

COBOUNDARY LIE BIALGEBRAS AND COMMUTATIVE SUBALGEBRAS OF UNIVERSAL ENVELOPING ALGEBRAS

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ABSTRACT. We solve a functional version of the problem of twist quantization of a coboundary Lie bialgebra (\mathfrak{g}, r, Z) . We derive from this the following results: (a) the formal Poisson manifolds \mathfrak{g}^* and G^* are isomorphic; (b) we construct an injective algebra morphism $S(\mathfrak{g}^*)^{\mathfrak{g}} \hookrightarrow U(\mathfrak{g}^*)$. When (\mathfrak{g}, r, Z) can be quantized, we construct a deformation of this morphism. In the particular case when \mathfrak{g} is quasitriangular and nondegenerate, we compare our construction with Semenov-Tian-Shansky's construction of a commutative subalgebra of $U(\mathfrak{g}^*)$. We also show that the canonical derivation of the function ring of G^* is Hamiltonian.

0. INTRODUCTION

Let (\mathfrak{g}, r, Z) be a coboundary Lie bialgebra over a field \mathbb{K} of characteristic 0. This means that \mathfrak{g} is a Lie bialgebra, the Lie cobracket δ of which is the coboundary of $r \in \wedge^2(\mathfrak{g})$: $\delta(x) = [x \otimes 1 + 1 \otimes x, r]$ for any $x \in \mathfrak{g}$. This condition means that $Z := \text{CYB}(r)$ belongs to $\wedge^3(\mathfrak{g})^{\mathfrak{g}}$ (here CYB is the l.h.s. of the classical Yang-Baxter equation). Quasi-triangular and triangular Lie bialgebras are particular cases of this definition.

It is an open question to construct a twist quantization of (\mathfrak{g}, r, Z) , i.e., a pair (J, Φ) , where $J \in U(\mathfrak{g})^{\otimes 2}[[\hbar]]$ and $\Phi \in U(\mathfrak{g})^{\otimes 3}[[\hbar]]$ are invertible (\hbar is a formal series), Φ is \mathfrak{g} -invariant, (J, Φ) satisfies a cocycle relation and deforms (r, Z) . If (J, Φ) satisfies these conditions, then Φ satisfies the pentagon relation, is \mathfrak{g} -invariant and deforms Z : such a Φ is called a quantization of Z . In [Dr3], Proposition 3.10, Drinfeld constructed a quantization Φ of Z . Any pair (J, Φ) can be made admissible (in the sense of [EH2]) and the associated formal functions (ρ, φ) then satisfy functional analogues of the pentagon and cocycle equations (this is explained in Section 5). We call this system of equations the functional analogue of twist quantization.

We describe the set of solutions of the functional analogue of twist quantization for (\mathfrak{g}, r, Z) . Namely, we derive from Drinfeld's result that Z can be lifted to an element $\varphi \in \mathfrak{m}_{\mathfrak{g}^*}^{\widehat{\otimes} 3}$ satisfying the functional pentagon relation (Proposition 1.1). We then prove that r can be lifted to an element $\rho \in \mathfrak{m}_{\mathfrak{g}^*}^{\widehat{\otimes} 2}$, such that (ρ, φ) satisfies the functional cocycle relation (Theorem 1.3); here $\mathfrak{m}_{\mathfrak{g}^*} \subset \mathcal{O}_{\mathfrak{g}^*}$ is the maximal ideal of the ring of formal functions on \mathfrak{g}^* . We show that all solutions are related by suitable gauge transformations.

The first corollary is that the formal Poisson manifolds \mathfrak{g}^* and G^* are isomorphic (Corollary 2.1).

In Section 3, we prove another corollary (Theorem 3.3): we construct an injection of algebras $S(\mathfrak{g}^*)^{\mathfrak{g}} \hookrightarrow U(\mathfrak{g}^*)$ (here \mathfrak{g}^* acts on $S(\mathfrak{g})$ by symmetric powers of the coadjoint action). This morphism is filtered, its associated graded is the canonical inclusion $S(\mathfrak{g}^*)^{\mathfrak{g}} \subset S(\mathfrak{g}^*)$. This way, we obtain a commutative subalgebra of $U(\mathfrak{g}^*)$. The fact that the graded subalgebra $S(\mathfrak{g}^*)^{\mathfrak{g}} \subset S(\mathfrak{g}^*)$ is Poisson commutative can be viewed as a classical limit of this situation. It can either be viewed as a corollary of the fact that $\mathcal{O}_G^{\mathfrak{g}} \subset \mathcal{O}_G$ is a Poisson commutative subalgebra or it can be proved directly (Lemma 3.2); here \mathcal{O}_G is the ring of formal functions on G , on which \mathfrak{g} acts by conjugation.

In Sections 6 and 7, assuming the existence of a twist quantization of \mathfrak{g} , we construct formal deformations of the algebra inclusions $S(\mathfrak{g}^*)^{\mathfrak{g}} \hookrightarrow U(\mathfrak{g}^*)$ and $\mathcal{O}_G^{\mathfrak{g}} \subset \mathcal{O}_{G^*}$. All these results use the theory of duality of QUE and QFSH algebras; this theory is recalled in Section 4.

In Section 8, we assume that \mathfrak{g} is quasitriangular. In that case, we show that $U(\mathfrak{g}^*)$ contains a family C_s of commutative subalgebras, indexed by $s \in \mathbb{K}$; this result may be viewed as a classical limit of Drinfeld's result about commutativity of twisted traces. We explain why only C_0 has an analogue in the general coboundary case. In [STS1], Semenov-Tian-Shansky defined an algebra morphism $U(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\text{STS}} U(\mathfrak{g}^*)$; we show that its image coincides with C_1 , and is therefore in general different from the image C_0 of the morphism in our construction.

Finally, in Section 9, we show that the canonical derivation of \mathcal{O}_{G^*} is Hamiltonian. This derivation is equal to $\hbar^{-1}(S^2 - \text{id})|_{\hbar=0}$, where S is the antipode of any quantization of \mathcal{O}_{G^*} .

Notation. We use the standard notation for the coproduct-insertion maps: we say that an ordered set is a pair of a finite set S and a bijection $\{1, \dots, |S|\} \rightarrow S$. For I_1, \dots, I_m disjoint ordered subsets of $\{1, \dots, n\}$, (U, Δ) a Hopf algebra and $a \in U^{\otimes m}$, we define

$$a^{I_1, \dots, I_m} = \sigma_{I_1, \dots, I_m} \circ (\Delta^{|I_1|} \otimes \dots \otimes \Delta^{|I_m|})(a),$$

with $\Delta^{(1)} = \text{id}$, $\Delta^{(2)} = \Delta$, $\Delta^{(n+1)} = (\text{id}^{\otimes n-1} \otimes \Delta) \circ \Delta^{(n)}$, and $\sigma_{I_1, \dots, I_m} : U^{\otimes \sum_i |I_i|} \rightarrow U^{\otimes n}$ is the morphism corresponding to the map $\{1, \dots, \sum_i |I_i|\} \rightarrow \{1, \dots, n\}$ taking $(1, \dots, |I_1|)$ to I_1 , $(|I_1| + 1, \dots, |I_1| + |I_2|)$ to I_2 , etc. When U is cocommutative, this definition depends only on the sets underlying I_1, \dots, I_m .

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1. SOLUTIONS OF THE FUNCTIONAL TWIST EQUATIONS

If \mathfrak{g} is a Lie algebra, we denote by $\mathcal{O}_{\mathfrak{g}^*} = \widehat{S}(\mathfrak{g})$ the formal series ring of functions on the formal neighborhood of 0 in \mathfrak{g}^* . We define by $\mathfrak{m}_{\mathfrak{g}^*} \subset \mathcal{O}_{\mathfrak{g}^*}$ the maximal ideal of this ring. If k is an integer ≥ 1 , we denote by $\mathcal{O}_{(\mathfrak{g}^*)^k} = \widehat{S}(\mathfrak{g})^{\widehat{\otimes} k}$ the 1 ring of formal functions functions on $(\mathfrak{g}^*)^k$, by $\mathfrak{m}_{(\mathfrak{g}^*)^k}$ its maximal ideal and by $\mathfrak{m}_{(\mathfrak{g}^*)^k}^i$ the i th power of this ideal.

If $f, g \in \mathfrak{m}_{(\mathfrak{g}^*)^k}^2$, then the series $f \star g = f + g + \frac{1}{2}\{f, g\} + \dots + B_n(f, g) + \dots$ is convergent, where $\sum_{i \geq 1} B_i(x, y)$ is the Baker-Campbell-Hausdorff series specialized to the Poisson bracket of $\mathfrak{m}_{(\mathfrak{g}^*)^k}^2$. The product \star defines a group structure on $\mathfrak{m}_{(\mathfrak{g}^*)^k}^2$.

If $f \in \mathcal{O}_{\mathfrak{g}^*}^{\widehat{\otimes} n}$ and P_1, \dots, P_m are disjoint subsets of $\{1, \dots, m\}$, one defines f^{P_1, \dots, P_m} as in the Introduction using the cocommutative coproduct of $\mathcal{O}_{\mathfrak{g}^*}$ (dual to the addition of \mathfrak{g}^*).

Let \mathfrak{g} be a Lie algebra and $Z \in \wedge^3(\mathfrak{g})^{\mathfrak{g}}$.

Proposition 1.1. *There exists $\varphi \in (\mathfrak{m}_{\mathfrak{g}^*}^{\widehat{\otimes} 3})^{\mathfrak{g}} \subset \mathfrak{m}_{(\mathfrak{g}^*)^3}^2$, the image of which under the map $\mathfrak{m}_{\mathfrak{g}^*}^{\widehat{\otimes} 3} \rightarrow (\mathfrak{m}_{\mathfrak{g}^*}/\mathfrak{m}_{\mathfrak{g}^*}^2)^{\otimes 3} = \mathfrak{g}^{\otimes 3} \xrightarrow{\text{Alt}} \wedge^3(\mathfrak{g})$ equals Z (here Alt is the total antisymmetrization map) and satisfying the functional pentagon equation*

$$\varphi^{1,2,3,4} \star \varphi^{12,3,4} = \varphi^{2,3,4} \star \varphi^{1,23,4} \star \varphi^{1,2,3}.$$

Such a φ (we call it a lift of Z) is unique up to the action of an element of $(\mathfrak{m}_{\mathfrak{g}^*}^{\widehat{\otimes} 2})^{\mathfrak{g}}$ by $\sigma \cdot \varphi = \sigma^{2,3} \star \sigma^{1,23} \star \varphi \star (-\sigma)^{12,3} \star (-\sigma)^{1,2}$.

¹ $\widehat{\otimes}$ is the completed tensor product, defined by $V_0[[x_1, \dots, x_n]] \widehat{\otimes} W_0[[y_1, \dots, y_n]] := V_0 \otimes W_0[[x_1, \dots, y_n]]$, where V_0, W_0 are vector spaces.

Proof. In [Dr3], Proposition 3.10, Drinfeld constructed a solution $\Phi \in U(\mathfrak{g})^{\otimes 3}[[\hbar]]$ of the pentagon equation

$$\Phi^{1,2,34}\Phi^{12,3,4} = \Phi^{2,3,4}\Phi^{1,23,4}\Phi^{1,2,3} \quad (1)$$

such that $\varepsilon^{(2)}(\Phi) = 1$ and $\Phi = 1^{\otimes 3} + O(\hbar)$ (here $\varepsilon^{(2)} = \text{id} \otimes \varepsilon \otimes \text{id}$; applying ε to the first and third factors of (1), we also get $\varepsilon^{(1)}(\Phi) = \varepsilon^{(3)}(\Phi) = 1$).

In [EH2], we stated that Φ can be transformed into an admissible solution Φ' of the same equations, using an invariant twist. In Appendix A, we explain why the proof given in [EH2] is wrong and we give a correct proof.

The classical limit of $\hbar \log(\Phi')$ then satisfies the functional pentagon equation. This gives the existence of φ . One can also construct φ directly using cohomological methods, as it will be done for ρ later.

