DUNKL OPERATOR AND QUANTIZATION OF $\mathbb{Z}_2$-SINGULARITY

GILLES HALBOUT AND XIANG TANG

Abstract. Let (X, ω) be a symplectic orbifold which is locally like the quotient of a $\mathbb{Z}_2$ action on $\mathbb{R}^n$. Let $A_X^{((\hbar))}$ be a deformation quantization of X constructed via the standard Fedosov method with characteristic class being ω. In this paper, we construct a universal deformation of the algebra $A_X^{((\hbar))}$ parametrized by codimension 2 components of the associated inertia orbifold $\widetilde{X}$. This partially confirms a conjecture of Dolgushev and Etingof (see [4]) in the case of $\mathbb{Z}_2$ orbifolds. To do so, we generalize the interpretation of Moyal star-product as a composition of symbols of pseudodifferential operators in the case where partial derivatives are replaced with Dunkl operators. The star-products we obtain can be seen as globalizations of symplectic reflection algebras ([6]).

1. Introduction

In this paper, we construct exotic deformation quantizations of symplectic orbifolds. Orbifolds provide a large class of examples of topological spaces which are obtained as quotients of manifolds by actions of compact groups. We consider a compact manifold M endowed with a symplectic structure ω and with a $\mathbb{Z}_2$ action which preserves the symplectic structure. Given these data one can construct a $\mathbb{Z}_2$-invariant (associative) star-product (using Fedosov method via a $\mathbb{Z}_2$ invariant connection for instance) with the characteristic class being ω. The restriction of the invariant star-product on $C^\infty(M)^{\mathbb{Z}_2}[[\hbar]]$ defines a deformation quantization of the orbifold $X = M/\mathbb{Z}_2$.

Let $(C^\infty(M)^{\mathbb{Z}_2}((\hbar)), \star)$ denote the star algebra on $M/\mathbb{Z}_2$ with the characteristic class being ω. In [4, Theorem 1.1] and [13, Theorem VII], the Hochschild cohomology of $(C^\infty(M)^{\mathbb{Z}_2}((\hbar)), \star)$ was computed to be equal to the cohomology of the corresponding inertia orbifold with coefficient in $\mathbb{C}((\hbar))$. In particular, Dolgushev and Etingof ([4]) conjectured[1] that deformations of the algebra $(C^\infty(M)^{\mathbb{Z}_2}((\hbar)), \star)$ are unobstructed.

Let γ be the non unital element in $\mathbb{Z}_2$ and M^γ be the γ fixed point subsets. The inertia orbifold $\widetilde{X}$ associated to the quotient $M/\mathbb{Z}_2$ is equal to $\widetilde{X} = M/\mathbb{Z}_2 \bigsqcup M^\gamma/\mathbb{Z}_2$. The Hochschild cohomology of $(C^\infty(M)^{\mathbb{Z}_2}((\hbar)), \star)$ is equal to

$$H^2(C^\infty(M)^{\mathbb{Z}_2}((\hbar)), C^\infty(M)^{\mathbb{Z}_2}((\hbar))) = H^2(M/\mathbb{Z}_2)((\hbar)) \bigoplus H^0(M_2^\gamma/\mathbb{Z}_2)((\hbar)),$$

where M_2^γ is the union of components of M^γ of codimension 2. The Dolgushev-Etingof conjecture implies that the algebra $(C^\infty(M)^{\mathbb{Z}_2}((\hbar)), \star)$ has a deformation coming from every γ fixed point component with codimension 2.

The aim of this paper is to prove that for every class in $H^0(M_2^\gamma/\mathbb{Z}_2)((\hbar))$, namely every codimension 2 component of the inertia orbifold $\widetilde{X}$, we are able to construct

[1] The original conjecture states for arbitrary orbifolds. In this paper, we focus on $\mathbb{Z}_2$-orbifolds.

a deformation of the algebra $(C^\infty(M)^{\mathbb{Z}_2}((\hbar)), \star)$. Moreover, there exists a universal deformation of $(C^\infty(M)^{\mathbb{Z}_2}((\hbar)), \star)$ parametrized by $H^0(M_2^\gamma/\mathbb{Z}_2)((\hbar_2))$. This partially confirms the conjecture of Dolgushev and Etingof in the case of $\mathbb{Z}_2$ orbifolds. Our result is not far away from the full Dolgushev-Etingof conjecture, and the detailed relations are explained in Remark 4.4. The Dolgushev-Etingof conjecture was proved by Etingof [7] when an orbifold is the cotangent bundle of a global quotient orbifold. It is the first time that we know that a large portion of this conjecture holds true for a large class of compact symplectic orbifolds.

In the case where M is $\mathbb{R}^{2n}$, the deformations we get are formal versions of symplectic reflection algebras ([6]) and our construction can be seen as a globalization of such algebras.

To globalize star-products on $\mathbb{R}^{2n}$, one should start with local formulas of the star-products like the Moyal product and Kontsevich star product [11]. Moyal product, which deforms the standard symplectic structure on $\mathbb{R}^{2n}$, can be described using composition of symbols of pseudodifferential operators on $\mathbb{R}^n$. One of the main ideas of the paper is to get a generalized Moyal product formula out of composition of symbols associated to difference-pseudodifferential operators. Following this approach, we replace partial derivatives with Dunkl operators to take into account the $\mathbb{Z}_2$ action and define local formulas for deformations of any non-commutative Poisson structures [10] associated with ω. In this sense, we can also view our construction as globalization of difference-pseudodifferential operators of "Dunkl type".

In Section 2, we recall general material on Dunkl operators and Dunkl pseudodifferential operators. This will allow us to construct an operator-symbol product formula in Section 3: we will get two families of $\mathbb{Z}_2$-local bilinear operators satisfying properties summarized in Theorem 3.10. Those operators will allow us to define a γ-local associative star product (Proposition 3.16) generalizing the standard Moyal star product. Interesting combinatorics appears in the associativity of the new star product. The proof of this main theorem is done in Section 5, using series expansions of pseudodifferential calculous and explicit computations.

Section 4 is devoted to globalization and thus to give a positive answer to Dolgushev-Etingof conjecture. The main idea there is to use Fedosov standard method on the complement of a tubular neighborhood of the $\mathbb{Z}_2$ fixed point submanifold of codimension 2. This can be done as the star product there is locally equivalent to the Moyal product. In the neighborhood of the fixed point submanifold of codimension 2, we use our generalized Moyal product and Fedosov's method of quantization of fixed point submanifolds. The fact that both the Moyal product and the generalized Moyal product are γ-local allows us to restrict the two deformations above on the intersections of the two open sets, which is diffeomorphic to the tubular neighborhood of the fixed point submanifold of codimension 2 with the fixed point submanifold removed. We are able to glue the two deformations on the intersection together to get a global deformation on $M/\mathbb{Z}_2$ as $\mathbb{Z}_2$ acts on the intersection freely.

Here are some remarks and questions for future directions.

(1) The fact that the group acting on M is $\mathbb{Z}_2$ is of major importance for our construction: if $M = \mathbb{R}^{2n}$, the $\mathbb{Z}_2$ action stabilizes the two corresponding copies of $\mathbb{R}^n$ and thus allows us to play with (Dunkl) operators. Such an idea was also used by Etingof [7] in his construction of universal deformation of the cotangent bundle of a global quotient orbifold. To extend our results to more general orbifolds, an important question to answer is how to quantize a symplectic orbifold when such a "polarization" of the symplectic orbifold does not exist.

(2) One could try to generalize our results to every $\mathbb{Z}_2$ invariant Poisson structure (and so deform the corresponding noncommutative Poisson structure). One would expect that with the help of the above mentioned polarization on a $\mathbb{Z}_2$ orbifold, we can play with the corresponding conjectural generalized Poisson sigma models to define the generalized Moyal products.

(3) Another natural question is to compute Hochschild cohomology (and K-theory) of our deformed algebra. It will be interesting to develop an algebraic index theorem for our deformed algebra. We hope to extract the information of singularities from the algebraic index theorem.

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2. Dunkl operator

In this section, we briefly review the theory of Dunkl operators, Dunkl transforms, and Dunkl pseudodifferential operators, which we will need in this paper. We will focus ourselves to a very special case in the theory of Dunkl operators. Most constructions and results we are reviewing go back to Dunkl's original work [5]. We refer readers to [14] and [3] for the proofs of the statements in this section.

Let $\mathbb{Z}_2 = \{1, \gamma\}$ be the group of two elements. It acts on the space $\mathbb{R}$ by reflection. We will use $C_c^\infty(\mathbb{R})$ to denote the space of compactly supported smooth functions on $\mathbb{R}$, and $\mathcal{S}(\mathbb{R})$ to denote the space of Schwartz functions on $\mathbb{R}$. For a real parameter $k \geq 0$, we consider the following differential-difference operator defined by

$$T_k(f)(x) = \frac{df}{dx}(x) + k\frac{f(x) - f(-x)}{x}, \quad f \in C^\infty(\mathbb{R}),$$

which is called Dunkl operator.

For the spectral of the operator T_k, one considers the following equation

$$\begin{cases} T_k(u)(x) & = -i\lambda u(x) \\ u(0) = 1 & \end{cases}$$

for $\lambda \in \mathbb{C}$.

The above equation actually has a unique solution $E_k(x, -i\lambda)$, called Dunkl kernel given by

$$E_k(x, -i\lambda) = j_{k-1/2}(i\lambda x) + \frac{\lambda x}{2k+1} j_{k+1/2}(i\lambda x),$$

where j_α is the "normalized first kind Bessel function of order α". From the above expression, one easily see that $E_k(x, -i\lambda)$ can be extended to a holomorphic function of variable $x \in \mathbb{C}, \lambda \in \mathbb{C}, \mathrm{Re}\, k \geq 0$. One can even show that for $x, \lambda \in \mathbb{R}$,

$$|E_k(x, i\lambda)| \leq 1.$$

We consider the following measure μ_k on $\mathbb{R}$ by

$$d\mu_k(x) = \frac{|x|^{2k}}{2^{k+1/2}\Gamma(k+1/2)} dx,$$

with $\Gamma(x)$ the Gamma function. It is not difficult to check that the Dunkl operator T_k is skew symmetric with respect to the L^2-norm associated to the measure μ_k, i.e.

$$\int_{\mathbb{R}} T_k(f)(x)\bar{g} d\mu_k(x) = -\int_{\mathbb{R}} f(x)\overline{T_k(g)}(x) d\mu_k(x).$$

For $1 \leq p < \infty$, define $L^p_k(\mathbb{R})$ to be the space of measurable complex valued functions on $\mathbb{R}$ such that

$$||f||_{p,k} = \left(\int_{\mathbb{R}} |f(x)|^p d\mu_k(x) \right)^{1/p} < \infty.$$

For $f \in L^1_k(\mathbb{R})$, define the Dunkl transform $\mathcal{F}_k$ of f by

$$\mathcal{F}_k(f)(\lambda) = \int_{\mathbb{R}} E_k(y, -i\lambda) f(y) d\mu_k(y).$$

When f is in $\mathcal{S}(\mathbb{R})$, then

(1) $\mathcal{F}_k(f) \in C^\infty(\mathbb{R})$, and $T_k \mathcal{F}_k(f) = -\mathcal{F}_k(ixf)$,
(2) $\mathcal{F}_k(T_k f) = i\lambda \mathcal{F}_k(f)$,
(3) the Dunkl transform leaves $\mathcal{S}(\mathbb{R})$ invariant,
(4) for all $f \in L^1_k(\mathbb{R})$ such that $\mathcal{F}_k(f) \in L^1_k(\mathbb{R})$, the inverse Dunkl transform is defined to be

$$\mathcal{F}_k^{-1}(f)(x) = \int_{\mathbb{R}} E_k(x, i\lambda) f(\lambda) d\mu_k(\lambda),$$

(5) for $f \in L^2_k(\mathbb{R})$, $||\mathcal{F}_k(f)||_{2,k} = ||f||_{2,k}$.

