

FORMALITY THEOREM FOR LIE BIALGEBRAS AND QUANTIZATION OF TWISTS AND COBOUNDARY r -MATRICES

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ABSTRACT. Let $(\mathfrak{g}, \delta_{\hbar})$ be a Lie bialgebra. Let $(U_{\hbar}(\mathfrak{g}), \Delta_{\hbar})$ a quantization of $(\mathfrak{g}, \delta_{\hbar})$ through Etingof-Kazhdan functor. We prove the existence of a L_{∞} -morphism between the Lie algebra $C(\mathfrak{g}) = \Lambda(\mathfrak{g})$ and the tensor algebra (without unit) $T_+U = T_+(U_{\hbar}(\mathfrak{g})[-1])$ with Lie algebra structure given by the Gerstenhaber bracket. When s is a twist for (\mathfrak{g}, δ) , we deduce from the formality morphism the existence of a quantum twist F . When $(\mathfrak{g}, \delta, r)$ is a coboundary Lie bialgebra, we get the existence of a quantization R of r .

0. INTRODUCTION

Let \mathbb{K} be a field of characteristic 0 and \hbar a formal parameter. Let $(\mathfrak{g}, [-, -], \delta_{\hbar})$ be a Lie bialgebra over \mathbb{K} (all our objects will be $\mathbb{K}[[\hbar]]$ -modules). Using Etingof-Kazhdan quantization functor, one can construct a quantization $(U_{\hbar}(\mathfrak{g}), \Delta_{\hbar})$ of $(\mathfrak{g}, \delta_{\hbar})$. Let us denote $C(\mathfrak{g}) = S(\mathfrak{g}[-1]) = \Lambda(\mathfrak{g})$ the free graded commutative algebra generated by \mathfrak{g} ; $(C(\mathfrak{g}), [-, -], \wedge, \delta_{\hbar})$ is a differential Gerstenhaber algebra. Let us also denote $T_+U = T_+(U_{\hbar}(\mathfrak{g})[-1])$ the tensor algebra (without unit) over $U_{\hbar}(\mathfrak{g})$ (when $\delta_{\hbar} = 0$, $U = U(\mathfrak{g})$, the enveloping algebra of \mathfrak{g}). More generally, we denote $T_+E = T_+(E[-1])$ the free tensor algebra (without unit) of a graded vector space E and ${}^cTE = T(E[1])$ the cofree tensor coalgebra of E . One can see the elements of \mathfrak{g} as invariant (under left action) vector fields on the manifold G where G is a connected group whose Lie algebra is \mathfrak{g} . In that framework, $C(\mathfrak{g})$ corresponds to the Gerstenhaber algebra $T_{\text{poly}}^{\text{inv}}$ of invariant multivector fields on G equipped with Schouten bracket. The space TU corresponds to the space $D_{\text{poly}}^{\text{inv}}$ of multidifferential operators. The space D_{poly} of Hochschild cochains carries a graded differential Lie algebra structure when equipped with the Hochschild coboundary and the Gerstenhaber bracket. Tamarkin proved in [Ta] that the space D_{poly} carries a G_{∞} structure. In this paper, for general Lie bialgebra case, we prove:

Proposition 0.1. *There exists a G_{∞} -structure on TU , whose underlying L_{∞} -structure is the one given by the differential graded Lie structure with deformed Gerstenhaber bracket and co-Hochschild differential.*

Theorem 0.2. *There exists φ , a L_{∞} -quasi-isomorphism between $C(\mathfrak{g})$ and T_+U for the corresponding Lie algebra structures (from Proposition 0.1) such that the associated morphism of complex φ^1 maps $v \in C(\mathfrak{g})$ to its alternation $\text{Alt}(v) \in T_+U \pmod{\hbar}$.*

Definitions of G_{∞} and L_{∞} -structures will be recalled in section 1 as well as the fact that G_{∞} -algebras have canonical underlying L_{∞} -structure. This theorem generalises a result of Calaque ([Ca]) when $\delta_{\hbar} = 0$ and answers to a conjecture of Tamarkin and Tsygan ([TT]).

The main result of this paper is the existence of quantization of classical twist.

Definition 0.3. [Dr3] Let $(\mathfrak{g}, \delta, [-, -])$ be a Lie bialgebra. A classical twist in \mathfrak{g} is an element $s \in \Lambda^2\mathfrak{g}$ such that

$$\text{CYB}(s) + \text{Alt}(\delta \otimes \text{id})(s) = 0,$$

where CYB is the l.h.s. of the classical Yang-Baxter equation.

This definition is motivated by the fact that if one can see \mathfrak{g} as a Lie quasi-bialgebra, the twisted Lie quasi-bialgebra $(\mathfrak{g}^s, \delta^s)$ is a Lie bialgebra in the class of (\mathfrak{g}, δ) ($(\mathfrak{g}_1, \delta_1)$ and $(\mathfrak{g}_2, \delta_2)$ are in the same class if there exists a Lie algebra isomorphism between the associated double, preserving the canonical forms and compatible with the canonical embeddings of Lie bialgebras into the doubles). We prove the following conjecture stated in [KPST]:

Theorem 0.4. *Any classical twist can be extended to a quantum twist, i.e., if (\mathfrak{g}, δ) is a Lie bialgebra and s is a classical twist, there exists a quantization $(U_{\hbar}(\mathfrak{g}), \Delta_{\hbar})$ of (\mathfrak{g}, δ) and $F \in U_{\hbar}(\mathfrak{g})^{\otimes 2}$ such that*

- (1) $F = 1 + O(\hbar)$ and $F - F^{21} = \hbar s + O(\hbar^2)$,
- (2) $(\Delta_{\hbar} \otimes \text{id})(F)F^{1,2} - (\text{id} \otimes \Delta_{\hbar})(F)F^{2,3} = 0$,
- (3) $(\varepsilon \otimes \text{id})(F) = (\text{id} \otimes \varepsilon)(F) = 1$.

Moreover, gauge equivalence classes of quantum twists for $U_{\hbar}(\mathfrak{g})$ are in bijection with gauge equivalence classes of \hbar -dependent classical twists $s_{\hbar} = \hbar s_1 + O(\hbar^2)$ for \mathfrak{g} .

Suppose now that (\mathfrak{g}, r, Z) is a (finite-dimensional) coboundary Lie bialgebra over \mathbb{K} . This means that \mathfrak{g} is a Lie bialgebra, the Lie cobracket δ is the coboundary of an element $r \in \Lambda^2(\mathfrak{g})$: $\delta(x) = [x \otimes 1 + 1 \otimes x, r]$ for any $x \in \mathfrak{g}$. This condition means that $Z := \text{CYB}(r)$ belongs to $\Lambda^3(\mathfrak{g})^{\mathfrak{g}}$. Quasi-triangular and triangular Lie bialgebras are particular cases of this definition.

Let us recall the definition of a quantization of (\mathfrak{g}, r, Z) in coboundary Hopf algebra ([Dr1]):

Definition 0.5. A algebra $(U_{\hbar}(\mathfrak{g}), R)$ is a coboundary quantization of (\mathfrak{g}, r, Z) if:

- (1) $(U_{\hbar}(\mathfrak{g}), \Delta_{\hbar})$ is a quantization of \mathfrak{g} ,
- (2) R is an invertible element of $U_{\hbar}(\mathfrak{g})^{\otimes 2}$ such that $R = 1 + \hbar r + O(\hbar^2)$ that satisfies:
- (3) $R^{1,2}(\Delta_{\hbar} \otimes \text{id})(R) = R^{2,3}(\text{id} \otimes \Delta_{\hbar})(R)$,
- (4) $(\varepsilon \otimes \text{id})(R) = (\text{id} \otimes \varepsilon)(R) = 1$,
- (5) $R^{2,1} = R^{-1}$.
- (6) R twists Δ_{\hbar} into $\Delta_{\hbar}^{\text{op}}$:

$$R\Delta_{\hbar}(a)R^{-1} = \Delta_{\hbar}^{2,1}(a), \quad a \in U_{\hbar}(\mathfrak{g}),$$

where $\Delta_{\hbar}^{2,1} = \Delta_{\hbar}^{\text{op}}$ is the opposite comultiplication.

Here the notation $R^{i,j}$ corresponds to the coproduct-insertion map and will be recalled at the end of this introduction. Using the formality map of Theorem 0.2, we prove the existence of a quantization of coboundary r -matrices r .