Let us prove uniqueness: let φ and φ' be two lifts of Z . The classes of φ and φ' are the same in $\mathfrak{m}_{\mathfrak{g}^*}^{\widehat{\otimes} 3} / (\mathfrak{m}_{\mathfrak{g}^*}^{\widehat{\otimes} 3} \cap \mathfrak{m}_{(\mathfrak{g}^*)_3}^3)$, as this space is 0. Let N be an integer ≥ 3 ; assume that we have found $\sigma_N \in (\mathfrak{m}_{\mathfrak{g}^*}^{\widehat{\otimes} 2})^{\mathfrak{g}}$ such that $\sigma_N \cdot \varphi$ and φ' are equal modulo $\mathfrak{m}_{\mathfrak{g}^*}^{\widehat{\otimes} 3} \cap \mathfrak{m}_{(\mathfrak{g}^*)_3}^N$. Write $\varphi' = \sigma_N \cdot \varphi + \psi$, with $\psi \in (\mathfrak{m}_{\mathfrak{g}^*}^{\widehat{\otimes} 3} \cap \mathfrak{m}_{(\mathfrak{g}^*)_3}^N)^{\mathfrak{g}}$. We will use the following lemma (see [EGH], p. 2477):

Lemma 1.2. *For any $k \geq 1$ and $n \geq 2$, $f, h \in \mathfrak{m}_{(\mathfrak{g}^*)^k}^2$ and $g \in \mathfrak{m}_{(\mathfrak{g}^*)^k}^n$, one has*

$$f \star (h + g) = f \star h + g, \quad (f + g) \star h = f \star h + g \text{ modulo } \mathfrak{m}_{(\mathfrak{g}^*)^k}^{n+1}.$$

Let $\bar{\psi}$ be the class of ψ in $(\mathfrak{m}_{\mathfrak{g}^*}^{\widehat{\otimes} 3} \cap \mathfrak{m}_{(\mathfrak{g}^*)_3}^N)^{\mathfrak{g}} / (\mathfrak{m}_{\mathfrak{g}^*}^{\widehat{\otimes} 3} \cap \mathfrak{m}_{(\mathfrak{g}^*)_3}^{N+1})^{\mathfrak{g}} = (S^{>0}(\mathfrak{g})^{\otimes 3})_N^{\mathfrak{g}}$. Then $\bar{\psi}^{1,2,34} + \bar{\psi}^{-1,2,3,4} = \bar{\psi}^{2,3,4} + \bar{\psi}^{-1,23,4} + \bar{\psi}^{-1,2,3}$, which means that $\bar{\psi}$ is a cocycle in the subcomplex $((S^{>0}(\mathfrak{g})^{\otimes \cdot})^{\mathfrak{g}}, d)$ of the co-Hochschild² complex $(S(\mathfrak{g})^{\otimes \cdot}, d)$. Using [Dr3], Proposition 3.11, one can prove that the k th cohomology group of this complex is $\wedge^k(\mathfrak{g})^{\mathfrak{g}}$ and that the antisymmetrization map coincides with the canonical map from the space of cocycles to the cohomology. For $N = 3$, the hypothesis implies that $\text{Alt}(\bar{\psi}) = 0$, so $\bar{\psi}$ is a coboundary of an element $\bar{\tau}_3 \in (S^{>0}(\mathfrak{g})^{\otimes 2})_3^{\mathfrak{g}}$. For $N > 3$, $\bar{\psi}$ is the a coboundary of an element $\bar{\tau}_N \in (S^{>0}(\mathfrak{g})^{\otimes 2})_N^{\mathfrak{g}}$, since the degree N part of the relevant cohomology group vanishes. We then set $\sigma_{N+1} = \sigma_N + \tau_N$, where $\tau_N \in (\mathfrak{m}_{\mathfrak{g}^*}^{\widehat{\otimes} 2} \cap \mathfrak{m}_{(\mathfrak{g}^*)_2}^N)^{\mathfrak{g}}$ is a lift of $\bar{\tau}_N$. Then $\sigma_{N+1} \cdot \varphi$ and φ' are equal modulo $\mathfrak{m}_{\mathfrak{g}^*}^{\widehat{\otimes} 3} \cap \mathfrak{m}_{(\mathfrak{g}^*)_3}^{N+1}$. The sequence $(\sigma_N)_{N \geq 3}$ has a limit σ . Then $\sigma \cdot \varphi = \varphi'$. \square

We now construct a lift of r :

Theorem 1.3. *There exists $\rho \in \mathfrak{m}_{\mathfrak{g}^*}^{\widehat{\otimes} 2}$, the image of which in $\mathfrak{g}^{\otimes 2}$ under the square of the projection $\mathfrak{m}_{\mathfrak{g}^*} \rightarrow \mathfrak{m}_{\mathfrak{g}^*} / \mathfrak{m}_{\mathfrak{g}^*}^2 = \mathfrak{g}$ equals r , and such that*

$$\rho^{1,2} \star \rho^{12,3} = \rho^{2,3} \star \rho^{1,23} \star \varphi. \quad (2)$$

Such a ρ (we call it a lift of r) is unique up to the action of $\mathfrak{m}_{\mathfrak{g}^*}$ by $\lambda \cdot \rho = \lambda^1 \star \lambda^2 \star \rho \star (-\lambda)^{12}$. We call equation (2) the functional cocycle equation.

Proof. Let us construct ρ by induction: we will construct a convergent sequence $\rho_N \in \mathfrak{m}_{\mathfrak{g}^*}^{\widehat{\otimes} 2}$ ($N \geq 2$) satisfying (2) in $\mathfrak{m}_{\mathfrak{g}^*}^{\widehat{\otimes} 3} / (\mathfrak{m}_{\mathfrak{g}^*}^{\widehat{\otimes} 3} \cap \mathfrak{m}_{(\mathfrak{g}^*)_3}^N)$. When $N = 3$, we take for ρ_2 any lift of r to $\mathfrak{m}_{\mathfrak{g}^*}^{\widehat{\otimes} 2}$; then equation (2) is automatically satisfied.

Let N be an integer ≥ 3 ; assume that we have constructed ρ_N in $\mathfrak{m}_{\mathfrak{g}^*}^{\widehat{\otimes} 2}$ satisfying equation (2) in $\mathfrak{m}_{\mathfrak{g}^*}^{\widehat{\otimes} 3} / (\mathfrak{m}_{\mathfrak{g}^*}^{\widehat{\otimes} 3} \cap \mathfrak{m}_{(\mathfrak{g}^*)_3}^N)$. Set $\alpha_N := \rho_N^{1,2} \star \rho_N^{12,3} - \rho_N^{2,3} \star \rho_N^{1,23} \star \varphi$. Then α_N belongs to $\mathfrak{m}_{\mathfrak{g}^*}^{\widehat{\otimes} 3} \cap \mathfrak{m}_{(\mathfrak{g}^*)_3}^N$,

²We denote by $S(\mathfrak{g})$ the symmetric algebra of \mathfrak{g} , by $S^{>0}(\mathfrak{g})$ is positive degree part; the index N means the part of total degree N .

and the following equalities hold in $\mathfrak{m}_{\mathfrak{g}^*}^{\otimes 4}/(\mathfrak{m}_{\mathfrak{g}^*}^{\otimes 4} \cap \mathfrak{m}_{(\mathfrak{g}^*)^4}^{N+1})$:

$$\begin{aligned}
\alpha_N^{12,3,4} &= \rho_N^{1,2} \star \alpha_N^{12,3,4} = \rho_N^{1,2} \star \rho_N^{12,3} \star \rho_N^{123,4} - \rho_N^{1,2} \star \rho_N^{3,4} \star \rho_N^{12,34} \star \varphi^{12,3,4} \\
&\quad (\text{using Lemma 1.2}) \\
&= (\alpha_N^{1,2,3} + \rho_N^{2,3} \star \rho_N^{1,23} \star \varphi^{1,2,3}) \star \rho_N^{123,4} - \rho_N^{3,4} \star \rho_N^{1,2} \star \rho_N^{12,34} \star \varphi^{12,3,4} \\
&= \alpha_N^{1,2,3} + \rho_N^{2,3} \star \rho_N^{1,23} \star \rho_N^{123,4} \star \varphi^{1,2,3} - \rho_N^{3,4} \star (\rho_N^{2,34} \star \rho_N^{1,234} \star \varphi^{1,2,34} + \alpha_N^{1,2,34}) \star \varphi^{12,3,4} \\
&\quad (\text{using Lemma 1.2, the invariance of } \varphi \text{ and the definition of } \alpha_N^{1,2,34}) \\
&= \alpha_N^{1,2,3} + \rho_N^{2,3} \star (\alpha_N^{1,23,4} + \rho_N^{23,4} \star \rho_N^{1,234} \star \varphi^{1,23,4}) \star \varphi^{1,2,3} \\
&\quad - \alpha_N^{1,2,34} - \rho_N^{3,4} \star \rho_N^{2,34} \star \rho_N^{1,234} \star \varphi^{1,2,34} \star \varphi^{12,3,4} \\
&\quad (\text{using the definition of } \alpha_N^{1,23,4} \text{ and Lemma 1.2}) \\
&= \alpha_N^{1,2,3} + \alpha_N^{1,23,4} + (\rho_N^{3,4} \star \rho_N^{2,34} \star \varphi^{2,3,4} + \alpha_N^{2,3,4}) \star \rho_N^{1,234} \star \varphi^{1,23,4} \star \varphi^{1,2,3} \\
&\quad - \alpha_N^{1,2,34} - \rho_N^{3,4} \star \rho_N^{2,34} \star \rho_N^{1,234} \star \varphi^{1,2,34} \star \varphi^{12,3,4} \\
&\quad (\text{using the definition of } \alpha_N^{2,3,4} \text{ and Lemma 1.2}) \\
&= \alpha_N^{1,2,3} + \alpha_N^{1,23,4} - \alpha_N^{1,2,34} + \alpha_N^{2,3,4}
\end{aligned}$$

(using Lemma 1.2, the invariance of φ and the fact that φ satisfies the functional pentagon equation).

Let us denote by $\bar{\alpha}_N$ the image of α_N in $(\mathfrak{m}_{\mathfrak{g}^*}^{\otimes 3} \cap \mathfrak{m}_{(\mathfrak{g}^*)^3}^N)/(\mathfrak{m}_{\mathfrak{g}^*}^{\otimes 3} \cap \mathfrak{m}_{(\mathfrak{g}^*)^3}^{N+1}) = (S^{>0}(\mathfrak{g})^{\otimes 3})_N$, then we get

$$\bar{\alpha}_N^{12,3,4} + \bar{\alpha}_N^{1,2,34} = \bar{\alpha}_N^{1,2,3} + \bar{\alpha}_N^{1,23,4} + \bar{\alpha}_N^{2,3,4}.$$

This means that α is a cocycle for the subcomplex $(S^{>0}(\mathfrak{g})^{\otimes \cdot}, d)$ of the co-Hochschild complex. Using [Dr3], Proposition 3.11, one proves that the k th cohomology group of this subcomplex is $\wedge^k(\mathfrak{g})$, and that antisymmetrization coincides with the canonical projection from the space of cocycles to the cohomology group. For $N = 3$, the equation $\text{CYB}(r) = Z$ implies $\text{Alt}(\bar{\alpha}_3) = 0$, hence $\bar{\alpha}_3$ is the coboundary of an element $\bar{\beta}_3 \in (S^{>0}(\mathfrak{g})^{\otimes 2})_3$. For $N > 3$, $\bar{\alpha}_N$ is the coboundary of an element $\bar{\beta}_N \in (S^{>0}(\mathfrak{g})^{\otimes 2})_N$, since the degree N part of the cohomology vanishes. We then set $\rho_{N+1} := \rho_N + \beta_N$, where $\beta_N \in \mathfrak{m}_{\mathfrak{g}^*}^{\otimes 2} \cap \mathfrak{m}_{(\mathfrak{g}^*)^2}^N$ is a representative of $\bar{\beta}_N$. Then ρ_{N+1} satisfies (2) in $\mathfrak{m}_{\mathfrak{g}^*}^{\otimes 3}/(\mathfrak{m}_{\mathfrak{g}^*}^{\otimes 3} \cap \mathfrak{m}_{(\mathfrak{g}^*)^3}^{N+1})$.

The sequence $(\rho_N)_{N \geq 2}$ has a limit ρ , which then satisfies (2).

The second part of the theorem can be proved either by analyzing the choices for $\bar{\beta}_N$ in the above proof, or following the proof of the previous proposition. \square

Remark 1.4. If φ is replaced by $\varphi' = \sigma \star \varphi$, then a solution of (2) is $\rho' = \rho \star (-\sigma)$.

2. ISOMORPHISM OF FORMAL POISSON MANIFOLDS $\mathfrak{g}^* \simeq G^*$

Let us assume that \mathfrak{g} is a finite dimensional coboundary Lie bialgebra. The following result was proved in [EEM] when \mathfrak{g} is quasitriangular; the result of [EEM] is itself a generalization of the formal version of the Ginzburg-Weinstein isomorphism ([GW, A, Bo]).

Corollary 2.1. *There exists an isomorphism of formal Poisson manifolds $\mathfrak{g}^* \simeq G^*$.*

Proof. Let $P : \wedge^2(\mathcal{O}_{\mathfrak{g}^*}) \rightarrow \mathcal{O}_{\mathfrak{g}^*}$ be the Poisson bracket on $\mathcal{O}_{\mathfrak{g}^*}$ corresponding to the Lie-Poisson³ Poisson structure on \mathfrak{g}^* . Then $(\mathcal{O}_{\mathfrak{g}^*}, m_0, P, \Delta_0)$ is a Poisson formal series Hopf (PFSH)

³or Kostant-Kirillov-Souriau, or linear

algebra; it corresponds to the formal Poisson-Lie group $(\mathfrak{g}^*, +)$ equipped with its Lie-Poisson structure.

Set ${}^\rho\Delta(f) = \rho \star \Delta_0(f) \star (-\rho)$ for any $f \in \mathcal{O}_{\mathfrak{g}^*}$. It follows from the fact that ρ satisfies the functional cocycle equation that $(\mathcal{O}_{\mathfrak{g}^*}, m_0, P, {}^\rho\Delta_0)$ is a PFSH algebra.

Let us denote by **PFSHA** and **LBA** the categories of PSFH algebras and Lie bialgebras. We have a category equivalence $c : \mathbf{PFSHA} \rightarrow \mathbf{LBA}$, taking $(\mathcal{O}, m, P, \Delta)$ to the Lie bialgebra $(\mathfrak{c}, \mu, \delta)$, where $\mathfrak{c} := \mathfrak{m}/\mathfrak{m}^2$ ($\mathfrak{m} \subset \mathcal{O}$ is the maximal ideal), the Lie cobracket of \mathfrak{c} is induced by $\Delta - \Delta^{2,1} : \mathfrak{m} \rightarrow \wedge^2(\mathfrak{m})$, and the Lie bracket of \mathfrak{c} is induced by the Poisson bracket $P : \wedge^2(\mathfrak{m}) \rightarrow \mathfrak{m}$. The inverse of the functor c takes $(\mathfrak{c}, \mu, \delta)$ to $\mathcal{O} = \widehat{S}(\mathfrak{c})$ equipped with its usual product; Δ depends only on δ and P depends on (μ, δ) .