3. Generalized pseudodifferential operators and Moyal type formula

Pseudo-differential operators associated to Dunkl operators in the case of $\mathbb{Z}_2$ have been studied by Dachraoui [2] and Abdelkefi-Amri-Sifi [1]. Let $D(\mathbb{R})$ be the algebra of differential operators on $\mathbb{R}$. In this section, our goal is to use the idea of operator-symbol calculus to define an associative deformation of the algebra $C^\infty(\mathbb{R}^2) \rtimes \mathbb{Z}_2$ and also $D(\mathbb{R}) \rtimes \mathbb{Z}_2$. When one restricts such a deformation to the subalgebra[2] $\mathrm{Poly}(\mathbb{R}^2) \rtimes \mathbb{Z}_2$, we obtain a Moyal type formula for the symplectic reflection algebra introduced by Etingof-Ginzburg [6] in the case of $\mathbb{Z}_2$ action on $\mathbb{R}^2$ by reflection.

[2] $\mathrm{Poly}(\mathbb{R}^2)$ denotes the algebra of polynomial functions on $\mathbb{R}^2$.

3.1. **Operator product.**

Definition 3.1. We say that a function $a(x,p) \in C^\infty(\mathbb{R}^2)$, a complex valued function on $\mathbb{R}^2$, belongs to the symbol class $\mathfrak{S}_0^m$ if for any $r, s \in \mathbb{N}$,

$$|\partial_p^r \partial_x^s a(x,p)| \leq C_{m,r,s}(1+|p|^2)^{(m-r)/2}.$$

Definition 3.2. Let $a \in \mathfrak{S}_0^m$, then define $\mathrm{Op}_k(a)$ a linear operator on $\mathcal{S}(\mathbb{R})$ by

$$\mathrm{Op}_k(a)(f)(x) = \int_{\mathbb{R}} a(x,p) E_k(x, ip) \mathcal{F}_k(f)(p) d\mu_k(p).$$

Dachraoui [2, Thm. 4.1] proves the following theorem :

Theorem 3.3. *([2]) Let $a \in \mathfrak{S}_0^m$, then the operator $\mathrm{Op}_k(a)$ associated to a is a linear continuous mapping from $\mathcal{S}(\mathbb{R})$ to itself.*

Remark 3.4. *For $a \in \mathfrak{S}_0^0$,* [1, Proposition 4.1] *proves that $\mathrm{Op}_k(a)$ defines a bounded operator on $L_k^p(\mathbb{R})$ for $1 < p < \infty$.*

Example 3.5. *For $a(x,p) = x^i p^j$, $\mathrm{Op}_k(a) = x^i T_k^j$. We remark that though polynomials are not in the symbol class $\mathfrak{S}_0^m$, for any polynomial a, $\mathrm{Op}_k(a)$ is a well defined linear operator on $\mathcal{S}(\mathbb{R})$, which is sufficient for our following developments.*

We consider the translation operator $\hat{\gamma} : f(x) \mapsto f(-x)$. It is easily seen that $\hat{\gamma}$ is an isometry on $L_k^2(\mathbb{R})$. We have the following observation :

Lemma 3.6. *For $a_j, b_j \in \mathrm{Poly}(\mathbb{R}^2), j = 0, \ldots, n$, if $\sum_j k^j(\mathrm{Op}_k(a_j) + \mathrm{Op}_k(b_j) \circ \hat{\gamma})$ is the zero operator for any $k \geq 0$, then $a_j = b_j = 0, j = 0, \ldots, n$.*

Proof. As $\mathrm{Op}_k(\sum_j k^j a_j) + \mathrm{Op}_k(\sum_j k^j b_j) \circ \hat{\gamma} = 0$, then

$$\begin{aligned} &\int_{\mathbb{R}} \big(\sum_j k^j a_j(x,p)\big) E_k(x, ip) \mathcal{F}_k(f)(p) d\mu_k(p) \\ &\quad + \int_{\mathbb{R}} \big(\sum_j k^j b_j(x,p)\big) E_k(x, ip) \mathcal{F}_k(\hat{\gamma}(f))(p) d\mu_k(p) = 0, \end{aligned} \tag{1}$$

for any $f \in \mathcal{S}(\mathbb{R})$.

We notice that $\mathcal{F}_k(\hat{\gamma}(f))(p) = \mathcal{F}_k(f)(-p)$, then Equation (1) becomes

$$\int_{\mathbb{R}} \sum_j k^j \big(a_j(x,p) E_k(x, ip) + b_j(x,-p) E_k(x, -ip)\big) \mathcal{F}_k(f)(p) d\mu_k(p) = 0,$$

for any $f \in \mathcal{S}(\mathbb{R})$. Therefore, we conclude that

$$\sum_j k^j (a_j(x,p) E_k(x, ip) + b_j(x,-p) E_k(x,-ip)) = 0,$$

for any $x, p \in \mathbb{R}$. If we consider the above equation at $k = 0$, then

$$a_0(x,p) \exp(ixp) + b_0(x,-p) \exp(-ixp) = 0.$$

From the above equation, we have that

$$\partial_x a(x,p) b(x,-p) - a(x,p) \partial_x b(x,-p) = -2ipa(x,p) b(x,-p).$$

By comparing the leading terms on both sides, we can quickly conclude that $a_0 = b_0 = 0$. By induction, we conclude that $a_j = b_j = 0$ for $j = 0, \dots, n$. □

To motivate the main result of this section, we introduce the following notion of a γ-local operator. (Recall that γ acts on $\mathbb{R}$ by reflection.)

Definition 3.7. A linear operator D on $C^\infty(\mathbb{R}^2)$ is called γ-local if for any $f \in C^\infty(\mathbb{R}^2)$, $D(f)(x,p)$ is determined completed by finitely many jets of f at (x,p), $(-x,p)$, $(x,-p)$, and $(-x,-p)$. In general, a k-linear operator D on $C^\infty(\mathbb{R}^2)$ is called γ-local, if for any $f_1, \dots, f_k \in C^\infty(\mathbb{R}^2)$, $D(f_1, \dots, f_k)(x,p)$ is determined by finitely many jets of $f_1, \dots, f_k$ at (x,p), $(-x,p)$, $(x,-p)$, and $(-x,-p)$.

Example 3.8. *Let us list some examples of γ-local operators.*

1. *Differential operators on $\mathbb{R}$ are γ-local.*
2. *The partial translation operator $\sigma_i : C^\infty(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$ for $i = 1, 2$ with $\sigma_1(f)(x,p) := f(-x,p)$ and $\sigma_2(f)(x,p) = f(x-p)$ are γ-local.*
3. *The difference operators $\tilde{\partial}_x, \tilde{\partial}_p : C^\infty(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$ with $\tilde{\partial}_x(f)(x,p) = (f(x,p) - f(-x,p))/x$ and $\tilde{\partial}_p(f)(x,p) = (f(x,p) - f(x,-p))/p$ are γ-local. We observe that $\partial_x + \tilde{\partial}_x$ (and $\partial_p + \tilde{\partial}_p$)is the Dunkl operator T_1 acting on the x-variable (and the p-variable), and is also Γ-local.*

Proposition 3.9. *The space of γ-local operators on $C^\infty(\mathbb{R}^2)$ is an associative algebra under composition.*

Proof. This is a straightforward check. □

The main result of this section can be summarized into the following Theorem.

Theorem 3.10. *There are 2 families of γ-local bilinear operators $C^1_{j,l}$ and $C^2_{j,l}$ on $C^\infty(\mathbb{R}^2)$ satisfying*

1. *For two polynomials a_1 and a_2 of degrees (m_1, n_1) and (m_2, n_2), $C^0_{j,l}(a_1, a_2)$ and $C^1_{j,l}(a_1, a_2)$ are again polynomials of degree $(m_1 + m_2 - j, n_1 + n_2 - j)$.*
2. *$C^0_{j,l}$ and $C^1_{j,l}$ vanish when $l > j$.*
3. *For two polynomials $a_1(x,p)$ and $a_2(x,p)$,*
$$\mathrm{Op}_k(a_1) \circ \mathrm{Op}_k(a_2) = \sum_{j,l} k^l \Big(\mathrm{Op}_k \left(C^0_{j,l}(a_1, a_2) \right) + \mathrm{Op}_k \left(C^1_{j,l}(a_1, a_2) \right) \circ \hat{\gamma} \Big).$$
We observe that for any given a_1, a_2, the above sum is actually finite and therefore well defined.

The proof of this theorem will be given in Section 5. In the left of this section, we will provide an explicit formula for each bilinear operator C^i_{jl}. In particular, when $l = 0$, $C^1_{j,0}$ vanishes and $C^0_{j,0}$ is the j-th component of the Moyal product,

$$C^1_{j,0}(a_1, a_2) = \frac{(-i)^j}{j!} \partial_p^j(a_1) \partial_x^j(a_2). \tag{2}$$

From this, we can see that the above operator-symbol calculus defines a deformation of the crossed production of $D(\mathbb{R}) \rtimes \mathbb{Z}_2$.

3.2. **A coproduct structure on** $\mathrm{Poly}(\mathbb{R})$**.** We consider a coproduct structure on the algebra of polynomials of one variable, which is useful in describing the operators C^i_{jl}.

Define Δ to be a linear map from $\mathrm{Poly}(\mathbb{R})$ to $\mathrm{Poly}(\mathbb{R}) \otimes_{\mathbb{C}} \mathrm{Poly}(\mathbb{R})$ by

$$\Delta(f)(x,y) := \frac{f(x)-f(y)}{x-y}.$$

Observe that $f(x)-f(y)$ is divisible by $x-y$, and therefore Δ is well defined.

The following is a list of properties of the operator Δ, which can be checked routinely.

Proposition 3.11. *The operator* $\Delta : \mathrm{Poly}(\mathbb{R}) \to \mathrm{Poly}(\mathbb{R}) \otimes \mathrm{Poly}(\mathbb{R})$ *satisfies the following properties.*

(1) *coassociative, i.e.*

$$(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta : \mathrm{Poly}(\mathbb{R}) \to \mathrm{Poly}(\mathbb{R}) \otimes \mathrm{Poly}(\mathbb{R}) \otimes \mathrm{Poly}(\mathbb{R});$$

(2) *Leibnitz rule, i.e.*

$$\Delta(fg) = (f \otimes 1)\Delta(g) + \Delta(f)(1 \otimes g);$$

(3) $\Delta(f)(x,x) = f'(x) = D(f)(x)$, *and* $\Delta(f)(x,-x) = (f(x)-f(-x))/2x = 1/2\tilde{D}(f)(x)$, *and* $T_k(f)(x) = (D + k\tilde{D})(f)(x)$*;*

(4) $\Delta(f)$ *is a symmetric function of* 2 *variables;*

(5) Δ *extends to be a linear map* $\Delta : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}) \hat{\otimes} C^\infty(\mathbb{R})$ *satisfying the same properties (1)-(4), where* $\hat{\otimes}$ *is the complete topological tensor product.*

Remark 3.12. *According to Proposition 3.11, (2), the operator* Δ *is a Hochschild cocycle of* $\mathrm{Poly}(\mathbb{R})$ *with coefficient in* $\mathrm{Poly}(\mathbb{R}) \otimes \mathrm{Poly}(\mathbb{R})$*. By the Koszul complex, we can compute that the Hochschild cohomology* $H^1(\mathrm{Poly}(\mathbb{R}), \mathrm{Poly}(\mathbb{R})^{\otimes 2})$ *is equal to* $\mathrm{Poly}(\mathbb{R})$*. Under this identification,* Δ *is mapped to the unit of* $\mathrm{Poly}(\mathbb{R})$*.*

Remark 3.13. *For* $\mathbb{R}^n$*, we can generalize* Δ *to a cocycle* $\Delta_n : \mathrm{Poly}(\mathbb{R}^n)^{\otimes n} \to \mathrm{Poly}(\mathbb{R}^n)^{\otimes 2}$ *by*

$$\Delta_n(f_1, \dots, f_n)(x,y)$$
$$:= \frac{(f_1(x_1,\dots,x_n)-f_2(y_1,x_2,\dots,x_n))(f_2(y_1,x_2,\dots,x_n)-f_2(y_1,y_2,x_3,\cdots,x_n))\cdots(f_n(y_1,\dots,y_{n-1},x_n)-f_n(y_1,\dots,y_n))}{(x_1-y_1)\cdots(x_n-y_n)},$$

where $x = (x_1, \dots, x_n)$ *and* $y = (y_1, \dots, y_n)$*.*

3.3. **Formulas for asymptotic expansion.** We will give explicit expressions for $C^i_{j,l}$, $i = 1, 2$, which involves interesting combinatorics.

