FORMALITY THEOREMS FOR HOCHSCHILD CHAINS IN THE LIE ALGEBROID SETTING

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Abstract. In this paper we prove Lie algebroid versions of Tsygan's formality conjecture for Hochschild chains both in the smooth and holomorphic settings. In the holomorphic setting our result implies a version of Tsygan's formality conjecture for Hochschild chains of the structure sheaf of any complex manifold and in the smooth setting this result allows us to describe quantum traces for an arbitrary Poisson Lie algebroid. The proofs are based on the use of Kontsevich's quasi-isomorphism for Hochschild cochains of $\mathbb{R}[[y^1, \dots, y^d]]$, Shoikhet's quasi-isomorphism for Hochschild chains of $\mathbb{R}[[y^1, \dots, y^d]]$, and Fedosov's resolutions of the natural analogues of Hochschild (co)chain complexes associated with a Lie algebroid.

Introduction

Lie algebroids and Lie groupoids provide a natural framework for developing analysis on differentiable foliations and manifolds with corners [19], [20], [21], [23], [33]. This motivates our interest to the natural analogues of Hochschild and cyclic (co)homological complexes in the setting of Lie algebroids and to the corresponding analogues of the Kontsevich-Tsygan formality conjectures. Thus the formality theorem for the differential graded Lie algebra (DGLA) of Hochschild cochains in the Lie algebroid setting [1] allows us to quantize an arbitrary Poisson Lie algebroid[1]. The formality of the DGLA module of Hochschild chains in the Lie algebroid setting would give a description of the quantum traces for Poisson Lie algebroids, and the formality of the cyclic complex in the setting of Lie algebroids would imply the algebraic index theorem [9], [22], [27] for the deformations associated with an arbitrary Poisson Lie algebroid.

An appropriate analogue of the Hochschild cochain (resp. chain) complex associated with a Lie algebroid E is the complex of E-polydifferential operators (resp. Hochschild E-chains) (see definitions 1.4 and 1.9 in the next section). It turns out that the complex of E-polydifferential operators is naturally a DGLA and the complex of E-chains is naturally a DG module over this DGLA. Due to the recent result [1] of the first author for any Lie algebroid E over a smooth manifold the DGLA of E-polydifferential operators is formal.

In this paper we use Kontsevich's [16] and Shoikhet's [25] formality theorems for $\mathbb{R}^d_{formal}$ and the 'Fedosov-like' [8] globalization technique [2, 6, 7, 22] to prove that for any Lie algebroid E over a smooth manifold (resp. holomorphic Lie algebroid over a complex manifold) the DGLA module of E-chains (resp. the sheaf of DGLA modules of E-chains) is formal. In the smooth setting this result allows

[1]According to the terminology of P. Xu [32] we have to call this object a triangular Lie bialgebroid. However, since we do not mention, the bialgebroid structure, we refer to this object as a Poisson Lie algebroid.

us to describe quantum traces for an arbitrary Poisson Lie algebroid and in the holomorphic setting our result implies a version of Tsygan's formality conjecture for Hochschild chains of the structure sheaf of any complex manifold.

Eliminating the sheaf of Hochschild E-chains in the holomorphic setting we get that for any holomorphic Lie algebroid E the sheaf of E-polydifferential operators is formal as a sheaf of DGLAs. In particular, this result implies Kontsevich's formality theorem for complex manifolds, the proof of which was formulated only for algebraic varieties (see paper [34] of A. Yekutieli).

The paper is organized as follows. In the first section we recall some basic facts about Lie algebroids and define algebraic structures on the complexes of E-polydifferential operators and E-polyjets of an algebroid E. We recall Kontsevich's [16] and Shoikhet's [25] formality theorems for $\mathbb{R}^d_{formal}$ and formulate our first result, the formality of the module of E-chains (see theorem 1.11 on page 10). The second section is devoted to the construction of the Fedosov resolutions of the sheaves of E-polydifferential operators, E-chains, E-polyvector fields and E-forms. It is the most technical part of the paper. Using these resolutions in section 3, we prove theorem 1.11. In the same section we apply this theorem to the description of quantum traces of Poisson Lie algebroids. In section 4 we prove Tsygan's formality conjecture for Lie algebroid chains in the holomorphic setting (see theorem 4.2 on page 28), which, in particular, gives us the formality theorem for Hochschild chains of the structure sheaf of an arbitrary complex manifold (see theorem 4.4). In the concluding section we mention an equivariant version of theorem 1.11 and raise some other questions.

Notation. We assume Einstein's convention for the summation over repeated indices and omit the symbol $\wedge$ referring to a local basis of exterior forms. The arrow $\succ\!\rightarrow$ denotes an L_∞-morphism of L_∞-algebras, the arrow $\succ\!\succ\!\rightarrow$ denotes a morphism of L_∞-modules, and the notation

$$\begin{array}{c}\mathcal{L}\\ \downarrow_{mod}\\ \mathcal{M}\end{array}$$

means that $\mathcal{M}$ is an L_∞-module over the L_∞-algebra $\mathcal{L}$. Throughout the paper (except section 4) M denotes a smooth real manifold and $\mathcal{O}_M$ denotes the structure sheaf of M. The abbreviation "DGLA" stands for "differential graded Lie algebra" and the abbreviation "DGA" stands for "differential graded associative algebra".