Then c restricts to a category equivalence $c_{\text{fd}} : \mathbf{PFSHA}_{\text{fd}} \rightarrow \mathbf{LBA}_{\text{fd}}$ of subcategories of finite-dimensional objects (in the case of **PFSH**, we say that \mathcal{O} is finite-dimensional iff $\mathfrak{m}/\mathfrak{m}^2$ is).

Let $\text{dual} : \mathbf{LBA}_{\text{fd}} \rightarrow \mathbf{LBA}_{\text{fd}}$ be the duality functor. It is a category antiequivalence; we have $\text{dual}(\mathfrak{g}, \mu, \delta) = (\mathfrak{g}^*, \delta^t, \mu^t)$. Then $\text{dual} \circ c_{\text{fd}} : \mathbf{PFSHA}_{\text{fd}} \rightarrow \mathbf{LBA}_{\text{fd}}$ is a category antiequivalence. Its inverse is the usual functor $\mathfrak{g} \mapsto U(\mathfrak{g})^*$. If G is the formal Poisson-Lie group with Lie bialgebra \mathfrak{g} , one sets $\mathcal{O}_G = U(\mathfrak{g})^*$.

Let us apply the functor c to $(\mathcal{O}_{\mathfrak{g}^*}, m_0, P, {}^\rho\Delta_0)$. We obtain $\mathfrak{c} = \mathfrak{m}/\mathfrak{m}^2 = \mathfrak{g}$; the Lie bracket is unchanged w.r.t. the case $\rho = 0$, so it is the Lie bracket of \mathfrak{g} ; the Lie cobracket is given by $\delta(x) = [r, x \otimes 1 + 1 \otimes x]$ since the reduction of ρ modulo $(\mathfrak{m}_{\mathfrak{g}^*})^2 \widehat{\otimes} \mathfrak{m}_{\mathfrak{g}^*} + \mathfrak{m}_{\mathfrak{g}^*} \widehat{\otimes} (\mathfrak{m}_{\mathfrak{g}^*})^2$ is equal to r .

Then applying $\text{dual} \circ c_{\text{fd}}$ to $(\mathcal{O}_{\mathfrak{g}^*}, m_0, P, {}^\rho\Delta_0)$, we obtain the Lie bialgebra \mathfrak{g}^* . So this PFSH algebra is isomorphic to the PFSH algebra of the formal Poisson-Lie group G^* . In particular, the Poisson algebras $\mathcal{O}_{\mathfrak{g}^*}$ and \mathcal{O}_{G^*} are isomorphic. It is easy to check that the map $\mathfrak{g} = \mathfrak{m}_{\mathfrak{g}^*}/\mathfrak{m}_{\mathfrak{g}^*}^2 \rightarrow \mathfrak{m}_{G^*}/\mathfrak{m}_{G^*}^2 = \mathfrak{g}$ induced by this isomorphism is the identity (here $\mathfrak{m}_{G^*} \subset \mathcal{O}_{G^*}$ is the maximal ideal). \square

Remark 2.2. When \mathfrak{g} is infinite dimensional, one can define \mathcal{O}_{G^*} as the image of \mathfrak{g} under **LBA** \rightarrow **PFSHA** and then show that the Poisson algebras \mathcal{O}_{G^*} and $\mathcal{O}_{\mathfrak{g}^*} = (\widehat{S}(\mathfrak{g}), \text{linear Poisson structure})$ are isomorphic.

3. THE MORPHISM $S(\mathfrak{g}^*)^{\mathfrak{g}} \hookrightarrow U(\mathfrak{g}^*)$

In this section, \mathfrak{g} is a finite dimensional coboundary Lie bialgebras. The following fact is well-known ([STS2]):

Lemma 3.1. $\mathcal{O}_G^{\mathfrak{g}} \subset \mathcal{O}_G$ is a Poisson commutative subalgebra.

Here the action of \mathfrak{g} on \mathcal{O}_G corresponds to adjoint action of G . We recall the proof: if $f, g \in \mathcal{O}_G$, then $\{f, g\} = m((\mathbf{L} - \mathbf{R})(r)(f \otimes g))$, where \mathbf{L}, \mathbf{R} are the infinitesimal left and right actions and m is the product map. If $\varphi \in \mathcal{O}_G^{\mathfrak{g}}$, then $\mathbf{L}(a)(\varphi) = \mathbf{R}(a)(\varphi)$ for any $a \in \mathfrak{g}$, therefore if $f, g \in \mathcal{O}_G^{\mathfrak{g}}$, then $(\mathbf{L} - \mathbf{R})(r)(f \otimes g) = 0$, hence $\{f, g\} = 0$.

The inclusion $\mathcal{O}_G^{\mathfrak{g}} \subset \mathcal{O}_G$ is a morphism of Poisson algebras with a decreasing filtration. By passing to the associated graded, we obtain:

Lemma 3.2. $S(\mathfrak{g}^*)^{\mathfrak{g}} \subset S(\mathfrak{g}^*)$ is a Poisson commutative subalgebra.

Another proof. If $\alpha, \beta \in \mathfrak{g}^*$, then $[\alpha, \beta] = \text{ad}^*(R(\beta))(\alpha) - \text{ad}^*(R(\alpha))(\beta)$, where $R : \mathfrak{g}^* \mapsto \mathfrak{g}$ is given by $R(\xi) = (\text{id} \otimes \xi)(r)$.

Let $f, g \in S(\mathfrak{g}^*)^{\mathfrak{g}}$ be of degrees k and ℓ . Write $f = \sum_{\alpha} a_1^{\alpha} \cdots a_k^{\alpha}$, $g = \sum_{\beta} b_1^{\beta} \cdots b_{\ell}^{\beta}$. Then

$$\{f, g\} = \sum_{\beta} \sum_{j=1}^{\ell} b_1^{\beta} \cdots \check{b}_j^{\beta} \cdots b_{\ell}^{\beta} \text{ad}^*(R(b_j^{\beta}))(f) - \sum_{\alpha} \sum_{i=1}^k a_1^{\alpha} \cdots \check{a}_i^{\alpha} \cdots a_k^{\alpha} \text{ad}^*(R(a_i^{\alpha}))(g).$$

When f and g are both invariant, this bracket vanishes. \square

We now prove that $S(\mathfrak{g}^*)^{\mathfrak{g}} \subset S(\mathfrak{g}^*)$ is also the associated graded of an inclusion of noncommutative algebras with an increasing filtration:

Theorem 3.3. *There exists a morphism of filtered algebras:*

$$\theta : S(\mathfrak{g}^*)^{\mathfrak{g}} \rightarrow U(\mathfrak{g}^*),$$

the associated graded morphism of which is the canonical inclusion $S(\mathfrak{g}^*)^{\mathfrak{g}} \subset S(\mathfrak{g}^*)$.

Proof. Let us denote by **FSHA** the category of formal series Hopf (FSH) algebras and by **FilAlg** the category of filtered algebras. There is a contravariant functor (restricted duality) $\mathbf{FSHA} \rightarrow \mathbf{FilAlg}$, defined by $\mathcal{O} \mapsto \mathcal{O}^\circ$, where $\mathcal{O}^\circ = \{\ell \in \mathcal{O}^* \mid \exists n \geq 0, \ell(\mathfrak{m}^n) = 0\} \subset \mathcal{O}^*$; here $\mathfrak{m} \subset \mathcal{O}$ is the maximal ideal of \mathcal{O} . The algebra structure of \mathcal{O}° is defined by $(\ell_1 \cdot \ell_2)(f) = (\ell_1 \otimes \ell_2)(\Delta(f))$; its filtration is defined by $(\mathcal{O}^\circ)_{\leq n} = \{\ell \in \mathcal{O}^* \mid \ell(\mathfrak{m}^{n+1}) = 0\}$.

Note that we have a category equivalence $\mathbf{FSHA} \rightarrow \mathbf{LCA}$, where **LCA** is the category of Lie coalgebras, taking \mathcal{O} to $\mathfrak{m}/\mathfrak{m}^2$, equipped with the cobracket induced by $\Delta - \Delta^{2,1}$. Then the composed functor $\mathbf{LCA} \rightarrow \mathbf{FSHA} \rightarrow \mathbf{FilAlg}$ is $\mathfrak{c} \mapsto U(\mathfrak{c}^*)$ (recall that \mathfrak{c}^* is a Lie algebra).

$(\mathcal{O}_{\mathfrak{g}^*}, \Delta_0) = (\widehat{S}(\mathfrak{g}), \Delta_0)$ is a graded FSH algebra. Its restricted dual is the graded algebra $S(\mathfrak{g}^*)$. Recall that $\mathcal{O}_{\mathfrak{g}^*}$ is also a Poisson algebra. We define the set of Poisson traces on $\mathcal{O}_{\mathfrak{g}^*}$ as the subspace of all $\ell \in \mathcal{O}_{\mathfrak{g}^*}^\circ$, such that $\ell(\{u, v\}) = 0$ for any $u, v \in \mathcal{O}_{\mathfrak{g}^*}$. Then $\{\text{Poisson traces on } \mathcal{O}_{\mathfrak{g}^*}\} \subset \mathcal{O}_{\mathfrak{g}^*}^\circ$ identifies with $S(\mathfrak{g}^*)^{\mathfrak{g}} \subset S(\mathfrak{g}^*)$; this is a graded subalgebra of $\mathcal{O}_{\mathfrak{g}^*}^\circ$. This defines a graded algebra structure on $\{\text{Poisson traces on } \mathcal{O}_{\mathfrak{g}^*}\}$.

Consider the FSH algebra $(\mathcal{O}_{\mathfrak{g}^*}, {}^\rho\Delta_0)$. It is isomorphic (as a filtered vector space) to $(\mathcal{O}_{\mathfrak{g}^*}, \Delta_0)$, and this isomorphism induces an algebra isomorphism between their associated graded FSH algebras. It follows that we have an isomorphism of filtered vector spaces between the filtered algebra $(\mathcal{O}_{\mathfrak{g}^*}, {}^\rho\Delta_0)^\circ$ and $S(\mathfrak{g}^*)$, and the associated graded of this morphism is an algebra isomorphism $\text{gr}((\mathcal{O}_{\mathfrak{g}^*}, {}^\rho\Delta_0)^\circ) \rightarrow S(\mathfrak{g}^*)$.

Recall that the vector spaces underlying $(\mathcal{O}_{\mathfrak{g}^*}, \Delta_0)^\circ$ and $(\mathcal{O}_{\mathfrak{g}^*}, {}^\rho\Delta_0)^\circ$ are the same (i.e., $\mathcal{O}_{\mathfrak{g}^*}^\circ$). We claim that the canonical inclusion $\{\text{Poisson traces on } \mathcal{O}_{\mathfrak{g}^*}\} \subset (\mathcal{O}_{\mathfrak{g}^*}, {}^\rho\Delta_0)^\circ$ is a morphism of filtered algebras. Indeed, let us denote by \cdot_ρ (resp., \cdot) the product of $(\mathcal{O}_{\mathfrak{g}^*}, {}^\rho\Delta_0)^\circ$ (resp., $(\mathcal{O}_{\mathfrak{g}^*}, \Delta_0)^\circ$). Let ℓ_1, ℓ_2 be Poisson traces on $\mathcal{O}_{\mathfrak{g}^*}$. Then for any $x \in \mathcal{O}_{\mathfrak{g}^*}$, we have $(\ell_1 \cdot_\rho \ell_2)(x) = (\ell_1 \otimes \ell_2)(\rho \star \Delta_0(f) \star (-\rho))$. Now Leibniz's rule implies that $(\ell_1 \otimes \ell_2)(\{u, v\}) = 0$ for any $u, v \in \mathcal{O}_{\mathfrak{g}^*}^{\widehat{\otimes} 2}$, therefore $(\ell_1 \cdot_\rho \ell_2)(x) = (\ell_1 \otimes \ell_2)(\Delta_0(x)) = (\ell_1 \cdot \ell_2)(x)$. So $\{\text{Poisson traces on } \mathcal{O}_{\mathfrak{g}^*}\} \subset (\mathcal{O}_{\mathfrak{g}^*}, {}^\rho\Delta_0)^\circ$ is an algebra morphism. Since the filtrations on the vector spaces underlying $(\mathcal{O}_{\mathfrak{g}^*}, \Delta_0)^\circ$ and $(\mathcal{O}_{\mathfrak{g}^*}, {}^\rho\Delta_0)^\circ$ are the same, and since the filtration on $\{\text{Poisson traces on } \mathcal{O}_{\mathfrak{g}^*}\}$ is induced by that of $(\mathcal{O}_{\mathfrak{g}^*}, \Delta_0)^\circ$, this morphism is filtered, and its associated graded is the canonical inclusion $S(\mathfrak{g}^*)^{\mathfrak{g}} \subset S(\mathfrak{g}^*)$.

Now the FSH algebra isomorphism $\mathcal{O}_{G^*} \simeq (\mathcal{O}_{\mathfrak{g}^*}, {}^\rho\Delta_0)$ (Corollary 2.1) induces a filtered algebra isomorphism $(\mathcal{O}_{\mathfrak{g}^*}, {}^\rho\Delta_0)^\circ \rightarrow \mathcal{O}_{G^*}^\circ = U(\mathfrak{g}^*)$. The fact that the associated graded of this morphism is the canonical inclusion $S(\mathfrak{g}^*)^{\mathfrak{g}} \subset S(\mathfrak{g}^*)$ follows from the fact that the completed graded of the FSH algebras \mathcal{O}_{G^*} and $(\mathcal{O}_{\mathfrak{g}^*}, {}^\rho\Delta_0)$ are both $(\mathcal{O}_{\mathfrak{g}^*}, \Delta_0)$.

