2}^{-1}(i^{-1}))) \otimes \kappa_{H1}(v) \right) \]
\[ = \kappa_{H2}(m_{(3,1)}^{11}) \circ \left( (\kappa_{H2}(\Xi_1^{-1}) \circ \kappa_{H2}(v)) \otimes \kappa_{H2}(v) \otimes (\kappa_{H2}(R_+), \kappa_{H2}^{-1}(i^{-1}))) \otimes \kappa_{H1}(v) \right). \]
Combining (17), (18) and (20), we get:

\[
\kappa_1^\Pi(m_1^{(3,1)}) \circ \left( (\kappa_1^\Pi(\Xi_f^1) \circ \kappa_3^\Pi(R_+) \otimes (\kappa_1^\Pi(\Xi_f^3) \circ \kappa_2^\Pi(\nu)) \otimes \kappa_{12}^v(\nu) \right) \\
= \kappa_1^\Pi(m_1) \circ \left( \kappa_1^\Pi(\nu) \otimes [\kappa_1^\Pi(R_+) \circ \kappa_1^\Pi(m_a) \circ (v \otimes (\kappa_1^\Pi(\nu) \circ \kappa_{23}^\Pi(\nu)))] \right). \tag{21}
\]

On the other hand, left multiplying (19) by \(\kappa_1^\Pi(\Xi_f^1) \circ \kappa_3^\Pi(R_+)\) using \(\kappa_1^\Pi(m_1)\), we get

\[
\kappa_1^\Pi(m_1^{(3,1)}) \circ \left( (\kappa_1^\Pi(\Xi_f^1) \circ \kappa_3^\Pi(R_+) \circ (\kappa_1^\Pi(\Xi_f^3) \circ \kappa_2^\Pi(\nu)) \otimes \kappa_{12}^v(\nu) \right) \\
= \kappa_1^\Pi(m_1) \circ \left( (\kappa_1^\Pi(\Xi_f^1) \circ \kappa_3^\Pi(R_+) \otimes \kappa_1^\Pi(\nu) \otimes (\kappa_1^\Pi(\nu) \circ \kappa_{23}^\Pi(\nu))) \right). \tag{22}
\]

Right composing (3) by \(i^{-1}\), applying \(\kappa_1^\Pi\) and right multiplying by \(\kappa_1^\Pi(R_+) \circ v\) using \(\kappa_1^\Pi(m_1)\), we get

\[
\kappa_1^\Pi(m_1) \circ \left( (\kappa_1^\Pi(\Xi_f^1) \circ \kappa_3^\Pi(R_+) \otimes \kappa_1^\Pi(m_a) \circ \left( v \otimes (\kappa_1^\Pi(\nu) \circ \kappa_{23}^\Pi(\nu))) \right) \right) \\
= \kappa_1^\Pi(m_1) \circ \left( \kappa_1^\Pi(\nu) \otimes \kappa_1^\Pi(m_a) \circ \left( v \otimes \kappa_{13}^\Pi(\nu) \otimes v \right) \right). \tag{23}
\]

Combining (21), (22) and (23), we get

\[
\kappa_1^\Pi(m_a) \circ \left( v \otimes (\kappa_1^\Pi(\nu) \circ \kappa_{13}^\Pi(\nu) \circ \kappa_{23}^\Pi(\nu)) \right) = \kappa_1^\Pi(m_a) \circ \left( \kappa_{13}^\Pi(\nu) \otimes v \right),
\]

i.e., (7).

We define the prop \(\text{LBA}_{f,f',f''}\) by generators \(\mu, \delta, f, f', f''\), where \(\mu \in \text{LBA}_{f,f',f''}(\lambda^2, \text{id})\), \(\delta \in \text{LBA}_{f,f',f''}(\lambda \delta^2, f, f', f'' \in \text{LBA}_{f,f',f''}(1, \lambda^2)\) and relations: \(\mu, \delta, f, f'\) satisfy the relations of \(\text{LBA}_{f,f'}\), and \((\mu, \delta, f, f', f'')\) satisfy the relation (4) satisfied by \((\mu, \delta, f, f')\).