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1. Algebraic structures associated with a Lie algebroid

1.1. **Lie algebroids and associated sheaves.** Let us recall the following

Definition 1.1. *A* Lie algebroid *over a smooth manifold M is a smooth vector bundle E of finite rank whose sheaf of sections is a sheaf of Lie algebras equipped with a map of sheaves of Lie algebras*

$$\rho : E \to TM \,.$$

The $\mathcal{O}_M$-module structure and the Lie algebra structure on the sheaf E are compatible in the following sense: for any open subset $U \subset M$, any function $f \in \mathcal{O}_M(U)$ and any sections $u, v \in \Gamma(U, E)$

$$[u, fv] = f[u, v] + \rho(u)(f)v \,. \tag{1.1}$$

The map ρ is called the anchor.

Example. The tangent bundle TM on M is the simplest example of a Lie algebroid. The bracket is the usual Lie bracket of vector fields and the anchor is the identity map $\mathrm{id} : TM \to TM$.

Definition 1.2. *The bundle ${}^E T^*_{poly}$ of E-*polyvector fields *is the exterior algebra of the bundle E with the shifted grading*

$$ {}^E T_{poly} = \bigoplus_{k \ge -1} {}^E T^k_{poly} \,, \qquad {}^E T^*_{poly} = \begin{cases} \wedge^{*+1} E \,, \; * \ge 0 \,, \\ \mathcal{O}_M \,, \; * = -1 \,. \end{cases} \tag{1.2}$$

It turns out that the Lie bracket $[\,,\,]$ on $\Gamma({}^E T^0_{poly}) = \Gamma(E)$ can be naturally extended to a Lie bracket on the whole vector space $\Gamma({}^E T^*_{poly})$ of E-polyvectors. Indeed, first, we defined a Lie bracket $[\,,\,]_{SN}$ on $\Gamma({}^E T^{-1}_{poly} \oplus {}^E T^0_{poly})$ as follows

$$\begin{aligned} &[f, g]_{SN} = 0, && \forall\, f, g \in \Gamma({}^E T^{-1}_{poly}) \,, \\ &[u, f]_{SN} = \rho(u) f, && \forall\, u \in \Gamma({}^E T^0_{poly}) \,, \; f \in \Gamma({}^E T^{-1}_{poly}) \,, \\ &[u, v]_{SN} = [u, v], && \forall\, u, v \in \Gamma({}^E T^0_{poly}) \,. \end{aligned} \tag{1.3}$$

Next, we extend $[\,,\,]_{SN}$ to $\Gamma({}^E T^*_{poly})$ by requiring the graded Leibniz rule with respect to the $\wedge$-product

$$[u, v \wedge w]_{SN} = [u, v]_{SN} \wedge w + (-1)^{k(l+1)} v \wedge [u, w]_{SN} \,, \tag{1.4}$$

$$\forall\, u \in \Gamma({}^E T^k_{poly}), \; v \in \Gamma({}^E T^l_{poly}), \; w \in \Gamma({}^E T^*_{poly}).$$

In the simplest example $E = TM$ the Lie bracket $[\,,\,]_{SN}$ coincides with the well known Schouten-Nijenhuis bracket of ordinary polyvector fields.

The exterior algebra $\wedge^* E^\vee$ of the dual bundle $E^\vee$ to E is a natural candidate for the bundle ${}^E\Omega^*_M$ of *E-differential forms* or just *E-forms* for short. The bundle ${}^E\Omega^*_M$ of E-forms is endowed with the following *E-de Rham differential*

$$\begin{aligned} {}^E d\omega(\sigma_0, \dots, \sigma_k) \;\; = \;\; & \textstyle\sum_i (-1)^i \rho(\sigma_i) \omega(\sigma_0, \dots, \hat{\sigma}_i, \dots, \sigma_k) \\ & + \textstyle\sum_{i<j} (-1)^{i+j} \omega([\sigma_i, \sigma_j], \sigma_0, \dots, \hat{\sigma}_i, \dots, \hat{\sigma}_j, \dots, \sigma_k) \,, \\ & \qquad \sigma_i \in \Gamma(E) \,. \end{aligned} \tag{1.5}$$

By the word *connection* on a vector bundle $\mathcal{B}$ over M we always mean *E-connection*, that is a linear operator

$$\nabla : \Gamma(M, \mathcal{B}) \to {}^E\Omega^1(M, \mathcal{B}) \tag{1.6}$$

satisfying the following equation

$$\nabla(fu) = {}^E d(f)u + f\nabla(u)$$

for any $f \in \mathcal{O}_M(M)$ and $u \in \Gamma(M, \mathcal{B})$.