We now compose the filtered algebra morphism $\{\text{Poisson traces on } \mathcal{O}_{\mathfrak{g}^*}\} \subset (\mathcal{O}_{\mathfrak{g}^*}, {}^\rho\Delta_0)^\circ$ with the filtered algebra isomorphism $(\mathcal{O}_{\mathfrak{g}^*}, {}^\rho\Delta_0)^\circ \rightarrow \mathcal{O}_{G^*}^\circ$ and obtain a filtered algebra morphism $S(\mathfrak{g}^*)^{\mathfrak{g}} \rightarrow U(\mathfrak{g}^*)$, whose associated graded is the canonical inclusion $S(\mathfrak{g}^*)^{\mathfrak{g}} \subset S(\mathfrak{g}^*)$.

The situation may be summarized as follows:

$$\begin{array}{ccc}
 & & (\mathcal{O}_{\mathfrak{g}^*}, \Delta_0)^\circ = S(\mathfrak{g}^*) \\
 & (a) \nearrow & \\
 S(\mathfrak{g}^*)^{\mathfrak{g}} = \{\text{Poisson traces on } \mathcal{O}_{\mathfrak{g}^*}\} & & (c) \uparrow \\
 & (b) \searrow & \\
 & & (\mathcal{O}_{\mathfrak{g}^*}, {}^\rho\Delta_0)^\circ \xrightarrow{(d)} \mathcal{O}_{G^*}^\circ = U(\mathfrak{g}^*)
 \end{array}$$

Here $S(\mathfrak{g}^*)^\natural$ and $S(\mathfrak{g}^*)$ are graded algebras, $(\mathcal{O}_{\mathfrak{g}^*}, {}^\rho\Delta_0)^\circ$ and $\mathcal{O}_{G^*}^\circ$ are filtered algebras; (a) is a morphism of graded algebras, (c) is an isomorphism of filtered vector spaces, (b) and (d) are morphisms of filtered algebras ((d) is an isomorphism). The associated graded of (c) is an isomorphism of graded algebras. \square

Remark 3.4. The restricted dual of the isomorphism $\mathcal{O}_{\mathfrak{g}^*} \rightarrow \mathcal{O}_{G^*}$ appearing in the above proof is an isomorphism of filtered vector spaces $\sigma : S(\mathfrak{g}^*) \rightarrow U(\mathfrak{g}^*)$, whose associated graded is the canonical isomorphism $S(\mathfrak{g}^*) \rightarrow \text{gr}(U(\mathfrak{g}^*))$. These properties are also satisfied by the symmetrization map Sym , however σ depends on ρ , so in general Sym and σ are different.

Remark 3.5. One can check that the morphism θ is independent on the choice of (ρ, φ) (these choices are described in Remark 1.4 and in Theorem 1.3).

4. DUALITY OF QUE AND QFSH ALGEBRAS

In this section, we recall some facts from [Dr1] (proofs can be found in [Gav]). Let us denote by **QUE** the category of quantized universal enveloping (QUE) algebras and by **QFSH** the category of quantized formal series Hopf (QFSH) algebras. We denote by **QUE**_{fd} and **QFSH**_{fd} the subcategories corresponding to finite dimensional Lie bialgebras.

We have contravariant functors **QUE**_{fd} \rightarrow **QFSH**_{fd}, $U \mapsto U^*$ and **QFSH**_{fd} \rightarrow **QUE**_{fd}, $\mathcal{O} \mapsto \mathcal{O}^\circ$. These functors are inverse to each other. U^* is the full topological dual of U , i.e., the space of all continuous (for the \hbar -adic topology) $\mathbb{K}[[\hbar]]$ -linear maps $U \rightarrow \mathbb{K}[[\hbar]]$. \mathcal{O}° the space of continuous $\mathbb{K}[[\hbar]]$ -linear forms $\mathcal{O} \rightarrow \mathbb{K}[[\hbar]]$, where \mathcal{O} is equipped with the \mathfrak{m} -adic topology (here $\mathfrak{m} \subset \mathcal{O}$ is the maximal ideal).

We also have covariant functors **QUE** \rightarrow **QFSH**, $U \mapsto U'$ and **QFSH** \rightarrow **QUE**, $\mathcal{O} \mapsto \mathcal{O}^\vee$. These functors are also inverse to each other. U' is a subalgebra of U , while \mathcal{O}^\vee is the \hbar -adic completion of $\sum_{k \geq 0} \hbar^{-k} \mathfrak{m}^k \subset \mathcal{O}[1/\hbar]$.

We also have canonical isomorphisms $(U')^\circ \simeq (U^*)^\vee$ and $(\mathcal{O}^\vee)^* \simeq (\mathcal{O}^\circ)'$.

If \mathfrak{a} is a finite dimensional Lie bialgebra and $U = U_\hbar(\mathfrak{a})$ is a QUE algebra quantizing \mathfrak{a} , then $U^* = \mathcal{O}_{A, \hbar}$ is a QFSH algebra quantizing the Poisson-Lie group A (with Lie bialgebra \mathfrak{a}), and $U' = \mathcal{O}_{A^*, \hbar}$ is a QFSH algebra quantizing the Poisson-Lie group A^* (with Lie bialgebra \mathfrak{a}^*). If now $\mathcal{O} = \mathcal{O}_{A, \hbar}$ is a QFSH algebra quantizing A , then $\mathcal{O}^\circ = U_\hbar(\mathfrak{a})$ is a QUE algebra quantizing \mathfrak{a} and $\mathcal{O}^\vee = U_\hbar(\mathfrak{a}^*)$ is a QFSH algebra quantizing \mathfrak{a}^* .

We now compute these functors explicitly in the case of cocommutative QUE and commutative QFSH algebras. If $U = U(\mathfrak{a})[[\hbar]]$ with cocommutative coproduct (where \mathfrak{a} is a Lie algebra), then U' is a completion of $U(\hbar\mathfrak{a}[[\hbar]])$; this is a flat deformation of $\widehat{S}(\mathfrak{a})$ equipped with its linear Lie-Poisson structure. If G is a formal group with function ring \mathcal{O}_G , then $\mathcal{O} := \mathcal{O}_G[[\hbar]]$ is a QFSH algebra, and \mathcal{O}^\vee is a commutative QUE algebra; it is a quantization of $(S(\mathfrak{g}^*), \text{commutative product, cocommutative coproduct, co-Poisson structure induced by the Lie bracket of } \mathfrak{g})$.

5. RELATION BETWEEN TWIST QUANTIZATION AND ITS FUNCTIONAL VERSION

Let us define a twist quantization of the coboundary Lie bialgebra (\mathfrak{g}, r, Z) as a pair (J, Φ) , $J \in U(\mathfrak{g})^{\otimes 2}[[\hbar]]$, $\Phi \in U(\mathfrak{g})^{\otimes 3}[[\hbar]]$, such that Φ is invariant, and (J, Φ) satisfies the twisted cycle relation

$$J^{1,2} J^{12,3} = J^{2,3} J^{1,23} \Phi, \quad (3)$$

$(\varepsilon \otimes \text{id})(J) = (\text{id} \otimes \varepsilon)(J) = 1$, $J = 1^{\otimes 2} + O(\hbar)$, $\Phi = 1^{\otimes 3} + O(\hbar)$, $\text{Alt}((J - 1^{\otimes 2})/\hbar) = r + O(\hbar)$, $\text{Alt}((\Phi - 1^{\otimes 3})/\hbar^2) = Z + O(\hbar)$. These conditions imply that Φ satisfies the pentagon relation, as well as $\varepsilon^{(i)}(\Phi) = 1^{\otimes 2}$, $i = 1, 2, 3$. (We know that such a twist quantization always exists

when \mathfrak{g} is triangular or quasi-triangular.) Our purpose is to relate twist quantization with its functional version.

The first step is to show that (J, Φ) can be transformed into an admissible pair, in a sense which we now precise.

Definition 5.1. *An element x in a QUE algebra U is admissible if $x \in 1 + \hbar U$, and if $\hbar \log x$ is in $U' \subset U$.*

We will use the isomorphism $U(\mathfrak{g})^{\otimes k}[[\hbar]] \simeq U(\mathfrak{g}^{\oplus k})[[\hbar]]$ to view $U(\mathfrak{g})^{\otimes k}[[\hbar]]$ as a QUE algebra.

Proposition 5.2. *Any twist quantization (J, Φ) of a coboundary Lie bialgebra (\mathfrak{g}, r, Z) is gauge equivalent to an admissible twist quantization (J', Φ') (i.e., such that J' and Φ' are admissible).*

Proof. Let us set $U = U(\mathfrak{g})[[\hbar]]$. According to Proposition A.1, one can find an invariant $F \in U^{\widehat{\otimes} 2}$, such that $F \in 1^{\otimes 2} + \hbar U_0^{\widehat{\otimes} 2}$ and $\Phi' := {}^F\Phi = F^{2,3}F^{1,23}\Phi(F^{1,2}F^{12,3})^{-1}$ is admissible. In particular, $\Phi' \in 1^{\otimes 3} + \hbar^2 U_0^{\widehat{\otimes} 3}$.

Then if we set $J_0 := JF$, we have $J_0^{1,2}J_0^{12,3} = J_0^{2,3}J_0^{1,23}\Phi'$, and $J_0 \in 1^{\otimes 2} + \hbar U_0^{\widehat{\otimes} 2}$. For any $u \in 1^{\otimes 3} + \hbar U_0$, ${}^uJ_0 := u^1u^2J_0(u^{12})^{-1}$ is such that $({}^uJ_0, \Phi')$ is a twist quantization of (\mathfrak{g}, r, Z) . It remains to find u such that $J' := {}^uJ_0$ is admissible.

We will construct u as a product $\cdots u_2u_1$, where $u_n \in 1 + \hbar^n U_0$, in such a way that if $J_n := {}^{u_n \cdots u_1}J_0$, then $\hbar \log(J_n) \in U_0^{\prime \widehat{\otimes} 2} + \hbar^{n+2}U_0^{\widehat{\otimes} 2}$.

We have already $\hbar \log(J_0) \in \hbar^2 U_0^{\widehat{\otimes} 2}$.

Expand $J_0 = 1^{\otimes 2} + \hbar j_1 + \cdots$, then $\text{Alt}(j_1) = r$. Moreover, the coefficient of \hbar in $J_0^{1,2}J_0^{12,3} = J_0^{2,3}J_0^{1,23}\Phi$ yields $d(j_1) = 0$, where $d : U(\mathfrak{g})_0^{\otimes 2} \rightarrow U(\mathfrak{g})_0^{\otimes 3}$ is the co-Hochschild differential. It follows that for some $a_1 \in U(\mathfrak{g})_0$, we have $j_1 = r + d(a_1)$. Then if we set $u_1 := \exp(\hbar a_1)$ and $J_1 = {}^{u_1}J_0$, we get $J_1 \in 1^{\otimes 2} + \hbar r + \hbar^2 U_0^{\widehat{\otimes} 2}$. Then $\hbar \log(J_1) \in \hbar^2 r + \hbar^3 U_0^{\widehat{\otimes} 2} \subset U_0^{\prime \widehat{\otimes} 2} + \hbar^3 U_0^{\widehat{\otimes} 3}$.

Assume that for $n \geq 2$, we have constructed u_1, \dots, u_{n-1} such that $\alpha_{n-1} := \hbar \log(J_{n-1}) \in U_0^{\prime \widehat{\otimes} 2} + \hbar^{n+1}U_0^{\widehat{\otimes} 2}$.