Define prop morphisms \(\kappa_{12}: \text{LBA}_{f,f',f''} \rightarrow \text{LBA}_{f,f',f''}\), \(\kappa_{123}: (\mu, \delta, f, f') \mapsto (\mu, \delta, f, f'), \kappa_{124}: (\mu, \delta, f, f') \mapsto (\mu, \delta, f, f') + f''\), \(\kappa_{14}: (\mu, \delta, f, f') \mapsto (\mu, \delta, f + f', f''), \kappa_{134}: (\mu, \delta, f, f') \mapsto (\mu, \delta, f + f', f'')\), \(\kappa_{24}: (\mu, \delta, f) \mapsto (\mu, \delta + \text{ad}(f), f', f'')\), \(\kappa_{234}: (\mu, \delta, f) \mapsto (\mu, \delta + \text{ad}(f), f', f'')\). Define prop morphisms \(\kappa_{ij}: \text{LBA} \rightarrow \text{LBA}_{f,f',f''}\) for \(i = 1, \ldots, 4\), by \(\kappa_1: (\mu, \delta) \mapsto (\mu, \delta), \kappa_2: (\mu, \delta) \mapsto (\mu, \delta + \text{ad}(f)), \kappa_3: (\mu, \delta) \mapsto (\mu, \delta + \text{ad}(f + f'))\).

Then \(\kappa_{ij} \circ \kappa_i = \kappa_i, \kappa_{ij} \circ \kappa_2 = \kappa_2, \kappa_{ij} \circ \kappa_3 = \kappa_3\) (where \(2 \leq i < j \leq 4\)) also have \(\kappa_{ij} \circ \kappa_1 = \kappa_1\) (where \(2 \leq i < j \leq 4\)); \(\kappa_234 \circ \kappa_1 = \kappa_234 \circ \kappa_2 = \kappa_234 \circ \kappa_3 = \kappa_234 \circ \kappa_3 = \kappa_3\) (where \(i = 1, 2, \kappa_{134} \circ \kappa_1 \circ \kappa_2 = \kappa_134 \circ \kappa_3 = \kappa_134 \circ \kappa_3 = \kappa_3\) (where \(i = 1, 2, 3, \kappa_{134} \circ \kappa_1 \circ \kappa_2 = \kappa_134 \circ \kappa_3 = \kappa_134 \circ \kappa_3 = \kappa_3\)).

Theorem 4.6.

\[
\kappa_1(m_a) \circ (\kappa_{134}^\Pi(v) \otimes \kappa_{1324}^\Pi(\nu)) = \kappa_1(m_a) \circ (\kappa_{124}^\Pi(v) \otimes (\kappa_{132}^\Pi(\nu) \circ \kappa_{234}^\Pi(\nu))). \tag{24}
\]
Proof. Applying \( \kappa_{134}^{(2,2)} \) to (6), we get
\[
\tilde{k}_1^{(2,2)}(m_a) \circ \tilde{k}_{14}^{(2,2)}(F) \otimes \kappa_{134}^{(2,2)}(\Delta_a \circ v)) = \kappa_{134}^{(2,2)}(m_a) \circ \left( \kappa_{134}^{(2,2)}(v) \otimes (\kappa_{134}^{(2,2)}(v) \otimes (\kappa_{134}^{(2,2)}(v) \otimes (\kappa_{134}^{(2,2)}(v))) \right).
\]
Using the fact that \( \kappa_{134}^{(1)}(v) = \kappa_{13}^{(1)}(F) \) and the image of (6) by \( \kappa_{123}^{(1)} \), we get
\[
\tilde{k}_1^{(2,2)}(m_a) \circ \left( \tilde{k}_{14}^{(2,2)}(F) \otimes \kappa_{134}^{(2,2)}(\Delta_a) \right) = \kappa_{134}^{(2,2)}(m_a) \circ \left( \kappa_{134}^{(2,2)}(v) \otimes (\kappa_{134}^{(2,2)}(v) \otimes (\kappa_{134}^{(2,2)}(v))) \right),
\]
which implies that
\[
\tilde{k}_1^{(2,2)}(m_a) \circ \left( \tilde{k}_{14}^{(2,2)}(F) \otimes \kappa_{134}^{(2,2)}(\Delta_a) \right) = \kappa_{134}^{(2,2)}(m_a) \circ \left( \kappa_{134}^{(2,2)}(v) \otimes (\kappa_{134}^{(2,2)}(v) \otimes (\kappa_{134}^{(2,2)}(v))) \right),
\]
where \( v_1 = \tilde{k}_1^{(2,2)}(m_a) \circ \kappa_{134}^{(2,2)}(F) \).

