Formality conjectures and deformation

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Résumé

Let M be a symplectic manifold over $\mathbb{R}$. Connes, Flato and Sternheimer constructed an invariant φ in the cyclic cohomology of M for any closed star-product. They compute this invariant in the de Rham complexe of M when $M = T^*V$. We complete this result by computing the image of φ in the de Rham complex for any symplectic manifold and any star-product and we show 1) that this invariant is a complete invariant of star-products 2) how this invariant is related to the general classification of Kontsevich. The proof uses the Riemann-Roch theorem for periodic cyclic chains of Nest-Tsygan. Finaly, we show that generalization of this invariant to any star-product over a Poisson manifold can be made by generalizing Kontsevitch formality conjectures.

§ 0 Introduction

Since their introduction by Bayen, Flato, Frodsal, Lichnerowicz and Sternheimer in 1975 (*cf.* [BFFLS1] and [BFFLS2]) existence and classification of star-products over a Poisson manifold has been a very important issue. Star-products were defined to give a mathematical framework for the analogy between classical and quantum equations in physics. Quantum mecanics was seen as a “deformation” of classical mecanics and instead of considering new objects such as operators, one still considers fonctions endowed with a new algebraic non-commutative structure. We will recall the definition of star-products in the first part.

The first results concerning existence and classifications have been given when the Poisson structure is regular of maximal rank, that is to say when the manifold is symplectic. Lecomte and De Wilde in 1983, Omori, Maeda and Yoshika in 1992 and Fedosov have given explicit constructions of such star-products and classifications depending on their constructions. We will discuss links between thoses classifications in the second part.

Nevertheless, it could be interesting to find a classification not depending on any construction. A good candidate for that was the invariant of Connes, Flato and Sternheimer who defined a cocycle for "closed star-product". In the third part, we will show how this cocycle can be defined for any star-product over a symplectic manifold. We will compute it in the De Rham complex using the index theorem of Nest and Tsygan (the computation was already done by Connes, Flato and Sternheimer when the manyfold is a cotangent bundle and the product given by composition of differential operators). Finaly, we show that this invariant is a complete invariant.

In the fourth part, we will use this invariant to relate the classification of Kontsevich with the one given in the symplectic case. This later result allows us to believe in a possible generalisation of the cocycle of Connes, Flato and Sternheimer. For that, we will have to generalize the index theorem of Nest and Tsygan.

In the fifth part, we will show how this generalization is related to a natural conjecture: the periodic cyclic complex of the deformed algebra is quasi-isomorphic to the corresponding Poisson-complex. This conjecture can be a consequense of generalisation of Kontsevich formality conjectures. We will show some possible ways to proove thoses conjectures: a first one will need an explicit homotopy formula. A second one will need extension of graphs definied by Kontsevich. In this part, we will also recall the basic definition of L_∞ morphisms.

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§ 1 Notations, definitions and first constructions

All along this note, M will denote a smooth manifold over a field $k = \mathbb{R}$ or $\mathbb{C}$ of dimension m. We will use the notation $\mathbb{A}$ for the space $C^\infty(M)[[h]]$ of series in the formal parameter h over the space of smooth functions over the manifold M. Let's recall the definition of star-products introduced in [BFFLS1] by Bayen, Flato, Frodsal, Lichnerowicz and Sternheimer :

Définition 1.1 *A star-product is a map* $\star : \mathbb{A}\times\mathbb{A} \to \mathbb{A}$, $\mathbb{R}[[h]]$*-bilinear, associative and satisfying, for all* f *and* g *in* $C^\infty(M)[[h]]$ *:*

$$f \star g = fg + hP_1(f,g) + h^2P_2(f,g) + \cdots$$

where the maps $P_i : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ *are bilinear maps, bidifferential and vanishing over constant functions.*

Two deformations $\star$ *et* $\star'$ *are called equivalent if there exist an isomorphism* $T = 1 + hT_1 + h^2T_2 + \cdots$ *(where the* T_i *are differential operators) such that :*

$$T(f \star g) = T(f)\star' T(g).$$

The space $\mathbb{A}$ *is then endowed with two algebraic structures :*

- we will write $\mathbb{A}_\cdot$ *when we use the canonical commutative mutiplication of functions*

- and we will write $\mathbb{A}_\star$ *when it is endowed with the star-product* $\star$*.*