Another operation defined on E-forms is a contraction with E-polyvector fields. For an E-polyvector field $u \in \Gamma({}^E T^k_{poly})$ we denote by ι_u the contraction with u. Using this contraction, the E-de Rham differential (1.5), and the Cartan-Weil formula

$$ {}^E L_u = {}^E d \circ \iota_u + (-1)^k \iota_u \circ {}^E d \tag{1.7}$$

we define the *E-Lie derivative* of E-forms by the E-polyvector field $u \in \Gamma({}^E T^k_{poly})$.

For our purposes it is more convenient to use the reversed grading in the bundle of E-forms. Thus we denote by

$$ {}^E A_* = {}^E \Omega^{-*}_M \,, \qquad {}^E A_0 = \mathcal{O}_M \tag{1.8}$$

the corresponding bundle with reversed grading and observe that ${}^E A_*$ is equipped with a structure of a graded module over the sheaf of graded Lie algebras ${}^E T^*_{poly}$ via the E-Lie derivative (1.7). Namely,

Lemma 1.3. *For any E-polyvector fields $u \in \Gamma({}^E T^k_{poly})$ and $v \in \Gamma({}^E T^l_{poly})$ one has*

$$ {}^E L_u \circ {}^E L_v - (-1)^{kl}\, {}^E L_v \circ {}^E L_u = {}^E L_{[u,v]_{SN}} \,. \tag{1.9}$$

Proof. First, it is immediate from the definition (1.7) that for any $u \in \Gamma({}^E T^k_{poly})$

$$ {}^E d \circ {}^E L_u = (-)^k\, {}^E L_u\, {}^E d \,. \tag{1.10}$$

Second, we claim that for any $u \in \Gamma({}^E T^k_{poly})$ and $v \in \Gamma({}^E T^l_{poly})$ we have

$$ {}^E L_u \circ \iota_v - (-1)^{k(l+1)} \iota_v \circ {}^E L_u = (-)^k \iota_{[u,v]_{SN}} \,. \tag{1.11}$$

Using (1.10) and (1.11) it is not hard to show that for any $u \in \Gamma({}^E T^k_{poly})$ and $v \in \Gamma({}^E T^l_{poly})$

$$ {}^E L_u({}^E d\iota_v + (-)^l \iota_v {}^E d) - (-)^{kl}({}^E d\iota_v + (-)^l \iota_v {}^E d){}^E L_u = ({}^E d\iota_{[u,v]_{SN}} + (-)^{k+l} \iota_{[u,v]_{SN}} {}^E d) \,.$$

Thus it suffices to prove that equation (1.11) holds.

The proof of (1.11) goes as follows. First, direct computations show that (1.11) holds for any sections u and v of the subsheaf ${}^E T^{-1}_{poly} \oplus {}^E T^0_{poly}$. Second, using the Leibniz rule (1.4) we prove the desired identity by induction on the degrees of E-polyvector fields u and v. In doing this, we need another simple identity

$$ {}^E L_{u_1 \wedge u_2} = {}^E L_{u_1} \iota_{u_2} - (-)^{k_1} \iota_{u_1} {}^E L_{u_2} \,, \qquad \forall\ u_i \in \Gamma({}^E T^{k_i}_{poly}) \,,$$

which follows easily from the fact that for any $u \in \Gamma({}^E T^k_{poly})$ and $v \in \Gamma({}^E T^l_{poly})$

$$ \iota_{u \wedge v} = \iota_u \circ \iota_v \,.$$

□

One can also define the $\mathcal{O}_M$-module $\mathcal{U}E$ of *E-differential operators* to be the sheaf of algebras locally generated by functions and E-vector fields. More precisely, $\mathcal{U}E$ is the sheaf associated with the following presheaf

$$ U \longmapsto T\Big(\mathcal{O}_M(U) \oplus \Gamma(U, E)\Big) \Big/ \left\{ \begin{array}{c} f \otimes g - fg,\ f \otimes u - fu, \\ u \otimes f - f \otimes u - \rho(u)f, \\ u \otimes v - v \otimes u - [u,v], \end{array} \right\} \tag{1.12}$$

$$f, g \in \mathcal{O}_M(U), \qquad u, v \in \Gamma(U, E)\,.$$

As a sheaf of $\mathcal{O}_M$-modules, $\mathcal{U}E$ is endowed with an increasing filtration

$$\mathcal{O}_M = \mathcal{U}E^0 \subset \mathcal{U}E^1 \subset \mathcal{U}E^2 \subset \cdots \subset \mathcal{U}E\,, \tag{1.13}$$

which is defined by assigning the degree 1 to the E-polyvectors.