Theorem 0.6. *Let (\mathfrak{g}, r, Z) be a coboundary Lie bialgebra. There exists a Hopf algebra $(U_{\hbar}(\mathfrak{g}), \Delta_{\hbar})$ and an element R in $U_{\hbar}(\mathfrak{g})^{\otimes 2}$ satisfying the first four properties of Definition 0.5.*

In section 1, we recall definitions of G_{∞} and L_{∞} -structures. G_{∞} -structures on a vector space A are defined on the cofree Lie coalgebra $S(\underline{TE}[1])$, where \underline{TE} is the quotient of the cofree tensor coalgebra without unit ${}^cT_+E$ of a vector space E by the image of the shuffle product $({}^cT_+E)^{\otimes 2} \rightarrow {}^cT_+E$. We also recall the existence of two exact functors: $\text{L-G} - G_{\infty}, \mathfrak{h} \rightarrow C(\mathfrak{h})[1]$ between the categories of differential Lie bialgebras and of differential Gerstenhaber algebras, viewed as G_{∞} -algebras and $\text{L-G}_{\infty}, \underline{TE} \rightarrow E$ between the categories of differential Lie bialgebras \underline{TE} and of G_{∞} -algebras.

In section 2, we recall Drinfeld duality between QUE (Quantum Universal Enveloping) and QFSH (Quantum Formal Series Hopf) algebras. In particular we recall the existence of functors $(-)'$: $\text{QUE} \rightarrow \text{QFSH}$ and $(-)^{\vee}$: $\text{QFSH} \rightarrow \text{QUE}$. We then recall Etingof-Kazhdan quantization/dequantization functors.

In section 3, we prove the existence of a bialgebra structure on ${}^cTT_+U$ coming from brace operations. Moreover, introducing a formal parameter ν and a suitable completion $\widehat{{}^cTT_+U}$ of ${}^cTT_+U$, we prove that $\widehat{{}^cTT_+U}[[\nu]]$ is a QFSH algebra deforming the shuffle/cofree (for product/coproduct) structure on ${}^cTT_+U$. Then using Etingof-Kazhdan dequantization functor DQ , we show that the QUE algebra $({}^cTT_+U)^\vee$ is sent to $\underline{{}^cTT_+U}$. So T_+U has a G_∞ -structure. This structure reduces to a differential Lie algebra structure with Gerstenhaber bracket and coHochschild differential defined using the coproduct Δ_\hbar .

In section 4, we prove the existence of a bialgebra quasi-isomorphism $\varphi_{\text{alg}} : U \rightarrow (\widehat{{}^cTT_+U})^\vee$ (where $\widehat{{}^cTT_+U}$ is $\widehat{{}^cTT_+U}[[\nu]]$ with identification $\nu = \hbar$) and of a Lie bialgebra quasi-isomorphism $\varphi_{\text{Lie}} : \underline{{}^cTE} \rightarrow \underline{{}^cTC}(\underline{{}^cTE})$ (for every graded vector space E).

In section 5, we recall the existence of an inverse map for any L_∞ -quasi-isomorphism between L_∞ -algebras.

In section 6, we deduce the existence of a L_∞ -quasi-isomorphism between $C(\mathfrak{g})$ and T_+U . The proof can be summarised in the following diagram:

$$\begin{array}{ccccc}
 \underline{{}^cTC}(\underline{{}^cTT_+U}) & C(\underline{{}^cTT_+U})[1] & = & C(\underline{{}^cTT_+U})[1] & \underline{{}^cTT_+U} & (\widehat{{}^cTT_+U})^\vee \\
 \uparrow \varphi_{\text{Lie}} & \xrightarrow{\text{L-}G_\infty} & \uparrow \varphi'_{G_\infty} & \uparrow \varphi_{G_\infty} & \xleftarrow{\text{L-}G_\infty} & \uparrow \varphi'_{\text{Lie}} & \xleftarrow{\text{DQ}} & \uparrow \varphi_{\text{alg}} \\
 \underline{{}^cTT_+U} & T_+U[1] & & C(\mathfrak{g})[1] & & \mathfrak{g} & & U = U_\hbar(\mathfrak{g}).
 \end{array}$$

Thus the composition of φ_{G_∞} with the inverse of φ'_{G_∞} gives the wanted quasi-isomorphism. From this, we prove Theorem 0.4 and Theorem 0.6.

In the last section, we make some remarks on possible applications and related open questions. In particular, we prove that the coHochschild complex (T_+U, b_{cH}) is quasi-isomorphic to the complex $(C(\mathfrak{g}), \delta_\hbar)$.

Notations. We use the standard notation for the coproduct-insertion maps: we say that an ordered set is a pair of a finite set S and a bijection $\{1, \dots, |S|\} \rightarrow S$. For I_1, \dots, I_m disjoint ordered subsets of $\{1, \dots, n\}$, (U, Δ) a Hopf algebra and $a \in U^{\otimes m}$, we define

$$a^{I_1, \dots, I_m} = \sigma_{I_1, \dots, I_m} \circ (\Delta^{(|I_1|)} \otimes \dots \otimes \Delta^{(|I_m|)})(a),$$

with $\Delta^{(1)} = \text{id}$, $\Delta^{(2)} = \Delta$, $\Delta^{(n+1)} = (\text{id}^{\otimes n-1} \otimes \Delta) \circ \Delta^{(n)}$, and $\sigma_{I_1, \dots, I_m} : U^{\otimes \sum_i |I_i|} \rightarrow U^{\otimes n}$ is the morphism corresponding to the map $\{1, \dots, \sum_i |I_i|\} \rightarrow \{1, \dots, n\}$ taking $(1, \dots, |I_1|)$ to I_1 , $(|I_1| + 1, \dots, |I_1| + |I_2|)$ to I_2 , etc. When U is cocommutative, this definition depends only on the sets underlying I_1, \dots, I_m .

Until the end of this paper, although we will often omit to mention it, we will always deal with graded structures.

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1. G_∞ -ALGEBRAS

1.1. Definitions. Let us recall definitions of L_∞ -algebras and L_∞ -morphisms. Let A be a graded vector space. We have denoted $T_+A = T_+(A[-1])$ the free tensor algebra (without unit) of A which, equipped with the coshuffle coproduct, is a bialgebra. We also have denoted $C(A) = S(A[-1])$ the free graded commutative algebra generated by $A[-1]$, seen as a quotient of T_+A . The coshuffle coproduct is still well defined on $C(A)$ which becomes a cofree cocommutative coalgebra on $A[-1]$. We also denote $\Lambda A = S(A[1])$, the analogous graded commutative algebra generated by $A[1]$ (in particular, for $A_1, A_2 \in A$, $A_1\Lambda A_2$ stands for the corresponding quotient of $A_1[1] \otimes A_2[1]$ in ΛA). We will use the notations $T_+^n A$, $\Lambda^n A$ and $C^n(A)$ for the elements of degree n .

Definition 1.1. A vector space A is endowed with a L_∞ -algebra (Lie algebra “up to homotopy”) structure if there are degree one linear maps $d^{1,\dots,1}: \Lambda^k A \rightarrow A[1]$ such that the associated coderivations (extended with respect to the cofree cocommutative structure on ΛA) $d: \Lambda A \rightarrow \Lambda A$, satisfy $d \circ d = 0$ where d is the coderivation

$$d = d^1 + d^{1,1} + \dots + d^{1,\dots,1} + \dots .$$

In particular, a differential Lie algebra $(A, b, [-, -])$ is a L_∞ -algebra with structure maps $d^1 = b[1]$, $d^{1,1} = [-, -][1]$ and $d^{1,\dots,1}: \Lambda^k A \rightarrow A[1]$ are 0 for $k \geq 3$. One can now define the generalisation of Lie algebra morphisms:

Definition 1.2. A L_∞ -morphism between two L_∞ -algebras $(A_1, d_1 = d_1^1 + \dots)$ and $(A_2, d_2 = d_2^1 + \dots)$ is a morphism of codifferential cofree coalgebras, of degree 0,

$$\varphi: (\Lambda A_1, d_1) \rightarrow (\Lambda A_2, d_2).$$

In particular $\varphi \circ d_1 = d_2 \circ \varphi$. As φ is a morphism of cofree cocommutative coalgebras, φ is determined by its image on the cogenerators, i.e., by its components: $\varphi^{1,\dots,1}: \Lambda^k A_1 \rightarrow A_2[1]$.