Once constructed a star-product on the algebra $\mathbb{A}_\cdot$*, we can define another structure : let's call* $[-,-]_\star$ *the commutator for the product* $\star$*,*

$$[f,g]_\star = f \star g - g \star f.$$

We can write $[-,-]_\star = h\{-,-\}_\star + O(h^2)$*. Thanks to the definition of* $\star$*, we easily check that* $\{-,-\}_\star$ *is a Poisson bracket. In other words, there exist a* 2*-tensor* $\pi \in \Gamma(M, \wedge^2TM)$ *satisfying* $[\pi,\pi]_S = 0$ *(*$[-,-]_S$ *is the Schouten bracket) such that :*

$$\{f,g\}_\star = < \pi, df \wedge dg > .$$

If now we start with a Poisson manifold (a manifold endowed with a Poisson bracket), we will ask the star-product to satisfy $\{f,g\}_\star = (P_1(f,g) - P_1(g,f))$ *for all* f *and* g *in* $C^\infty(M)$ *or, without lost of generality,*

$$P_1 = \frac{1}{2}\{-,-\}.$$

First answers to that problem were given when the Poisson structure is regular of maximal rank, in other words, when the Poisson manifold is symplectic. The symplectic structure is built using the canonical isomorphism that define $\pi : \Gamma(M, \wedge TM) \xrightarrow{s_\pi} \Omega^\cdot(M)$ *: the symplectic form* ω

is then the image $s_\pi(\pi)$. In that case, the quantization problem becomes easier thank to the Darboux theorem: localy, a symplectic manifold (M,ω) is isomorphic to $(\mathbb{R}^{2n=m},\omega_0)$ where ω_0 is a constant symplectic form. We know (cf. [Ve] et [FLS]) that, for such a canonical symplectic manifold, there exist a unique (up to equivalence) deformation: the Moyal-Weyl product, which we denote by $\star_{MW}$.

The main problem, in the symplectic case, was to glue all those local deformations together. Though a construction was given by Gutt for the cotangent bundle of a Lie group (cf. [Gu2]), we had to wait till 1983 to have the proof of existence of star-products over a general symplectic manifold.

§ 2 Exitences and classifications in the symplectic case

The first proof of existence was given by De Wilde and Lecomte (cf. [DWL]), following works of Neroslavsky and Vlassov where we supposed the third de Rham cohomology group was trivial (cf. [NV]). In that construction, coefficients of the star-products were given by induction using very thin results on Hochschild cohomology and properties of the Gerstenhaber bracket.

De Wilde and Lecomte also gave a classification of deformations. This classification depends strongly on their construction, and tells that equivalence classes of star-products are in bijection with de Rham cohomology classes. More precisely, we have a one to one correspondance:

$$\{\star\}_{/\sim} \overset{\sim}{\rightarrow} \hbar[\omega] + \hbar^2 H^2(M)[[\hbar]].$$

In 1991, Omori, Y. Maeda, A. Yoshioka (cf. [OMY]) gave another proof of existence. Following, the same kind of approach, Fedosov (cf. [Fe1]) gave a more geometrical proof. The proof starts with the construction of a bundle over the manifold M, the Weyl bundle $\mathbb{W}$; its fiber over a point $m \in M$ is the symetric algebra generated by h and the elements of $T^*_m M$, $\mathbb{W}_m = S(T^*_m M)[[h]]$. Since the manifold M is symplectic, we can endowe each fiber with a constant symplectic structure, and thus also with the Moyal-Weyl star-product. So, we get a star-product (defined on each fiber) on the bundle $\mathbb{W}$. To get a star-product back on the manifold M, Fedosov uses a "flat" connection (which is also a derivation). Taking horizontal sections of this connection, we get an algebra $(\mathbb{A}_\nabla)$ isomorphic to $(C^\infty(M)[[h]]$ and thus by structure transportation, we build the product on M. This diagram summarizes *the construction:*

$$\begin{tikzcd}
\mathbb{W} \ : \mathbb{W}_m = S(T_m^*M)[[h]] \arrow[r] & \textit{star-product on each fiber } (\mathbb{W}, \star_{MW}) \arrow[d, "\nabla=\textit{flat connection}"', "+\ \textit{derivation}"] \\
M \ : \ (C^\infty(M)[[h]], \star) \arrow[u] & (\mathbb{A}_\nabla = \{f \in \Gamma(M, \mathbb{W}) | \nabla f = 0\}, \star_{MW}) \arrow[l, "\simeq"']
\end{tikzcd}$$

Moreover Fedosov showed that, up to equivalence, any star-product over a symplectic manifold can be constructed in that way. He also gave a classification of those star-products in term of de Rham cohomology classes using the Weyl curvature of the connection $\nabla : \theta = curv(\nabla) \in -\frac{1}{\hbar}\omega + \Omega^2(M)[[\hbar]]$.