In the terminology of [24] E is a sheaf of Lie-Rinehart algebras over the structure sheaf $\mathcal{O}_M$ and $\mathcal{U}E$ is the universal enveloping algebroid of E. Thus, besides the fact that $\mathcal{U}E$ is a sheaf of algebras, $\mathcal{U}E$ is also equipped with a coassociative $\mathcal{O}_M$-linear coproduct $\Delta : \mathcal{U}E \to \mathcal{U}E \otimes_{\mathcal{O}_M} \mathcal{U}E$ which is defined as follows

$$\begin{gathered} \Delta(1) = 1 \otimes 1\,, \\ \Delta(u) = u \otimes 1 + 1 \otimes u, \qquad \Delta(PQ) = \Delta(P)\Delta(Q)\,, \\ \forall\ u \in \Gamma(E),\ P, Q \in \Gamma(\mathcal{U}E)\,. \end{gathered} \tag{1.14}$$

Notice that, in the simplest example $E = TM$ of the Lie algebroid $\mathcal{U}E$ is the sheaf of differential operators on M .

1.2. **Algebraic structures on E-polydifferential operators and E-polyjets.**

We start this subsection with the following

Definition 1.4. *The bundle* ${}^E D^*_{poly}$ *of* E-polydifferential operators *is the tensor algebra of the bundle* $\mathcal{U}E$ *with a shifted grading:*

$${}^E D_{poly} = \bigoplus_{k \ge -1} {}^E D^k_{poly}\,, \qquad {}^E D^*_{poly} = \begin{cases} \otimes^{*+1}_{\mathcal{O}_M} \mathcal{U}E\,,\ * \ge 0\,, \\ \mathcal{O}_M\,,\ * = -1\,. \end{cases}$$

It is easy to see that in the case $E = TM$ the sheaf ${}^E D^*_{poly}$ is the sheaf of polydifferential operators on M.

Using the coproduct (1.14) in $\mathcal{U}E$ we endow the graded sheaf ${}^E D^*_{poly}$ of E-polydifferential operators with a Lie bracket $[,]_G$. To introduce this bracket we first define the following bilinear product of degree 0

$$\bullet\ : {}^E D_{poly} \otimes {}^E D_{poly} \to {}^E D_{poly}\,,$$

$$\begin{gathered} P \bullet Q = \sum_{i=0}^{|P|} (-1)^{i|Q|} \big(1^{\otimes i} \otimes \Delta^{(|Q|)} \otimes 1^{\otimes |P|-i}\big)(P) \cdot (1^{\otimes i} \otimes Q \otimes 1^{\otimes |P|-i})\,, \\ P \bullet f = \sum_{i=0}^{|P|} (-1)^i (1^{\otimes i} \otimes \rho \otimes 1^{\otimes |P|-i})(P)(1^{\otimes i} \otimes f \otimes 1^{\otimes |P|-i})\,, \\ f \bullet g = 0\,, \qquad f \bullet P = 0 \end{gathered} \tag{1.15}$$

for any $P, Q \in \Gamma({}^E D^{\ge 0}_{poly})$ and $f, g \in \Gamma({}^E D^{-1}_{poly}) = \mathcal{O}_M$. Here $\Delta^{(n)} = (\Delta \otimes 1^{\otimes n-1}) \circ \cdots \circ \Delta$, $\Delta^{(0)}$ is by convention the identity map, and ρ denotes the representation of $\mathcal{U}E$ on $\mathcal{O}_M$ induced via the anchor map.

Although the bilinear product is not associative, the graded commutator

$$[P, Q]_G = P \bullet Q - (-1)^{|P||Q|} Q \bullet P\,, \qquad P, Q \in \Gamma({}^E D^*_{poly}) \tag{1.16}$$

defines a graded Lie bracket between the E-polydifferential operators.

It is not hard to see that in the case $E = TM$ the above bracket reduces to the well known Gerstenhaber bracket [12] between polydifferential operators on M.

Let us now define the vector bundle of *E-polyjets*.

Definition 1.5. *The bundle* ${}^E J_*^{poly}$ *of* E-polyjets *is the following graded bundle placed in nonnegative degrees*

$$ {}^E J^{poly} = \bigoplus_{k\geq 0} {}^E J_k^{poly} \,, \qquad {}^E J_*^{poly} := Hom_{\mathcal{O}_M}(\mathcal{U}E^{\otimes_{\mathcal{O}_M}^{*+1}}, \mathcal{O}_M)\,. $$

Since the sheaf ${}^E D_{poly}^*$ of E-polydifferential operators is an ind-finite dimensional graded vector bundle the sheaf ${}^E J_*^{poly}$ of E-polyjets is a profinite dimensional graded vector bundle. Furthermore, the sheaf ${}^E J_*^{poly}$ is endowed with a canonical flat connection ∇^G which is called the *Grothendieck connection* and defined by the formula

$$ \nabla^G_\sigma(j)(P) := \rho(\sigma)(j(P)) - j(\sigma \bullet P)\,, \tag{1.17} $$

where $\sigma \in \Gamma(E)$, $j \in \Gamma({}^E J_k^{poly})$, $P \in \Gamma({}^E D_{poly}^k)$, and the operation $\bullet$ is defined in (1.15).