Let us denote by $\bar{\alpha}$ the image of the class of α_{n-1} in $U(\mathfrak{g})_0^{\otimes 2}/(U(\mathfrak{g})_0^{\otimes 2})_{\leq n+1}$ under the isomorphism of this space with $(U_0^{\prime \widehat{\otimes} 2} + \hbar^{n+1}U_0^{\widehat{\otimes} 2})/(U_0^{\prime \widehat{\otimes} 2} + \hbar^{n+2}U_0^{\widehat{\otimes} 2})$ (see Lemma A.2). Let $\alpha \in U(\mathfrak{g})_0^{\otimes 2}$ be a representative of $\bar{\alpha}$, then $\alpha_{n-1} = \alpha' + \hbar^{n+1}\alpha$, where $\alpha' \in U_0^{\prime \widehat{\otimes} 2} + \hbar^{n+2}U_0^{\widehat{\otimes} 2}$. Let us set $\varphi' := \hbar \log(\Phi')$, then the twist equation gives

$$(-\alpha' - \hbar^{n+1}\alpha)^{1,23} \star_{\hbar} (-\alpha' - \hbar^{n+1}\alpha)^{2,3} \star_{\hbar} (\alpha' + \hbar^{n+1}\alpha)^{1,2} \star_{\hbar} (\alpha' + \hbar^{n+1}\alpha)^{12,3} = \varphi',$$

where \star_{\hbar} is defined as in Appendix A. According to Lemma A.3, the image of this equality in $(U^{\widehat{\otimes} 3} + \hbar^{n+1}U^{\prime \widehat{\otimes} 3})/(U^{\widehat{\otimes} 3} + \hbar^{n+2}U^{\prime \widehat{\otimes} 3}) \simeq U(\mathfrak{g})^{\otimes 3}/(U(\mathfrak{g})^{\otimes 3})_{\leq n+1}$ is $d(\bar{\alpha})$, where d is the co-Hochschild differential on $U(\mathfrak{g})_0^{\otimes \cdot}/(U(\mathfrak{g})_0^{\otimes \cdot})_{\leq n+1}$. Since $n \geq 2$, the relevant cohomology group vanishes, so $\bar{\alpha} = d(\bar{\beta})$, where $\bar{\beta} \in U(\mathfrak{g})_0/(U(\mathfrak{g})_0)_{\leq n+1}$. Let $\beta \in U(\mathfrak{g})_0$ be a representative of $\bar{\beta}$ and set $u_n := \exp(\hbar^n \beta)$, $J_n := {}^{u_n}J_{n-1}$, $\alpha_n := \hbar \log(J_n)$. Then

$$\alpha_n = (\hbar^{n+1}\beta)^1 \star_{\hbar} (\hbar^{n+1}\beta)^2 \star_{\hbar} \alpha_{n-1} \star_{\hbar} (-\hbar^{n+1}\beta)^{12}.$$

According to Lemma A.3, the image of α_n in

$$(U_0^{\widehat{\otimes} 2} + \hbar^{n+1}U_0^{\prime \widehat{\otimes} 2})/(U_0^{\widehat{\otimes} 2} + \hbar^{n+2}U_0^{\prime \widehat{\otimes} 2}) \simeq U(\mathfrak{g})_0^{\otimes 2}/(U(\mathfrak{g})_0^{\otimes 2})_{\leq n+1}$$

is $\bar{\alpha} - d(\bar{\beta}) = 0$. So α_n belongs to $U_0^{\widehat{\otimes} 2} + \hbar^{n+2}U_0^{\prime \widehat{\otimes} 2}$, as required. This proves the induction step. \square

If now (J', Φ') is an admissible twist quantization, then $\rho := \hbar \log(J')|_{\hbar=0}$ and $\varphi := \hbar \log(\Phi')|_{\hbar=0}$ are formal functions on $\mathfrak{m}_{\mathfrak{g}^*}^{\widehat{\otimes} 2}$ and $\mathfrak{m}_{\mathfrak{g}^*}^{\widehat{\otimes} 3}$, solutions of the functional twist equation.

6. QUANTIZATION OF $\mathcal{O}_G^{\mathfrak{g}} \subset \mathcal{O}_G$

Using a (non necessarily admissible) twist quantization, we construct a formal noncommutative deformation of the inclusion of algebras of Lemma 3.1:

Proposition 6.1. *We have an injective algebra morphism $\mathcal{O}_G^{\mathfrak{g}}[[\hbar]] \hookrightarrow \mathcal{O}_{G,\hbar}$ deforming $\mathcal{O}_G^{\mathfrak{g}} \subset \mathcal{O}_G$, where $\mathcal{O}_{G,\hbar}$ is a quantization of the PFSH algebra \mathcal{O}_G and $\mathcal{O}_G^{\mathfrak{g}}[[\hbar]]$ is the trivial deformation of the commutative algebra $\mathcal{O}_G^{\mathfrak{g}}$ (it is also commutative).*

Proof. Let us first construct the QFSH algebra $\mathcal{O}_{G,\hbar}$. For $x \in U(\mathfrak{g})[[\hbar]]$, set ${}^J\Delta_0(x) = J\Delta_0(x)J^{-1}$, where Δ_0 is the usual cocommutative coproduct. Then $U_{\hbar}(\mathfrak{g}) = (U(\mathfrak{g})[[\hbar]], m_0, {}^J\Delta_0)$ is a quantization of the Lie bialgebra \mathfrak{g} (here m_0 is the product on $U(\mathfrak{g})$). The dual $\mathcal{O}_{G,\hbar} := U_{\hbar}(\mathfrak{g})^*$ of this QUE algebra is a QFSH algebra quantizing the PFSH algebra \mathcal{O}_G . The product in this QFSH algebra is defined by $(f \star g)(x) = (f \otimes g)(J\Delta_0(x)J^{-1})$ for $f, g \in U_{\hbar}(\mathfrak{g})^*$ and $x \in U_{\hbar}(\mathfrak{g})$.

On the other hand, the FSH algebra \mathcal{O}_G is equal to $U(\mathfrak{g})^*$, and its product is defined by $(fg)(x) = (f \otimes g)(\Delta_0(x))$ for $f, g \in U(\mathfrak{g})^*$ and $x \in U(\mathfrak{g})$.

We say that $f \in U(\mathfrak{g})^*$ is a trace iff $f(xy) = f(yx)$ for any $x, y \in U(\mathfrak{g})$. Then the inclusion $\{\text{traces on } U(\mathfrak{g})\} \subset U(\mathfrak{g})^*$ identifies with $\mathcal{O}_G^{\mathfrak{g}} \subset \mathcal{O}_G$. In the same way, we define $\{\text{traces on } U(\mathfrak{g})[[\hbar]]\}$; this is a subalgebra of $U(\mathfrak{g})[[\hbar]]^* \simeq \mathcal{O}_G[[\hbar]]$, which identifies with $\mathcal{O}_G^{\mathfrak{g}}[[\hbar]]$.

The canonical map $\{\text{traces on } U(\mathfrak{g})[[\hbar]]\} \rightarrow U_{\hbar}(\mathfrak{g})^*$ is an algebra morphism. Indeed, if f_1, f_2 are traces on $U(\mathfrak{g})[[\hbar]]$, then $f_1 \otimes f_2$ is a trace on $U(\mathfrak{g})^{\otimes 2}[[\hbar]]$, so $(f_1 \star f_2)(x) = (f_1 \otimes f_2)(J\Delta_0(x)J^{-1}) = (f_1 \otimes f_2)(\Delta_0(x)) = (f_1 f_2)(x)$ for any $x \in U(\mathfrak{g})[[\hbar]]$, so $f_1 \star f_2 = f_1 f_2$. So we have obtained an algebra morphism $\mathcal{O}_G^{\mathfrak{g}}[[\hbar]] \rightarrow U_{\hbar}(\mathfrak{g})^* = \mathcal{O}_{G,\hbar}$. It is clearly a deformation of the canonical inclusion $\mathcal{O}_G^{\mathfrak{g}} \subset \mathcal{O}_G$. \square

7. QUANTIZATION OF $S(\mathfrak{g}^*)^{\mathfrak{g}} \hookrightarrow U(\mathfrak{g}^*)$

Assume now that (J, Φ) is an admissible twist quantization. We will construct a formal deformation of the inclusion of algebras of Theorem 3.3.

Theorem 7.1. *There is an injective algebra morphism:*

$$\theta_{\hbar} : S(\mathfrak{g}^*)^{\mathfrak{g}}[[\hbar]] \hookrightarrow U_{\hbar}(\mathfrak{g}^*),$$

where $U_{\hbar}(\mathfrak{g}^*)$ is a quantization of \mathfrak{g}^* . Its reduction modulo \hbar coincides with the morphism $S(\mathfrak{g}^*)^{\mathfrak{g}} \hookrightarrow U(\mathfrak{g})$ from Theorem 3.3.

Proof. Recall that $U(\mathfrak{g})[[\hbar]]'$ is a cocommutative QFSH algebra; we denote by m_0, Δ_0 its product and coproduct.

Since $(\varepsilon \otimes \text{id})(J) = (\text{id} \otimes \varepsilon)(J) = 1$, we have $\hbar \log(J) \in \mathfrak{m}_0^{\widehat{\otimes} 2}$, where $\mathfrak{m}_0 \subset U(\mathfrak{g})[[\hbar]]'$ is the kernel of the counit. According to [EH1], Proposition 3.1, this implies that the inner automorphism $z \mapsto JzJ^{-1}$ of $U(\mathfrak{g})^{\otimes 2}[[\hbar]]$ restricts to an automorphism of $U(\mathfrak{g})^{\otimes 2}[[\hbar]]'$.

We then equip $U(\mathfrak{g})[[\hbar]]'$ with the coproduct ${}^J\Delta : x \mapsto J\Delta_0(x)J^{-1}$. Then $(U(\mathfrak{g})[[\hbar]]', m_0, {}^J\Delta_0)$ is a QFSH algebra. Its classical limit is the PFSH algebra $(\mathcal{O}_{\mathfrak{g}^*}, m_0, P, {}^\rho\Delta_0)$. We have seen that this PFSH algebra is isomorphic to \mathcal{O}_{G^*} , hence $(U(\mathfrak{g})[[\hbar]]', m_0, {}^J\Delta_0)$ is a quantization of \mathcal{O}_{G^*} .

It now follows from Section 4 that $(U(\mathfrak{g})[[\hbar]]', m_0, {}^J\Delta_0)^\circ$ is a quantization of $U(\mathfrak{g}^*)$, which we denote by $U_{\hbar}(\mathfrak{g}^*)$.

Let us say that $\varphi \in (U(\mathfrak{g})[[\hbar]]')^\circ$ is a trace if $\varphi(xy) = \varphi(yx)$ for any $x, y \in U(\mathfrak{g})[[\hbar]]'$. Then $\{\text{traces on } U(\mathfrak{g})[[\hbar]]'\} \subset (U(\mathfrak{g})[[\hbar]]')^\circ$ is a subalgebra. Indeed, if ℓ_1, ℓ_2 are traces then $\ell_1 \otimes \ell_2$ is also a trace, so for $x, y \in U(\mathfrak{g})[[\hbar]]'$, we have $(\ell_1 \ell_2)(xy) = (\ell_1 \otimes \ell_2)(\Delta(x)\Delta(y)) = (\ell_1 \ell_2)(yx)$. This inclusion identifies with the inclusion $(\mathcal{O}_G[[\hbar]]^\vee)^{\mathfrak{g}} \subset \mathcal{O}_G[[\hbar]]^\vee$. Indeed, the Drinfeld functors have the property that $(U')^\circ = (U^*)^\vee$ for any QUE algebra U .

Now we show that the map $\{\text{traces on } U(\mathfrak{g})[[\hbar]]'\} \subset (U(\mathfrak{g})[[\hbar]]', {}^J\Delta_0)^\circ$ is also an algebra morphism. Indeed, let \cdot_J be the product of the latter algebra. If ℓ_1, ℓ_2 are traces and $x, y \in U(\mathfrak{g})[[\hbar]]$, then $(\ell_1 \cdot_J \ell_2)(x) = (\ell_1 \otimes \ell_2)(J\Delta_0(x)J^{-1}) = (\ell_1 \otimes \ell_2)(\Delta_0(x)) = (\ell_1 \ell_2)(x)$, so $\ell_1 \cdot_J \ell_2 = \ell_1 \ell_2$. So we have constructed an algebra morphism $(\mathcal{O}_G[[\hbar]]^\vee)^\mathfrak{g} \rightarrow U_{\hbar}(\mathfrak{g})$. It is clearly a deformation of the morphism constructed in Theorem 3.3.

Recall that $\mathcal{O}_G[[\hbar]]^\vee$ is the \hbar -adic completion of $\sum_{k \geq 0} \hbar^{-k} \mathfrak{m}_G^k \subset \mathcal{O}_G((\hbar))$.⁴ Then $(\mathcal{O}_G[[\hbar]]^\vee)^\mathfrak{g}$ is a topologically free $\mathbb{K}[[\hbar]]$ -commutative algebra; its specialization at $\hbar = 0$ is $\mathcal{O}_G[[\hbar]]^\vee / \hbar \mathcal{O}_G[[\hbar]]^\vee \simeq S(\mathfrak{g}^*)$.

The action of \mathfrak{g} on \mathcal{O}_G induces an action of \mathfrak{g} on $(\mathcal{O}_G[[\hbar]]^\vee)^\mathfrak{g}$. Then $(\mathcal{O}_G[[\hbar]]^\vee)^\mathfrak{g}$ is the \hbar -adic completion of $\sum_{k \geq 0} \hbar^{-k} (\mathfrak{m}_G^k)^\mathfrak{g}$. We have an inclusion of topologically free $\mathbb{K}[[\hbar]]$ -algebras $(\mathcal{O}_G[[\hbar]]^\vee)^\mathfrak{g} \subset \mathcal{O}_G[[\hbar]]^\vee$.

Now the dual of the symmetrization map induces an algebra isomorphism $\widehat{S}(\mathfrak{g}^*) = \mathcal{O}_{\mathfrak{g}} \simeq \mathcal{O}_G$ (dual to the exponential map $\mathfrak{g} \rightarrow G$). This isomorphism induces a \mathfrak{g} -equivariant isomorphism of $(\mathcal{O}_G[[\hbar]]^\vee)^\mathfrak{g}$ with the \hbar -adic completion of $\sum_{k \geq 0} \hbar^{-k} \mathfrak{m}_{\mathfrak{g}}^k \subset \mathcal{O}_{\mathfrak{g}}((\hbar))$. So we have an algebra isomorphism $(\mathcal{O}_G[[\hbar]]^\vee)^\mathfrak{g} \simeq S(\mathfrak{g}^*)[[\hbar]]$. It restricts to an isomorphism $(\mathcal{O}_G[[\hbar]]^\vee)^\mathfrak{g} \simeq S(\mathfrak{g}^*)^\mathfrak{g}[[\hbar]]$.