Gutt and Deligne, in the language of algebraic geometry (cf. [De]), have compared the two constructions and classifications of Lecomte-De Wilde and Fedosov. One can also find in [Ha1] an explicit comparison: starting with a given star-product, its classes in the Lecomte-De Wilde and Fedosov classifications are related by the operator $h^3.\partial_h$, *namely:*

$$\begin{tikzcd}
\textit{Lecomte-De Wilde: } \{\star\}_{/\sim} \arrow[r, "\simeq"] \arrow[d, phantom, "="] & \hbar[\omega] + \hbar^2 H^2(M)[[\hbar]] \\
\textit{Fedosov: } \{\star\}_{/\sim} \arrow[r, "\theta"] & -\frac{1}{\hbar}[\omega] + H^2(M)[[\hbar]] \arrow[u, "h^3.\partial_h"']
\end{tikzcd}$$

Now we have shown links between different constructions and their classifications, a natural question arise: is there a way to classify star-products over a symplectic manifold not depending on any construction? Next paragraph will answer to that question.

§ 3 A complete invariant in the symplectic case

A good candidate for classifying star-products is the invariant defined by Connes, Flato and Sternheimer in [CFS]. This invariant, φ *was defined for closed star-products i. e. those for which the map*

$$\mathrm{Tr} \ : \ \mathbb{A}_\star \to k[[h, h^{-1}], \ f \mapsto \frac{1}{h^n}\int_M f$$

is a trace. To construct this invariant, they used the map $\sigma : C^\infty(M) \times C^\infty(M) \to C^\infty(M)[[\hbar]]$, $\sigma(f,g) = f \star g - fg$ *which takes into account the fact that identity map,* $\mathbb{A}_\cdot \to \mathbb{A}_\star$ *is not an algebra homomorphism.*

The map φ is a cocycle in the cyclic complex defined as follow:

$$\varphi :=\sum_{k\geq 0}\frac{1}{k!}\varphi_{2k}$$

where $\varphi_{2k}(f_0, f_1, \dots, f_{2k}) = \mathrm{Tr}(f_0 \star \sigma(f_1, f_2) \star \cdots \star \sigma(f_{2k-1}, f_{2k}))$.
Some questions remained opened:

1) Is it possible to define a similar invariant for any (non necessarly closed) star-product?

2) We know (cf. [Co]) that the periodic complex is quasi-isomorphic to the de Rham complex. What is the image of φ in this complex? (when $M = T^*V$ and the star-product is given by composition of differential opperators on V, Connes, Flato and Sternheimer showed that the image is Todd(V))

3) Is the invariant a complete invariant? (In other words, do different classes of star-product give different classes in the de Rham cohomology?)

Thanks to [Gu1] and [OMY], we know that, for any star-product $\star$, there exist a unique (up to a constant in $k[[h,h^{-1}]]$) trace $\mathrm{Tr}_{\mathrm{can}}$ *on* $\mathbb{A}_\star$. *So, for any star product, one can copy the work of [CFS], replacing* Tr *with* $\mathrm{Tr}_{\mathrm{can}}$. *That answers positively to the first question.*

Calculation of φ in the de Rham complex was made in [Ha2]:

as we have seen in the previous part, any star-product is equivalent to a one constructed using a connection ∇ (with corresponding Weyl curvature θ). Let's call $\mathbb{A}_\nabla$ the corresponding deformed algebra. Proof strongly used the index theorem of Nest-Tsygan (cf. [NT1], [NT2] and also [Fe2], [BNT]): the following diagram is commutative up to homotopy:

$$\begin{tikzcd}
\check{C}^{\cdot}(M, CC^{per}_{\cdot}(\mathbb{A}_\nabla)_{\mathbb{C}}) \arrow[r, "``h=0''"] \arrow[d, "\mu"'] & \check{C}^{\cdot}(M, CC^{per}_{\cdot}(\mathbb{A}_{\cdot})_{\mathbb{C}}) \arrow[d, "HKR"] \\
\check{C}^{\cdot}(M, \Omega^{\cdot}(M)[\hbar^{-1}, \hbar]]) & \check{C}^{\cdot}(M, \Omega^{\cdot}(M)[\hbar^{-1}, \hbar]]) \arrow[l, "\_\cup \hat{A}(TM)\cup e^{-\theta}"]
\end{tikzcd}$$

where "$h = 0$" is the canonical projection, HKR is the Hochschild-Kostant-Rosenberg quasi-isomorphism (cf. [HKR]), and $CC^{per}_{\cdot}$ (resp. $\check{C}^{\cdot}$) denote the cyclic periodic (resp. Chech) complex. The map μ is a quasi-isomorphism of complex, defined in [NT1] and called "trace density" as it satisfies:

$$\int_M \circ\, \mu = \mathrm{Tr}_{\mathrm{can}}\,.$$

Using work of [NT3], we can build a map $J : \check{C}^\cdot(M, CC^{per}_\cdot(\mathbb{A}_\cdot)_\mathbb{C}) \to \check{C}^\cdot(M, CC^{per}_\cdot(\mathbb{A}_\nabla)_\mathbb{C})$ *satisfying*

$$\text{“}h = 0\text{”} \circ J = \mathrm{Id} \quad \textit{and} \quad \frac{\int_M e^{\frac{\omega}{\hbar}}}{\int_M \hat{A}(TM)e^{-\theta}} \int_M \circ\, \mu \circ J = \varphi.$$

It is finaly obvious that the image of φ *in the de Rham complex is* $\frac{\int_M e^{\frac{\omega}{\hbar}}}{\int_M \hat{A}(TM)e^{-\theta}} \hat{A}(TM)e^{-\theta}$.