We start with considering the linear equation

$$y_0 + y_1 + \cdots + y_l = j - l. \tag{3}$$

Let $P_{j-l,l}$ be the set of integer solutions to Equation (3) where y_0, y_l are nonnegative and $y_1, \dots, y_{l-1}$ are positive.

Let $D(f)(x) = f'(x)$ and $\tilde{D}(f)(x) = (f(x) - f(-x))/x$.

For an element $\nu \in P_{m,n}$, define B_ν a linear operator on $\mathrm{Poly}(\mathbb{R})$ by

$$B_\nu(f)(x) := D^{y_n} \circ \tilde{D} \circ D^{y_{n-1}} \cdots D^{y_1} \circ \tilde{D} \circ D^{y_0}(f)(x).$$

For $n \in \mathbb{N}$, define n_0 to be the number of positive even numbers less than or equal to n, and n_1 to be the number of positive odd numbers less than or equal to n. Obviously,

$n = n_0 + n_1$. Given $\nu \in P_{m,n}$, we define $\Lambda_0 = y_0 + \sum_{\text{even } i} y_i$, and $\Lambda_1 = \sum_{\text{odd } i} y_i$. We have $\Lambda_0 + \Lambda_1 = m$. Define A_ν a linear operator on Poly($\mathbb{R}$) by

$$\Delta^{m+n}(f)(\underbrace{x, \dots, x}_{\Lambda_0+n_0+1}, \underbrace{-x, \dots, -x}_{\Lambda_1+n_1}).$$

By the associativity of Δ (Prop. 3.11, (1)), define $\Delta^k(f) = (\Delta \otimes 1 \otimes \cdots \otimes 1) \cdots (\Delta \otimes 1)\Delta(f)$. And according to Prop. 3.11 (4), $\Delta^k(f)$ is a symmetric function of $k+1$ variables.

In order to define $C^i_{j,l}$, which are bilinear operators on Poly($\mathbb{R}^2$), we lift A_ν and B_ν on Poly($\mathbb{R}^2$) by applying A_ν on the variable p and B_ν on the variable x.

Now we are ready to define $C^i_{j,l}$.

I. $C^0_{j,l}$. The bilinear operator $C^0_{j,l}$ vanishes if l is odd, and when l is even,

$$C^0_{j,l}(a_1, a_2) := (-i)^j \sum_{\nu \in P_{j-l,l}} A_\nu(a_1)(x,p) B_\nu(a_2)(x,p).$$

II. $C^1_{j,l}$. The bilinear operator $C^1_{j,l}$ vanishes if l is even, and when l is odd,

$$C^1_{j,l}(a_1, a_2)(x,p) := (-i)^j \sum_{\nu \in P_{j-l,l}} A_\nu(a_1)(x,p) B_\nu(a_2)(x,-p).$$

We point out that with the expression of C^i_{ij}, Theorem 3.10, (1) follows obviously by the definition of A_ν and B_ν. Furthermore, one notices that if $j - l < l - 1$, then $P_{j-l,l}$ is an empty set, and therefore $C^i_{j,l}$ vanishes. This gives a stronger version of Theorem 3.10, (2).

From the above discussion, we are left to prove part (3) of Theorem 3.10. This is an interesting application of operator-symbol calculus and the detail will be in Section 5. In particular, we will explain how we obtain the operators A_ν and B_ν.

3.4. **A "Moyal" formula.** Motivated by the result of Theorem 3.10, we introduce the following algebra.

Definition 3.14. Define the following product $\star$ on $C^\infty(\mathbb{R}^2) \rtimes_{\mathbb{C}} \mathbb{Z}_2[[\hbar_1, \hbar_2]]$ by

(1) $\star$ is $\mathbb{C}[[\hbar_1, \hbar_2]]$ linear;
(2) For $a_1, a_2 \in C^\infty(\mathbb{R}^2)$, $a_1 \star a_2$ is defined by

$$a_1 \star a_2 = \sum_{j,l} \hbar_1^j \hbar_2^l (C^0_{j,l}(a_1, a_2) + C^1_{j,l}(a_1, a_2)\gamma).$$

As we have explained at the end of Section 3.1, when $\hbar_2 = 0$, the above product $\star$ reduces back the standard Moyal product. Hence, we can view $(C^\infty(\mathbb{R}^2) \rtimes \mathbb{Z}_2[[\hbar_1, \hbar_2]], \star)$ as a deformation of the crossed product of the Weyl algebra $\mathbb{W}_2$ with $\mathbb{Z}_2$. Furthermore, we point out that as $C^i_{j,l} = 0$ when $l > j$, we can allow $\hbar_2$ be a complex number in $\mathbb{C}$ rather than a formal parameter. In this way, we can also view $(C^\infty(\mathbb{R}^2) \rtimes \mathbb{Z}_2[[\hbar_1, \hbar_2]], \star)$ as a formal deformation quantization of the crossed product algebra $C^\infty(\mathbb{R}^2) \rtimes \mathbb{Z}_2$ along the noncommutative Poisson structure $\pi + \hbar_2 \pi\gamma$ on $C^\infty(\mathbb{R}^2) \rtimes \mathbb{Z}_2$ as we introduced in [10].

Lemma 3.15. *For any $(x_0, p_0) \in \mathbb{R}^2$, and $f \in C^\infty(\mathbb{R}^2)$, given $m, n \in \mathbb{N} \cup \{0\}$, there is a polynomial $g_{m,n} \in \mathrm{Poly}(\mathbb{R}^2)$ such that $\partial_x^i \partial_p^j f$ agrees with $\partial_x^i \partial_p^j g_{m,n}$ at (x_0, p_0), $(x_0, -p_0)$, $(-x_0, p_0)$, and $(-x_0, -p_0)$ for $0 \le i \le m, 0 \le j \le n$.*

Proof. We divide our proofs into 4 different situations according to (x_0, p_0).

(1) $x_0 = p_0 = 0$,
(2) $x_0 \neq 0$ and $p_0 = 0$,
(3) $x_0 = 0$ and $p_0 \neq 0$,
(4) $x_0 \neq 0$ and $p_0 \neq 0$.

Case (1). For any $m, n \in \mathbb{N}$, define

$$g_{m,n} = \sum_{0 \le i \le m, 0 \le j \le n} \frac{1}{i!j!} \partial_x^i \partial_p^j (f)(0,0) x^i p^j.$$

It is easy to check $\partial_x^i \partial_p^j g_{m,n}$ agrees $\partial_x^i \partial_p^j f$ at $(0,0)$ for $0 \le i \le m, 0 \le j \le n$.
Case (2) and (3). The proof for these two cases are exactly same. Therefore, we will only prove Case (2). Define

$$g_1 = \sum_{0 \le i \le m, 0 \le j \le n} \frac{1}{i!j!} \partial_x^i \partial_p^j (f)(x_0, 0)(x - x_0)^i p^j.$$

Define $g_{m,n} = g_1 + (x - x_0)^{m+1} g_2$ where g_2 is some polynomial to be determined. It is easy to check that $\partial_x^i \partial_p^j g_{m,n}(x_0, 0)$ agrees with $\partial_x^i \partial_p^j f(x_0, 0)$. We proceed to look for g_2 such that $\partial_x^i \partial_p^j g_{m,n}(-x_0, 0)$ agrees with $\partial_x^i \partial_p^j f(-x_0, 0)$. We write

$$g_2 = \sum_{1 \le s \le m, 1 \le t \le n} 1/s!t! a_{st} (x + x_0)^s p^t.$$

We need to solve a_{st}. From the requirement that $\partial_x^i \partial_p^j g_{m,n}(-x_0, 0) = \partial_x^i \partial_p^j f(-x_0, 0)$, we know that

$$\partial_x^i \partial_p^j (g_1)(-x_0, 0) + \begin{pmatrix} i \\ k \end{pmatrix} \partial_x^{i-k} (x - x_0)^{m+1} \partial_x^k \partial_p^j g_2(-x_0, 0) = \partial_x^i \partial_p^j f(-x_0, 0). \tag{4}$$

If we order a_{st} lexicographically, then it is not difficult to see that the above equations for $1 \le i \le m, 1 \le j \le n$ define a system of linear equations for variable a_{st}. We notice that in Eq. (4), the leading term is a_{ij} with coefficient $(-2x_0)^{m+1}$. When i and j vary, we have a system of linear equations whose coefficient matrix is an upper triangular matrix with a nonzero number $(-2x_0)^{m+1}$ at every entry of the diagonal. This implies that we have a unique solution for a_{st}, and therefore a solution for $g_{m,n}$.
Case (4). Following the proof of Case (2), we construct g step by step. Firstly, define g_0 to be

$$g_0 = \sum_{0 \le i \le m, 0 \le j \le n} \frac{1}{i!j!} \partial_x^i \partial_p^j f(x_0, y_0)(x - x_0)^i (p - p_0)^j.$$

We now look for g_1 of the form $\sum_{0 \le i \le m, 0 \le j \le n} 1/i!j! a_{ij} (x + x_0)^i (p - p_0)^j$ such that $\partial_x^i \partial_p^j (g_0 + (x - x_0)^{m+1} g_1)$ agrees with $\partial_x^i \partial_p^j f$ at both (x_0, p_0) and $(-x_0, p_0)$ for $0 \le i \le m, 0 \le j \le n$. We notice that it is always true that $\partial_x^i \partial_p^j (g_0 + (x - x_0)^{m+1} g_1)(x_0, p_0) = \partial_x^i \partial_p^j f(x_0, p_0)$ for $0 \le i \le m, 0 \le j \le n$. By the same arguments as in the proof of Case (2), we can find a unique family a_{ij} such that $\partial_x^i\partial_p^j(g_0+(x-x_0)^{m+1}g_1)(-x_0,p_0)$ is same to $\partial_x^i\partial_p^j(f)(-x_0,p_0)$ for $0\leq i\leq m, 0\leq j\leq n$.

We next look for g_2 of the form $\sum_{0\leq i\leq m, 0\leq j\leq n} 1/i!j!b_{ij}(x-x_0)^i(p+p_0)^j$ such that $\partial_x^i\partial_p^j(g_0+(x-x_0)^{m+1}g_1+(p-p_0)^{n+1}g_2)$ agrees with $\partial_x^i\partial_p^j f$ at (x_0,p_0), $(-x_0,p_0)$, $(x_0,-p_0)$ for $0\leq i\leq m, 0\leq j\leq n$. Again, it not difficult to check that the partial derivatives of these two functions agree at (x_0,p_0) and $(-x_0,p_0)$ no matter what g_2 is like. With the above arguments, we know that there exists a unique solution for b_{st} such that the derivatives of the two functions agree at $(x_0,-p_0)$.

Continuing the above procedure, we look for g_3 of the form $\sum_{0\leq i\leq m,0\leq j\leq n} 1/i!j!c_{ij}(x+x_0)^i(p+p_0)^j)$ such that $\partial_x^i\partial_p^j(g_0+(x-x_0)^{m+1}g_1+(p-p_0)^{n+1}g_2+(x-x_0)^{m+1}(p-p_0)^{n+1}g_3)$ agrees with $\partial_x^i\partial_p^j f$ at $(x_0,p_0),(-x_0,p_0),(x_0,p_0),(-x_0,-p_0)$ for $0\leq i\leq m, 0\leq j\leq n$. Again the two functions have the same derivatives at $(x_0,p_0),(-x_0,p_0),(x_0,-p_0)$ no matter what g_3 is like. The same arguments as in the proof of Case (2) shows that there is a unique solution for c_{ij}.