Composing its inverse with the morphism $(\mathcal{O}_G[[\hbar]]^\vee)^\mathfrak{g} \rightarrow U_{\hbar}(\mathfrak{g}^*)$, we get the announced morphism $S(\mathfrak{g}^*)^\mathfrak{g}[[\hbar]] \rightarrow U_{\hbar}(\mathfrak{g}^*)$. \square

8. THE QUASITRIANGULAR CASE

A quasitriangular Lie bialgebra (QTLBA) is a pair (\mathfrak{g}, r') , where \mathfrak{g} is a Lie algebra and $r' \in \mathfrak{g}^{\otimes 2}$ is such that $\text{CYB}(r') = 0$ and $t := r' + r'^{2,1} \in S^2(\mathfrak{g})^\mathfrak{g}$. Any QTLBA gives rise to a coboundary Lie bialgebra (\mathfrak{g}, r, Z) , where $r = (r' - r'^{2,1})/2$ and $Z = [t^{1,2}, t^{2,3}]/4$. We call a QTLBA *nondegenerate* if \mathfrak{g} is finite dimensional and t is nondegenerate.

Let $D : \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ be the composition of the Lie cobracket $\delta : \mathfrak{g}^* \rightarrow \wedge^2(\mathfrak{g}^*)$ with the Lie bracket of \mathfrak{g}^* . It is a derivation and a coderivation, and it induces a derivation of $U(\mathfrak{g}^*)$, which we also denote by D (or sometimes $D_{\mathfrak{g}^*}$).

Proposition 8.1. *For any scalar s , $C_s := \text{Ker}(\delta - s(D \otimes \text{id}) \circ \Delta_0)$ is a commutative subalgebra of $U(\mathfrak{g}^*)$.*

Proof. The condition $\ell \in C_s$ means that for any $u, v \in \mathcal{O}_{G^*}$, we have $\ell(\{u, v\} - sD^*(u)v) = 0$ (here D^* is the derivation of \mathcal{O}_{G^*} dual to the coderivation D).

Let ℓ_1, ℓ_2 belong to C_s . Then for any $u, v \in \mathcal{O}_{G^*}$,

$$\begin{aligned} (\ell_1 \ell_2)(\{u, v\} - sD^*(u)v) &= (\ell_1 \otimes \ell_2)(\{\Delta(u), \Delta(v)\} - s\Delta(D^*(u))\Delta(v)) \\ &= (\ell_1 \otimes \ell_2)(\{u^{(1)}, v^{(1)}\} \otimes u^{(2)}v^{(2)} + u^{(1)}v^{(1)} \otimes \{u^{(2)}, v^{(2)}\} \\ &\quad - sD^*(u^{(1)})v^{(1)} \otimes u^{(2)}v^{(2)} - u^{(1)}v^{(1)} \otimes sD^*(u^{(2)})v^{(2)}) = 0, \end{aligned}$$

hence $\ell_1 \ell_2 \in C_s$. Here Δ is the coproduct of \mathcal{O}_{G^*} .

Moreover, we constructed in [EGH] an element $\varrho \in \mathfrak{m}_{G^*}^{\otimes 2}$, such that $\Delta'(u) = \varrho \star \Delta(u) \star (-\varrho)$ for any $u \in \mathcal{O}_{G^*}$; if $(U_{\hbar}(\mathfrak{g}), \mathcal{R})$ is any quantization of (\mathfrak{g}, r') , then $\hbar \log(\mathcal{R}) \in \mathfrak{m}_{\hbar}^{\otimes 2}$, where $\mathfrak{m}_{\hbar} \subset U_{\hbar}(\mathfrak{g})'$ is the augmentation ideal, and the reduction of $\hbar \log(\mathcal{R}) \bmod \hbar$ equals ϱ . Then it follows from $(S^2 \otimes S^2)(\mathcal{R}) = \mathcal{R}$ that $(S_{\mathcal{O}}^2 \otimes S_{\mathcal{O}}^2)(\hbar \log \mathcal{R}) = \hbar \log \mathcal{R}$, where S is the antipode of $U_{\hbar}(\mathfrak{g})$ and $S_{\mathcal{O}} = S|_{U_{\hbar}(\mathfrak{g})'}$ is the antipode of $U_{\hbar}(\mathfrak{g})' \subset U_{\hbar}(\mathfrak{g})$; since the specialization for $\hbar = 0$ of $\hbar^{-1}(S_{\mathcal{O}}^2 - \text{id})$ is D^* , we get $(D^* \otimes \text{id} + \text{id} \otimes D^*)(\varrho) = 0$.

⁴ $\mathcal{O}_G[[\hbar]]^\vee$ may be also be viewed as the formal Rees algebra associated to the decreasing filtration $\mathcal{O}_G \supset \mathfrak{m}_G \supset \mathfrak{m}_G^2 \cdots$.

Then if $\ell_1, \ell_2 \in C_s$, then $(\ell_2 \ell_1)(u) = (\ell_1 \otimes \ell_2)(\Delta'(u)) = (\ell_1 \otimes \ell_2)(\varrho \star \Delta(u) \star (-\varrho)) = (\ell_1 \otimes \ell_2)(\Delta(u)) + \sum_{n \geq 1} (1/n!) (\ell_1 \otimes \ell_2)(\{\varrho, \{\varrho, \dots, \{\varrho, \Delta(u)\}\})$. Now if $f \in \mathcal{O}_{G^*}^{\widehat{\otimes} 2}$, then $(\ell_1 \otimes \ell_2)(\{\varrho, f\}) = s(\ell_1 \otimes \ell_2)((D^* \otimes \text{id} + \text{id} \otimes D^*)(\varrho)f) = 0$. It follows that $\ell_2 \ell_1 = \ell_1 \ell_2$. \square

Remark 8.2. If A is a quasitriangular Hopf algebra with antipode S , set $C_{s,A} := \{\ell \in A^* | \forall a, b \in A, \ell(ab) = \ell(bS^{-2s}(a))\}$ for any $s \in \mathbb{Z}$. Then it follows from [Dr2] that $C_{s,A}$ is a commutative algebra, and that we have isomorphisms $C_s \simeq C_{s+2}$ for any $s \in \mathbb{Z}$. The isomorphism takes $\ell \in C_s$ to $\bar{\ell} \in C_{s+2}$ defined by $\bar{\ell}(x) = \ell(xu^{-1}S(u))$, where $u = m \circ (\text{id} \otimes S)(R)$ (m, R are the product and R -matrix of A). The definition of $C_{s,A}$ can be generalized to $s \in \mathbb{K}$ when $A = (U_{\hbar}(\mathfrak{g}), \mathcal{R})$ is a quasitriangular QUE Hopf algebra. Define $U_{\hbar}(\mathfrak{g}^*)$ as $(U_{\hbar}(\mathfrak{g})')^{\circ} = (U_{\hbar}(\mathfrak{g}^*))^{\vee} \supset U_{\hbar}(\mathfrak{g}^*)$. Then $C_{s,\hbar} := \{\ell \in (U_{\hbar}(\mathfrak{g})')^{\circ} | \forall a, b \in U_{\hbar}(\mathfrak{g})', \ell(ab) = \ell(b(S^2)^{-s}(a))\}$ is a commutative subalgebra of $U_{\hbar}(\mathfrak{g}^*)$, and its reduction modulo \hbar is contained in C_s . In this case, $u^{-1}S(u)$ does not necessarily belong to $U_{\hbar}(\mathfrak{g})'$, therefore $C_{s,\hbar}$ and $C_{s+2,\hbar}$ are not necessarily isomorphic.

Remark 8.3. If (\mathfrak{g}, r, Z) is a coboundary Lie bialgebra, then r is D -invariant iff $(\mu \otimes \text{id})(Z)$ is symmetric (where μ is the Lie bracket of \mathfrak{g}). Otherwise, if we set $\varrho := \rho^{2,1} \star (-\rho)$, then $(D^* \otimes \text{id} + \text{id} \otimes D^*)(\varrho) \neq 0$, so unless $s = 0$, one cannot prove that C_s is commutative.

For each nondegenerate QTLBA (\mathfrak{g}, r') , Semenov-Tian-Shansky defined an algebra morphism $\Theta : Z(U(\mathfrak{g})) \rightarrow U(\mathfrak{g}^*)$, where $Z(A)$ denotes the center of an algebra A ([STS1]). Let us recall the construction of Θ . There are unique Lie algebra morphisms $L, R : \mathfrak{g}^* \rightarrow \mathfrak{g}$, defined by $L(\ell) = (\ell \otimes \text{id})(r')$, $R(\ell) = -(\text{id} \otimes \ell)(r')$ for any $\ell \in \mathfrak{g}^*$. We denote by $\alpha : U(\mathfrak{g}^*) \rightarrow U(\mathfrak{g})$ the composed map $U(\mathfrak{g}^*) \xrightarrow{\Delta_0} U(\mathfrak{g}^*)^{\otimes 2} \xrightarrow{L \otimes (S_0 \circ R)} U(\mathfrak{g})^{\otimes 2} \xrightarrow{m_0} U(\mathfrak{g})$. Here m_0, Δ_0 are the standard product and coproduct maps, we still denote by L, R the algebra morphisms induced by L, R , and S_0 denotes the antipode of $U(\mathfrak{g})$. The associated graded of the map α is the isomorphism $S(\mathfrak{g}^*) \rightarrow S(\mathfrak{g})$ induced by t , hence α is an isomorphism. Then $\Theta : Z(U(\mathfrak{g})) \rightarrow U(\mathfrak{g}^*)$ is defined as the restriction of α^{-1} to $Z(U(\mathfrak{g}))$; one can prove that it is an algebra morphism.

We will show, together with Proposition 8.8:

Proposition 8.4. $\text{Im}(\Theta) = C_1 \subset U(\mathfrak{g}^*)$. The associated graded of C_1 (for the degree filtration of $U(\mathfrak{g}^*)$) is $S(\mathfrak{g}^*)^{\mathfrak{g}}$.

Remark 8.5. Let θ be as in Theorem 3.3. The image of $\theta : S(\mathfrak{g}^*)^{\mathfrak{g}} \rightarrow U(\mathfrak{g}^*)$ is $\{\text{Poisson traces on } \mathcal{O}_{G^*}\}$, i.e., this is $\text{Ker}(\delta)$, where $\delta : U(\mathfrak{g}^*) \rightarrow \wedge^2 U(\mathfrak{g}^*)$ is the co-Poisson map of $U(\mathfrak{g}^*)$. So the images of Θ and θ do not coincide. \square

Let us now construct a deformation Θ_{\hbar} of Θ . The following lemma is proved in [Dr2].

Lemma 8.6. Let (A, Δ, R) be a quasitriangular Hopf algebra with antipode S . Define a linear map $\alpha_A : A^* \rightarrow A$ by $\alpha_A(\ell) = (\ell \otimes \text{id})(R^{2,1}R)$. Then α_A induces an algebra morphism $C_{1,A} \rightarrow Z(A)$.

Lemma 8.7. Assume moreover that A is finite dimensional and $R^{2,1}R$ is nondegenerate. Then the map $C_1 \rightarrow Z(A)$ is a linear isomorphism. Its inverse induces an algebra morphism $\Theta_A : Z(A) \rightarrow A^*$.

Proof. We have to check that if $\ell \in A^*$ is such that $\alpha_A(\ell) \in Z(A)$, then ℓ is a trace. The condition $\alpha_A(\ell) \in Z(A)$ means that for any $a \in A$, we have $(\ell \otimes \text{id})([R^{2,1}R, 1 \otimes a]) = 0$. It follows that for any $a \in A$, we have $S^{-1}(a^{(4)})(\ell \otimes \text{id})([R^{2,1}R, a^{(2)}S^{-1}(a^{(1)}) \otimes a^{(3)}]) = 0$. Since $R^{2,1}R$ commutes with the image of Δ_A , $(\ell \otimes \text{id})((a^{(2)} \otimes S^{-1}(a^{(4)})a^{(3)})[R^{2,1}R, S^{-1}(a^{(1)}) \otimes 1]) = 0$. Therefore $(\ell \otimes \text{id})((a^{(2)} \otimes 1)R^{2,1}R(S^{-1}(a^{(1)}) \otimes \text{id})) = \varepsilon(a)(\ell \otimes \text{id})(R^{2,1}R)$.

Since $R^{2,1}R$ is nondegenerate, this means that for any $b \in A$, we have $\ell(a^{(2)}bS^{-1}(a^{(1)})) = \varepsilon(a)\ell(b)$. Replacing $a \otimes b$ by $a^{(1)} \otimes S(a^{(2)})b$, we get $\ell(bS^{-1}(a)) = \ell(S(a)b)$, so that $\ell \in C_1$. \square

The QUE algebra version of these lemmas is 1), 2) of the following proposition. Let (\mathfrak{g}, r') be a QTLBA and let $(U_{\hbar}(\mathfrak{g}), \Delta, \mathcal{R})$ be a quantization of (\mathfrak{g}, r') .