Let E be a graded vector space. Let us denote ${}^cT(E)$ the cofree tensor coalgebra of E with coproduct Δ' . Equipped with the shuffle product \bullet (defined on the cogenerators ${}^cT(E) \otimes {}^cT(E) \rightarrow {}^cT(E)$ as $\text{pr} \otimes \varepsilon + \varepsilon \otimes \text{pr}$, where $\text{pr}: {}^cT(E) \rightarrow E$ is the projection and ε is the counit), it is a bialgebra. Let ${}^cT_+(E)$ be the augmentation ideal. We denote $\underline{{}^cT}(E) = {}^cT_+(E)/({}^cT_+(E) \bullet {}^cT_+(E))$ the quotient by the shuffles. It has a graded cofree Lie coalgebra structure (with coproduct $\delta = \Delta' - \Delta'^{\text{op}}$). Then $S(\underline{{}^cT}(E)[1])$ has a structure of cofree coGerstenhaber algebra (i.e. equipped with cofree coLie and cofree cocommutative coproducts satisfying compatibility condition). We use the notation $\underline{{}^cT}^n(E)$ for the elements of degree n .

Definition 1.3. A vector space E is endowed with a G_∞ -algebra (Gerstenhaber algebra “up to homotopy”) structure if there are degree one linear maps $d^{p_1,\dots,p_k}: \underline{{}^cT}^{p_1} E \Lambda \dots \Lambda \underline{{}^cT}^{p_k} E \subset \Lambda^{k,c} \underline{{}^cT} E \rightarrow E[1]$ such that the associated coderivation (extended with respect to the cofree coGerstenhaber structure on $\Lambda \underline{{}^cT}(E)$) $d: \Lambda \underline{{}^cT}(E) \rightarrow \Lambda \underline{{}^cT}(E)$ satisfies $d \circ d = 0$ where d is the coderivation

$$d = d^1 + d^{1,1} + \dots + d^{p_1,\dots,p_k} + \dots .$$

In particular we have

Remark 1.4. If $(E, b, [-, -], \wedge)$ is a differential Gerstenhaber algebra, then $E[1]$ is a G_∞ -algebra with structure maps $d^1 = b[1]$, $d^{1,1} = [-, -][1]$, $d^2 = \wedge[1]$ and other $d^{p_1,\dots,p_k}: \underline{{}^cT}^{p_1}(E[1]) \Lambda \dots \Lambda \underline{{}^cT}^{p_k}(E[1]) \rightarrow E[2]$ are 0.

One can finally define the generalisation of Gerstenhaber algebra morphisms:

Definition 1.5. A G_∞ -morphism between two G_∞ -algebras $(E_1, d_1 = d_1^1 + d_1^2 + \dots)$ and $(E_2, d_2 = d_2^1 + d_2^2 + \dots)$ is a morphism of differential coGerstenhaber coalgebras, of degree 0,

$$\varphi : (\Lambda^c \underline{T}(E_1), d_1) \rightarrow (\Lambda^c \underline{T}(E_2), d_2).$$

In particular $\varphi \circ d_1 = d_2 \circ \varphi$. As φ is a morphism of cofree coGerstenhaber coalgebras, φ is determined by its image on the cogenerators, i.e., by its components: $\varphi^{p_1, \dots, p_k} : \underline{T}^{p_1}(E_1) \wedge \dots \wedge \underline{T}^{p_k}(E_1) \rightarrow E_2[1]$.

1.2. Functors L-G $-G_\infty$ and L- G_∞ . Let $(\mathfrak{h}, \delta, d)$ be a differential Lie bialgebra. Let $C(\mathfrak{h}) = S(\mathfrak{h}[-1])$ be the free graded commutative algebra generated by \mathfrak{h} . Recall from the previous subsection that $C(\mathfrak{h})$ is also a cofree coalgebra and that coderivations $C(\mathfrak{h}) \rightarrow C(\mathfrak{h})$ are defined by their images in \mathfrak{h} . Thus, one easily checks that the coderivation $[-, -] : C(\mathfrak{h}) \rightarrow C(\mathfrak{h})$ extending the Lie bracket (with degree shifted by one) defines a Lie algebra structure on $C(\mathfrak{h})$ and that $(C(\mathfrak{h}), [-, -], \wedge)$ is a Gerstenhaber algebra, where \wedge is the commutative product: $(\alpha, \beta) \mapsto \alpha \wedge \beta = \alpha \Lambda \beta$ on $C(\mathfrak{h})$. Moreover, one can extend maps $d : \mathfrak{h} \rightarrow \mathfrak{h}$ and $\delta : \mathfrak{h} \rightarrow S^2(\mathfrak{h}[-1])$ on the free commutative algebra $C(\mathfrak{h})$ so that $(C(\mathfrak{h}), [-, -], \wedge, d + \delta)$ is a differential Gerstenhaber algebra. In fact, the differential δ is the Chevalley Eilenberg differential: the space $C(\mathfrak{h}) = S^*(\mathfrak{h}[-1])$ is isomorphic to the standard complex $(\Lambda^*(\mathfrak{h}))[-*]$ and δ is simply the differential given by the underlying Lie algebra structure of \mathfrak{h} . Moreover any morphism $f : \mathfrak{h}_1 \rightarrow \mathfrak{h}_2$ can be extended into a morphism $C(f) : C(\mathfrak{h}_1) \rightarrow C(\mathfrak{h}_2)$ of free commutative algebras thanks to the inclusion $\mathfrak{h}_2 \subset C(\mathfrak{h}_2)$. This morphism is easily seen to be a differential Gerstenhaber algebra morphism. Thus, we have defined a functor L-G from differential Lie bialgebras to differential Gerstenhaber algebras which sends \mathfrak{h} to $C(\mathfrak{h})$. This functor is exact. As the differential δ and d anticommutes by construction, the complex $(C(\mathfrak{h}), d, \delta)$ is a (first quadrant) bicomplex. Hence a quasi-isomorphism $(\mathfrak{h}_1, d_1) \rightarrow (\mathfrak{h}_2, d_2)$ induces a quasi-isomorphism $(C(\mathfrak{h}_1), d_1, \delta_1) \rightarrow (C(\mathfrak{h}_2), d_2, \delta_2)$. Then, thanks to Remark 1.4, we get a functor L-G $-G_\infty : \mathfrak{h} \rightarrow C(\mathfrak{h})[1]$ from differential Lie bialgebras to G_∞ -algebras.

Let us now define the functor L- G_∞ . Consider the category CFDLB of differential Lie bialgebras which are cofree as a Lie coalgebra. In other words we are interested in cofree Lie coalgebra $\underline{T}(E)$ on a graded vector space E together with a differential ℓ and a cobracket L on $\underline{T}(E)$ that makes it a differential Lie bialgebra. As $\underline{T}(E)$ is cofree, the differential is uniquely determined by its restriction to cogenerators $l^p : \underline{T}^p(E) \rightarrow E$. Similarly, the Lie bracket is uniquely determined by maps $L^{p_1, p_2} : \underline{T}^{p_1}(E) \wedge \underline{T}^{p_2}(E) \rightarrow E$. Now, this data determines on E a structure of G_∞ -algebra with structure maps given by $d^{p_1, \dots, p_k} : \underline{T}^{p_1}(E) \wedge \dots \wedge \underline{T}^{p_k}(E) \rightarrow E$, with $d^{p_1, \dots, p_k} = 0$ for $k > 2$ and $d^{p_1, p_2} = L^{p_1, p_2}$ and $d^p = l^p$ (with degrees shifted by one). In fact, according to Definition 1.3, a G_∞ -structure on E is given by a differential d on $\Lambda^c \underline{T}(E)$ which as a space is isomorphic to the standard Chevalley-Eilenberg complex of the differential Lie algebra $(\underline{T}(E), \ell, L)$. The differential defined above is simply the Chevalley-Eilenberg differential. In particular d is the sum $d = d^1 + d^2$ with $d^1 = \sum_{p \geq 1} l^p$ and $d^2 = \sum_{p_1, p_2 \geq 1} L^{p_1, p_2}$ and $(\Lambda^c \underline{T}(E), d^1, d^2)$ is a bicomplex. Moreover, a morphism $\varphi : \underline{T}(E_1) \rightarrow \underline{T}(E_2)$ of differential Lie bialgebras is determined by its restriction to cogenerator of the cofree Lie coalgebra structure, that is to say by maps: $\varphi^n : \underline{T}^n(E_1) \rightarrow E_2$. It determines a G_∞ -morphism $E_1 \rightarrow E_2$ (with the G_∞ -structures defined above) defined by maps $\varphi^n : \underline{T}^n(E_1) \rightarrow E_2$, other being 0. This is simply the functoriality of the Chevalley-Eilenberg complex. Thus we have defined a functor from CFDLB to the category of G_∞ -algebras. This functor is exact. A quasi-isomorphism of differential Lie bialgebras $(\underline{T}(E_1), \ell_1) \rightarrow (\underline{T}(E_2), \ell_2)$ induces a quasi-isomorphism $(\Lambda^c \underline{T}(E_1), d_1^1) \rightarrow (\Lambda^c \underline{T}(E_2), d_2^1)$, hence, as $(\Lambda^c \underline{T}(E), d^1, d^2)$ is a (first quadrant) bicomplex, a quasi-isomorphism $(\Lambda^c \underline{T}(E_1), d_1) \rightarrow (\Lambda^c \underline{T}(E_2), d_2)$.