For this connection we have the following standard

Proposition 1.6. *Let* χ *be a map of sheaves*

$$ \chi : {}^E J_k^{poly} \to \begin{cases} {}^E J_{k-1}^{poly}\,, & \text{if} \quad k > 0\,, \\ \mathcal{O}_M\,, & \text{if} \quad k = 0 \end{cases} $$

defined by the formula

$$ \chi(a)(P) = a(1 \otimes P)\,, \quad P \in \Gamma({}^E D_{poly}^{k-1})\,, \quad a \in \Gamma({}^E J_k^{poly})\,. \tag{1.18} $$

The restriction of the map χ *to the* ∇^G*-flat* E*-polyjets gives the isomorphism of sheaves*

$$ \chi : \ker \nabla^G \cap {}^E J_k^{poly} \xrightarrow{\sim} \begin{cases} {}^E J_{k-1}^{poly}\,, & \text{if} \quad k > 0\,, \\ \mathcal{O}_M\,, & \text{if} \quad k = 0\,. \end{cases} \tag{1.19} $$

Proof. To see that the map (1.19) is surjective one has to notice that for any E-polyjet b of degree $k-1$ (resp. a function $b \in \Gamma(\mathcal{O}_M)$) the equations

$$ a(1 \otimes P) = b(P)\,, \quad P \in \Gamma({}^E D_{poly}^{k-1}) $$

and

$$ a(u \cdot Q \otimes P) = \rho(u) a(Q \otimes P) - a(Q \otimes (\Delta^{(k-1)}(u) \cdot P))\,, \tag{1.20} $$

$$ Q \in \Gamma(\mathcal{U}E)\,, \qquad u \in \Gamma(E) $$

define a ∇^G-flat E-polyjet a of degree k (resp. a ∇^G-flat E-jet a).

On the other hand, if a is a ∇^G-flat E-polyjet of degree k equation (1.20) is automatically satisfied. Thus a is uniquely determined by its image $\chi(a)$. □

Let t be the cyclic permutation acting on the sheaf ${}^E J_*^{poly}$ of E-polyjets

$$ t(a)(P_0 \otimes \cdots \otimes P_l) := a(P_1 \otimes \cdots \otimes P_l \otimes P_0)\,, \tag{1.21} $$

$$ a \in \Gamma({}^E J_l^{poly})\,, \qquad P_i \in \Gamma(\mathcal{U}E)\,. $$

Using this operation and the bilinear product (1.15) we define the map

$$ {}^E S : {}^E D_{poly}^k \otimes {}^E J_l^{poly} \to {}^E J_{l-k}^{poly}\,, $$

$$ P \otimes a \mapsto {}^E S_P(a) \text{ such that} $$

$$(1.22)\quad {}^E S_P(a)(Q) = a(Q \bullet P) + \sum_{j=1}^{k} (-1)^{lj} t^j(a)\big((\Delta^{(k)} \otimes 1^{\otimes(l-k)})(Q) \cdot (P \otimes 1^{\otimes(l-k)})\big)\,,$$

$$\text{for } P \in \Gamma({}^E D^k_{poly})\,, \qquad a \in \Gamma({}^E J^{poly}_l)\,, \qquad Q \in \Gamma({}^E D^{l-k}_{poly})\,.$$

Due to the following proposition the map (1.22) defines an action of the sheaf of graded Lie algebras ${}^E D^*_{poly}$ of E-polydifferential operators on the graded sheaf ${}^E J^{poly}_*$ of E-polyjets. Namely,

Proposition 1.7. *For any pair $P_1, P_2 \in \Gamma({}^E D^*_{poly})$ of E-polydifferential operators and any E-polyjet $a \in \Gamma({}^E J^{poly}_*)$*

$$(1.23)\qquad {}^E S_{P_1}\, {}^E S_{P_2}(a) - (-)^{|P_1||P_2|}\, {}^E S_{P_2}\, {}^E S_{P_1}(a) = {}^E S_{[P_1,P_2]_G}(a)\,.$$

Moreover, the action (1.22) is compatible with the Grothendieck connection (1.17)

$$(1.24)\qquad \nabla^G_u\left({}^E S_P(a)\right) = {}^E S_P(\nabla^G_u(a))\,, \quad u \in \Gamma(E)\,, \quad P \in \Gamma({}^E D^*_{poly})\,.$$