Proposition 8.8. 1) *The linear map $U_{\hbar}(\mathfrak{g})^* \rightarrow U_{\hbar}(\mathfrak{g})$, $\ell \mapsto (\ell \otimes \text{id})(\mathcal{R}\mathcal{R}^{2,1})$ extends to a map $\alpha_{\hbar} : U_{\hbar}(\mathfrak{g}^*) \rightarrow U_{\hbar}(\mathfrak{g})$.*

2) *If (\mathfrak{g}, r') is nondegenerate, then α_{\hbar} is a linear isomorphism, and it restricts to an algebra isomorphism $C_{1,\hbar} \rightarrow Z(U_{\hbar}(\mathfrak{g}))$.*

3) *Proposition 8.4 is true.*

Proof. Let us prove 1). Define $L_{\hbar}, R'_{\hbar} : U_{\hbar}(\mathfrak{g})^* \rightarrow U_{\hbar}(\mathfrak{g})$ by $L_{\hbar}(\xi) = (\xi \otimes \text{id})(\mathcal{R})$, $R'_{\hbar}(\xi) = (\text{id} \otimes \xi)(\mathcal{R})$. According to [EH1], $\hbar \log(\mathcal{R}) \subset (U_{\hbar}(\mathfrak{g}'_0)^{\otimes 2} \subset U_{\hbar}(\mathfrak{g}'_0) \widehat{\otimes} \hbar U_{\hbar}(\mathfrak{g})_0)$, so that $\log(\mathcal{R}) \in U_{\hbar}(\mathfrak{g}'_0) \widehat{\otimes} U_{\hbar}(\mathfrak{g})_0$. According to [EGH], appendix, the image of $\log(\mathcal{R})$ in $(\mathfrak{m}_{G^*}/\mathfrak{m}_{G^*}^2) \widehat{\otimes} U(\mathfrak{g})_0$ (by reduction mod \hbar followed by projection) is r' . It follows that $\mathcal{R} \in U_{\hbar}(\mathfrak{g})' \widehat{\otimes} U_{\hbar}(\mathfrak{g})$, therefore L_{\hbar} extends to a map $U_{\hbar}(\mathfrak{g}^*) \rightarrow U_{\hbar}(\mathfrak{g})$; this map is necessarily a QUE algebra morphism. The quasitriangularity identities imply that the image of \mathcal{R} in $\mathcal{O}_{G^*} \widehat{\otimes} U(\mathfrak{g})$ has the form $\exp(\rho)$, where $\rho \in \mathfrak{m}_{G^*} \otimes \mathfrak{g}$ is a lift of r . It follows that the reduction mod \hbar of L_{\hbar} is the morphism induced by $\mathfrak{g}^* \rightarrow \mathfrak{g}$, $\ell \mapsto (\ell \otimes \text{id})(r)$. In the same way, R'_{\hbar} extends to a (anti)morphism $U_{\hbar}(\mathfrak{g}^*) \rightarrow U_{\hbar}(\mathfrak{g})$.

Define $\alpha_{\hbar} : U_{\hbar}(\mathfrak{g}^*) \rightarrow U_{\hbar}(\mathfrak{g})$, by $x \mapsto m \circ (L_{\hbar} \otimes R'_{\hbar}) \circ \Delta$. Then α_{\hbar} extends $\ell \mapsto (\ell \otimes \text{id})(\mathcal{R}\mathcal{R}^{2,1})$.

Let us prove 2). The reduction mod \hbar of α_{\hbar} is α , which is a linear isomorphism; hence α_{\hbar} is a linear isomorphism. The second part is proved as Lemma 8.6.

Let us prove Proposition 8.4. Assume that $U_{\hbar}(\mathfrak{g})$ is as in [EK], hence $U_{\hbar}(\mathfrak{g}) \simeq U(\mathfrak{g})[[\hbar]]$ as algebras. Then $Z(U_{\hbar}(\mathfrak{g})) \simeq Z(U(\mathfrak{g}))[[\hbar]]$. 2) implies that α induces an isomorphism $(\text{mod } \hbar)(C_{1,\hbar}) \rightarrow Z(U(\mathfrak{g}))$; here $(\text{mod } \hbar)$ is the reduction modulo \hbar . On the other hand, $(\text{mod } \hbar)(C_{1,\hbar}) \subset C_1$, therefore $\Theta(Z(U(\mathfrak{g}))) \subset C_1$.

The map $\delta - (D \otimes \text{id}) \circ \Delta_0 : U(\mathfrak{g}^*) \rightarrow U(\mathfrak{g}^*)^{\otimes 2}$ is filtered, and its associated graded is the dual $\delta : S(\mathfrak{g}^*) \rightarrow \wedge^2(S(\mathfrak{g}^*))$ of the Poisson bracket of $S(\mathfrak{g})$. We have a surjective morphism $S(\mathfrak{g})_{\mathfrak{g}} = S(\mathfrak{g})/\{\mathfrak{g}, S(\mathfrak{g})\} \rightarrow S(\mathfrak{g})/\{S(\mathfrak{g}), S(\mathfrak{g})\}$ to the cokernel of this Poisson bracket, hence $\text{Ker}(\delta) \hookrightarrow (S(\mathfrak{g})_{\mathfrak{g}})^* = S(\mathfrak{g}^*)^{\mathfrak{g}}$. We have $\text{gr}(C_1) \subset \text{Ker}(\delta)$, hence $\text{gr}(C_1) \subset S(\mathfrak{g}^*)^{\mathfrak{g}}$. Now since Θ is filtered and its associated graded takes $\text{gr}(Z(U(\mathfrak{g}))) \simeq S(\mathfrak{g})^{\mathfrak{g}}$ to $S(\mathfrak{g}^*)^{\mathfrak{g}}$, we get $\text{gr}(C_1) = S(\mathfrak{g}^*)^{\mathfrak{g}}$ and $\Theta(Z(U(\mathfrak{g}))) = C_1$. \square

We denote by $\Theta_{\hbar} : Z(U_{\hbar}(\mathfrak{g})) \rightarrow U_{\hbar}(\mathfrak{g}^*)$ the algebra morphism inverse to α_{\hbar} . Θ_{\hbar} is the QUE algebra version of Θ_A defined above.

The image of Θ_{\hbar} is $C_{1,\hbar}$. When the quantization is an in [EK], $U_{\hbar}(\mathfrak{g}) \simeq U(\mathfrak{g})[[\hbar]]$, so this image is not the same as that of θ_{\hbar} , which is $\{\text{traces on } U_{\hbar}(\mathfrak{g})'\} = C_{0,\hbar}$. Therefore in this case, the images of θ_{\hbar} and Θ_{\hbar} do not coincide.

9. ON THE CANONICAL DERIVATION OF \mathcal{O}_{G^*}

Let $(\mathfrak{a}, \mu_{\mathfrak{a}}, \delta_{\mathfrak{a}})$ be a finite dimensional Lie bialgebra. Then \mathcal{O}_A is a Poisson-Lie group, dual to $U(\mathfrak{a})$. Set $D_{\mathfrak{a}} := \mu_{\mathfrak{a}} \circ \delta_{\mathfrak{a}}$, then $D_{\mathfrak{a}}$ is a derivation of $U(\mathfrak{a})$, such that if $U_{\hbar}(\mathfrak{a})$ is any quantization of $U(\mathfrak{a})$ with antipode S , then $D_{\mathfrak{a}} = \hbar^{-1}(S^2 - \text{id})|_{\hbar=0}$ (see [Dr2]). It follows that the dual derivation $D_{\mathfrak{a}}^*$ of \mathcal{O}_A has the same property.

When $\mathfrak{a} = (\mathfrak{g}, r')$ is a quasitriangular Lie bialgebra, $D_{\mathfrak{a}}$ is inner, given by $D_{\mathfrak{a}}(x) = -[\mu(r'), x]$ for any $x \in U(\mathfrak{g})$; here μ is the Lie bracket of \mathfrak{g} (see [Dr2]).

Proposition 9.1. *If \mathfrak{g} is a nondegenerate quasitriangular Lie bialgebra, then the derivation $D_{\mathfrak{g}^*}^*$ of \mathcal{O}_{G^*} is inner, i.e., there exists a function $h \in \mathcal{O}_{G^*}$ such that $D_{\mathfrak{g}^*}^*(f) = \{h, f\}$ for any $f \in \mathcal{O}_{G^*}$.*

Proof. We assume that \mathfrak{g} is the double $\mathfrak{a}_+ \oplus \mathfrak{a}_-$ of a Lie bialgebra \mathfrak{a}_+ (here $\mathfrak{a}_- = \mathfrak{a}_+^*$); the general case is similar. Then \mathfrak{g}^* is (as a Lie algebra) the direct sum $\mathfrak{a}_+ \oplus \mathfrak{a}_-$. Let A_\pm be the formal groups corresponding to \mathfrak{a}_\pm . The morphism $\alpha : U(\mathfrak{g}^*) \rightarrow U(\mathfrak{g})$ is now $U(\mathfrak{a}_+) \otimes U(\mathfrak{a}_-) \rightarrow U(\mathfrak{g})$, $x_+ \otimes x_- \mapsto x_- S(x_+)$. The dual morphism $\alpha^* : \mathcal{O}_G \rightarrow \mathcal{O}_{G^*}$ takes $F \in \mathcal{O}_G$ to $f \in \mathcal{O}_{G^*}$ given by $f(g_+, g_-) := F(g_- g_+^{-1})$.

Lemma 9.2. *Let $D_{\mathfrak{g}}^*$, $D_{\mathfrak{g}^*}^*$ be the canonical derivations of \mathcal{O}_G and \mathcal{O}_{G^*} . Then $\alpha^* \circ D_{\mathfrak{g}}^* = D_{\mathfrak{g}^*}^* \circ \alpha^*$. Moreover, $D_{\mathfrak{g}}^* = \mathbf{L}_{\mu(r')} - \mathbf{R}_{\mu(r')}$, where μ is the Lie bracket of \mathfrak{g} and $\mathbf{L}_a f(g) = (d/d\varepsilon)|_{\varepsilon=0} F(e^{\varepsilon a} g)$, $\mathbf{R}_a f(g) = (d/d\varepsilon)|_{\varepsilon=0} F(g e^{\varepsilon a})$.*

Proof of Lemma. $D_{\mathfrak{g}^*}^*$ is a coderivation, so $\Delta_0 : U(\mathfrak{g}^*) \rightarrow U(\mathfrak{g}^*)^{\otimes 2}$ intertwines $D_{\mathfrak{g}^*}^*$ and $D_{\mathfrak{g}^*}^* \otimes \text{id} + \text{id} \otimes D_{\mathfrak{g}^*}^*$; L and R are Lie bialgebra morphisms, so they intertwine $D_{\mathfrak{g}^*}^*$ and $D_{\mathfrak{g}}^*$; S commutes with $D_{\mathfrak{g}}^*$; and $D_{\mathfrak{g}}^*$ is a derivation, so m_0 intertwines $D_{\mathfrak{g}}^* \otimes \text{id} + \text{id} \otimes D_{\mathfrak{g}}^*$ with $D_{\mathfrak{g}}^*$. Hence $\alpha \circ D_{\mathfrak{g}^*}^* = D_{\mathfrak{g}}^* \circ \alpha$. The first part follows.

According to [Dr2], $D_{\mathfrak{g}}(x) = -[\mu(r'), x]$, which implies the second part. \square

In [STS2], the image of the Poisson bracket on G^* under the formal isomorphism $\alpha : G^* \rightarrow G$ dual to α^* was computed. Let $f, h \in \mathcal{O}_{G^*}$ and $F = (\alpha^*)^{-1}(f)$, $H = (\alpha^*)^{-1}(h)$, then

$$\begin{aligned} (\alpha^*)^{-1}(\{f, h\})(g) &= \langle (d_{\mathbf{R}} - d_{\mathbf{L}})F(g) \otimes d_{\mathbf{L}}H(g), r' \rangle + \langle (d_{\mathbf{R}} - d_{\mathbf{L}})F(g) \otimes d_{\mathbf{R}}H(g), (r')^{2,1} \rangle \\ &= \langle (d_{\mathbf{L}} - d_{\mathbf{R}})F(g), L(d_{\mathbf{R}}H(g)) - R(d_{\mathbf{L}}H(g)) \rangle \end{aligned} \quad (4)$$

where $g \in G$, $d_{\mathbf{L}}F(g), d_{\mathbf{R}}F(g) \in \mathfrak{g}^*$ are the left and right differentials defined by $\langle d_{\mathbf{L}}F(g), a \rangle = (\mathbf{L}_a F)(g)$, $\langle d_{\mathbf{R}}F(g), a \rangle = (\mathbf{R}_a F)(g)$ for any $a \in \mathfrak{g}$.

Lemma 9.3. *There exists a function $H(g) \in \mathcal{O}_G$ such that $L(d_{\mathbf{R}}H(g)) - R(d_{\mathbf{L}}H(g)) = \mu(r')$.*

Proof of lemma. We prove this when \mathfrak{g} is the double $\mathfrak{a}_+ \oplus \mathfrak{a}_-$ of a Lie bialgebra \mathfrak{a}_+ . Then set $a = (a_+, a_-)$ where $a_\pm \in \mathfrak{a}_\pm$. We have $\mathfrak{g}^* = \mathfrak{a}_+ \oplus \mathfrak{a}_-$, and we should solve: $d_{\mathbf{R}}H_a(g) = \mu(r')_- + u_+(g)$, $d_{\mathbf{L}}H_a(g) = \mu(r')_+ + u_-(g)$, where $u_\pm(g)$ are functions $G \rightarrow \mathfrak{a}_\pm$. Now $d_{\mathbf{L}}H(g) = \text{Ad}(g)(d_{\mathbf{R}}H(g))$, hence $\mu(r')_+ + u_-(g) = \text{Ad}(g)(\mu(r')_- + u_+(g))$. Let us decompose $g = g_- g_+^{-1}$, where $g_\pm \in A_\pm = \exp(\mathfrak{a}_\pm)$, we get $\text{Ad}(g_+^{-1})(u_+(g)) - \text{Ad}(g_-^{-1})(u_-(g)) = \text{Ad}(g_-^{-1})(\mu(r')_+) - \text{Ad}(g_+^{-1})(\mu(r')_-)$. Therefore

$$u_+(g) = \text{Ad}(g_+)(\text{Ad}(g_-^{-1})(\mu(r')_+) - \text{Ad}(g_+^{-1})(\mu(r')_-))_+$$

and the condition is

$$d_{\mathbf{R}}H(g) = \text{Ad}(g_+)(\text{Ad}(g_+^{-1})(\mu(r')_-))_- + \text{Ad}(g_+)(\text{Ad}(g_-^{-1})(\mu(r')_+))_+,$$

i.e.,

$$\mathbf{R}_\alpha H_a(g) = \langle \mu(r')_-, \text{Ad}(g_+)((\text{Ad}(g_+^{-1})(\alpha))_+) \rangle + \langle \mu(r')_+, \text{Ad}(g_-)((\text{Ad}(g_+^{-1})(\alpha))_-) \rangle \quad (5)$$

for any $\alpha \in \mathfrak{g}$. Let us denote by $A_\alpha(g)$ the r.h.s. of (5).