Untill the end of the paper, we will use the notations TE for $T(E[-1])$ and \underline{TE} for $\underline{T}(E[1])$.

2. ETINGOF-KAZHDAN FUNCTORS

2.1. QUE and QFSH algebras. We recall some facts from [Dr1] (proofs and definitions can be found in [Gav]). Let us denote by **QUE** the category of quantized universal enveloping (QUE) algebras and by **QFSH** the category of quantized formal series Hopf (QFSH) algebras. Let us recall the definition of FSH and QFSH algebras:

Definition 2.1. A FSH algebra is a Hopf algebra of power series isomorphic as an algebra to $\mathbb{K}[[\{u_i | i \in J\}]]$ (for some set J).

There is an equivalence of categories between the category of FSH algebra and the category of Lie coalgebra (LC algebra), sending $\mathcal{O}_{\mathfrak{h}}$ to $\mathfrak{h} = \mathcal{O}_{\mathfrak{h}_+} / \mathcal{O}_{\mathfrak{h}_+}^2$ where $\mathcal{O}_{\mathfrak{h}_+}$ is the maximal ideal of $\mathcal{O}_{\mathfrak{h}}$.

Definition 2.2. A QFSH algebra is a Hopf algebra H , which is a topologically free $\mathbb{K}[[\hbar]]$ -module, such that $H_0 := H/\hbar H$ is isomorphic to a FSH algebra

Let us give an example of a FSH algebra, very important in this paper: let V be a vector space and cTV (defined in the previous section) the cofree coalgebra, equipped with the shuffle product. Let us now complete cTV . The algebra cTV is a graded algebra with V being the set of elements of degree 1. Let us denote $\mathcal{M}_{{}^cTV}$ the set of elements of degree ≥ 1 . Finally, we denote $\widehat{{}^cTV}$ the commutative cofree bialgebra, $\mathcal{M}_{{}^cTV}$ -adic completion of cTV .

Proposition 2.3. $\widehat{{}^cTV}$ is the FSH algebra $\mathcal{O}_{{}^cTV}$ associated with the Lie coalgebra $\underline{{}^cTV} = {}^cT_+V / ({}^cT_+V)^2$, which is the cofree Lie coalgebra over V .

Proof. Let us prove that ${}^cTV = S(\underline{{}^cTV})$: let us recall that TV^* is isomorphic as a Hopf algebra to $U(\text{Lie } V^*)$ which itself is isomorphic as coalgebra to $S(\text{Lie } V^*)$. Taking the dual, we get that cTV is isomorphic as an algebra to $S((\text{Lie } V^*)^*)$. Thus the maximal ideal $\mathcal{M}_{{}^cTV}$ is isomorphic to $S((\text{Lie } V^*)^*)_+$ and so $\widehat{{}^cTV}$ is isomorphic to $\widehat{S}((\text{Lie } V^*)^*)$ which is a FSH algebra. Let us conclude with proving that $(\text{Lie } V^*)^*$ is isomorphic to ${}^cT_+V / ({}^cT_+V)^2$ which will prove that $\widehat{{}^cTV}$ is the FSH algebra $\mathcal{O}_{{}^cTV}$. The cofree Lie coalgebra $(\text{Lie } V^*)^*$ is clearly the coalgebra associated with the FSH algebra $\widehat{S}((\text{Lie } V^*)^*)$ (as $\widehat{S}_+((\text{Lie } V^*)^*) / \widehat{S}_+((\text{Lie } V^*)^*)^2 = (\text{Lie } V^*)^*$). Moreover, if one applies the functor described before the proposition, but this time to $\widehat{{}^cTV}$, we get that $(\text{Lie } V^*)^*$ is isomorphic to $\mathcal{M}_{\widehat{{}^cTV}} / \mathcal{M}_{\widehat{{}^cTV}}^2 = \widehat{{}^cT}_+V / (\widehat{{}^cT}_+V)^2 = {}^cT_+V / ({}^cT_+V)^2$.

Note that the Lie coalgebra $\underline{{}^cTV} = {}^cT_+V / ({}^cT_+V)^2$ is sometimes called the Lyndon algebra. \square

We have covariant functors **QUE** \rightarrow **QFSH**, $U \mapsto U'$ and **QFSH** \rightarrow **QUE**, $\mathcal{O} \mapsto \mathcal{O}^\vee$. These functors are also inverse to each other.

U' is a subalgebra of U defined as follows: for any ordered subset $\Sigma = \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ with $i_1 < \dots < i_k$, define the morphism $j_\Sigma : U^{\otimes k} \rightarrow U^{\otimes n}$ by $j_\Sigma(a_1 \otimes \dots \otimes a_k) := b_1 \otimes \dots \otimes b_n$ with $b_i := 1$ if $i \notin \Sigma$ and $b_{i_m} := a_m$ for $1 \leq m \leq k$; then set $\Delta_\Sigma := j_\Sigma \circ \Delta^{(k)}$, $\Delta_\emptyset := \Delta^{(0)}$, and

$$\delta_\Sigma := \sum_{\Sigma' \subset \Sigma} (-1)^{n-|\Sigma'|} \Delta_{\Sigma'}, \quad \delta_\emptyset := \epsilon.$$

We shall also use the notation $\delta^{(n)} := \delta_{\{1,2,\dots,n\}}$, $\delta^{(0)} := \delta_\emptyset$, and the useful formula

$$\delta^{(n)} = (id_V - \epsilon)^{\otimes n} \circ \Delta^{(n)}.$$

Finally, we define

$$U' := \{ a \in U \mid \delta^{(n)}(a) \in \hbar^n U^{\otimes n} \} \quad (\subseteq U)$$

and endow it with the induced topology.

On the other way, \mathcal{O}^\vee is the \hbar -adic completion of $\sum_{k \geq 0} \hbar^{-k} \mathcal{M}^k \subset \mathcal{O}[1/\hbar]$ (here $\mathcal{M} \subset \mathcal{O}$ is the maximal ideal).

2.2. The functor DQ. In [GH], a generalisation of Etingof-Kazhdan theorem ([EK]) was proven in an appendix by Enriquez and Etingof:

Theorem 2.4. *We have an equivalence of categories*

$$DQ_\Phi : \text{DGQUE} \rightarrow \text{DGLBA}_\hbar$$

from the category of differential graded quantized universal enveloping super-algebras to that of differential graded Lie super-bialgebras such that if $U \in \text{Ob}(\text{DGQUE})$ and $\mathfrak{a} = DQ(U)$, then $U/\hbar U = \mathbb{U}(\mathfrak{a}/\hbar \mathfrak{a})$, where \mathbb{U} is the universal algebra functor, taking a differential graded Lie super-algebra to a differential graded super-Hopf algebra.

Here Φ is a Drinfeld associator. We will use any of these functor and denote it DQ.

3. BIALGEBRA STRUCTURE ON ${}^c TT_+ U$

Here, we will define a bialgebra structure on ${}^c TT_+ U$. Following [GV, GJ, Ka], one can construct a bialgebra structure on the space of Hochschild cochains of an algebra using the brace operations. In our case, we will firstly generalise the definition of brace operations for a general Hopf algebra. More precisely, let $(H, \Delta_\hbar, \times)$ be a Hopf algebra (in our case H will be the Etingof-Kazhdan quantization $U_\hbar(\mathfrak{g})$ of the Lie bialgebra \mathfrak{g}). We will define a brace structure on the cofree tensor coalgebra ${}^c TT_+ H$ of the free tensor algebra $T(H[-1])$ without unit. To distinguish the two tensor products, we denote \otimes the tensor product on $T_+ H$ and \boxtimes the tensor product on ${}^c TT_+ H$.