Proof. It is not hard to show that for any $a \in \Gamma({}^E J^{poly}_n)$

$$(1.25)\qquad {}^E S_{P_1}\, {}^E S_{P_2}(a) = {}^E S_{P_1 \bullet P_2}(a) + H(P_1,P_2)(a) + (-)^{|P_1||P_2|} H(P_2,P_1)(a)\,,$$

where[2]

$$H(P_1,P_2) : {}^E J^{poly}_* \to {}^E J^{poly}_{*-|P_1|-|P_2|}$$

is a graded $\mathcal{O}_M$-linear endomorphism of the sheaf ${}^E J^{poly}_*$ defined by the following formula

$$(H(P_1,P_2)(a))(Q) =$$

$$\sum_{i,j} (-)^{i|P_1|+j|P_2|} a \Big[(1^{\otimes i} \otimes \Delta^{|P_1|} \otimes 1^{\otimes(j-i-|P_1|-1)} \otimes \Delta^{|P_2|} \otimes 1^{\otimes(n-j-|P_2|)}(Q)) \cdot$$
$$(1^{\otimes i} \otimes P_1 \otimes 1^{\otimes(j-i-|P_1|-1)} \otimes P_2 \otimes 1^{\otimes(n-j-|P_2|)})\Big] +$$
$$\sum_{k,l} (-)^{k|P_2|+l(n-|P_2|)} t^l(a) \Big[(\Delta^{|P_1|} \otimes 1^{k+l-|P_1|-1} \otimes \Delta^{|P_2|} \otimes 1^{\otimes n-k-l-|P_2|}\ (Q)) \cdot$$
$$P_1 \otimes 1^{\otimes(k+l-|P_1|-1)} \otimes P_2 \otimes 1^{\otimes(n-k-l-|P_2|)}\Big]\,,$$

the sums run over all i, j, k, l satisfying the conditions

$$0 \le i \le j - |P_1| - 1, \qquad j \le n - |P_2|\,,$$
$$1 \le l \le |P_1|, \qquad |P_1| - l + 1 \le k \le n - |P_2| - l\,,$$

and

$$Q \in \Gamma({}^E D^{n-|P_1|-|P_2|}_{poly})\,.$$

Equation (1.25) obviously implies identity (1.23).

Equation (1.24) follows immediately from the fact that the coproduct (1.14) is compatible with the multiplication of the E-differential operators and the fact that the Grothendieck connection (1.17) commutes with the cyclic permutation (1.21). □

[2]Formula (1.25) is essentially borrowed from paper [13] of E. Getzler.

Notice that an element $1 \otimes 1 \in \Gamma({}^E D^1_{poly})$ is distinguished by the following remarkable identity $[1 \otimes 1, 1 \otimes 1]_G = 0$. Using this observation we define the following differentials

$$\partial = [1 \otimes 1, \ ]_G : {}^E D^*_{poly} \to {}^E D^{*+1}_{poly}, \qquad \mathfrak{b} := {}^E S_{1\otimes 1} : {}^E J^{poly}_* \to {}^E J^{poly}_{*-1} \tag{1.26}$$

on the sheaf of E-polydifferential operators and the sheaf of E-polyjets.

From the definition of the differentials (1.26) and equation (1.23), we see that ∂ is compatible with the Lie bracket (1.16) and the differential $\mathfrak{b}$ is compatible with the action (1.22) in the sense of the following equation

$$\mathfrak{b}\left({}^E S_P(a)\right) = {}^E S_{\partial P}(a) + (-)^{|P|} \ {}^E S_P(\mathfrak{b}(a)) .$$
$$\forall \ a \in \Gamma({}^E J^{poly}_*) , \quad P \in \Gamma({}^E D^*_{poly}) .$$

Thus, $({}^E D^*_{poly}, \partial, [,]_G)$ is a sheaf of differential graded Lie algebras (DGLA for short) and $({}^E J^{poly}_*, \mathfrak{b}, {}^E S)$ is a sheaf of differential graded modules (DG modules for short) over ${}^E D^*_{poly}$.

We would like to mention that the tensor product of sections (over $\mathcal{O}_M$) turns the sheaf ${}^E D_{poly}[-1]^*$ with the shifted grading into a sheaf of graded associative algebras. Moreover, it is not hard to see that the differential ∂ (1.26) is compatible with this product. Thus ${}^E D^*_{poly}$ can be also viewed as a sheaf of DG associative algebras (DGA).

The complex of sheaves $(\Gamma({}^E J^{poly}_*), \mathfrak{b})$ is not a good candidate for the Hochschild chain complex in the Lie algebroid setting. Indeed, if our Lie algebroid E is TM then the complex $(\Gamma({}^E J^{poly}_*), \mathfrak{b})$ boils down to the Hochschild chain complex of $C^\infty(M)$ without the zeroth term and the action (1.22) does not coincide with the standard action of Hochschild cochains on Hochschild chains (see eq. (3.4) in [7]). To cure these problems simultaneously we introduce a graded sheaf ${}^E C^{poly}_*$ of $\mathcal{O}_M$-modules placed in non-positive degrees

$$ {}^E C^{poly}_* = \begin{cases} \mathcal{O}_M , & * = 0 , \\ {}^E J^{poly}_{-*-1} & * \le 0 , \end{cases} \tag{1.27}$$

and the following $\mathbb{R}$-linear isomorphism of sheaves

$$\varrho : {}^E C^{poly}_* \to \ker \nabla^G \cap {}^E J^{poly}_{-*} \tag{1.28}$$

obtained by inverting the map (1.19).