Applying \( \kappa_{123}^{(1)} \) to (6), we get
\[
\tilde{k}_1^{(2,2)}(m_a) \circ \left( \tilde{k}_{14}^{(2,2)}(F) \otimes \kappa_{134}^{(2,2)}(\Delta_a) \right) = \kappa_{134}^{(2,2)}(m_a) \circ \left( \kappa_{134}^{(2,2)}(v) \otimes (\kappa_{134}^{(2,2)}(v) \otimes (\kappa_{134}^{(2,2)}(v))) \right),
\]
and applying \( \kappa_{234}^{(2,2)} \) to the same identity, we get
\[
\tilde{k}_2^{(2,2)}(m_a) \circ \left( \tilde{k}_{14}^{(2,2)}(F) \otimes \kappa_{134}^{(2,2)}(\Delta_a) \right) = \kappa_{134}^{(2,2)}(m_a) \circ \left( \kappa_{134}^{(2,2)}(v) \otimes (\kappa_{134}^{(2,2)}(v) \otimes (\kappa_{134}^{(2,2)}(v))) \right),
\]
Since \( \tilde{k}_2(m_a) = \tilde{k}_{12}^{(2,2)}(i) \circ \tilde{k}_1(m_a) \circ \left( \tilde{k}_{12}^{(2,2)}(i) \right)^{-1} \), we get
\[
\tilde{k}_1^{(2,2)}(m_a) \circ \left( \tilde{k}_{14}^{(2,2)}(F) \otimes \kappa_{134}^{(2,2)}(\Delta_a) \right) = \kappa_{134}^{(2,2)}(m_a) \circ \left( \kappa_{134}^{(2,2)}(v) \otimes (\kappa_{134}^{(2,2)}(v) \otimes (\kappa_{134}^{(2,2)}(v))) \right),
\]
Right multiplying this identity by \( \kappa_{12}^{(2,2)}(F) \) using \( \kappa_{1}^{(2,2)}(m_a) \), and using \( \kappa_{13}^{(2,2)}(m_a) \circ \left( \kappa_{13}^{(2,2)}(i) \right)^{-1} \), we get
\[
\kappa_{13}^{(2,2)}(m_a) \circ \left( \tilde{k}_{14}^{(2,2)}(F) \otimes \kappa_{134}^{(2,2)}(\Delta_a) \right) = \kappa_{134}^{(2,2)}(m_a) \circ \left( \kappa_{134}^{(2,2)}(v) \otimes (\kappa_{134}^{(2,2)}(v) \otimes (\kappa_{134}^{(2,2)}(v))) \right),
\]
Right multiplying (26) by \( \kappa_{13}^{(2,2)}(\Delta_a) \circ \kappa_{134}^{(2,2)}(v) \) using \( \kappa_{1}^{(2,2)}(m_a) \), we then get
\[
\tilde{k}_1^{(2,2)}(m_a) \circ \left( \tilde{k}_{14}^{(2,2)}(F) \otimes \kappa_{134}^{(2,2)}(\Delta_a) \right) = \kappa_{134}^{(2,2)}(m_a) \circ \left( \kappa_{134}^{(2,2)}(v) \otimes (\kappa_{134}^{(2,2)}(v) \otimes (\kappa_{134}^{(2,2)}(v))) \right),
\]
where \( v_1 = \tilde{k}_1^{(2,2)}(m_a) \circ \kappa_{134}^{(2,2)}(F) \).