Definition 3.1. We define brace operations on ${}^c TT_+ H$ by extending the following maps given on the cogenerators of the cofree coalgebra ${}^c TT_+ H$:

- (1) $B^0 = 0$,
- (2) $B^1 = b_{cH}$ (the coHochschild coboundary on $T_+ H$),
- (3) $B^2 : \alpha \boxtimes \beta \mapsto \alpha \otimes \beta$,
- (4) $B^n = 0$ for $n > 2$,
- (5) $B^{0,1} = B^{1,0} = \text{id}$,
- (6) $B^{0,n} = B^{n,0} = 0$ for $n \geq 1$,
- (7) $B^{1,n} : (\alpha, \beta_1 \boxtimes \dots \boxtimes \beta_n) \mapsto$

$$\sum_{\substack{0 \leq i_1, \dots, i_m + k_m \leq n \\ i_l + k_l \leq i_{l+1}}} (-1)^\varepsilon \alpha^{1, \dots, i_1+1, \dots, i_1+k_1, \dots, i_m+1, \dots, i_m+k_m, \dots, n} \times \\ 1^{\otimes i_1} \otimes \beta_1 \otimes 1^{\otimes i_2 - (i_1 + k_1)} \otimes \beta_2 \otimes \dots \otimes \beta_n \otimes 1^{\otimes n - (i_m + k_m)},$$

where $k_s = |\beta_s|$, $n = |\alpha| + \sum_s k_s - m$ and $\varepsilon = \sum_s (k_s - 1) i_s$,

- (8) $B^{k,l} = 0$ for $k > 1$.

Operations (2), (3) and (4) define a differential d and (5), (6), (7) and (8) define a product \star deforming the shuffle product.

Note that, when $H = U(\mathfrak{g})$, the enveloping algebra of a Lie algebra \mathfrak{g} , $T(H[-1])$ can be seen as the space of invariant multidifferential operators over the Lie group corresponding to \mathfrak{g} and in that case, our definition coincides with those of [GV, GJ, Ka].

Still following [GV, GJ, Ka], we have:

Theorem 3.2. *The brace operations of Definition 3.1 defines a differential bialgebra structure on the cofree tensor coalgebra ${}^cTT_+H$, with product \star extending $\sum B^{p_1, p_2}$ and differential d extending $\sum B^p$.*

Proof. To prove the associativity of \star , one has to check the following equation for $\alpha, \beta_1, \dots, \beta_l, \gamma_1, \dots, \gamma_l \in TH$:

$$\begin{aligned} \sum_{0 \leq i_1 \leq \dots \leq i_l \leq m} (-1)^\varepsilon \{\alpha\} \{\gamma_1, \dots, \gamma_{i_1}, \{\beta_1\} \{\gamma_{i_1+1}, \dots\}, \dots, \gamma_{i_l}, \{\beta_l\} \{\gamma_{i_l+1}, \dots\}, \dots, \gamma_m\} \\ = \{\{\alpha\} \{\beta_1, \dots, \beta_l\}\} \{\gamma_1, \dots, \gamma_m\}, \end{aligned}$$

where $\varepsilon = \sum_{p=1}^l (|\beta_p| - 1) \sum_{q=1}^{i_p} (|\gamma_q| - 1)$ and $\{\alpha\} \{\beta_1, \dots, \beta_l\}$ is $B^{1,l}(\alpha, \beta_1 \boxtimes \dots \boxtimes \beta_l)$. This equation is a consequence of the associativity, coassociativity and compatibility of \times and Δ :

$$(\Delta_{\hbar}^{(k)} \otimes \text{id}^{\otimes l} \otimes \Delta_{\hbar}^{(m)})(\alpha) = (\Delta_{\hbar}^{(k)} \otimes \text{id}^{\otimes l+m}) \circ (\text{id}^{\otimes l+1} \otimes \Delta_{\hbar}^{(m)})(\alpha)$$

and

$$\begin{aligned} (\Delta_{\hbar}^{(k)} \otimes \text{id}^{\otimes n+p-2})((\Delta_{\hbar}^{(n)} \otimes \text{id}^{\otimes p-1})(\alpha) \times (\beta \otimes 1^{\otimes p-1})) &= (\Delta_{\hbar}^{(n+k-1)} \otimes \text{id}^{\otimes p-1})(\alpha) \\ &\times (\Delta_{\hbar}^{(k)} \otimes \text{id}^{\otimes n+p-2})(\beta \otimes 1^{\otimes p-1}), \end{aligned}$$

For $\alpha, \beta \in T_+H$ of degree p and n .

One can then notice that the map d is the commutator, with respect to the product \star , of the element $1 \otimes 1 \in T_+H \subset {}^cTT_+H$. Thus d is compatible with the multiplication. Finally, d is a differential as $[1 \otimes 1, 1 \otimes 1]_{\star} = 0$. \square

Let us now complete ${}^cTT_+H$ as in section 2 with $V = T_+H$. We get a commutative cofree bialgebra $\widehat{{}^cTT_+H}$, the $\mathcal{M}_{{}^cTT_+H}$ -adic completion of ${}^cTT_+H$ (where $\mathcal{M}_{{}^cTT_+H}$ is the maximal ideal of ${}^cTT_+H$). Let us consider the free $\mathbb{K}[[\nu]]$ -module $\widehat{{}^cTT_+H}[[\nu]]$. One can now replace the operations $B^{p,q}$ of Definition 3.1 with $\mathbb{K}[[\nu]]$ -linear operations $\nu^{p+q-1}B^{p,q}$. Those operations are well defined on the completion $\widehat{{}^cTT_+H}[[\nu]]$ as this space is complete for the grading induced by the degree in ${}^cTT_+H = {}^cTV$ plus the \hbar -adic valuation and because the operations we just defined are homogeneous for this grading. Thus we get a morphism of differential bialgebra

$$\begin{aligned} I_{\nu} : ({}^cTT_+H, \star, \Delta, d) &\rightarrow ({}^cTT_+H[[\nu]][\nu^{-1}], \star_{\nu}, \Delta_{\nu}, d_{\nu}) \\ x &\mapsto \nu^{-|x|}x, \end{aligned} \tag{1}$$

where $|x|$ is the degree in cT . The morphism I_{ν} extends to $I_{\nu} : ({}^cTT_+H[[\nu]], \star, \Delta, d) \rightarrow ({}^cTT_+H[[\nu]][\nu^{-1}], \star_{\nu}, \Delta_{\nu}, d_{\nu})$ which restricts to

$$I'_{\nu} : (\widehat{\oplus} \nu^n {}^cT^n T_+H[[\nu]], \star, \Delta, d) \rightarrow (\widehat{{}^cTT_+H}[[\nu]][\nu^{-1}], \star_{\nu}, \Delta_{\nu}, d_{\nu}) \tag{2}$$

We have now:

Proposition 3.3. *The algebra $(\widehat{{}^cTT_+H}[[\nu]], \star_{\nu}, \Delta_{\nu}, d_{\nu})$ is a QFSHA.*

Proof. The algebra $\widehat{{}^cTT_+H}[[\nu]]$ is clearly a deformation of $\widehat{{}^cTT_+H}$ which is a FSH algebra, by Theorem 2.3, isomorphic to $\mathcal{O}_{{}^cTT_+H}$, the space of functions over the formal group associated with $(\underline{{}^cTT_+H})_+ = {}^cT_+T_+H / ({}^cT_+T_+H)_+^2$. \square

One easily checks that

Proposition 3.4. *The underlying differential Lie bialgebra structure on $\underline{{}^cTT_+H}$ is given by the Gerstenhaber bracket*

$$[\alpha, \beta]_G = B^{1,1}(\alpha, \beta) - (-1)^{(|\alpha|-1)(|\beta|-1)} B^{1,1}(\beta, \alpha)$$

and coHochschild differential

$$b_{\text{cH}}(\alpha) = [1 \otimes 1, \alpha]_G,$$

for $\alpha, \beta \in TH$ and then naturally extended on ${}^c\underline{TT}_+H$ using the cofree Lie cobracket (see section 1).

Remark 3.5. Let now H be the QUE algebra $U = U_{\hbar}(\mathfrak{g})$. We have proven that T_+U can be equipped with a G_∞ -structure (see section 1). Since the cofree Lie coalgebras are rigid, the differential Lie bialgebra corresponding to ${}^c\widehat{TT}_+U[[\nu]]$ through Etingof-Kazhdan dequantization functor DQ is isomorphic to ${}^c\underline{TT}_+U[[\nu]]$ as a $\mathbb{K}[[\nu]]$ -Lie coalgebra, and is therefore free.