By the same time, we answered to the third question: as the class of φ *depends (and depends only) on the curvature* θ *classifying star-products on* M*, we see that the invariant of Connes, Flato and Sternheimer is a complete invariant.*

Moreover, we checked that this classification is compatible whith the one given by Kontsevich (cf. [Ko]) in term of classes of Poisson tensor fields for deformations on a general Poisson manifold. A natural question was then to generalize the invariant of [CFS] for any star-product over a Poisson manifold.

§ 4 Generalisation of the invariant

As we have seen in the last paragraph, the invariant φ *can be read in the commutative diagram of Nest and Tsygan and satisfies, up to a constant,* $\varphi = \int_M \circ\, \mu \circ J$*. So, to generalize* φ*, we will have to generalize the diagram. We will keep the notations of the previous paragraph.*

The curvature θ *defines a generalized symplectic form and so is non degenerated. It defines thus an isomorphism* $s'_\theta : T^*M[[\hbar]] \to TM[[\hbar]]$*, analogus of the inverse of the isomorphism* s_π *defined in the first paragraph. Let* $\pi_\theta \in \hbar\pi + \hbar^2\Gamma(M, \wedge^2 TM)[[\hbar]]$ *be the image of the form* $-\theta$ *through this isomorphism. We will also note* $D_\theta : \Omega^\cdot(M)[\hbar^{-1}, \hbar]] \to \Omega^{2n-\cdot}(M)[\hbar^{-1}, \hbar]]$ *the Poincaré duality induced by the form* $\frac{\theta^n}{n!}$*. Taking the left composition of the “trace density”* μ *by* D_θ*, one obtains a quasi-isomorphism of complexes* $\mu_\star = D_\theta \circ \mu$ *:*

$$\check{C}^\cdot(M, CC^{per}_\cdot(\mathbb{A}_\star)_{\mathbb{C}[\hbar^{-1},\hbar]]}, b_\star + B) \to \check{C}^\cdot(M, \Omega^\cdot(M)[\hbar^{-1}, \hbar]], \mathrm{d} + L_{\pi_\theta})$$

where $L_{\pi_\theta} = [d, i_{\pi_\theta}]$ *is the Poisson derivative (cf. [Br] et [Kos]). One checks that*

$$L_{\pi_\theta} \circ L_{\pi_\theta} = L_{[\pi_\theta, \pi_\theta]_S} = 0 \ \textit{et}\ D_\theta \circ \mathrm{d} = L_{\pi_\theta} \circ D_\theta.$$

So we have :

Proposition 4.1 *The following diagram is commutative up to homotopy:*

$$\begin{tikzcd}
\check{C}^{\cdot}(M, CC^{per}_{\cdot}(\mathbb{A}_\nabla)_{\mathbb{C}}, b_\star + B) \arrow[r, "``h=0''"] \arrow[d, "\mu_\star"'] & \check{C}^{\cdot}(M, CC^{per}_{\cdot}(\mathbb{A}_{\cdot})_{\mathbb{C}}, b + B) \arrow[d, "HKR"] \\
\check{C}^{\cdot}(M, \Omega^{\cdot}(M)[\hbar^{-1}, \hbar]], d + L_{\pi_\theta}) & \check{C}^{\cdot}(M, \Omega^{\cdot}(M)[\hbar^{-1}, \hbar]], d) \arrow[l, "\_\cup \hat{A}(TM)"]
\end{tikzcd}$$

PREUVE : *Notice that the map* $e^{-i_{\pi_\theta}} : (\Omega^{\cdot}(M), d) \to (\Omega^{\cdot}(M), d + L_{\pi_\theta})$ *is a morphism of complexes. Indeed,*

$$e^{-i_{\pi_\theta}}(d) e^{i_{\pi_\theta}} = e^{\mathrm{ad}(-i_{\pi_\theta})} d = d + L_{\pi_\theta}.$$

To proove the proposition, we just have to check that the morphism of complexes

$$e^{i_{\pi_\theta}} \circ D_\theta \circ e^{-i_{\pi_\theta}} \ : \ (\Omega^{\cdot}(M), d) \to (\Omega^{\cdot}(M), d)$$

correspond to the produit by the form $e^{-\theta}$*. For that, let's calculate the image of* 1 *through that map:*

$$\begin{aligned} e^{-i_{\pi_\theta}}(1) &= 1 \\ D_\theta(1) &= \frac{\theta^n}{n!} \\ e^{i_{\pi_\theta}}\left(\frac{\theta^n}{n!}\right) &= e^{-\theta}. \end{aligned}$$