There exists a unique \( w \in \Pi_{f, n}(\tilde{k}_{14}^{(2,2)}, S_{\tilde{k}_{14}^{(2,2)}}) \), of the form \( w = 1 + \text{degree} > 0 \), such that \( v_1 = \kappa_{1}^{(2,2)}(m_a) \circ (w \otimes v_2) \). Then (25) and (27) imply that
\[
\tilde{k}_1^{(2,2)}(m_a) \circ \left( \tilde{k}_{14}^{(2,2)}(F) \otimes \kappa_{134}^{(2,2)}(\Delta_a) \right) = \kappa_{134}^{(2,2)}(m_a) \circ \left( \kappa_{134}^{(2,2)}(v) \otimes (\kappa_{134}^{(2,2)}(v) \otimes (\kappa_{134}^{(2,2)}(v))) \right),
\]
where \( w' := \tilde{k}_{14}^{(2,2)}(i) \circ w \) satisfies \( (w')^{\otimes 2} = \kappa_{12}^{(2,2)}(\Delta_a) \circ w' \).

Identity (24) now follows from:
Proposition 4.7. If \( x \in \Pi_{f,f',f''}(1,2,1,2) \) is of the form \( X = 1 + \text{degree} > 0 \) and \( x^{\Pi 2} = \tilde{k}_4(\Delta_a) \circ x \), then \( x = 1 \).

Proof of Proposition. \( \text{LBA}_{f,f',f''} \) is equipped with a prop automorphism \( \iota \), where \( \iota^2 = \text{id} \), uniquely defined by \( (\mu, \delta, f, f', f'') \mapsto (\mu, \delta + \text{ad}(f + f' + f''), -f'', -f', -f) \). Then \( \iota \circ \tilde{k}_4 = \tilde{k}_4 \).

Set \( y := \iota(y) \), then \( \tilde{k}_4(y) = y^{\Pi 2} \).

The prop \( \text{LBA}_{f,f',f''} \) is equipped with a degree, such that \( \deg(\mu) = 0 \) and \( \deg(\delta) = \deg(f) = \deg(f') = \deg(f'') = 1 \). We then decompose \( y = 1 + y \) for this degree. Assume that we showed \( y_1 = \ldots = y_{n-1} = 0 \). We then get: \( y_n \otimes 1 + 1 \otimes y_n = \text{the degree n part of} \) \( \tilde{k}_n(\Delta_n) \circ y_n \), i.e., \( = \Delta_0 \circ y_n \). According to the computation of the co-Hochschild cohomology of the complex \( S^{\otimes 0} \to S \to S^{\otimes 2} \to \ldots \) of Schur functors, we get \( y_n \in \Pi_{f,f',f''}(1,2,1,2) \subset \Pi_{f,f',f''}(1,2,1,2) \).

The degree \( n + 1 \) part of the equation \( \tilde{k}_n(\Delta_n) \circ y \) then yields (degree \( n + 1 \) part of \( \Delta_0 \circ y_{n+1} + \tilde{k}_n(\Delta_n) \circ y_n \)) \( = y_n \otimes 1 + 1 \otimes y_{n+1} \). Antisymmetrizing, we get \( \delta \circ y_n = 0 \).

We then show:

Lemma 4.8. The map \( \text{LBA}_{f,f',f''}(1,1,2) \to \text{LBA}_{f,f',f''}(1,1,2), \) \( y \mapsto \delta \circ y \) is injective.

Proof of Lemma. As in \([EnH]\), we will construct a retraction of this map. As in \([EnH]\), one shows that \( \text{LBA}_{f,f',f''}(F,G) \) is the cokernel of \( \text{LBA}(C \otimes D \otimes F, G) \to \text{LBA}(C \otimes F, G), \)

\( x \mapsto x \circ ((\text{id}_C \otimes \text{id}_D) \circ \Delta_C) \otimes \text{id}_F \) where \( C = S(\Lambda^2 \otimes \Lambda^2 \otimes \Lambda^2), D = \Lambda^3 \otimes \Lambda^3 \otimes \Delta_C : C \to C^{\otimes 2} \) is induced by the coalgebra structure of \( S \), and \( p \in \text{LBA}(C,D) = \oplus_{k \geq 0} \text{LBA}(S^k \circ (\Lambda^2 \otimes \Lambda^2 \otimes \Lambda^2), \Lambda^3 \otimes \Lambda^3 \otimes \Lambda^3) \) has nonzero components for \( k = 1,2 \) only; the \( k = 1 \), this component specializes to \( \Lambda^3(a)^{\otimes 3} \to \Lambda^3(a)^{\otimes 3} \).