In summary, we have fund a function $g_{m,n}=g_0+(x-x_0)^{m+1}g_1+(p-p_0)^{n+1}g_2+(x-x_0)^{m+1}(p-p_0)^{n+1}g_3$ such that $\partial_x^i\partial_p^j f$ agrees with $\partial_x^i\partial_p^j g_{m,n}$ at (x_0,p_0), $(-x_0,p_0)$, $(x_0,-p_0)$, and $(-x_0,-p_0)$ for $0\leq i\leq m, 0\leq j\leq n$. □

Proposition 3.16. *The product $\star$ is associative on $C^\infty(\mathbb{R}^2)\rtimes\mathbb{Z}_2[[\hbar_1,\hbar_2]]$. For $e^{i\theta}\in U(1)$, the map $x\mapsto e^{i\theta}x, p\mapsto e^{-i\theta}p$ defines a $U(1)$ action on the algebra $(C^\infty(\mathbb{R}^2)\rtimes\mathbb{Z}_2[[\hbar_1,\hbar_2]],\star)$*

Proof. We observe that $\mathrm{Poly}(\mathbb{R}^2)\rtimes\mathbb{Z}_z$ is closed under $\star$. If a_i $(i=1,2,3)$ are monomials of degrees (m_i,n_i), then $\sum_l k^l(C^0_{j,l}(a_1,a_2)+C^1_{j,l}(a_1,a_2)\gamma)$ is the degree (m_1+m_2-j, n_1+n_2-j) in the expansion of $\mathrm{Op}_k(a_1)\circ\mathrm{Op}_k(a_2)$. Therefore,

$$\sum_{j_1+j_2=j}\sum_l k^l \sum_{l_1+l_2=l}\Big(\mathrm{Op}_k\left(C^0_{j_1,l_1}(C^0_{j_2,l_2}(a_1,a_2),a_3)+C^1_{j_1,l_1}(C^1_{j_2,l_2}(a_1,a_2),\gamma(a_3))\right)$$
$$+\mathrm{Op}_k\left(C^0_{j_1,l_1}(C^1_{j_2,l_2}(a_1,a_2),\gamma(a_3))+C^1_{j_1,l_1}(C^0_{j_2,l_2}(a_1,a_2),a_3)\right)\gamma\Big)$$

is the degree $(m_1+m_2+m_3-j, n_1+n_2+n_3-j)$ component of the expansion of $(\mathrm{Op}_k(a_1)\circ\mathrm{Op}_k(a_2))\circ\mathrm{Op}_k(a_3)$.

As the composition between operators on $\mathcal{S}(\mathbb{R})$ is associative, by Theorem 3.10 and Lemma 3.6, we conclude that the product $\star$ on $\mathrm{Poly}(\mathbb{R}^2)\rtimes\mathbb{Z}_2$ is associative by comparing components with degree $(m_1+m_2+m_3-j, n_1+n_2+n_3-j)$ and power k^l in the expansions of $(\mathrm{Op}_k(a_1)\circ\mathrm{Op}_k(a_2))\circ\mathrm{Op}_k(a_3)$ and $\mathrm{Op}_k(a_1)\circ(\mathrm{Op}_k(a_2)\circ\mathrm{Op}_k(a_3))$.

To prove that $\star$ is associative on $C^\infty(\mathbb{R}^2)\rtimes\mathbb{Z}_2$, it is sufficient to check that $\star$ is associative at every point (x,p) up to any $\hbar_1^j\hbar_2^l$. We notice that $C^i_{j,l}(a_1,a_2)(x,p)$ is determined by the values of $a_1,\partial_p a_1,\dots,\partial_p^j a_1$ at (x,p) and $(x,-p)$, together with values of $a_2,\partial_x a_2,\dots,\partial_x^j a_2$ at $(x,p),(-x,p),(x,-p)$, and $(-x,-p)$. Therefore to check $((a_1\star a_2)\star a_3)(x,p)$ agrees with $(a_1\star(a_2\star a_3))(x,p)$ up to degree $\hbar_1^j\hbar_2^l$, it sufficient to check $(b_1\star b_2)\star b_3(x,p)$ agrees with $b_1\star(b_2\star b_3)(x,p)$ up to degree $\hbar_1^j\hbar_2^l$ for polynomials b_1,b_2,b_3 where the values of $\partial_x^s\partial_p^t b_i$ at $(x,p),(-x,p),(x,-p),(-x,-p)$ agree with the corresponding values of $\partial_x^s\partial_p^t a_i$ for $i=1,2,3, 1\leq s,t\leq j$. Hence by the associativity of $\star$ on $\mathrm{Poly}(\mathbb{R}^2)\rtimes\mathbb{Z}_2$ and Lemma 3.15, we conclude that $\star$ is associative on $C^\infty(\mathbb{R}^2)\rtimes\mathbb{Z}_2$.

For the action of $t = e^{i\theta}$, we notice that $t = e^{i\theta}$ acts on operators A_ν and B_ν with eigenvalues t^{-j} and t^j. Therefore, one can quickly check that C^i_{jl} is a $U(1)$ invariant bilinear operator on $C^\infty(\mathbb{R}^2)\rtimes\mathbb{Z}_2[[\hbar_1,\hbar_2]]$ for any i, j, l. Therefore, $U(1)$ acts on $C^\infty(\mathbb{R}^2)\rtimes\mathbb{Z}_2[[\hbar_1,\hbar_2]]$ by algebra automorphisms. □

Remark 3.17. *The algebra $(C^\infty(\mathbb{R}^2)\rtimes\mathbb{Z}_2[[\hbar_1,\hbar_2]],\star)$ is the formal version of a symplectic reflection algebra [6] for $\mathbb{Z}_2$ action on the standard symplectic vector space $\mathbb{R}^2$. Theorem 3.10 gives an operator interpretation of this symplectic reflection algebra and furthermore a Moyal type expansion formula.*

Let $P = (1+\gamma)/2 \in C^\infty(\mathbb{R}^2)\rtimes\mathbb{Z}_2$. Consider the subspace of $A = (C^\infty(\mathbb{R}^2)\rtimes\mathbb{Z}_2[[\hbar_1,\hbar_2]],\star)$ defined by $P\star A\star P = P\star C^\infty(\mathbb{R}^2)\rtimes\mathbb{Z}_2[[\hbar_1,\hbar_2]]\star P$. In [6], Etingof and Ginzburg proved that $P\star A\star P$ is Morita equivalent to A. In particular, one can quickly check that the space $P\star A\star P$ as a vector space is isomorphic to $C^\infty(\mathbb{R}^2)^{\mathbb{Z}_2}[[\hbar_1,\hbar_2]]$. Via the natural identification,

$$a \in C^\infty(\mathbb{R}^2)^{\mathbb{Z}_2}[[\hbar_1,\hbar_2]] \mapsto aP \in C^\infty(\mathbb{R}^2)\rtimes\mathbb{Z}_2[[\hbar_1,\hbar_2]],$$

$C^\infty(\mathbb{R}^2)^{\mathbb{Z}_2}[[\hbar_1,\hbar_2]]$ is equipped with a star-product which we will again denote by $\star$. We call this algebra Dunkl-Weyl algebra $\mathbb{D}_2$, which is called the spherical subalgebra by Etingof and Ginzburg [6]. By Proposition 3.16, we conclude that the Dunkl-Weyl algebra $\mathbb{D}_2$ is an associative algebra with a natural $U(1)$ action.

Notation: We use $\mathbb{C}((\hbar_1,\hbar_2))$ to denote the space of all series of the form

$$\sum_{s\le i,t\le j}\hbar_1^i\hbar_2^j$$

for some $s,t\in\mathbb{Z}$. In the later applications, we many times will work with the algebra $\mathbb{D}_2\otimes_{\mathbb{C}[[\hbar_1,\hbar_2]]}\mathbb{C}((\hbar_1,\hbar_2))$, which will be denoted by $\mathbb{D}_2((\hbar_1,\hbar_2))$.

4. Quantization of $\mathbb{Z}_2$-orbifold

In this section, we consider deformation quantization of $\mathbb{Z}_2$-orbifolds. Let M be a symplectic manifold with a symplectic $\mathbb{Z}_2$ action. As $\mathbb{Z}_2$ is finite, we can always find a $\mathbb{Z}_2$ invariant symplectic connection on M. Using Fedosov's method, we can construct a $\mathbb{Z}_2$ invariant star-product $\star$ on $C^\infty(M)[[\hbar]]$. (As our construction is local, it works more generally for an orbifold which locally is a quotient of a $\mathbb{Z}_2$ action.) The restriction of the invariant star-product on $C^\infty(M)^{\mathbb{Z}_2}[[\hbar]]$ defines a deformation quantization of the orbifold $X = M/\mathbb{Z}_2$. We use $A^{((\hbar))}_{M/\mathbb{Z}_2}$ to denote the quantized algebra on $M/\mathbb{Z}_2$ with the characteristic class equal to ω (with $A^{((\hbar))}_{M/\mathbb{Z}_2}$, we refer to the algebra $C^\infty(M)^{\mathbb{Z}_2}\otimes_{\mathbb{C}}\mathbb{C}((\hbar))$ with the extended star-product $\star$).

According to [4, Theorem 1.1] and [13, Theorem VII], the Hochschild cohomology of $A^{((\hbar))}_{M/\mathbb{Z}_2}$ is equal to the cohomology of the corresponding inertia orbifold with coefficient in $\mathbb{C}((\hbar))$. In the case of $M/\mathbb{Z}_2$, the corresponding inertia orbifold is defined to be $\tilde{X} := M/\mathbb{Z}_2 \sqcup M^\gamma/\mathbb{Z}_2$, where M^γ is the fixed submanifold of the group element γ in $\mathbb{Z}_2$. If M^γ has several components maybe of different dimensions, we will take the disjoint union of all components. We use ℓ to denote the codimension of M^γ in M, and ℓ is a locally constant function on X. We point out that $\mathbb{Z}_2$ acts on M^γ trivially, but we will view M^γ as an orbifold with a global stabilizer group $\mathbb{Z}_2$. We have

$$H^\bullet(A^{((\hbar))}_{M/\mathbb{Z}_2}, A^{((\hbar))}_{M/\mathbb{Z}_2}) = H^{\bullet-\ell}(\tilde{X}, \mathbb{C}((\hbar))). \tag{5}$$

Looking at Equation (5), we conclude that the second Hochschild cohomology of $A^{((\hbar))}_{M/\mathbb{Z}_2}$ is equal to a direct sum of $H^2(M/\mathbb{Z}_2, \mathbb{C}((\hbar)))$ and $H^0(M^\gamma_2/\mathbb{Z}_2, \mathbb{C}((\hbar)))$ for the components M^γ_2 of M^γ with codimension 2 (we have degree 0 cohomology on M^γ_2 because of the degree shifting in Equation (5)). From the experience of deformation quantization of a symplectic manifold, we know that the component $H^2(M/\mathbb{Z}_2, \mathbb{C}((\hbar)))$ of $H^2(A^{((\hbar))}_{M/\mathbb{Z}_2}, A^{((\hbar))}_{M/\mathbb{Z}_2})$ corresponds to isomorphism classes of $\mathbb{Z}_2$ invariant deformation quantizations on M. In the following of this section, we construct deformations of $A^{((\hbar))}_{M/\mathbb{Z}_2}$ corresponding to $H^0(M^\gamma_2/\mathbb{Z}_2, \mathbb{C}((\hbar)))$. This gives a partial positive answer to [4, Conjecture 1] in the case of $\mathbb{Z}_2$ orbifolds. We construct a deformation of $A^{((\hbar))}_{M/\mathbb{Z}_2}$ in 3 steps,

(1) Dunkl-Weyl algebra bundle,
(2) Quantization of punctured disk bundle,
(3) Global quantization.