Due to propositions 1.7 the action (1.22) and the differential $\mathfrak{b}$ (1.26) commute with the Grothendieck connection ∇^G. Thus, the ∇^G-flat E-polyjets form a sheaf of DG submodule of $({}^E J^{poly}_*, \mathfrak{b}, {}^E S)$ over the sheaf of DGLAs $({}^E D^*_{poly}, \partial, [,]_G)$. Combining this observation with proposition 1.6 we conclude that the isomorphism (1.28) allows us to endow the sheaf (1.27) with a structure of a sheaf of DG modules over the sheaf of DGLAs ${}^E D^*_{poly}$. Namely,

Proposition 1.8. *The map*

$${}^E R_\bullet : {}^E D^k_{poly} \otimes {}^E C^{poly}_l \to {}^E C^{poly}_{k+l} , \tag{1.29}$$

given by the formula

$${}^E R_P(a) = \chi \circ {}^E S_P(\varrho(a)), \quad P \in \Gamma({}^E D^k_{poly}), \quad a \in \Gamma({}^E C^{poly}_l) , \tag{1.30}$$

and the differential

$$\mathfrak{b}(a) = \chi \circ {}^E S_{1\otimes 1}(\varrho(a)) : \Gamma({}^E C^{poly}_*) \to \Gamma({}^E C^{poly}_{*+1}) \tag{1.31}$$

turn ${}^E C_*^{poly}$ *(1.27) into a sheaf of DG modules over the sheaf of DGLAs* ${}^E D_{poly}^*$ *.* □

Remark 1. Since the map ϱ is NOT $\mathcal{O}_M$-linear the DGLA module structure (1.30), (1.31) on ${}^E C_*^{poly}$ is only $\mathbb{R}$-linear unlike the DGLA module structure (1.22) (1.26) on the sheaf ${}^E J_*^{poly}$.

Remark 2. It is not hard to see that in the case $E = TM$ the global sections of the sheaf ${}^E C_*^{poly}$ give the jet version [30] of the homological Hochschild complex of the algebra $\mathcal{O}_M$ of functions on M.

The second remark motivates the following definition:

Definition 1.9. *We refer to the sheaf* ${}^E C_*^{poly}$ *of DG modules over the sheaf of DGLAs* ${}^E D_{poly}^*$ *of* E*-polydifferential operators as* the sheaf of the Hochschild E-chains *or just* E-chains *for short.*

The cohomology of the complexes ${}^E D_{poly}^*$ and ${}^E C_*^{poly}$ are described by Hochschild-Kostant-Rosenberg type theorems. The original version of this theorem [15] says that the module of Hochschild homology of a smooth affine algebra is isomorphic to the module of exterior forms of the corresponding affine variety. In [3] A. Connes proved an analogous statement for the algebra of smooth functions on any compact real manifold, and in [29], N. Teleman was able to get rid of the assumption of compactness. The similar question about Hochschild cohomology turns out to be tractable if we replace the Hochschild cochains by polydifferential operators. We believe that the cohomology of this complex was originally computed by J. Vey [31]. All these computations correspond to the case when $E = TM$. In our general case we have the following proposition:

Proposition 1.10. *The natural maps*

$$
\begin{array}{rcl}
\mathcal{V} : ({}^E T_{poly}^*, 0) & \longrightarrow & ({}^E D_{poly}^*, \partial) \\
v_0 \wedge \cdots \wedge v_k & \longmapsto & \dfrac{1}{(k+1)!} \displaystyle\sum_{\sigma \in S_{k+1}} \epsilon(\sigma) v_{\sigma_0} \otimes \cdots \otimes v_{\sigma_k}
\end{array} \tag{1.32}
$$

and

$$
\begin{array}{rcl}
\mathfrak{C} : ({}^E C_*^{poly}, \mathfrak{b}) & \longrightarrow & ({}^E A_*, 0) \\
a & \longmapsto & (v \mapsto \varrho(a)(\mathcal{V}(v)))
\end{array} \tag{1.33}
$$

are quasi-isomorphisms of complexes of sheaves. The induced structure of the graded Lie algebra on ${}^E T_{poly}^*$ *coincides with the one given by the bracket (1.4) and the induced structure of the graded module on* ${}^E A_*$ *coincides with the one given by the Lie derivative (1.7).*

Remark. Recall that ${}^E A_*$ is the sheaf (1.8) of E-forms with reversed grading.