\[ (f_a, f'_a, f''_a) \mapsto ((\delta_a \otimes \text{id}_a)(f_a) + c.p., (\delta_a \otimes \text{id}_a)(f'_a) + c.p., (\delta_a \otimes \text{id}_a)(f''_a) + c.p.), \]

where c.p. means cyclic permutation, and for \( k = 2 \) is specializes to \( S^2(\Lambda^3(a)^{\otimes 3}) \to \Lambda^3(a)^{\otimes 3} \),

\[ (f_a, f'_a, f''_a)^{\otimes 2} \mapsto ([f_a^{12}, f_a^{13}] + c.p., [f_a^{12}, f_a^{13} + f_a^{23}] + [f_a^{12}, f_a^{13}] + c.p., [f_a^{12}, f_a^{13} + f_a^{23}] + [f_a^{12}, f_a^{13}] + c.p.). \]

Since left and right compositions commute, we have a commutative diagram,

\[ \begin{array}{ccc}
\text{LBA}(C \otimes D, \text{id}) & \xrightarrow{\delta_{\otimes \text{id}}} & \text{LBA}(C \otimes D, \Lambda^2) \\
\downarrow & & \downarrow \\
\text{LBA}(C, \text{id}) & \xrightarrow{\delta_{\otimes \text{id}}} & \text{LBA}(C, \Lambda^2)
\end{array} \]

whose vertical cokernel is the map \( \text{LBA}_{f,f',f''}(1,1,2) \to \text{LBA}_{f,f',f''}(1,1,2), \) \( y \mapsto \delta \circ y \).

For any Schur functor \( A \), we will construct a retraction \( r_A : \text{LBA}(A, \Lambda^2) \to \text{LBA}(A, \text{id}) \) of the map \( \text{LBA}(A, \text{id}) \to \text{LBA}(A, \Lambda^2) \), such that the diagram

\[ \begin{array}{ccc}
\text{LBA}(C \otimes D, \Lambda^2) & \xrightarrow{r_{\otimes \text{id}}} & \text{LBA}(C \otimes D, \text{id}) \\
\downarrow & & \downarrow \\
\text{LBA}(C, \Lambda^2) & \xrightarrow{r} & \text{LBA}(C, \text{id})
\end{array} \] \hspace{1cm} (28)

commutes. The vertical cokernel of this map is then the desired retraction.

We have \( \text{LBA}(A, \text{id}) = \oplus_{Z \in \text{Irr} \text{(Sch)}} \text{LCA}(A, Z) \otimes \text{LCA}(Z, \text{id}) \). As in \([EnH]\), one shows that \( \text{LCA}(Z, \text{id} \otimes Z) \) is 1-dimensional, and one constructs an element \( \delta_Z \in \text{LCA}(Z, \text{id} \otimes Z) \), such that the component \( (Z', Z'') = (\text{id}, Z) \) of the map \( \text{LCA}(Z, \text{id}) \to \text{LBA}(Z, \Lambda^2) \subset \text{LBA}(Z, \Lambda^2) \), \( \oplus_{Z', Z'' \in \text{Irr} \text{(Sch)}} \text{LCA}(Z, Z' \otimes Z') \otimes \text{LCA}(Z', Z'' \otimes \text{id}) \otimes \text{LCA}(Z'', \text{id}) \) is \( \lambda \mapsto \delta_Z \otimes \text{id}_{\text{id}} \otimes \lambda \).

It follows that the component \( Z \mapsto (Z', Z'') = (\text{id}, Z) \) of the map \( \oplus_{Z \in \text{Irr} \text{(Sch)}} \text{LCA}(A, Z) \otimes \text{LCA}(Z, \text{id}) \) \( \simeq \text{LBA}(A, \Lambda^2) \subset \text{LBA}(A, \text{id}) \) is \( \kappa \otimes \lambda \mapsto (\lambda \otimes \delta_Z) \otimes \text{id}_{\text{id}} \otimes \kappa \).
Dually to [EnH], we construct a retraction of the map \( \lambda \mapsto \lambda \circ \delta \); it gives rise to the section \( r_A \). One then proves the commutativity of (28) as in [EnH].