We briefly explain the strategy before we go into the details of the construction. In the first step, we will quantize the normal bundle of the γ fixed point submanifold with codimension 2. Quantization of normal bundle of a fixed point submanifold has been considered by Fedosov [9] and Kravchenko [12]. Here the new input is that along the fiber direction of the normal bundle, we will use the Dunkl-Weyl algebra introduced at the end of Section 3. The main result will be that with the new algebra $\mathbb{D}_2((\hbar_1, \hbar_2))$, the construction of Fedosov [9] and Kravchenko [12] has a natural generalization and we obtain a flat connection on the associated Dunkl-Weyl algebra bundle. This first step can be viewed as a quantization of a tubular neighborhood of the γ fixed point submanifold with codimension 2. In order to extend this quantization of a tubular neighborhood of the γ fixed point submanifolds, in Step 2, we restrict the quantization we obtained in Step 1 to a punctured tubular neighborhood of the γ fixed point submanifold with the zero section removed. We are allowed to restrict this quantization because of the locality of the product $\star$ on $\mathbb{D}_2((\hbar_1, \hbar_2))$ discussed in Section 3, Theorem 3.10. An important property of the punctured tubular neighborhood is that the $\mathbb{Z}_2$ action on it is free, and there is no fixed point. Therefore, quantizations of such a punctured neighborhood can be classified by Fedosov's theory without any extra contribution from the fixed point submanifold. In Step 3, we will extend the quantization obtained in Step 1 of the tubular neighborhood of the γ fixed point submanifold with codimension 2 to the whole orbifold. Here the key is that with the study in Step 2, we can regularize the quantization obtained in Step 1 on the punctured tubular neighborhood. Namely, it is isomorphic to some standard quantization of the punctured tubular neighborhood using Fedosov's construction via the characteristic classes developed by Fedosov [8] and Kravchenko [12]. We point out the above strategy is possible to be generalized by replacing the Dunkl-Weyl algebra $\mathbb{D}_2((\hbar_1, \hbar_2))$ by the spherical subalgebra of other symplectic reflection algebras [6] if we know the product is "local".

4.1. **Dunkl-Weyl algebra bundle.** We consider the collection of connected components of M^γ which are of codimension 2, and we denote it by M_2^γ. The symplectic orthogonal space of TM_2^γ in $TM|_{M_2^\gamma}$ defines a normal bundle N of M_2^γ in M. N inherits a $\mathbb{Z}_2$ action from the $\mathbb{Z}_2$ action on M. The restriction of the symplectic form ω to N makes N a $\mathbb{Z}_2$ equivariant symplectic vector bundle with the symplectic structure ω^N. We will fix a global $\mathbb{Z}_2$ invariant compatible almost complex structure on M. (Such an almost complex structure always exists.) An invariant almost complex structure makes N into a $\mathbb{Z}_2$ equivariant hermitian line bundle. In particular, the corresponding principal bundle P associated to N is a principal $U(1)$ bundle. By Proposition 3.16, $U(1)$ naturally acts on the Dunkl-Weyl algebra $(\mathbb{D}_2((\hbar_1,\hbar_2)),\star)$. Therefore, we define the following Dunkl-Weyl algebra bundle over M_2^γ by

$$\mathcal{V} := P\times_{U(1)}\mathbb{D}_2((\hbar_1,\hbar_2)).$$

We have constructed a bundle $\mathcal{V}$ of infinitely dimensional algebras over a symplectic manifold M_2^γ. The hermitian connection on the principal bundle P induces a connection on the Dunkl-Weyl algebra bundle. We exhibit this connection in local coordinates. Let x^ν $(\nu = 1,\dots,2n-2)$ be coordinates on M_2^γ and $z,\bar z$ be coordinates along the fiber direction. The hermitian connection ∇^N on N can be written as

$$\nabla^N_{\frac{\partial}{\partial x^\nu}}\partial_z = i\Gamma_\nu(x)\partial_z,\qquad \nabla^N_{\frac{\partial}{\partial x^\nu}}\partial_{\bar z} = -i\Gamma_\nu(x)\partial_{\bar z},$$

where Γ_ν is a real valued function on M_2^γ.

The induced connection ∂^N on $\mathcal{V}$ is defined by

$$\partial^N\xi = dx^i\otimes\big(\frac{\partial\xi}{\partial x^i}+\frac{i}{2\hbar_1}[\Gamma_i z\bar z,\xi]_\star\big),\qquad \xi\in\Gamma(\mathcal{V}),$$

where $[\ ,\ ]_\star$ is the star-commutator.

Let $R^N_{\nu_1\nu_2}$ be the curvature tensor associated to the hermitian connection ∇. Then one can quickly compute that

$$\partial^N\circ\partial^N(\xi) = \frac{1}{2\hbar_1}[dx^{\nu_1}\wedge dx^{\nu_2}R_{\nu_1\nu_2}z\bar z,\xi]_\star.$$

We remark that because N is a complex 1-dim vector bundle, $\hbar_2$ does not appear in the above curvature expression although it does show up in general in the star-product.

Let $\mathcal{W}$ be the Weyl algebra (with coefficient in $\mathbb{C}((\hbar_1))$) bundle associated to the symplectic form ω^0 on M_2^γ. Following Fedosov's method [8] and Kravchenko's modification [12], we construct a flat connection D on the associated bundle

$$\wedge^\bullet T^*M_2^\gamma\otimes\mathcal{W}\otimes\mathcal{V}.$$

We remark that $\mathcal{W}$ is a bundle of algebras with respect to the ring $\mathbb{C}((\hbar_1))$, and $\mathcal{V}$ is an algebra with respect to the ring $\mathbb{C}((\hbar_1,\hbar_2))$. The tensor product between $\mathcal{W}$ and $\mathcal{V}$ is taken over the ring $C^\infty(M)((\hbar_1))$. Though our construction is essentially a repetition of the ones in [12], since the Dunkl-Weyl algebra is a new ingredient, we recall the construction of the flat connection on $\wedge^\bullet T^*M_2^\gamma\otimes\mathcal{W}\otimes\mathcal{V}$ briefly.

Let ∇^T be a symplectic connection on TM_2^γ with respect to the symplectic form ω_0. If Γ_{ij}^k be the Christoeffel symbol associated to the connection ∇^T on TM_2^γ, then

$$\partial^T\eta = d\eta + \frac{i}{2\hbar_1}[\omega^0_{il}\Gamma^l_{jk}y^iy^jdx^k, \eta]_*,\ \eta\in\mathcal{W}$$

defines a connection on the Weyl algebra bundle $\mathcal{W}$, where $y^i, i = 1, \dots 2n-2$ are coordinates along the fiber direction of TM_2^γ and $[\ ,\ ]_*$ is the commutator with respect to the star-product $*$ on $\mathcal{W}$. Accordingly, $\partial := \partial^T\otimes 1 + 1\otimes\partial^N$ defines a connection on the bundle $\mathcal{W}\otimes\mathcal{V}$. It is a straightforward computation to find

$$\partial^2 a = \frac{i}{\hbar_1}[R^T\otimes 1 + 1\otimes R^N, a],\ a\in\Gamma^\infty(\mathcal{W}\otimes\mathcal{V}),$$

where R^T is the curvature form of ∂^T.

Define $\delta : \wedge^\bullet T^*M_2^\gamma\otimes\mathcal{W}\otimes\mathcal{V}\longrightarrow\wedge^\bullet T^*M_2^\gamma\otimes\mathcal{W}\otimes\mathcal{V}$ by $\delta(a) = \sum_{i=1}^{2n-2}dx^i\partial a/\partial y^i$. Then the same proof as [12, Thm. 5.5] proves that there is a flat connection D on $\wedge^\bullet T^*M_2^\gamma\otimes\mathcal{W}\otimes\mathcal{V}$ of the form

$$D = d + \frac{i}{\hbar_1}[\gamma,\cdot] = \partial + \delta + \frac{i}{\hbar_1}[r,\cdot],$$

where r is an element in $T^*M_2^\gamma\otimes\mathcal{W}\otimes\mathcal{V}$. The key point in the Kravchenko's proof of [12, Theorem 5.5] is that one has to modify the definition of the operator δ to compensate the existence of the curvature form R^N in the expression of ∂^2 because $i/\hbar_1 R^N$ will contribute an extra term of degree -1 in Fedosov's iteration procedure of constructing a flat connection. This also applies to our construction with the following observation.

By the definition of the star-product on the Dunkl-Weyl algebra, one can easily check that for $j\le 2$, C^0_{jl} and C^1_{jl} vanishes on $z\bar z$. Therefore, we have for an arbitrary $f\in\mathbb{D}_2$, as f is $\mathbb{Z}_2$ invariant,

$$\begin{aligned}&[z\bar z, f]_\star\\ =&\hbar_1\Big(\{z\bar z, f\} + \hbar_2\big(\frac{(z\bar z + z\bar z)}{2\bar z}\frac{(f(z,-\bar z) - f(-z,-\bar z))}{2z} - \frac{(f(z,\bar z) - f(z,-\bar z))}{2\bar z}\frac{(-z\bar z - z\bar z)}{2z}\big)\Big)\\ =&\hbar_1\{z\bar z, f\},\end{aligned}$$

where $\{\ .\ \}$ is the Poisson structure $\{f,g\} = i(\partial_{\bar z}f\partial_z g - \partial_{\bar z}g\partial_z f)$. This shows that on the bundle $\mathcal{V}$ with the fiberwise algebra isomorphic to $\mathbb{D}_2((\hbar_1,\hbar_2))$, the curvature of the connection ∂^N which is equal to the star-commutator of R^N with respect to the product on $\mathbb{D}_2((\hbar_1,\hbar_2))$ acts as same as the Poisson commutator associated to the restriction of the symplectic form ω^N along the fiber direction of N. With this observation, we can repeat the exactly same construction as those in Kravchenko's proof of [12, Theorem 5.5]. And we conclude that there is a flat connection D on the bundle $\wedge^\bullet T^*M_2^\gamma\otimes\mathcal{W}\otimes\mathcal{V}$ whose Weyl curvature is equal to ω, the pullback of the symplectic form on M to the normal bundle N.

4.2. **Quantization of punctured disk bundle.** With the above flat connection D on $\mathcal{W}\otimes\mathcal{V}$, we consider flat sections with respect to the connection D. The space $\mathcal{A}_D$ of flat sections is isomorphic $C^\infty(M_2^\gamma,\mathcal{V})$ as a vector space. Furthermore, including the isomorphism between $\mathbb{D}_2((\hbar_1,\hbar_2))$ with $\mathrm{Poly}(\mathbb{R}^2)^{\mathbb{Z}_2}((\hbar_1,\hbar_2))$ as vector spaces, we conclude that $\mathcal{A}_D$ is isomorphic to the space of functions on M_2^γ with value in the associated bundle $P \otimes_{U(1)} \mathrm{Poly}(\mathbb{R}^2)^{\mathbb{Z}_2}((\hbar_1, \hbar_2))$. The later space can be viewed as the space $C^\infty(N)^{\mathbb{Z}_2}((\hbar_1, \hbar_2))$ of $\mathbb{Z}_2$-invariant smooth functions on N. The identification between $C^\infty(N)^{\mathbb{Z}_2}((\hbar_1, \hbar_2))$ and $\mathcal{A}_D$ equips $C^\infty(N)^{\mathbb{Z}_2}((\hbar_1, \hbar_2))$ with a new associative product, which is a deformation of the standard commutative product on $C^\infty(N)^{\mathbb{Z}_2}((\hbar_1, \hbar_2))$.

Via the exponential map with respect to some $\mathbb{Z}_2$ invariant metric on N, we can identify a tubular neighborhood B_ϵ of M_2^γ in M with an ϵ neighborhood N_ϵ of the zero section in N for some $\epsilon > 0$. As was discussed in Theorem 3.10, the star-product on $\mathbb{D}_2((\hbar_1, \hbar_2))$ is γ-local. Furthermore, the product on the standard Weyl algebra is also local. These locality results imply that the deformed product on $C^\infty(N)^{\mathbb{Z}_2}((\hbar_1, \hbar_2))$ is γ-local, and therefore, we are allowed to restrict $(C^\infty(N)^{\mathbb{Z}_2}((\hbar_1, \hbar_2)), \star)$ to the ϵ neighborhood N_ϵ of the zero section in N, which defines an associative deformation of $\mathbb{Z}_2$-invariant functions on N_ϵ. Finally, pushing forward along the exponential map, we obtain a deformation of $\mathbb{Z}_2$-invariant smooth functions on B_ϵ, namely $(C^\infty(B_\epsilon)^{\mathbb{Z}_2}((\hbar_1, \hbar_2)), \star)$.