We notice that, $i_{\pi_\theta}(\theta) = -n$ *and we can demonstrate by induction that* $i_{\pi_\theta}(\theta^n) = -n\theta^{n-1}$*. One proove then (once again by induction) that* $\frac{i^k_{\pi_\theta}}{k!}(\frac{\theta^n}{n!}) = -\frac{\theta^{n-k}}{(n-k)!}$*, and thus* $e^{i_{\pi_\theta}}(\frac{\theta^n}{n!}) = e^{-\theta}$*. The proposition is then a simple consequence of the algebraic index theorem (cf. last paragraph).* □

Suppose now that the Poisson structure $\{.,.\}$ *over* M *is not regular anymore. Let* $\star$ *be a deformation of* $(M, \{.,.\})$ *and* $\pi_\hbar$ *a generalised 2-tensor field in* $\hbar\pi + \hbar^2\Gamma(M, \wedge^2 TM)[[\hbar]]$ *satisfying* $[\pi_\hbar, \pi_\hbar]_S = 0$*. Once again, we hope to construct a quasi-isomorphism of complexes*

$$\mu_{\pi_\hbar} \ : \ (CC^{per}_{\cdot}(\mathbb{A}_\star), b_\star + B) \to (\Omega^{\cdot}(M), d + L_{\pi_\hbar}).$$

Existence of such a map is still a conjecture but would allow us to get the wanted classification of deformations. We thus want to generalize the algebraic Riemann-Roch theorem, and then to define the generalized $\hat{A}$*-genius of the Poisson manifold* $\hat{A}_\pi(M)$*: following the latest works* *of Tsygan, we hope that this Chern class would be defined to make this new diagram still commuting:*

$$\begin{tikzcd}
\check{C}^{\cdot}(M, CC_{\cdot}^{per}(\mathbb{A}_{\nabla})_{\mathbb{C}}, b_{\star}+B) \arrow[r, "``h=0''"] \arrow[d, "\mu_{\pi_\hbar}"'] & \check{C}^{\cdot}(M, CC_{\cdot}^{per}(\mathbb{A}_{\cdot})_{\mathbb{C}}, b+B) \arrow[d, "HKR"] \\
\check{C}^{\cdot}(M, \Omega^{\cdot}(M)[\hbar^{-1}, \hbar]], d+L_{\pi_\hbar}) & \check{C}^{\cdot}(M, \Omega^{\cdot}(M)[\hbar^{-1}, \hbar]], d). \arrow[l, "\_\cup \hat{A}_{\pi}(M)"]
\end{tikzcd}$$

Thus, one would find again the fact (cf. [Ko]) that equivalence classes of star-products are in one to one bijection with Gauge equivalence classes of generalized 2-vector tensor $\pi_\hbar$.

As we can see, the main difficulty is to build the morphism $\mu_{\pi_\hbar}$. Thanks to the technic of retracts by deformation perturbation (cf. [Ka]), we only have to proove existence of a quasi-isomorphism

$$\mu^0_{\pi_\hbar} \;:\; (C_{\cdot}(\mathbb{A}_{\star}), b_{\star}) \to (\Omega^{\cdot}(M), L_{\pi_\hbar}).$$

Cohomology of the complex $(\Omega^{\cdot}(M), L_{\pi_\hbar})$ is not yet well known but one could find in [Ma] some very interesting results. In the next paragraph we will show that construction of such a morphism is a consequence of the existence of a L_∞-homomorphism. Nevertheles, for the moment, we can only give the first terms of such an homomorphism, the existence of which is still a conjecture.

§ 5 Formality conjectures

Let's start with a definition:

Définition 5.1 *Let $\mathfrak{g}_1^{\cdot}$ and $\mathfrak{g}_2^{\cdot}$ be two $\mathbb{N}$-graded Lie algebras; a L_∞-homomorphism is given by maps $\rho_i \;:\; \wedge^i \mathfrak{g}_1 \to \mathfrak{g}_2$ of degree $1-i$:*

$$\begin{cases}
\rho_0 \in \mathfrak{g}_2^1 \\
\rho_1 \;:\; \mathfrak{g}_1^l \to \mathfrak{g}_2^l \\
\rho_2 \;:\; \mathfrak{g}_1^l \wedge \mathfrak{g}_1^m \to \mathfrak{g}_2^{l+m-1} \\
\cdots \\
\rho_i \;:\; \mathfrak{g}_1^{l_1} \wedge \cdots \wedge \mathfrak{g}_i^{l_i} \to \mathfrak{g}_2^{l_1+\cdots+l_i+1-i}
\end{cases}$$