This ends the proof of the lemma, and therefore also of Proposition 4.7 and Theorem 4.6.

We now draw the consequences of the results of the previous Subsection for the quantization of twists of Lie bialgebras.

Let \( a = (a, \mu_a, \delta_a) \) be a Lie bialgebra. Its quantization is \( Q(a) = (S(a)[[\hbar]], m(a), \Delta(a)) \), where \( m(a) := \mu_a(\hbar \delta_a, \hbar \delta_a) \), \( \Delta(a) := \Delta_a(\mu_a, h \delta_a) \). We set \( F(a, f_a) := F(\mu_a, h \delta_a, h f_a) \), \( i(a, f_a) := i(\mu_a, h \delta_a, h f_a) \).

Then \( F(a, f_a) \in Q(a)^{\otimes 2} \) and \( i(a, f_a) : Q(a) \to Q(a_{f_a}) \) are such that

\[
F(a, f_a)(\otimes 1) \ast (\Delta(a) \otimes id)(F(a, f_a)) = (1 \otimes F(a, f_a)) \ast (id \otimes \Delta(a))(F(a, f_a))
\]

(where the product \( m(a) \) is denoted \( \ast \)).

Assume that \( f'_a \) is a twist of \( a_{f_a} \). Then Theorem 4.4 implies that \( v(a, f_a, f'_a) := v(\mu_a, h \delta_a, h f_a, h f'_a) \) satisfies

\[
F(a, f_a + f'_a) = v(a, f_a, f'_a)^{\otimes 2} \ast (i(a, f_a)^{\otimes 2})^{-1}(F(a_{f'_a}, f'_a)) \ast F(a, f_a) \ast \Delta(a)(v(a, f_a, f'_a))^{-1},
\]

\[
i(a, f_a + f'_a) = i(a_{f'_a}, f'_a) \ast i(a, f_a) \circ \Delta(v(a, f_a, f'_a))^{-1}
\]

(in both equalities, \( m(a) \) in understood; it is denoted \( \ast \) in the first equality).

Finally, Theorem 4.6 implies that if \( f''_a \) is a twist of \( a_{f_a + f'_a} \), then

\[
v(a, f_a + f'_a, f''_a) \ast v(a, f_a, f'_a) = v(a, f_a, f'_a + f''_a) \ast i(a, f_a)^{-1}(v(a_{f'_a}, f'_a, f''_a)).
\]

5. Quantization of \( \Gamma \)-Lie bialgebras

5.1. Assume that \((a, \theta, f)\) is a \( \Gamma \)-Lie bialgebra. We construct its quantization as follows. Set \( A = S(a) \otimes k[\Gamma[[\hbar]]] \). We set \( |x| := x \otimes 1, [x \otimes x'] := (x \otimes \gamma) \otimes (x' \otimes \gamma') \in A^{\otimes 2} \).

There are unique linear maps \( m : A^{\otimes 2} \to A \) and \( \Delta : A \to A^{\otimes 2} \), such that

\[
m : [x \gamma][x' \gamma'] \mapsto [x \ast i(a, f_a)^{-1}(\theta_a(x')) \ast v(a, f_a, \gamma_2(\theta_a(f_a)))^{-1}, \gamma \gamma']
\]

\[
\Delta : [x \gamma] \mapsto [\Delta(a)(x) \ast F(a, f_a)^{-1}], \gamma_2.
\]

The unit for \( A \) is \([1, e] \), and the counit is the map \( |x| \mapsto \delta_{\gamma, e} \in (x) \) (recall that \( \ast \) denotes the product \( m(a) \) on \( S(a)[[\hbar]] \) or its tensor square).

Proposition 5.1. This defines a bialgebra structure on \( A \), quantizing the co-Poisson bialgebra structure induced by \((a, \theta, f)\).

Proof. This follows from the above relations on twists.

5.2. Propic version. The quantization of \( \Gamma \)-Lie bialgebras has a propic version, which we now describe.

Define \( L\Gamma \) as the prop with generators \( \mu \in LA(\lambda^2, \text{id}) \) and \( \theta \in LA(\text{id}, \text{id})^\times \), and relations: Jacobi identity on \( \mu, \Gamma \to L\Gamma(\text{id}, \text{id})^\times, \gamma \mapsto \theta_\gamma \) is a group morphism, and \( \lambda^2(\theta_\gamma) \circ \mu \circ \theta_\gamma^{-1} = \mu \).