4. BIALGEBRA QUASI-ISOMORPHISMS

4.1. **A bialgebra quasi-isomorphism** $\varphi_{\text{alg}} : U \rightarrow (\widehat{{}^cTT}_+U)^\vee$. We will first prove

Proposition 4.1. *Let U be a QUE algebra. One can define a bialgebra quasi-isomorphism $\varphi_{\text{alg}} : U \rightarrow \widehat{{}^cTT}_+U$ from the bialgebra $(U, \Delta_{\hbar}, \times)$ to the bialgebra $({}^cTT_+U, \Delta, \star)$ whose structure was described in the previous section.*

Proof. Since ${}^cTT_+U$ is a cofree coalgebra, a coalgebra map φ_{alg} is uniquely determined by its restriction $U \rightarrow T_+U$ to cogenerators of ${}^cTT_+U$. We define φ_{alg} to be as follows: its first component is $\text{id} - \eta\varepsilon : U \rightarrow U$ ($\eta : \mathbb{K} \rightarrow U$ is the counit map); the components $U \rightarrow (T_+U)_k$, $k \geq 2$ are all zero.

In [Ta2], this morphism was described as follows. Let

$$\begin{aligned} \delta^{(n)} : U &\rightarrow U^{\otimes n}, \\ x &\mapsto (\text{id} - \eta\varepsilon)^{\otimes n} \circ \Delta^{(n)}(x). \end{aligned}$$

Let $\bar{\delta} : U \rightarrow TU$ be the direct sum of all $\delta^{(n)}$, $n \geq 0$. We compose $\bar{\delta}$ with the isomorphism of vector spaces $TU \rightarrow {}^cTU$, then with cT of the canonical inclusion $U \subset T_+U$. The resulting map $U \xrightarrow{\bar{\delta}} TU \simeq {}^cTU \hookrightarrow {}^cTT_+U$ is equal to φ_{alg} .

As both $U \rightarrow {}^cTU$ and ${}^cTU \rightarrow {}^cTT_+U$ are coalgebra morphisms, to check that φ_{alg} is an algebra morphism, one only has to check that the following diagram commutes after canonical projection $Pr : {}^cTT_+U \rightarrow T_+U[1]$:

$$\begin{array}{ccc} U \otimes U & \xrightarrow{\phi_{\text{alg}} \otimes \phi_{\text{alg}}} & {}^cTT_+U \otimes {}^cTT_+U \\ \downarrow \times & & \downarrow \star \\ U & \xrightarrow{\phi_{\text{alg}}} & {}^cTT_+U \xrightarrow{Pr} T_+U[1], \end{array}$$

that is to say we have $Pr \circ (\star \circ \phi_{\text{alg}} \otimes \phi_{\text{alg}}) = Pr \circ (\phi_{\text{alg}} \circ \times)$. Let us check this equality for $\mu \otimes \eta \in U \otimes U$. It is obvious if one of the elements μ or η is in \mathbb{K} , so let us consider the image of $\mu \otimes \eta \in \mathcal{M} \otimes \mathcal{M}$ through the two paths of this diagram (here $\mathcal{M} = \text{Ker}(U \xrightarrow{\varepsilon} \mathbb{K})$). Using the notations of section 3, we have $\phi_{\text{alg}}(\alpha) = \sum_n \alpha^{[1]} \boxtimes \dots \boxtimes \alpha^{[n]}$, where we use a variant of Sweedler's notation $\delta^{(n)}(\alpha) = \alpha^{[1]} \otimes \dots \otimes \alpha^{[n]}$, and $\phi_{\text{alg}}(\beta) = \sum_n \beta^{[1]} \boxtimes \dots \boxtimes \beta^{[n]}$. Here $\alpha^{[1]}, \beta^{[1]} \in U \subset T_+U$. Note that $B^{1, k-j+1}(\alpha^{[i]}, \beta^{[j]} \boxtimes \beta^{[j+1]} \boxtimes \dots \boxtimes \beta^{[k]})$ is zero if $k - j > 1$ and is $\alpha^{[i]} \times \beta^{[j]}$ otherwise and that $B^{p>1, q} = 0$. So we get

$$Pr(\phi_{\text{alg}}(\alpha) \star \phi_{\text{alg}}(\beta)) = Pr(\alpha \times \phi_{\text{alg}}(\beta)) = \alpha \times \beta,$$

which is the commutation property.

Let us show now that ϕ_{alg} is a quasi-isomorphism of complexes. Recall that the differential d on ${}^cTT_+U$ is defined as extension of $B^1 + B^2$ (cf. definition 3.1). Let us prove the following lemma

Lemma 4.2. *We have $H.({}^cTT_+U, d) \simeq U$ in degree 1 and is 0 for degree ≥ 2 .*

Proof. ${}^cTT_+U$ has a bigrading by (\deg_{cT}, \deg_{T_+}) , where $\deg_{cT}(\boxtimes_{i=1}^n x_i) = n$, for any $x_i \in T_+U$, and $\deg_{T_+}(\boxtimes_{i=1}^n x_i) = \sum_{i=1}^n \deg_{T_+}(x_i)$, and $\deg_{T_+}(x) = k$ if $x = y_1 \otimes \cdots \otimes y_k$, where $k \geq 1$, and $y_1, \dots, y_k \in U$. The differential d is the sum $d = d^1 + d^2$ where d^1 and d^2 correspond respectively to the maps B^1 and B^2 and have degree $(-1, 0)$ and $(0, 1)$. Thus ${}^cTT_+U$ has a structure of bicomplexe. Explicitly, in the nongraded case,

$$d^2(\boxtimes_{i=1}^n x_i) = \sum_{i=1}^n (-1)^{i+1} x_1 \boxtimes \cdots \boxtimes x_i x_{i+1} \boxtimes \cdots \boxtimes x_n,$$

where $x_i \in T_+U$, and $x_i x_{i+1}$ is the concatenation product in T_+U . The differential $d^1: {}^cTT_+U \rightarrow {}^cTT_+U$ is the coalgebra differential whose composition with the projection ${}^cTT_+U \rightarrow T_+U$ is $\boxtimes_{i=1}^n x_i \rightarrow 0$ if $n \neq 1$, and whose restriction to ${}^cT^1T_+U = T_+U \rightarrow T_+U$ is the co-Hochschild differential of the coalgebra U . Let us compute the homology with respect to d^2 : (${}^cTT_+U \simeq \prod_{n=1}^{\infty} {}^cT^nT_+U, d^2$) is the Hochschild complex associated to T_+U and T_+U is a free associative algebra then

$$H.({}^cTT_+U, d^2) \simeq H.({}^cT^nT_+U, d^2) \simeq U,$$

concentrated in degree 1. So $H.({}^cTT_+U, d^1 + d^2) \simeq \text{Ker}(U, d^1) = U$ as $d^1(H.(T_+U)) \cong 0$. \square

Finally, we check that ϕ_{alg} is a morphism of complexes. As before, it is enough to check on the cogenerators that $Pr(d(\phi_{\text{alg}}(\alpha))) = 0$ for $\alpha \in U$. Using again the variant of the Sweedler's notation, we have

$$\phi_{\text{alg}}(\alpha) = \varepsilon(\alpha)1_{{}^cTT_+U} + (\alpha - \varepsilon(\alpha)1_U) + (\alpha^{[1]} \boxtimes \alpha^{[2]}) + \text{terms of } {}^cT\text{-degree} \geq 2.$$

Now

$$d^2(\phi_{\text{alg}}(\alpha)) = -(\alpha^{[1]} \otimes \alpha^{[2]}) + \text{terms of } {}^cT\text{-degree} \geq 2$$

$$d^1(\phi_{\text{alg}}(\alpha)) = ((\alpha - \varepsilon(\alpha)1_U)^{[1]} \otimes (\alpha - \varepsilon(\alpha)1_U)^{[2]}) + \text{terms of } {}^cT\text{-degree} \geq 2.$$

So $Pr((d^1 + d^2)(\phi_{\text{alg}}(\alpha))) = 0$. \square

Let $U' \subset U$ (see section 2 for the definition of U'). Now we want to check that we have a bialgebra quasi-isomorphism $\varphi_{\text{alg}} : (U', \times) \rightarrow (\widehat{{}^cTT_+U}, \star_{\hbar})$ of QFSH algebra, where $(\widehat{{}^cTT_+U}, \star_{\hbar})$ is $(\widehat{{}^cTT_+U}[[\nu]], \star_{\nu}) / (\nu = \hbar)$ ($\widehat{{}^cTT_+U}[[\nu]]$ is the free $\mathbb{K}[[\hbar]]$ -module defined in the previous section: we changed the operations $B^{p,q}$ into $\nu^{p+q-1}B^{p,q}$). Let us first check that $\phi_{\text{alg}}(U') \subset \widehat{\nu}^n {}^cT^n T_+ H[[\nu]]$: recall that

$$\phi_{\text{alg}}(\alpha) = \varepsilon(\alpha)1_{{}^cTT_+U} + (\alpha - \varepsilon(\alpha)1_U) + (\alpha^{[1]} \boxtimes \alpha^{[2]}) + \cdots + (\alpha^{[1]} \boxtimes \cdots \boxtimes \alpha^{[n]}) + \cdots.$$