satisfying, if $\rho = \sum_k \rho_k$ and ∂^{Lie} is the Lie differential,

$$[\rho, \rho] = \rho \circ \partial^{Lie}$$

where $[\rho,\rho](g_1\wedge\cdots\wedge g_l)=$

$$\sum_{i=0}^{l}\sum_{\sigma\in S_l}(-1)^{\{\sigma\}}[\rho_i(g_{\sigma(1)}\wedge\cdots\wedge g_{\sigma(i)}),\rho_{l-i}(g_{\sigma(i+1)}\wedge\cdots\wedge g_{\sigma(j)})]$$

and $\{\sigma\}$ *is such that* $g_{\sigma(1)}\wedge\cdots\wedge g_{\sigma(l)}=(-1)^{\{\sigma\}}g_1\wedge\cdots\wedge g_l$.

Using this definition, on can easily write the formality theorem of Kontsevich.

Théorème 5.2 *[Ko] Let* M *be a differential manifold. There exist a* L_∞*-homomorphism,* $K_\cdot$ *between the two graded Lie algebras* $\mathfrak{g}_1=(\Gamma(M,\wedge^{\cdot-1}TM),[.,.]_S)$ *and* $\mathfrak{g}_2=(C^{\cdot-1}(C^\infty(M),C^\infty(M)),[.,.]_G)$ *such that* K_0 *is egual to* m*, the commutative multiplication on* $C^\infty(M)$ *and* K_1 *is the morphism of Hochschild-Kostant-Rosenberg* HKR.

Corollaire 5.3

- *If* $(M,\{.,.\})$ *is a Poisson manifold,* π *the associated tensor field and* $K_\cdot$ *a* L_∞*-homomor-phism like in the previous theorem, the map*

$$m_\star=\sum_l\frac{1}{l!}K_l(\hbar\pi\wedge\cdots\wedge\hbar\pi)\ \in C^2(C^\infty(M),C^\infty(M))[[\hbar]]$$

is a deformation of $(M,\{.,.\})$.

- *If* $\mathbb{A}_\star$ *is the algebra* $(C^\infty(M)[[\hbar]],m_\star)$ *deformed using such an homomorphism. The map* $\phi_\pi:(\Gamma(M,\wedge^\cdot TM),[\pi,.]_S)\to(C^\cdot(\mathbb{A}_\star,\mathbb{A}_\star)[[\hbar]],\partial_\star)$,

$$\Lambda\mapsto\sum_l\frac{1}{l!}K_{l+1}(\hbar\pi\wedge\cdots\wedge\hbar\pi,\Lambda)$$

is a morphism of complexes (the differential $\partial_\star=[m_\star,.]_G$ *is the Hochschild coboundary defined over the algebra* $\mathbb{A}_\star$*).*

PREUVE : *Condition* $[\pi,\pi]_S=0$ *shows that*

$$\partial^{Lie}(\hbar\pi\wedge\cdots\wedge\hbar\pi)=0.$$

The corollary is then an easy consequence of theoreme 5.2. □

Now, we want to proove a similar result for the Hochschild homology. Let's recall that for all unitary algebra A*, the space* $C_\cdot(A)$ *is a module over the Lie algebra* $(C^\cdot(A,A),[.,.]_G)$ *(cf. [NT3] or [Ha2]). Let* C_1*,* C_2 *be elements of* $C^\cdot(A,A)$ *and* $\alpha\in C_\cdot(A)$ *; with the notations of [NT3] or [Ha2] we have :*

$$\begin{aligned}C_1\cdot\alpha&=L_{C_1}(\alpha)=(-1)^{|\alpha||C_1|}(\alpha\bullet_2(1,C_1))\\ [C_1,C_2]_G\cdot\alpha&=C_1\cdot(C_2\cdot(\alpha))-C_2\cdot(C_1\cdot(\alpha)).\end{aligned}$$

In the same way, the space $\Omega^{\cdot}(M)$ *is a module over the Lie algebra* $(\Gamma(M,\wedge^{\cdot}TM,[.,.]_S)$*. Let* Λ_1*,* Λ_2 *be elements of* $\Gamma(M,\wedge^{\cdot}TM)$ *and* $\alpha\in\Omega^{\cdot}(M)$ *;*

$$\begin{array}{rl}\Lambda_1{\cdot}\alpha & = L_{\Lambda_1}(\alpha)=[d,i_{C_1}](\alpha)\\ {[\Lambda_1,\Lambda_2]_S}{\cdot}\alpha & =\Lambda_1{\cdot}(\Lambda_2{\cdot}(\alpha))-\Lambda_2{\cdot}(\Lambda_1{\cdot}(\alpha)).\end{array}$$