Define \( L\Gamma \) as the prop with generators \( \mu \in L\Gamma(\lambda^2, \text{id}), \delta \in L\Gamma(\text{id}, \lambda^2), \theta_\gamma \in L\Gamma(\text{id}, \text{id}) \) and \( f_\gamma \in L\Gamma(1, \lambda^2) \), and relations: \((\mu, \delta) \) satisfy the relations of the prop \( L\Gamma \), \((\mu, \theta_\gamma, f_\gamma) \) satisfy the relations of \( L\Gamma \); for each \( \gamma \in \Gamma \), \((\mu, \delta, f_\gamma) \) satisfy the defining relations of \( L\Gamma \), as well as \( \lambda^2(\theta_\gamma) \circ \delta \circ \theta_\gamma^{-1} = \delta + \text{ad}(f_\gamma) \), and for each pair \( \gamma, \gamma' \in \Gamma \), \( f_{\gamma \gamma'} = f_\gamma + \lambda^2(\theta_\gamma) \circ f_\gamma \).

Define the prop \( Bialg_\Gamma \) of \( \Gamma \)-bialgebras as follows. When \( \Gamma \) is finite, in addition to the generators \( m, \Delta, \varepsilon, \eta \) of \( \text{Bialg} \), it has generators \( e_\gamma \in Bialg_\Gamma(\text{id}, \text{id}) \), and the additional relations
are \(\sum_{\gamma \in \Gamma} e_{\gamma} = \text{id}_{\mathbf{id}}, e_{\gamma} \circ e_{\gamma'} = \delta_{\gamma\gamma'} e_{\gamma}, m \circ (e_{\gamma} \boxtimes e_{\gamma'}) = e_{\gamma\gamma'} \circ m, \Delta \circ e_{\gamma} = e_{\gamma2} \circ \Delta, e_{\gamma} \circ \eta = \delta_{\gamma\eta} \eta, \varepsilon \circ e_{\gamma} = \delta_{\varepsilon\varepsilon}.\)

In general, \(\text{Bialg}_{\Gamma}\) is defined as follows. If \(S\) is a set, define \(\text{Sch}_S\) as the category of polynomial Schur functors \(\text{ Vect}^S \to V\) of the form \((V_s)_{s \in S} \to \otimes_{s \in S} M(Z_s) \otimes (\otimes_{s \in S} Z_s(V_s))\), where \(Z_s\) are almost all \(1\) (the unit Schur functor \(1(V) = k\)). Then \(\text{ Sch}_S\) is a symmetric tensor category. We define a \(S\)-prop as a symmetric tensor category \(P\) together with a natural transformation \(\text{ Sch}_S \to P\), which is the identity on objects. A \(S\)-prop may be defined by generators and relations. Then \(\text{Bialg}_{\Gamma\langle \text{prop}\rangle}\) is the \(\Gamma\)-prop defined by generators \(m_{\gamma,\gamma'} \in \text{Bialg}_{\Gamma\langle \text{prop}\rangle}(\mathbf{id}, \mathbf{id}, \mathbf{id}, \mathbf{id})\), \(\Delta_{\gamma} \in \text{Bialg}_{\Gamma\langle \text{prop}\rangle}(\mathbf{id}, \mathbf{id}, \mathbf{id})\), \(\varepsilon_{\gamma} \in \text{Bialg}_{\Gamma\langle \text{prop}\rangle}(\mathbf{id}, \mathbf{id})\), and we set \(\text{Bialg}_{\Gamma\langle \text{prop}\rangle}(\Gamma)\) := \(\text{Bialg}_{\Gamma\langle \text{prop}\rangle}(\Gamma^{\langle \text{prop}\rangle})\).