We next look at the space $B_\epsilon^* := B_\epsilon - M_2^\gamma$ of punctured neighborhood, which is diffeomorphic to $N_\epsilon - M_2^\gamma$, the punctured disk bundle via the exponential map. As the star product on the Dunkl-Weyl algebra $\mathbb{D}_2$ is γ-local, we can restrict the algebra $(C^\infty(B_\epsilon)^{\mathbb{Z}_2}((\hbar_1, \hbar_2)), \star)$ to the punctured neighborhood, which is denoted by

$$(C^\infty(B_\epsilon^*)^{\mathbb{Z}_2}((\hbar_1, \hbar_2)), \star).$$

We observe that as the $\mathbb{Z}_2$ action on B_ϵ^* is free, the space of $\mathbb{Z}_2$-invariant functions on B_ϵ^* can be identified with the space of functions on the quotient $B_\epsilon^*/\mathbb{Z}_2$, which is a smooth manifold. Hence the algebra $(C^\infty(B_\epsilon^*)^{\mathbb{Z}_2}((\hbar_1, \hbar_2)), \star)$ can be viewed as a deformation quantization of the quotient $B_\epsilon^*/\mathbb{Z}_2$. On the other hand, the $\mathbb{Z}_2$-invariant symplectic form ω on M restricts to define a symplectic form on the quotient $B_\epsilon^*/\mathbb{Z}_2$. As $B_\epsilon^*/\mathbb{Z}_2$ is a smooth symplectic manifold, one can apply the standard Fedosov construction of a deformation quantization on $B_\epsilon^*/\mathbb{Z}_2$, and therefore obtain an associative algebra $(C^\infty(B_\epsilon^*)^{\mathbb{Z}_2}((\hbar_1, \hbar_2)), \star_{\mathrm{F}})$ with the characteristic class equal to ω.

Proposition 4.1. *The two algebras* $(C^\infty(B_\epsilon^*)^{\mathbb{Z}_2}((\hbar_1, \hbar_2)), \star)$ *and* $(C^\infty(B_\epsilon^*)^{\mathbb{Z}_2}((\hbar_1, \hbar_2)), \star_F)$ *are isomorphic.*

Proof. We consider an intermediate algebra to relate the above two algebras. We look at the normal bundle N over M_2^γ. The restriction of the symplectic form ω to each fiber makes N into a symplectic vector bundle. We consider the associated Weyl algebra bundle to $\mathcal{V}_{\mathrm{W}}$ by $\mathcal{V}_{\mathrm{W}} := P \times_{U(1)} \mathbb{W}_2^{\mathbb{Z}_2}$, where $\mathbb{W}_2$ is the tensor of the Weyl algebra W_2 (with coefficient in $\mathbb{C}((\hbar_1))$) on $\mathbb{R}^2$ with $\mathbb{C}((\hbar_2))$ and $\mathbb{W}_2^{\mathbb{Z}_2}$ is the $\mathbb{Z}_2$ invariant subalgebra of $\mathbb{W}_2$. Similar to what we have done in Section 4.1, we can construct a flat connection D_{W} on the bundle $\wedge^\bullet T^* M_2^\gamma \otimes \mathcal{W} \otimes \mathcal{V}_{\mathrm{W}}$. The space of flat sections with respect to the flat connection D_{W} is isomorphic to $C^\infty(N)^{\mathbb{Z}_2}((\hbar_1, \hbar_2))$. Therefore we obtain an associative algebra $(C^\infty(N)^{\mathbb{Z}_2}((\hbar_1, \hbar_2)), \star_{\mathrm{W}})$ as a deformation quantization of $C^\infty(N)^{\mathbb{Z}_2}((\hbar_1, \hbar_2))$. As the Weyl algebra $\mathbb{W}_2$ has a local product, the algebra $(C^\infty(N)^{\mathbb{Z}_2}((\hbar_1, \hbar_2)), \star_{\mathrm{W}})$ restricts to the punctured disk bundle $N_\epsilon - M_2^\gamma$. And via the exponential map with respect to the $\mathbb{Z}_2$ invariant riemannian metric, $(C^\infty(N)^{\mathbb{Z}_2}((\hbar_1, \hbar_2)), \star_{\mathrm{W}})$ restricts to define a deformation quantization of the punctured tubular neighborhood, $(C^\infty(B_\epsilon^*)^{\mathbb{Z}_2}((\hbar_1, \hbar_2)), \star_{\mathrm{W}})$.

Now we compare the algebra $(C^\infty(B_\epsilon^*)^{\mathbb{Z}_2}((\hbar_1,\hbar_2)),\star_{\mathrm{W}})$ with $(C^\infty(B_\epsilon^*)^{\mathbb{Z}_2}((\hbar_1,\hbar_2)),\star_{\mathrm{F}})$. According to [9, Sec. 5] and [12, Thm 5.6], these two algebras are isomorphic as they have the same characteristic class ω. To compare the algebra $(C^\infty(B_\epsilon^*)^{\mathbb{Z}_2}((\hbar_1,\hbar_2)),\star)$ with $(C^\infty(B_\epsilon^*)^{\mathbb{Z}_2}((\hbar_1,\hbar_2)),\star_{\mathrm{W}})$, we see that the procedure to obtain these two algebras are different only at one step, where $\mathcal{V}$ is $P\times_{U(1)}\mathbb{D}_2((\hbar_1,\hbar_2))$ and $\mathcal{V}_{\mathrm{W}}$ is $P\times_{U(1)}\mathbb{W}_2^{\mathbb{Z}_2}$. We have pointed out that the product on $\mathbb{D}_2((\hbar_1,\hbar_2))$ is γ-local and the product on $\mathbb{W}_2^{\mathbb{Z}_2}$ is local. Therefore, the restrictions of $(C^\infty(N)^{\mathbb{Z}_2}((\hbar_1,\hbar)),\star)$ and $(C^\infty(N)^{\mathbb{Z}_2}((\hbar_1,\hbar_2)),\star_{\mathrm{W}})$ to $N_\epsilon - M_2^\gamma$ can be constructed via the flat connections D and D_{W} on the bundle $\wedge^\bullet T^*M_2^\gamma\otimes\mathcal{W}\otimes(P\times_{U(1)}\mathbb{D}_2((\hbar_1,\hbar_2))|_{D_\epsilon^*})$ and $\wedge^\bullet T^*M_2^\gamma\otimes\mathcal{W}\otimes(P\times_{U(1)}\mathbb{W}_2^{\mathbb{Z}_2}|_{D_\epsilon^*})$, where D_ϵ^* is the puncture disk of radius ϵ in $\mathbb{R}^2$.

The algebras $\mathbb{D}_2((\hbar_1,\hbar_2))|_{D_\epsilon^*}$ and $\mathbb{W}_2^{\mathbb{Z}_2}|_{D_\epsilon^*}$ are both deformation quantization of the punctured disk $D_\epsilon^*/\mathbb{Z}_2$. In fact, if we look at the algebra $\mathbb{D}_2((\hbar_1,\hbar_2))|_{D_\epsilon^*}$ more carefully, we notice that the product of this algebra is an expression of $f,g\in C^\infty(D_\epsilon^*)^{\mathbb{Z}_2}$ of power series $\hbar_1$ and $\hbar_2$. In particular, if we look at $f\star g$ as a formal power series of $\hbar_2$, the 0-th power term is exactly the product on $\mathbb{W}_2^{\mathbb{Z}_2}|_{D_\epsilon^*}$. Therefore, we can view $\mathbb{D}_2((\hbar_1,\hbar_2))|_{D_\epsilon^*}$ as a formal deformation quantization of the algebra $W_2^{\mathbb{Z}_2}|_{D_\epsilon^*}$, where $W_2^{\mathbb{Z}_2}$ is the subspace of $\mathbb{Z}_2$ invariant elements in the Weyl algebra W_2 (with coefficient in $\mathbb{C}((\hbar_1))$). As we can identify $W_2^{\mathbb{Z}_2}|_{D_\epsilon^*}$ as a deformation quantization of the quotient $D_\epsilon^*/\mathbb{Z}_2$, its Hochschild cohomology of $W_2^{\mathbb{Z}_2}|_{D_\epsilon^*}$ can be computed using the result of [13]. In particular, the second Hochschild cohomology of $W_2^{\mathbb{Z}_2}|_{D_\epsilon^*}$ is equal to the degree 2 de Rham cohomology of $D_\epsilon^*/\mathbb{Z}_2$ with coefficient in $\mathbb{C}((\hbar_1))$. As $D_\epsilon^*/\mathbb{Z}_2$ is homotopic to a circle, its degree 2 de Rham cohomology is zero. This implies that $\mathbb{D}_2((\hbar_1,\hbar_2))|_{D_\epsilon^*}$ must be a trivial deformation of $W_2^{\mathbb{Z}_2}|_{D_\epsilon^*}$. Furthermore, as $U(1)$ is compact, by the standard averaging trick, we can obtain a $U(1)$ equivariant isomorphism from $\mathbb{D}_2((\hbar_1,\hbar_2))|_{D_\epsilon^*}$ to $\mathbb{W}_2^{\mathbb{Z}_2}|_{D_\epsilon^*}=W_2^{\mathbb{Z}_2}[\hbar_2^{-1},\hbar_2]]$.

With the above $U(1)$ equivariant isomorphism between the algebras on each fiber, we have an natural isomorphism of bundles $\mathcal{V}$ and $\mathcal{V}_{\mathrm{F}}$ which accordingly identifies the connections ∂_N on the corresponding bundles. Noticing that Fedosov's construction of flat connection is canonical with respect to the choice of a symplectic connection, we can easily check that the construction of flat connections D and D_{W} are actually compatible with respect to this isomorphism of bundles. Hence, we can conclude that there is an isomorphism between $(C^\infty(B_\epsilon^*)^{\mathbb{Z}_2}((\hbar_1,\hbar_2)),\star)$ and $(C^\infty(B_\epsilon^*)^{\mathbb{Z}_2}((\hbar_1,\hbar_2)),\star_{\mathrm{W}})$ as flat sections of D and D_{W}.

We conclude that $(C^\infty(B_\epsilon^*)^{\mathbb{Z}_2}((\hbar_1,\hbar_2)),\star)$ is isomorphic to $(C^\infty(B_\epsilon^*)^{\mathbb{Z}_2}((\hbar_1,\hbar_2)),\star_{\mathrm{W}})$ and therefore is isomorphic to $(C^\infty(B_\epsilon^*)^{\mathbb{Z}_2}((\hbar_1,\hbar_2)),\star_{\mathrm{F}})$. □

4.3. **Global algebra.** In this subsection, we construct the algebra promised at the beginning of this section, which is a deformation of $A_{M/\mathbb{Z}_2}^{((\hbar_1))}$.

Recall that M_2^γ is a disjoint union of fixed point submanifolds of M which are of codimension 2. Fix a $\mathbb{Z}_2$-invariant almost complex structure on M, which also defines a $\mathbb{Z}_2$ invariant metric on M. We choose a sufficiently small ϵ such that the ϵ tubular neighborhood of each component of M_2^γ in M does not intersect with each other. We use B_ϵ to denote the disjoint union of the ϵ tubular neighborhood of each component of M_2^γ. Furthermore, we denote M^- to be the open complement $M-M_2^\gamma$ to the closed subset M_2^γ. In this way, we have the orbifold as a union of two open subsets $B_\epsilon/\mathbb{Z}_2$ and $M^-/\mathbb{Z}_2$, and the intersection of these two open sets is $B_\epsilon^*/\mathbb{Z}_2$, the ϵ punctured neighborhood of M^γ in $M/\mathbb{Z}_2$.