The results follows from the fact that for $\alpha \in U'$, $(\alpha^{[1]} \boxtimes \cdots \boxtimes \alpha^{[n]}) \in O(\hbar^n)$. If one now composes the map ϕ_{alg} with the map I'_{\hbar} (see (2) for $\hbar = \nu$), one gets the bialgebra quasi-isomorphism of QFSH algebra $\varphi_{\text{alg}} : (U', \times) \rightarrow (\widehat{{}^cTT_+U}, \star_{\hbar})$. This can be summarized in the

following diagram:

$$\begin{array}{ccc}
 ({}^cTT_+U, \star) & \supset & (\widehat{\oplus} \nu^n {}^cT^m T_+H, \star) \xrightarrow{I'_h} (\widehat{c}TT_+U, \star_h) \\
 \uparrow \phi_{\text{alg}} & & \uparrow \phi_{\text{alg}} \quad \nearrow \varphi_{\text{alg}} \\
 (U, \times) & \supset & (U', \times).
 \end{array}$$

Finally, applying to φ_{alg} the derived Drinfeld functor $(-)^{\vee}$, we get a bialgebra quasi-isomorphism $\varphi_{\text{alg}}: U \rightarrow (\widehat{c}TT_+U)^{\vee}$.

Remark 4.3. At this point one may want to compute explicitly the classical limit φ_{alg}^0 of this morphism. Let U_0 be the universal enveloping algebra of \mathfrak{g} . It is not hard to check that $\varphi_{\text{alg}}^0: \mathfrak{g} \rightarrow {}^cTT_+U_0$ is defined on the cogenerators of the cofree coLie coalgebra ${}^cTT_+U_0$ by $\varphi_{\text{alg}}^0: \mathfrak{g} \rightarrow T_+U_0$, $x \mapsto x \in T_+U_0$. It is the composition of $\bar{\delta}^0: \mathfrak{g} \rightarrow {}^cT\mathfrak{g}$, $x \mapsto x + \sum_{k \geq 2} \bar{\delta}_k^0(x)$ (where $\bar{\delta}_k^0$ is built using iterates of δ) with $\underline{c}T$ of the inclusion: $\mathfrak{g} \rightarrow T_+U_0$. We will use a general version of this map in the next subsection. Although this map is a Lie bialgebra morphism (which can be constructed without using any quantization) it is clearly not a quasi-isomorphism $(\mathfrak{g}, 0) \rightarrow (\underline{c}TT_+U_0, b)$ (where b is the coHochschild coboundary). So one cannot use such a map for the purpose of this paper. Using the “naive” dequantization, we encounter the same problem as one would have when taking the “naive” classical limit of the brace structure to get a G_{∞} structure on T_+U_0 or T_{poly} in Tamarkin ([Ta]). Doing so, one misses the operation $B^2 = \cup$ which is a key tool in this paper or in [Ta] to get quasi-isomorphisms.

4.2. A Lie bialgebra quasi-isomorphism $\varphi_{\text{Lie}}: \underline{c}TA \rightarrow \underline{c}TC(\underline{c}TA)$. Let A be a vector space. Suppose now that the cofree Lie coalgebra $\underline{c}TA$ has a structure $(\underline{c}TA, \delta, [-, -], d)$ of a differential Lie bialgebra. Using the functor L-G $-G_{\infty}$ (see section 1), one gets a differential Gerstenhaber algebra $(C(\underline{c}TA), [-, -], \wedge, d + \delta)$. One can extend the structure maps on the cofree Lie coalgebra $\underline{c}TC(\underline{c}TA)$ and one gets a differential cofree Lie bialgebra $(\underline{c}TC(\underline{c}TA), \delta', [-, -], d + \delta + \wedge)$ (we will set $d^1 = d + \delta$ and $d^2 = \wedge$). We can now prove the existence of a differential Lie bialgebra quasi-isomorphism $\varphi_{\text{Lie}}: \underline{c}TA \rightarrow \underline{c}TC(\underline{c}TA)$. As $\underline{c}TC(\underline{c}TA)$ is the cofree Lie coalgebra, let us extend the inclusion $\text{inc}: \underline{c}TA \rightarrow C(\underline{c}TA)$ to a morphism of Lie coalgebra $\varphi_{\text{Lie}}: \underline{c}TA \rightarrow \underline{c}TC(\underline{c}TA)$. It is the composition $\varphi_{\text{Lie}} = {}^cTi \circ \bar{\delta}$ of a map

$$\begin{array}{ccc}
 \bar{\delta}: \underline{c}TA & \rightarrow & \underline{c}T(\underline{c}TA), \\
 x & \mapsto & x + \sum_{k \geq 2} \bar{\delta}_k(x),
 \end{array}$$

where $\bar{\delta}_k$ is built using iterates of δ , with ${}^cTi: \underline{c}T(\underline{c}TA) \rightarrow \underline{c}TC(\underline{c}TA)$ which is $\underline{c}T$ of the inclusion $i: \underline{c}TA[-1] \rightarrow C(\underline{c}TA)$. We can reproduce the proof of the previous subsection (all structures are much more simpler) and easily check that $\varphi_{\text{Lie}}: \underline{c}TA \rightarrow \underline{c}TC(\underline{c}TA)$ is a differential Lie bialgebra quasi-isomorphism. It is obviously a Lie bialgebra morphism. Moreover, for $\alpha \in \underline{c}TA$, we get after projection on the cogenerators,

$$(d^1 + d^2)(\varphi_{\text{alg}}(\alpha)) = d^1(\alpha) + d^2(Ti\delta(\alpha)) = d(\alpha) - \delta(\alpha) + \delta(\alpha) = d(\alpha).$$

The fact that φ_{Lie} is a quasi-isomorphism can be proved as in the previous section.

5. INVERSION OF FORMALITY MORPHISMS

Let us recall Theorem 4.4 of Kontsevich ([Ko]):

Theorem 5.1. *Let \mathfrak{g}_1 and \mathfrak{g}_2 be two L_∞ -algebras and \mathcal{F} be a L_∞ -morphism from \mathfrak{g}_1 to \mathfrak{g}_2 . Assume that \mathcal{F} is a quasi-isomorphism. Then there exists an L_∞ -morphism from \mathfrak{g}_2 to \mathfrak{g}_1 inducing the inverse isomorphism between associated cohomology of complexes.*

Remark 5.2. We know from private communications the existence of a similar G_∞ -version of this theorem. This result would imply the existence of a corresponding G_∞ -morphism in Theorem 0.2.

6. PROOF OF THE MAIN THEOREMS

6.1. Proof of Theorem 0.2. Let $(\mathfrak{g}, \delta_{\hbar})$ be a Lie bialgebra. We write $\delta_{\hbar} = \hbar\delta_1 + \hbar^2\delta_2 + \dots$. Let $(U_{\hbar}(\mathfrak{g}), \Delta_{\hbar})$ be the Etingof-Kazhdan canonical quantization of $(\mathfrak{g}, \delta_{\hbar})$. We denote $U = U_{\hbar}(\mathfrak{g})$ for short. In section 3, we proved the existence of a bialgebra structure on ${}^cTT_+U$ and thanks to section 4, we have a bialgebra quasi-isomorphism $\varphi_{\text{alg}}: U \rightarrow ({}^cTT_+U)^\vee$. Thanks to Etingof-Kazhdan dequantization functor (see section 2), and the fact that $({}^cTT_+U)^\vee$ is a QUE algebra quantizing ${}^c\underline{TT}_+U$ (see section 3), we get a Lie bialgebra quasi-isomorphism $\varphi'_{\text{Lie}}: \mathfrak{g} \rightarrow {}^c\underline{TT}_+U$, a differential Lie bialgebra. Using the exact functor $\text{L-G} - G_\infty$ (see section 1), we get a quasi-isomorphism of differential Gerstenhaber algebras $\varphi_{\text{Ger}_\infty}: C(\mathfrak{g})[1] \rightarrow C({}^c\underline{TT}_+U)[1]$.

According to section 4, we also have a differential Lie bialgebra quasi-isomorphism $\varphi_{\text{Lie}}: {}^c\underline{TT}_+U \rightarrow {}^c\underline{TC}({}^c\underline{TT}_+U)$. This quasi-isomorphism is sent to a G_∞ -quasi-isomorphism $\varphi'_{G_\infty}: T_+U[1] \rightarrow C({}^c\underline{TT}_+U)[1]$ using the functor L-G_∞ defined in section 1.