Let us compute $\mathbf{R}_\alpha A_\beta - \mathbf{R}_\beta A_\alpha$, for $\alpha, \beta \in \mathfrak{g}$. Recall that $g = g_- g_+^{-1}$, then we have $\mathbf{R}_\alpha(g) = g\alpha$, so $\mathbf{R}_\alpha(g_\pm^{-1}) = \pm(\text{Ad}(g_\pm^{-1})(\alpha))_\pm g_\pm^{-1}$. After computations, we find:

$$\mathbf{R}_\alpha A_\beta - \mathbf{R}_\beta A_\alpha = A_{[\beta, \alpha]} + B_{\alpha, \beta},$$

where

$$\begin{aligned} B_{\alpha, \beta}(g) &= -\langle [(\text{Ad}^*(g_+^{-1})(\beta))_+, (\text{Ad}^*(g_+^{-1})(\alpha))_+], (\text{Ad}(g_+^{-1})(\mu(r')_-))_- \rangle \\ &\quad + \langle [(\text{Ad}^*(g_+^{-1})(\beta))_-, (\text{Ad}^*(g_+^{-1})(\alpha))_-], (\text{Ad}(g_-^{-1})(\mu(r')_+))_+ \rangle. \end{aligned}$$

Now for $u, v \in \mathfrak{a}_+$, we have

$$\begin{aligned} \langle [u, v], (\text{Ad}(g_+^{-1})(\mu(r')_-))_- \rangle &= \langle [u, v], \text{Ad}(g_+^{-1})(\mu(r')_-) \rangle \\ &= \langle [\text{Ad}^*(g_+)(u), \text{Ad}^*(g_+)(v)], \mu(r')_- \rangle = \langle [\text{Ad}^*(g_+)(u), \text{Ad}^*(g_+)(v)], \mu(r') \rangle \\ &= \langle \text{Ad}^*(g_+)(u) \otimes \text{Ad}^*(g_+)(v), \delta(\mu(r')) \rangle = 0, \end{aligned}$$

since $\delta(\mu(r')) = 0$ (see [Dr2]). In the same way, the second term of $B_{\alpha, \beta}(g)$ vanishes. Hence the system (5) has a solution (it is unique if we impose that H vanishes at the origin). \square

End of proof of Proposition 9.1. Now if $h = -\alpha^*(H)$ with H as in Lemma 9.3 and for any $f \in \mathcal{O}_{G^*}$, we have

$$\begin{aligned} D_{\mathfrak{g}^*}(f) &= \alpha^*(D_{\mathfrak{g}}^*(F)) = \alpha^*((\mathbf{L}_{\mu(r')} - \mathbf{R}_{\mu(r')})(F)) \\ &= \alpha^*(\langle (d_{\mathbf{L}} - d_{\mathbf{R}})(F)(g), R(d_{\mathbf{L}}H(g)) - L(d_{\mathbf{R}}H(g)) \rangle) = \{h, f\}. \end{aligned}$$

\square

APPENDIX A. PROOF THAT ASSOCIATORS CAN BE MADE ADMISSIBLE

In [EH2], Proposition 3.2, 2) should read ‘‘Assume that $x \in U'$ and for any trees R , etc.’’. This affects Proposition 4.5 in [EH2], because the proof implicitly relies on the statement of Proposition 3.2 of [EH2] without the assumption $x \in U'$. Below we prove a particular case of Proposition 4.5 of [EH2] (the general case is similar).

Proposition A.1. *Let \mathfrak{g} be a Lie algebra and let $\Phi \in U(\mathfrak{g})^{\otimes 3}[[\hbar]]$ be an invariant solution of the pentagon equation, such that $\varepsilon^{(i)}(\Phi) = 1^{\otimes 2}$ for $i = 1, 2, 3$, $\Phi = 1^{\otimes 3} + O(\hbar)$ and $\text{Alt}(\Phi) = O(\hbar^2)$. Then there exists an invariant twist $F \in U(\mathfrak{g})^{\otimes 2}[[\hbar]]$, such that $\varepsilon^{(i)}(F) = 1$, $i = 1, 2$, $F = 1^{\otimes 2} + O(\hbar)$, and ${}^F\Phi = F^{2,3}F^{1,23}\Phi(F^{1,2}F^{12,3})^{-1}$ is admissible, i.e., $\hbar \log({}^F\Phi) \in (U(\mathfrak{g})[[\hbar]])^{\otimes 3}$.*

Proof. We will construct F as a product $\cdots F_2 F_1$, where F_n belongs to $1^{\otimes 2} + \hbar^i U_0^{\otimes 2}$ and is such that if $\Phi_n := F_n \cdots F_1 \Phi$, then $\hbar \log(\Phi_n) \in (U_0')^{\otimes 2} + \hbar^{n+2} U_0^{\otimes 3}$. Here $U = U(\mathfrak{g})[[\hbar]]$ and the index 0 denotes the augmentation ideals.

We first construct F_1 . Expand $\Phi = 1^{\otimes 3} + \hbar \phi_1 + \cdots$, then $d(\phi_1) = 0$ and $\text{Alt}(\phi_1) = 0$, hence $\phi_1 = d(\psi_1)$, where $\psi_1 \in (U_0^{\otimes 2})^{\mathfrak{g}}$ (here d is the co-Hochschild differential of $(U_0^{\otimes \cdot})^{\mathfrak{g}}$). We then set $F_1 = 1^{\otimes 2} + \hbar \psi_1$; we get $\Phi_1 = 1^{\otimes 3} + \hbar^2 \phi_2' + \cdots$. Then $\hbar \log(\Phi_1) \in \hbar^3 U_0^{\otimes 3}$.

Now $d(\phi_2') = 0$, so there exists $\psi_2 \in (U_0^{\otimes 2})^{\mathfrak{g}}$ such that $\phi_2' = Z + d(\psi_2)$, where $Z \in \wedge^3(\mathfrak{g})[[\hbar]]$. Set $F_2 := 1^{\otimes 2} + \hbar^2 \psi_2$, we get $\hbar \log(\Phi_2) \in \hbar^3 Z + \hbar^4 U_0^{\otimes 3} \subset (U_0')^{\otimes 3} + \hbar^4 U_0^{\otimes 3}$.

Let $n \geq 3$. Assume that we have constructed F_1, \dots, F_{n-1} and let us construct F_n . By assumption, $\Phi_{n-1} \in 1^{\otimes 3} + \hbar U_0^{\otimes 3}$ is such that $\varphi_{n-1} := \hbar \log(\Phi_{n-1}) \in (U_0')^{\otimes 3} + \hbar^{n+1} U_0^{\otimes 3}$.

Lemma A.2. *The quotient $(U' + \hbar^n U)/(U' + \hbar^{n+1} U)$ identifies with $U(\mathfrak{g})/U(\mathfrak{g})_{\leq n}$. In the same way, the quotient $(U_0^{\otimes k} + \hbar^n U_0^{\otimes k})/(U_0^{\otimes k} + \hbar^{n+1} U_0^{\otimes k})$ identifies with $U(\mathfrak{g})_0^{\otimes k}/(U(\mathfrak{g})_0^{\otimes k})_{\leq n}$ and the quotient of \mathfrak{g} -invariant subspaces $(U_0^{\otimes k} + \hbar^n U_0^{\otimes k})^{\mathfrak{g}}/(U_0^{\otimes k} + \hbar^{n+1} U_0^{\otimes k})^{\mathfrak{g}}$ identifies with $(U(\mathfrak{g})_0^{\otimes k})^{\mathfrak{g}}/(U(\mathfrak{g})_0^{\otimes k})_{\leq n}^{\mathfrak{g}}$.*

The inverse of the first isomorphism takes the class of $\beta \in U(\mathfrak{g})$ to the class of $\hbar^n \beta \in U' + \hbar^n U$. Let $\bar{\alpha} \in (U(\mathfrak{g})_0^{\otimes 3})^{\mathfrak{g}}/(U(\mathfrak{g})_0^{\otimes 3})_{\leq n+1}^{\mathfrak{g}}$ be the image of the class of φ_{n-1} under the above isomorphism. Let $\alpha \in (U(\mathfrak{g})_0^{\otimes 3})^{\mathfrak{g}}$ be a representative of $\bar{\alpha}$. Then we have $\varphi_{n-1} = \varphi + \hbar^{n+1} \alpha$, where $\varphi \in (U_0')^{\otimes 3} + \hbar^{n+2} U_0^{\otimes 3}$. Now φ_{n-1} satisfies the pentagon equation, so we get

$$\begin{aligned} (-\varphi - \hbar^{n+1} \alpha)^{1,2,3,4} \star_{\hbar} (\varphi + \hbar^{n+1} \alpha)^{2,3,4} \star_{\hbar} (\varphi + \hbar^{n+1} \alpha)^{1,2,3,4} \star_{\hbar} (\varphi + \hbar^{n+1} \alpha)^{1,2,3} \\ \star_{\hbar} (-\varphi - \hbar^{n+1} \alpha)^{12,3,4} = 0, \end{aligned} \quad (6)$$

where $a \star_{\hbar} b$ is the CBH product for the Lie bracket $[a, b]_{\hbar} = [a, b]/\hbar$.

Lemma A.3. *Assume that $n \geq 2$. If $f_1, f_2 \in (U'_0)^2 + \hbar^{n+1}U_0$ and $g, h \in \hbar^n U_0$, then $(f_1 + g) \star_{\hbar} (f_2 + h) = g + h$ modulo $(U'_0)^2 + \hbar^{n+1}U_0$.*

Proof of Lemma. The contribution of the degree 1 part of the CBH series is $f_1 + g + f_2 + h$, which gives $g + h$ modulo $(U'_0)^2 + \hbar^{n+1}U_0$.

We now prove that $[(U'_0)^2 + \hbar^n U_0, (U'_0)^2 + \hbar^n U_0]_{\hbar} \subset (U'_0)^2 + \hbar^{n+1}U_0$. Indeed, we have $[U'_0, U'_0]_{\hbar} \subset U'_0$, hence $[(U'_0)^2, (U'_0)^2]_{\hbar} \subset (U'_0)^2$; $[\hbar^n U_0, \hbar^n U_0]_{\hbar} \subset \hbar^{2n-1}U_0 \subset \hbar^{n+1}U_0$; and $[U'_0, \hbar^n U_0]_{\hbar} \subset \hbar^n U_0$ since $U'_0 \subset \hbar U_0$, so that $[(U'_0)^2, \hbar^n U_0]_{\hbar} \subset \hbar^n(U_0 U'_0 + U'_0 U_0) \subset \hbar^{n+1}(U_0)^2$ again because $U'_0 \subset \hbar U_0$.

It follows that the contributions of all the higher degree parts of the CBH series belong to $(U'_0)^2 + \hbar^{n+1}U_0$. This implies the lemma. \square

End of proof of Proposition A.1. Lemma A.3 implies that the image of (6) in $(U' \widehat{\otimes}^4 + \hbar^{n+1}U \widehat{\otimes}^4)/(U' \widehat{\otimes}^4 + \hbar^{n+2}U \widehat{\otimes}^4) = U(\mathfrak{g})^{\otimes 4}/(U(\mathfrak{g})^{\otimes 4})_{\leq n+2}$ gives $d(\bar{\alpha}) = 0$, where

$$d : U(\mathfrak{g})^{\otimes 3}/(U(\mathfrak{g})^{\otimes 3})_{\leq n+2} \rightarrow U(\mathfrak{g})^{\otimes 4}/(U(\mathfrak{g})^{\otimes 4})_{\leq n+2}$$

is the map induced by the co-Hochschild differential.

According to [Dr3], the cohomology of the complex $C^2 \rightarrow C^3 \rightarrow C^4$ vanishes, where $C^k = (U(\mathfrak{g})_0^{\otimes k})^{\mathfrak{g}}/(U(\mathfrak{g})_0^{\otimes k})^{\mathfrak{g}}_{\leq n+2}$.

It follows that there exists $\bar{\beta} \in C^2$, such that $\bar{\alpha} = d(\bar{\beta})$. Let $\beta \in (U(\mathfrak{g})_0^{\otimes 2})^{\mathfrak{g}}$ be a representative of $\bar{\beta}$. Set $F_n := \exp(\hbar^n \beta)$ and $\Phi_n = F_n \Phi_{n-1}$.

We get

$$\varphi_n = f_n^{2,3} \star_{\hbar} f_n^{1,2,3} \star_{\hbar} \varphi_{n-1} \star_{\hbar} (-f_n^{12,3}) \star_{\hbar} (-f_n^{1,2}),$$

where $f_n = \hbar^{n+1}\beta$. According to Lemma A.3, the class of φ_n in $((U'_0)^{\widehat{\otimes}^3} + \hbar^{n+1}U_0^{\widehat{\otimes}^3})/((U'_0)^{\widehat{\otimes}^3} + \hbar^{n+2}U_0^{\widehat{\otimes}^3})$ is $\bar{\alpha} - d(\bar{\beta}) = 0$, hence $\varphi_n \in (U'_0)^{\widehat{\otimes}^3} + \hbar^{n+2}U_0^{\widehat{\otimes}^3}$. This proves the induction step. \square

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