When A *is the non deformed algebra* $\mathbb{A}_{\cdot}$*, one can construct two Lie algebras :* $(C^{\cdot}(\mathbb{A}_{\cdot},\mathbb{A}_{\cdot})\oplus C_{\cdot}(\mathbb{A}_{\cdot}),[.,.]'_G)$ *and* $(\Gamma(M,\wedge^{\cdot}TM)\oplus\Omega^{\cdot}(M),[.,.]'_S)$ *where*

$$\begin{array}{lcll}[.,.]'_G & = & [.,.]_G & \text{over } C^{\cdot}(\mathbb{A}_{\cdot},\mathbb{A}_{\cdot})\\ {[.,.]'_G} & = & 0 & \text{over } C_{\cdot}(\mathbb{A}_{\cdot})\\ {[C,\alpha]'_G} & = & C{\cdot}\alpha & \text{for } C \text{ in } C^{\cdot}(\mathbb{A}_{\cdot},\mathbb{A}_{\cdot}) \text{ and } \alpha \text{ in } C_{\cdot}(\mathbb{A}_{\cdot})\\ {[.,.]'_S} & = & [.,.]_S & \text{over } \Gamma(M,\wedge^{\cdot}TM)\\ {[.,.]'_S} & = & 0 & \text{over } \Omega^{\cdot}(M)\\ {[\Lambda,\alpha]'_S} & = & \Lambda{\cdot}\alpha & \text{for } \Lambda \text{ in } \Gamma(M,\wedge^{\cdot}TM) \text{ and } \alpha \text{ in } \Omega^{\cdot}(M)\end{array}$$

Conjecture 5.4 *There exists a* L_∞*-homomorphism,* $F_{\cdot}$*, between the two graded Lie algebras* $\mathfrak{g}_1{}'=(\Gamma(M,\wedge^{\cdot}TM)\oplus\Omega^{\cdot}(M),[.,.]'_S)$ *and* $\mathfrak{g}_2{}'=(C^{\cdot}(\mathbb{A}_{\cdot},\mathbb{A}_{\cdot})\oplus C_{\cdot}(\mathbb{A}_{\cdot}),[.,.]'_G)$ *such that* $F_0=m$*, commutative multiplication in* $C^\infty(M)$ *and*

$$F_i=K_i+I_{i-1}$$

where $I_l : \wedge^l\Gamma(M,\wedge^{\cdot}TM)\otimes\Omega^{\cdot}(M)\to C_{\cdot}(\mathbb{A}_{\cdot})$ *and* $K_i:\Gamma(M,\wedge^{\cdot}TM)\to C^{\cdot}(\mathbb{A}_{\cdot})$ *is a* L_∞*-homomorphism like in theorem 5.2.*

Corollaire 5.5 *Let* $(M,\{.,.\})$ *be a Poisson manifold,* π *the associated tensor field,* $K_{\cdot}$ *a* L_∞*-homomorphism like in theoreme 5.2,* $F_{\cdot}$ *a* L_∞*-homomorphism like in the last conjecture and* $m_\star=\sum_l\frac{1}{l!}K_l(\hbar\pi\wedge\cdots\wedge\hbar\pi)$ *the star-product associated to* $K_{\cdot}$*. The map*

$$\mu^*_\pi\ :\ (\Omega^{\cdot}(M),\hbar L_\pi)\to(C_{\cdot}(\mathbb{A}_\star),b_\star),\ \alpha\mapsto\sum_l\frac{1}{l!}I_l(\hbar\pi\wedge\cdots\wedge\hbar\pi\otimes\alpha)$$

is a complex homomorphism.

PREUVE : *The corollary is once again a straitforward consequence of theorem 5.2, the previous conjecture conjecture and the fact that* $\partial^{Lie}(\hbar\pi\wedge\cdots\wedge\hbar\pi)=0$. ☐

We will now study necessary conditions that must satisfy the first terms of the sequence $(I_l)_{l\geq0}:\ \wedge^l\Gamma(M,\wedge^{\cdot}TM)\otimes\Omega^{\cdot}(M)\ \to\ C_{\cdot}(\mathbb{A}_{\cdot})$*. First of all, we have to choose* I_0 *so that, for each coordinate system* $(x_1,\ldots,x_n)$*,*

$$I_0(f\,dx_{i_1}\wedge\cdots\wedge dx_{i_l})=f\otimes\mathrm{Alt}(x_{i_1}\otimes\cdots\otimes x_{i_l}).$$