Then any \(E\)-quantization functor gives rise to a prop morphism \(\text{Bialg}_{\Gamma} \to S(\text{LBA}_{\Gamma})^\Gamma\) with suitable classical limit properties. A group morphism \(\Gamma \to \Gamma'\) gives rise to a commutative diagram

\[
\begin{array}{ccc}
\text{Bialg}_{\Gamma} & \to & S(\text{LBA}_{\Gamma})^\Gamma \\
\downarrow & & \downarrow \\
\text{Bialg}_{\Gamma'} & \to & S(\text{LBA}_{\Gamma'})^\Gamma'
\end{array}
\]

We have therefore a quantization functor \(\{\text{group Lie bialgebras}\} \to \{\text{quasicocommutative group bialgebras}\}\) (where both sides are full subcategories of \{co-Poisson cocommutative bialgebras\} and \{quasicocommutative bialgebras\}).

### 5.3. Quantization of quasitriangular \(\Gamma\)-Lie bialgebras

We defined a quasitriangular \(\Gamma\)-Lie bialgebra as a triple \((a, r_a, \theta_a)\), where \((a, r_a)\) is a quasitriangular Lie bialgebra (i.e., \(r_a \in a^{\otimes 2}\) satisfies the classical Yang-Baxter identity, and \(t_a := r_a + r_a^{21}\) is \(a\)-invariant), and \(\theta_a : \Gamma \to \text{Aut}(a, t_a)\) be an action of \(\Gamma\) by Lie algebra automorphisms of \(a\), preserving \(t_a\). It gives rise to a \(\Gamma\)-Lie bialgebra, with \(\delta(x) = [a, x^1 + x^2]\) and \(f_t := \theta_{\gamma2}(a) - r_a\).

In that case a quantization can be constructed directly: we set \(A = U(a) \rtimes \Gamma[[h]]\), the product is undeformed, and the coproduct is \(\Delta(x) = J(hr_a)\Delta_0(x) J(hr_a)^{-1}\) (\(\Delta_0\) is the standard coproduct).

Denote by \(q_{\Gamma}\) the prop of quasitriangular \(\Gamma\)-Lie bialgebras, and by \(\tilde{q}_{\Gamma}\) its completion. We have a natural prop morphism \(\text{LBA}_{\Gamma} \to q_{\Gamma}\). We claim that the prop morphisms \(\text{Bialg}_{\Gamma} \to S(\tilde{q}_{\Gamma})^\Gamma\) (the above direct construction) and the composed morphism \(\text{Bialg}_{\Gamma} \to S(q_{\Gamma})^\Gamma\) are equivalent (i.e., can be obtained from each other using an inner automorphism of \(S(q_{\Gamma})^\Gamma\)).

This is a consequence of the following statement on twists. Let \((a, r_a)\) be a quasitriangular Lie bialgebra and let \(f_a \in \Lambda^2(a)\) be a twist. According to [EK], there exists an invertible \(j(a, r_a) : U(a)[[h]] \to S(a)[[h]]\), such that \(m(a) = j(a, r_a) \circ m_0 \circ (j(a, r_a)^{\otimes 2})^{-1}, \Delta(a) = j(a, r_a)^{\otimes 2} \circ \text{Ad}(J(hr_a)) \circ \Delta_0 \circ j(a, r_a)^{-1}\). Then one proves that \(J(a, r_a + f_a) = j(a, f_a) \circ j(a, r_a) \circ \text{Ad}(v(a, r_a)^{-1})\), and \(J(a, r_a + f_a) = (v(a, r_a) \otimes v(a, r_a)) \cdot (j(a, r_a)^{\otimes 2})^{-1}(F(a, f_a)) \cdot J(a, r_a) \cdot \Delta_0(v(a, r_a))^{-1}\) (here \(*\) is the undeformed product on \(U(a)^{\otimes 2}[[h]]\)).

### 5.4. Open questions

Let \(a\) be a simple Lie algebra and let \(\tilde{W}\) be its extended Weyl group. One expects that the only possible quantization of \(U(a) \times \tilde{W}\) is the Majid-Sokolikman algebra of \(a\). When \(a\) is a Mac-Moody Lie algebra, one expects that if \(Q\) is any \(E\) quantization functor, then \(Q(a, \tilde{W})\) is the Majid-Sokolikman algebra of \(a\).

Both statements are analogues of well-known results [Dr, EK6].
REFERENCES