We construct an algebra $\mathfrak{A}_{M/\mathbb{Z}_2}^{((\hbar_1,\hbar_2))}$ as follows. On $B_\epsilon/\mathbb{Z}_2$, this algebra is isomorphic to $(C^\infty(B_\epsilon)^{\mathbb{Z}_2}((\hbar_1,\hbar_2)),\star)$ which, via the exponential map, can be identified as the restriction to the bundle N_ϵ of the space of flat sections of the Dunkl-Weyl algebra introduced in Section 4.1. On $M^-/\mathbb{Z}_2$, the algebra $\mathfrak{A}_{M/\mathbb{Z}_2}^{((\hbar_1,\hbar_2))}$ is isomorphic to the Fedosov quantization[3] $\mathcal{A}_{M^-}^{((\hbar_1,\hbar_2))}$ of M^- with the Weyl curvature being ω. By Proposition 4.1, the restriction of $(C^\infty(B_\epsilon)^{\mathbb{Z}_2}((\hbar_1,\hbar_2)),\star)$ to $B_\epsilon^*/\mathbb{Z} = M^-/\mathbb{Z}_2\cap B_\epsilon/\mathbb{Z}_2$ is isomorphic to $(C^\infty(B_\epsilon^*)_2^{\mathbb{Z}}((\hbar_1,\hbar_2)),\star_{\mathrm{F}})$, which is the restriction of $(\mathcal{A}_{M^-}^{((\hbar_1,\hbar_2))})^{\mathbb{Z}_2}$ to $B_\epsilon^*/\mathbb{Z}_2$. We define $\mathfrak{A}_{M/\mathbb{Z}_2}^{((\hbar_1,\hbar_2))}$ to be the algebra defined by gluing $(C^\infty(B_\epsilon)^{\mathbb{Z}_2}((\hbar_1,\hbar_2)),\star)$ and $(\mathcal{A}_{M^-}^{((\hbar_1,\hbar_2))})^{\mathbb{Z}_2}$ via the isomorphism on $B_\epsilon^*/\mathbb{Z}_2$.

We summarize the above construction into the following theorem.

Theorem 4.2. *The algebra $\mathfrak{A}_{M/\mathbb{Z}_2}^{((\hbar_1,\hbar_2))}$ is a nontrivial deformation of the algebra $\mathcal{A}_{M/\mathbb{Z}_2}^{((\hbar_1))}$.*

Proof. We look at the product on the algebra $\mathfrak{A}^{\hbar_1,\hbar_2}$. If f and g are two elements of $C^\infty(M/\mathbb{Z}_2)$, $f\star g$ can be written as a formal power series of $\hbar_2$. From the construction in Section 4.1, it is not difficult to see that the $\hbar_2^0$ component is exactly the product of the algebra $\mathcal{A}_{M/\mathbb{Z}_2}^{((\hbar_1))}$. Furthermore, from the local computation on B_ϵ, we can see that as $\mathbb{D}_2((\hbar_1,\hbar_2))$ is a nontrivial deformation of the invariant Weyl algebra $\mathbb{W}_2^{\mathbb{Z}_2}$, the algebra $\mathfrak{A}_{M/\mathbb{Z}_2}^{((\hbar_1,\hbar_2))}$ is a nontrivial deformation. □

Remark 4.3. *In the construction of the algebra $\mathfrak{A}^{((\hbar_1,\hbar_2))}$, we have chosen an ϵ neighborhood B_ϵ of M_2^γ. We point out that different choices of ϵ give rise to isomorphic algebras $\mathfrak{A}^{((\hbar_1,\hbar_2))}$. One can easily check the $U(1)$ equivariant isomorphism from $\mathbb{D}_2((\hbar_1,\hbar_2))|_{D_\epsilon^*}$ to $\mathbb{W}_2^{\mathbb{Z}_2}|_{D_\epsilon^*}$ constructed in Proposition 4.1 can be made to be compatible with the restriction map to $B_{\epsilon'}$ with $\epsilon' < \epsilon$, as the operators appearing in the isomorphism are all γ-local operators. This compatibility with respect to the restriction map assures that the outcome algebra $\mathfrak{A}^{((\hbar_1,\hbar_2))}$ are all isomorphic for different choices of ϵ.*

We have used an almost complex structure and therefore a compatible riemannian metric to identify the ϵ neighborhood B_ϵ with the ϵ neighborhood N_ϵ of the zero section of N. Our algebra $\mathfrak{A}^{((\hbar_1,\hbar_2))}$ does seem to depend on the choice of almost complex structures since we are taking the normal ordering of the operator symbol calculus in Definition 3.1 and also in Definition 3.14 of the Dunkl-Weyl algebra $\mathbb{D}_2((\hbar_1,\hbar_2))$. The analogous well-known phenomena is that wick and anti-wick deformation quantization of an almost Kähler manifold depends on the choices of almost complex structures. We plan to discuss this dependence of almost complex structures in the future.

Remark 4.4. *We explain how far we are away from a full proof of the Dolgushev-Etingof conjecture in the case of a $\mathbb{Z}_2$ orbifold. In this section, we have constructed a deformation of the algebra $A_{M/\mathbb{Z}_2}^{((\hbar_1))}$ along the direction of the union M_2^γ of all codimension 2 components in the inertia orbifold $\widetilde{M/\mathbb{Z}_2}$. Furthermore, we observe that our constructions are*

[3]The algebra $\mathcal{A}_{M^-}^{((\hbar_1,\hbar_2))}$ is defined to $\mathcal{A}_{M^-}^{((\hbar_1))}\otimes\mathbb{C}((\hbar_2))$.

local with respect to every connected component M_2^γ. Such an observation allows us to construct a deformation of $A_{M/\mathbb{Z}_2}^{((\hbar_1))}$ for every connected component of M_2^γ. If we associate a formal parameter c_i for every component of M_2^γ, we actually have constructed a universal deformation of $A_{M/\mathbb{Z}_2}^{((\hbar_1))}$ parametrized by codimension 2 components in $\widetilde{M/\mathbb{Z}_2}$. This is the main part of the Dolgushev-Etingof conjecture [4].

We are not able to prove the full conjecture of Dolgushev-Etingof for $\mathbb{Z}_2$ orbifolds because in our construction, in particular the proof of Proposition 4.1, we have used crucially two extra assumptions. One is that we have assumed the characteristic class of $A_{M/\mathbb{Z}_2}^{((\hbar_1))}$ is ω, the other is that the parameter $\hbar_2$ is formal. However, the Dolgushev-Etingof conjecture [4] *does not require these two assumptions. The first assumption allows us to compare the Fedosov quantizations of the normal bundle N of the fixed point submanifold M_2^γ and a tubular neighborhood B_ϵ. If we change the characteristic class ω by a class t in $H^2(M/\mathbb{Z}_2)((\hbar_1))$, it is hard to realize the information of t on the normal bundle N, which prevents us from comparing the corresponding Fedosov quantizations. The second assumption that $\hbar_2$ is formal allows us to apply homological algebra arguments to show that $\mathbb{D}_2((\hbar_1,\hbar_2))|_{D_\epsilon^*}$ is isomorphic to $\mathbb{W}_2^{\mathbb{Z}_2}|_{D_\epsilon^*}$. If we are able to prove that the constructed isomorphism in power series of $\hbar_2$ is convergent, then we can actually allow $\hbar_2$ to be a number in $\mathbb{C}$. We do not have solutions to avoid these two assumptions now, and hope to address these problems in a future publication.*

We finally remark that in this section, we have been working with the global quotient orbifold, the quotient of a symplectic manifold M by a $\mathbb{Z}_2$ action. Our construction does generalize for general orbifolds which is locally either diffeomorphic to $\mathbb{R}^n$ or the quotient of $\mathbb{R}^n$ by the linear $\mathbb{Z}_2$ action.

5. Proof Theorem 3.10

We will prove Theorem 3.10 in 2 steps. In the first step, we work with Dunkl pseudo-differential operators of the forms introduced in Definition 3.2 to derive an asymptotic expansion of the symbol of the product of two operators. The asymptotic expansion of the symbol we obtain in Step I may contain a sum of infinitely many terms in a fixed symbol class. In the second step, we will rewrite the asymptotic expansion of the symbol obtained in Step I into the expressions introduced in Section 3.3.

5.1. **Step I.** Let a, b be two polynomials on $\mathbb{R}^2$. To compute the asymptotic expansion of $Op_k(a)\circ Op_k(b)$ we need to study the following integral

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} d\mu_k(p)d\mu_k(y)d\mu_k(p_1)E_k(x,ip)a(x,p)E_k(y,-ip)E_k(y,ip_1)b(y,p_1)E_k(z,-ip_1).$$

Since $a(x,p)$ is a polynomial, we take the Taylor expansion of $a(x,p)$ with respect to p, i.e. $a(x,p)=\sum_\alpha p^\alpha/\alpha!\partial_p^\alpha a(x,0)$.

We insert the Taylor expansion into the above equation of $Op_k(a)\circ Op_k(b)$.

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} d\mu_k(p)d\mu_k(y)d\mu_k(p_1)\sum_\alpha E_k(x,ip)\frac{p^\alpha}{\alpha!}\partial_p^\alpha a(x,0)E_k(y,-ip)$$

$$E_k(y,ip_1)b(y,p_1)E_k(z,-ip_1).$$

Recall that when applying the variable y, we have $T_k(E_k(y,-ip)) = (-ip)E_k(y,-ip)$. Hence we can replace $pE_k(y,-ip)$ by $iT_k(E_k(y,-ip))$ in the above integral, and obtain

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} d\mu_k(p)d\mu_k(y)d\mu_k(p_1)\sum_{\alpha} E_k(x,ip)\frac{p^{\alpha-1}}{\alpha!}\partial_p^\alpha a(x,0)iT_k(E_k(y,-ip))$$
$$E_k(y,ip_1)b(y,p_1)E_k(z,-ip_1).$$

As T_k is a skew adjoint operator on $L^2_k(\mathbb{R})$, we can rewrite the above equation as

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} d\mu_k(p)d\mu_k(y)d\mu_k(p_1)\sum_{\alpha} E_k(x,ip)\frac{p^{\alpha-1}}{\alpha!}\partial_p^\alpha a(x,0)E_k(y,-ip)$$
$$(-i)T_k\big(E_k(y,ip_1)b(y,p_1)\big)E_k(z,-ip_1).$$

We apply

$$\begin{aligned}&T_k\big(E_k(y,ip_1)b(y,p_1)\big)\\ =&ip_1E_k(y,ip_1)b(y,p_1)+E_k(y,ip_1)\partial_y b(y,p_1)+E_k(-y,ip_1)k\tilde{\partial}_y b(y,p_1)\end{aligned}$$

to the above equation, and obtain

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} d\mu_k(p)d\mu_k(y)d\mu_k(p_1)\sum_{\alpha} E_k(x,ip)\frac{p^{\alpha-1}}{\alpha!}\partial_p^\alpha a(x,0)E_k(y,-ip)$$
$$\big(p_1E_k(y,ip_1)b(y,p_1)-iE_k(y,ip_1)\partial_y b(y,p_1)-E_k(-y,ip_1)k\tilde{\partial}_y b(y,p_1))\big)E_k(z,-ip_1).$$

Substituting p_1 by $-p_1$, we obtain the following expression

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} d\mu_k(p)d\mu_k(y)d\mu_k(p_1)\sum_{\alpha} E_k(x,ip)\frac{p^{\alpha-1}}{\alpha!}\partial_p^\alpha a(x,0)E_k(y,-ip)$$
$$E_k(y,ip_1)\big((p_1-i\partial_y)b(y,p_1)-ik\tilde{\partial}_y b(y,-p_1)\hat{\gamma})\big)E_k(z,-ip_1),$$

where $\hat{\gamma}$ is an operator on variable z changing z to $-z$. In summary, we have seen above that an extra variable p on $a(x,p)$ in the integral of $Op_k(a)\circ Op_k(b)$ is equivalent to apply $p_1 - i\partial_y - ik\sigma_2\tilde{\partial}\hat{\gamma}$ on b, where σ_2 is mapping $b(x,p)$ to $b(x,-p)$.