Let's notice that the map $I_0 : (\Omega^{\cdot}(M), 0) \to (C_{\cdot}(\mathbb{A}_{\cdot}), b)$ *is a morphism of complexes "inverse" of* HKR, $(C_{\cdot}(\mathbb{A}_{\cdot}), b) \to (\Omega^{\cdot}(M), 0)$:

$$HKR \circ I = \mathrm{Id} \;\; \textit{in } C_{\cdot}(\mathbb{A}_{\cdot}) \textit{ and } I \circ HKR = \mathrm{Id} \;\; \textit{in } H_{\cdot}(\mathbb{A}_{\cdot}).$$

Now, let's build the map $I_1 : \Gamma(M, \wedge^{\cdot} TM) \otimes \Omega^{\cdot}(M) \to C_{\cdot}(\mathbb{A}_{\cdot})$. *This map must satisfy:*

$$\forall \Lambda \in \Gamma(M, \wedge^{\cdot} TM),\ \alpha \in \Omega^{\cdot}(M),\ I_0(\Lambda \cdot \alpha) - K_1(\Lambda) \cdot I_0(\alpha) = m \cdot I_1(\Lambda, \alpha).$$

If one looks at that condition on a local coordinate system $(x_1, ..., x_n)$ *in the particular case where* $\Lambda = g_1 \partial_{x_1}$ *and* $\alpha = g_2\, dx_1$, *one find:*

$$\begin{array}{lcl}
K_1(\Lambda) & = & g_1 \partial_{x_1} \quad (\in C^1(C^\infty(M), C^\infty(M)),\ f \mapsto g \frac{\partial f}{\partial x_1}) \\
I_0(\alpha) & = & g_2 \otimes x_1 \\
\Lambda \cdot \alpha & = & g_2\, dg_1 + g_1 \frac{\partial g_2}{\partial x_1}\, dx_1 \\
I_0(\Lambda \cdot \alpha) & = & g_2 \sum_i \frac{\partial g_1}{\partial x_i} \otimes x_i + g_1 \frac{\partial g_2}{\partial x_1} \otimes x_1 \\
K_1(\Lambda) \cdot I_0(\alpha) & = & g_1 \frac{\partial g_2}{\partial x_1} \otimes x_1 + g_2 \otimes g_1 \\
m \cdot & = & L_m = -b \quad \textit{(Hochschild boudary)}
\end{array}$$

We thus search a chain $I_1(\Lambda, \alpha) = \tilde{I}_1(g_1, g_2)$ *in* $C_2(C^\infty(M))$ *such that*

$$b\tilde{I}_1(g_1, g_2) = g_2 \otimes g_1 - g_2 \sum_i \frac{\partial g_1}{\partial x_i} \otimes x_i.$$

The chain I_1 *can be seen as an homotopy between complexes of Hochschild and of de Rham in degree* 1. *Reciprocally, it is easy to see that constructing such an homotopy is enough to define the map* I_1.

Proposition 5.6 *The map* I_1 *defined like in the conjecture 5.4 exists if and only if there exist une homotopy between complexes of Hochschild and of de Rham in degree* 1.

Homotopy formulas have been written by Connes in [Co]. We can also find an explicit one in [Ha3] for polynomial functions. Such a formula could help us to proove conjecture 5.4. for the dual of a Lie algebra.

Recently Tsygan has developed another approach: instead of conjecture 5.4, he states an onther conjecture. Thanks to theorem 5.2, one can replace $C^{\cdot}(\mathbb{A}_{\cdot})$ *with* $\Gamma(M, \wedge^{\cdot} TM)$ *and see* $C_{\cdot}(\mathbb{A}_{\cdot})$ *as a module over* $\Gamma(M, \wedge^{\cdot} TM)$. *The new formality conjecture if then*

Conjecture 5.7 *There exists a* L_∞*-homomorphism,* F', *between the two graded Lie algebras* $\mathfrak{g}_1{}' = (\Gamma(M, \wedge^{\cdot} TM) \oplus C_{\cdot}(\mathbb{A}_{\cdot}), [.,.]'_G)$ *and* $\mathfrak{g}_2{}' = (\Gamma(M, \wedge^{\cdot} TM) \oplus \Omega^{\cdot}(M), [.,.]'_S)$ *such that*

$$F'_i = \mathrm{Id} + I'_{i-1}$$

where $I'_l : \wedge^l \Gamma(M, \wedge^{\cdot} TM) \otimes C_{\cdot}(\mathbb{A}_{\cdot}) \to \Omega^{\cdot}(M)$.

The problem is now to build the maps I'_l*. To do that, an optimistic approach would be to copy the proof of Kontsevich of theorem 5.2: we would like to define new graph acting on forms instead of functions, graph which allowed us to define maps* $T_l : \wedge^l\Gamma(M, \wedge^{\cdot} TM) \otimes C_{\cdot}(\Omega^{\cdot}(M)) \to \Omega^{\cdot}(M)$*. The maps* I'_l *would be then just particular cases of maps* T_l*.*

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