Finally, in section 5, we recalled the existence of an inverse map for any L_∞ -quasi-isomorphism between L_∞ -algebras so the L_∞ -restriction of the quasi-isomorphism φ'_{G_∞} is invertible. Then one can define $\varphi: C(\mathfrak{g}) \rightarrow T_+U$ as the composition of the L_∞ -restriction of $\varphi_{\text{Ger}_\infty}$ with the inverse of the L_∞ -restriction of φ'_{G_∞} .

One has to show now that φ^1 maps $v \in C(\mathfrak{g})$ to $\text{Alt}(v) \in T_+U \pmod{\hbar}$. Let us replace \mathfrak{g} with $\hbar\mathfrak{g}$ in the previous construction. Let $\hbar v$ be an element of $\hbar\mathfrak{g}$. Let us still call $\hbar v$ its image in the quantization U . By definition of φ_{alg} , $\varphi_{\text{alg}}(\hbar v) = \hbar v \pmod{\hbar^2}$ (once again, we still use the same notation for its image in $({}^cTT_+U)^\vee$). So $\varphi'_{\text{Lie}}(\hbar v) = \hbar v \pmod{\hbar^2}$ ($\hbar v$ on the right hand side is the image of $\hbar v$ in ${}^c\underline{TT}_+U$). Note that here appeared highly non trivial terms in $O(\hbar^2)$. So $\varphi_{\text{Ger}_\infty}(\hbar v_1 \wedge \dots \wedge \hbar v_n) = \hbar^n \text{Alt}(v_1 \otimes \dots \otimes v_n) \pmod{\hbar^{n+1}}$.

Moreover, it is clear by definition, that $\varphi_{\text{Lie}}: {}^c\underline{TT}_+U \rightarrow {}^c\underline{TC}({}^c\underline{TT}_+U)$ is the identity map $\text{id} \pmod{\hbar}$ when restricted to T_+U thus so is the corresponding map of complexes $\varphi'^1_{G_\infty}: T_+U \rightarrow C({}^c\underline{TT}_+U)$. Then $\varphi^1: C^n(\hbar\mathfrak{g}) \rightarrow T_+U$ is $\text{Alt} \pmod{\hbar^{n+1}}$. \square

6.2. Proof of Theorem 0.4. Suppose that $(\mathfrak{g}, \delta, [-, -])$ is a (finite-dimensional) Lie bialgebra over \mathbb{K} , and s is a classical twist. Recall that this means that $s \in \Lambda^2\mathfrak{g}$ satisfies

$$\text{CYB}(s) + \text{Alt}(\delta \otimes \text{id})(s) = 0,$$

i.e. $\hbar s \in \hbar\Lambda^2\mathfrak{g}[[\hbar]]$ is a Maurer-Cartan element in the differential graded Lie algebra $(\hbar\Lambda\mathfrak{g}[[\hbar]], \hbar\delta, [-, -]): [s, s] + \delta_{\hbar}(s) = 0$. Let us write the L_∞ -morphism φ of Theorem 0.2: $\varphi = \sum_{k \geq 1} \varphi^{1, \dots, 1}$, where $\varphi^{1, \dots, 1}: \Lambda^n\mathfrak{g} \rightarrow T_+U$. Let us define $F = 1 \otimes 1 + \sum_{k \geq 1} \frac{1}{n!}(\Lambda^n s)$. By definition of L_∞ -morphism, we get

$$(\Delta_{\hbar} \otimes \text{id})(F)F^{1,2} - (\text{id} \otimes \Delta_{\hbar})(F)F^{2,3} = 0.$$

So F is a Maurer-Cartan element and as Maurer-Cartan equation for F is equivalent to the twist equation, this proves Theorem 0.4. Using the formality morphism, one similarly prove the one to one correspondence between equivalence classes of quantum twists and of classes of \hbar -dependent classical twists. \square

6.3. Proof of Theorem 0.6. Suppose now that $(\mathfrak{g}, \delta, [-, -], r, Z)$ is a (finite-dimensional) coboundary Lie bialgebra over \mathbb{K} . This means that the Lie cobracket δ_{\hbar} is the coboundary of $r \in \Lambda^2(\mathfrak{g})$: $\delta(x) = [x \otimes 1 + 1 \otimes x, r]$ for any $x \in \mathfrak{g}$. Let $r' = -\hbar r$. This means that r' is a mauer-Cartan element in $\hbar\Lambda\mathfrak{g}[[\hbar]]$: $[r', r'] + \delta_{\hbar}(r') = 0$. Using again the L_{∞} -morphism φ of Theorem 0.2, we define $R' = 1 \otimes 1 + \sum_{k \geq 1} \frac{1}{k!} (\Lambda^k r')$ which satisfies

$$(\Delta_{\hbar} \otimes \text{id})(R')R'^{1,2} - (\text{id} \otimes \Delta_{\hbar})(R')R'^{2,3} = 0.$$

Let us now define $R = R'^{-1}$. We have a solution of Theorem 0.6. \square

Note that $-r$ can be seen as a classical twist twisting δ_{\hbar} into $-\delta_{\hbar}$.

7. CONCLUDING REMARKS

Let us consider the L_{∞} -quasi-isomorphism of Theorem 0.2. It induces a quasi-isomorphism of complexes:

Theorem 7.1. *Let $(\mathfrak{g}, \delta_{\hbar})$ be a Lie bialgebra. Let $(U_{\hbar}(\mathfrak{g}), \Delta_{\hbar})$ be the associated Etingof-Kazhdan quantization. The restriction φ of the L_{∞} -quasi-isomorphism of Theorem 0.2 defines a quasi-isomorphism between the coHochschild complex $(T(U_{\hbar}(\mathfrak{g})), b_{\text{cH}})$ (with differential associated to the coproduct Δ_{\hbar}) and $(C(\mathfrak{g}), \delta_{\hbar})$ the exterior product over the Lie algebra \mathfrak{g} with differential given by the Lie cobracket.*

The theorem generalises well known theorem for $\delta_{\hbar} = 0$ (see [Dr3], Proposition 2.2).

Let us discuss some possible generalisations:

Remark 7.2. The main tool used in that paper is Etingof-Kazhdan quantization/dequantization theorem. Suppose now that one can prove an analogue of this theorem for Lie bialgebroids and Hopf algebroid (which is still a conjecture). Using the same framework, one would get a L_{∞} -quasi-isomorphism (or even G_{∞} -quasi-isomorphism, cf. Remark 5.2) between the exterior power of a any Lie bialgebroid and the tensor product of the associated quantized Hopf algebroid. In the case the Lie algebroid is the algebroid of tensor fields over a manifold M , this would give directly a global formality theorem between tensor fields and multidifferential operators.

Remark 7.3. To answer completely to ‘‘Drinfeld last unsolved problem’’ ([Dr2]), one should also check the last two conditions of Definition 0.5: $R^{-1} = R^{2,1}$ and $R\Delta_{\hbar}(-)R^{-1} = \Delta_{\hbar}^{2,1}(-)$. Those properties may not be satisfied for every quantization $U_{\hbar}(\mathfrak{g})$ of $(\mathfrak{g}, \delta_{\hbar})$. The last property should be a consequence of the following conjecture:

Conjecture 7.4. Let $(\mathfrak{g}_1, \delta_{1,\hbar})$ be a Lie bialgebra, s be a classical twist, twisting \mathfrak{g}_1 into $(\mathfrak{g}_2, \delta_{2,\hbar})$, and F a quantization of s ; then F twists $(U_{\hbar}(\mathfrak{g}_1), \Delta_1)$ into $(U_{\hbar}(\mathfrak{g}_2), \Delta_2)$ where $U_{\hbar}(\mathfrak{g}_i)$ are the quantization of \mathfrak{g}_i through Etingof-Kazhdan functor.

Indeed, if $(\mathfrak{g}, \delta, r, Z)$ is a coboundary Lie bialgebra, twisting \mathfrak{g} with r , one gets $(\mathfrak{g}, -\delta)$. Moreover, using Etingof-Kazhdan quantization functor, one gets that if $(U_{\hbar}(\mathfrak{g}), \Delta_{\hbar})$ is the quantization of (\mathfrak{g}, δ) then $(U_{\hbar}(\mathfrak{g}), \Delta_{\hbar}^{\text{op}})$ is the quantization of $(\mathfrak{g}, -\delta)$ (the proof of this result will be part of a forthcoming paper with B. Enriquez). So if the conjecture is true, the quantization F of s would twist Δ_{\hbar} into $\Delta_{\hbar}^{\text{op}}$ which is the last condition needed to answer Drinfeld’s problem.

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