By induction with respect to the power α, we have the following expression

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} d\mu_k(p)d\mu_k(y)d\mu_k(p_1)E_k(x,ip)a(x,p)E_k(y,-ip)E_k(y,ip_1)\\ &\qquad b(y,p_1)E_k(z,-ip_1)\\ =&\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} d\mu_k(p)d\mu_k(y)d\mu_k(p_1)E_k(x,ip)E_k(y,-ip)E_k(y,ip_1)\\ &\qquad\sum_{\alpha}\frac{1}{\alpha!}\partial_p^\alpha a(x,0)[p_1-i\partial_y-ik\sigma_2\tilde{\partial}_y\hat{\gamma}]^\alpha b(y,p_1)E_k(z,-ip_1).\end{aligned}$$

Integrating over variable p, we have the integral on the right hand side equal to

$$\int_{\mathbb{R}}\int_{\mathbb{R}} d\mu_k(y)d\mu_k(p)E_k(x,ip_1)$$
$$\sum_{\alpha}\frac{1}{\alpha!}\partial_p^\alpha a(x,0)[p_1-i\partial_y-ik\sigma_2\tilde{\partial}_y\hat{\gamma}]^\alpha b(y,p_1)|_{y=x}E_k(z,-ip_1).$$

Therefore, we conclude that

$$Op_k(a)\circ Op_k(b) = Op_k(\sum_\alpha \frac{1}{\alpha!}\partial_p^\alpha a(x,0)[p_1 - i\partial_y - ik\sigma_2\tilde{\partial}_y\hat{\gamma}]^\alpha b(y,p_1)|_{y=x}). \tag{6}$$

We remark that the above sum is finite as a is a polynomial.

5.2. **Step II.** In this step, we aim to understand the expansion formula

$$\sum_\alpha \frac{1}{\alpha!}\partial_p^\alpha a(x,0)[p_1 - i\partial_y - ik\sigma_2\tilde{\partial}_y\hat{\gamma}]^\alpha b(y,p_1)|_{y=x}$$

obtained in the previous subsection.

We look at the power $[p_1 - i\partial_y - ik\sigma_2\tilde{\partial}_y\hat{\gamma}]^\alpha$. Define $A = -i\partial_y$, and $B = -ik\sigma_2\tilde{\partial}_y\hat{\gamma}$. We observe the following commuting relations,

$$Ap_1 = p_1A, \qquad p_1B = -Bp_1.$$

With these relations, we write $[p_1 - i\partial_y - ik\sigma_2\tilde{\partial}_y\hat{\gamma}]^\alpha$ as

$$\sum_{\nu\in P_{m,n}} c_\nu(-i)^{m+n}k^n p_1^{\alpha-m-n}B_\nu\sigma_2^n\hat{\gamma}^n,$$

where $P_{m,n}$ is the set of solutions to Eq. (3) introduced in Section 3.3, and B_ν is the operator introduced in Section 3.3 as compositions of ∂_y and $\tilde{\partial}_y$, and c_ν is number determined by $\nu\in P_{m,n}$.

We study the number c_ν more carefully. c_ν is the number of the term

$$(-i)^{m+n}k^n p_1^{\alpha-m-n}B_\nu\sigma_2^n\hat{\gamma}^n$$

appearing in the expansion $[p_1 - i\partial_y - ik\sigma_2\tilde{\partial}_y\hat{\gamma}]^\alpha$. When we write out the expansion of $[p_1 - i\partial_y - ik\sigma_2\tilde{\partial}_y\hat{\gamma}]^\alpha$, it is a sum of monomials of the form $p_1^{x_0}\partial_y^{\nu_0}\tilde{\partial}_y p_1^{x_1}\partial_y^{\nu_1}\tilde{\partial}_y\cdots\tilde{\partial}_y p_1^{x_n}\partial_y^{\nu_n}\sigma_2^n\hat{\gamma}^n$, where $\nu=(\nu_0,\dots,\nu_n)$ is a fixed element of $P_{m,n}$, and $x_0,\dots,x_n$ are nonnegative integers with $x_0+\cdots+x_n = \alpha-m-n$. We remark that as p_1 commutes with ∂_y, we do not need to count the relative positions between ∂_y and p_1. Therefore, totally there are

$$\prod_{j=0}^n \binom{y_i+x_i}{x_i}$$

number of the term $p_1^{x_0}\partial_y^{\nu_0}\tilde{\partial}_y p_1^{x_1}\partial_y^{\nu_1}\tilde{\partial}_y\cdots\tilde{\partial}_y p_1^{x_n}\partial_y^{\nu_n}\sigma_2^n\hat{\gamma}^n$ in the power $[p_1 - i\partial_y - ik\sigma_2\tilde{\partial}_y\hat{\gamma}]^\alpha$. Furthermore, as σ_2 changes the sign of p_1, when we move σ_2 to the right end, we need to count the change of signs. Therefore, in front of the term $p_1^{x_0}\partial_y^{\nu_0}\tilde{\partial}_y p_1^{x_1}\partial_y^{\nu_1}\tilde{\partial}_y\cdots\tilde{\partial}_y p_1^{x_n}\partial_y^{\nu_n}\sigma_2^n\hat{\gamma}^n$, there should be a sign

$$(-1)^{x_1+2x_2+\cdots+nx_n}.$$

In summary, for $\nu\in P_{m,n}$, c_ν is equal to

$$\sum_{x_0+\cdots+x_n=\alpha-m-n}\prod_{j=0}^n \binom{y_j+x_j}{x_j}(-1)^{jx_j}$$

Separating j from even to odd, we have c_ν equal to

$$\sum_{x_0+\cdots+x_n=\alpha-m-n}\left[\prod_{j \text{ is even}}\binom{y_j+x_j}{x_j}\right]\left[\prod_{j \text{ is odd}}\binom{y_j+x_j}{x_j}(-1)^{x_i}\right]$$

$$=\sum_{s+t=\alpha-m-n}\left[\sum_{x_0+x_2+\cdots+x_{\text{even}}=s}\prod_{j \text{ is even}}\binom{y_j+x_j}{x_j}\right]\left[\sum_{x_1+\cdots+x_{\text{odd}}=t}\prod_{j \text{ odd}}\binom{y_j+x_j}{x_j}(-1)^{x_i}\right]$$

To evaluate the above number, we introduce the generating function $1/(1-x)^{s+1}$. The Taylor expansion of $1/(1-x)^{1+s}$ and $1/(1+x)^{1+s}$ at 0 with $|x|<1$ is

$$\frac{1}{(1-x)^{1+s}}=\sum_{k=0}^{\infty}\binom{s+k}{k}x^k,$$
$$\frac{1}{(1+x)^{1+s}}=\sum_{k=0}^{\infty}\binom{s+k}{k}(-1)^k x^k.$$

In summary, if we denote $\Lambda_0=\nu_0+\sum_{\text{even i}}\nu_i$ and $\Lambda_1=\sum_{\text{odd i}}\nu_i$, then c_ν is the coefficient of the term $x^{\alpha-m-n}$ of the Taylor expansion of the following function

$$\frac{1}{(1-x)^{\Lambda_0+n_0+1}(1+x)^{\Lambda_1+n_1}}, \tag{7}$$

where n_0 (and n_1) is the number of positive even (odd) numbers less than or equal to n.

We next consider the expansion

$$\sum_\alpha \frac{1}{\alpha!}\partial_p^\alpha a(x,0)[p_1-i\partial_y-ik\sigma_2\tilde{\partial}_y\hat{\gamma}]^\alpha b(y,p_1)|_{y=x}.$$

By inserting the expansion of the power $[p_1-i\partial_y-ik\sigma_2\tilde{\partial}_y\hat{\gamma}]^\alpha$, we have the above expansion equal to

$$\sum_\alpha\sum_{m,n,\nu}\frac{(-i)^{m+n}k^n}{\alpha!}\partial_p^\alpha a(x,0)c_\nu p_1^{\alpha-m-n}B_\nu\sigma_2^n(b)\hat{\gamma}^n$$
$$=\sum_{m,n}(-i)^{m+n}k^n\Big(\sum_{\nu\in P_{m,n}}\sum_\alpha\frac{c_\nu}{p_1^{m+n}}\frac{p_1^\alpha}{\alpha!}\partial_p^\alpha a(x,0)\Big)B_\nu\sigma_2^n(b)\hat{\gamma}.$$

We notice that the terms $B_\nu\sigma_2^n(b)\hat{\gamma}^k$ are independent of α and ν, then we are left to deal with $\frac{c_\nu}{p_1^{m+n}}\frac{p_1^\alpha}{\alpha!}\partial_p^\alpha a(x,0)$ for the sum over α.

Considering the above interpretation of c_ν, if we introduce an auxiliary variable $t\in\mathbb{C}-\{0\}$, then we have $\sum_\alpha\frac{c_\nu}{p_1^{m+n}}\frac{p_1^\alpha}{\alpha!}\partial_p^\alpha a(x,0)$ equal to the t^0 term of the product between

$$\frac{t^{m+n}}{(1-tp_1)^{\Lambda_0+n_0+1}(1+tp_1)^{\Lambda_1+n_1}}$$

and $a(x,1/t)$ for $|tp_1|<1$. We remark that by $1/(1-tp_1)^{\Lambda_0+n_0+1}(1+tp_1)^{\Lambda+n_1}$ we really mean the Taylor expansion with respect to variable tp_1 as $|p_1t|<1$, and by $a(x,1/t)$ we mean the Taylor expansion of a with respect to the variable $1/t$. As a is assumed to be a polynomial, its Taylor expansion with respect to variable $1/t$ only has finitely many terms, and $a(x,1/t)$ is an element in $\mathbb{C}[x]((t))$. This assures the product between $a(x,1/t)$ and $t^{m+n}/(1-tp_1)^{\Lambda_0+n_0+1}(1+tp_1)^{\Lambda_1+n_1}$ well defined, as a product of two Laurent series of variable t. In conclusion, we conclude that $\sum_\alpha \frac{c_\nu}{p_1^{m+n}}\frac{p_1^\alpha}{\alpha!}\partial_p^\alpha a(x,0)$ is equal to the t^0 component of the product $t^{m+n}a(x,1/t)1/(1-tp_1)^{\Lambda_0+n_0+1}(1+tp_1)^{\Lambda_1+n_1}$.

Now we relate the above explanation of $\sum_\alpha \frac{c_\nu}{p_1^{m+n}}\frac{p_1^\alpha}{\alpha!}\partial_p^\alpha a(x,0)$ with the operator Δ in Proposition 3.11. Let f be a polynomial of one variable. Then $\Delta(f)(q_1,q_2)=(f(q_1)-f(q_2))/(q_1-q_2)$ is equal to the t^0 component of the product between $f(1/t)$ and $t/(1-tq_1)(1-tq_2)$ for $|tq_1|<1$ and $|tq_2|<1$. In this identification we have viewed $1/(1-tq_1)(1-tq_2)$ as a Taylor series of variables t,q_1,q_2. Now applying the same trick, we can identify $\Delta^2(f)(q_1,q_2,q_3)$ as the t^0 component of the product $f(1/t)t^2/(1-tq_1)(1-tq_2)(1-tq_3)$ for $|tq_i|<1$, $i=1,2,3$. Extending this procedure, we have that in general, for $k\in\mathbb{N}$, $\Delta^k(f)(q_1,\ldots,q_k)$ is the t^0 component of the product $f(1/t)t^k/(1-tq_1)\cdots(1-tq_k)$ for $|tq_i|<1$, $i=1,\ldots,k$. Finally, comparing this interpretation of $\Delta^k(f)$, we conclude that $\sum_\alpha \frac{c_\nu}{p_1^{m+n}}\frac{p_1^\alpha}{\alpha!}\partial_p^\alpha a(x,0)$ is equal to $\Delta^{m+n}(a)$ evaluating at

$$\underbrace{(x,p_1)\times\cdots\times(x,p_1)}_{\Lambda_0+n_0+1}\times\underbrace{(x,-p_1)\times\cdots\times(x-p_1)}_{\Lambda_1+n_1}.$$

This is exactly the expression of the operator A_ν, introduced in Section 3.3, on the function a with respect to the variable p_1.

In summary, from the above expression about the symbol of the operator $Op_k(a)\circ Op_k(b)$, we can quickly check property (1)-(3) in Theorem 3.10.

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Institut de Mathématiques et de Modélisation de Montpellier, Université de Montpellier 2, CC5149, Place Eugène Bataillon, F-34095 Montpellier CEDEX 5, France
E-mail address: halbout@math.univ-montp2.fr

Department of Mathematics, Washington University, St. Louis, Missouri, USA, 63130
E-mail address: xtang@math.wustl.edu