Proof. In order to prove theorem 1.11 we do not need this proposition in full generality. It essentially suffices to restrict ourselves to the case of the tangent Lie algebroid $T\mathbb{R}^d \to \mathbb{R}^d$. This particular case is well known. So we may regard proposition 1.10 as a corollary of theorem 1.11 □

1.3. **The formality of the DGLA module of E-chains.** Unfortunately, the maps (1.32) and (1.33) respect neither the Lie brackets nor the actions. This defect can be cured using the notion of Lie algebras and their modules *up to homotopy* (see [14] for a detailed discussion of the general theory and its applications, and [7, section 2] for a quick review of the notions and results we need). The first result of this paper is the following theorem:

Theorem 1.11. *For any C^∞ Lie algebroid (E, M, ρ) one can construct a commutative diagram of sheaves of DGLAs and DGLA modules over M*

$$
\begin{array}{ccccccc}
{}^E T^*_{poly} & \succ\!\!\rightarrow & \mathcal{L}_1 & \succ\!\!\rightarrow & \mathcal{L}_2 & \leftarrow\!\!\prec & {}^E D^*_{poly} \\
\downarrow_{mod} & & \downarrow_{mod} & & \downarrow_{mod} & & \downarrow_{mod} \\
{}^E A_* & \succ\!\succ\!\!\rightarrow & \mathcal{M}_1 & \leftarrow\!\!\prec\!\prec & \mathcal{M}_2 & \leftarrow\!\!\prec\!\prec & {}^E C^{poly}_* ,
\end{array}
\tag{1.34}
$$

in which the horizontal arrows in the upper row are quasi-isomorphisms of sheaves of DGLAs and the horizontal arrows in the lower row are quasi-isomorphisms of L_∞-modules. The terms ($\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{M}_1$, $\mathcal{M}_2$) and the quasi-isomorphisms of the diagram (1.34) are functorial for isomorphisms of pairs (E, ∂^E), where E is a C^∞ Lie algebroid and ∂^E is a torsion free E-connection on E.

The proof of this theorem occupies the next two sections.

1.4. **Formality theorems for the Hochschild complexes of $\mathbb{R}[[y^1, \dots, y^d]]$.** In order to prove theorem 1.11 we construct the Fedosov resolutions of the sheaves of DGLAs ${}^E T^*_{poly}$ and ${}^E D^*_{poly}$ and of the sheaves of DGLA modules ${}^E A_*$ and ${}^E C^{poly}_*$. These resolutions allow us to reduce the problem to the case of the Lie algebroid $T\mathbb{R}^d \to \mathbb{R}^d$. For the latter case the desired result follows from the combination of Kontsevich's [16] and Shoikhet's [25] formality theorems.

First, we recall the required version of Kontsevich's formality theorem. Let $M = \mathbb{R}^d_{formal}$ be the formal completion of $\mathbb{R}^d$ at the origin. In other words we set $\mathcal{O}_M = \mathbb{R}[[y^1, \dots, y^d]]$ and $E = \mathrm{Der}(\mathcal{O}_M)$. Let us denote by $T^*_{poly}(\mathbb{R}^d_{formal})$ and $D^*_{poly}(\mathbb{R}^d_{formal})$ the DGLA of polyvector fields and polydifferential operators on $\mathbb{R}^d_{formal}$, respectively, then

Theorem 1.12 (Kontsevich, [16])**.** *There exists an L_∞-quasi-isomorphism $\mathcal{K}$ from $T^*_{poly}(\mathbb{R}^d_{formal})$ to $D^*_{poly}(\mathbb{R}^d_{formal})$ such that*

1. *The first structure map $\mathcal{K}_1$ is Vey's quasi-isomorphism (1.32) of complexes $\mathcal{V}$.*
2. *$\mathcal{K}$ is $GL_d(\mathbb{R})$-equivariant.*
3. *If $n > 1$ then for any vector fields $v_1, \dots, v_n \in T^0_{poly}(\mathbb{R}^d_{formal})$*
$$\mathcal{K}_n(v_1, \dots, v_n) = 0$$
4. *If $n > 1$ then for any vector field $v \in T^0_{poly}(\mathbb{R}^d_{formal})$ linear in the coordinates y^i and any polyvector fields $\chi_2, \dots, \chi_n \in T^*_{poly}(\mathbb{R}^d_{formal})$*
$$\mathcal{K}_n(v, \chi_2, \dots, \chi_n) = 0.$$

We denote by

$$A^*(\mathbb{R}^d_{formal}) = \mathbb{R}[[y^1, \dots, y^d]] \otimes \bigwedge(\mathbb{R}^d)$$

the complex of exterior forms on $\mathbb{R}^d_{formal}$ with the vanishing differential and by

$$J^{poly}_*(\mathbb{R}^d_{formal}) = \mathbb{R}[[y^1, \dots, y^d]]^{\hat{\otimes}\,(*+1)}$$

the complex of Hochschild chains of $\mathbb{R}[[y^1,\dots,y^d]]$, where the notation $\hat{\otimes}$ stands for the tensor product completed in the adic topology on $\mathbb{R}[[y^1,\dots,y^d]]$.

Using the Lie derivative (1.7) of exterior forms by a polyvector field, we can regard $A^*(\mathbb{R}^d_{formal})$ as a graded module over the graded Lie algebra $T^*_{poly}(\mathbb{R}^d_{formal})$. Furthermore, the action of Hochschild cochains on Hochschild chains (see formula (3.4) in [7]) allows us to regard $J^{poly}_*(\mathbb{R}^d_{formal})$ as a DG modules over the DGLA $D^*_{poly}(\mathbb{R}^d_{formal})$. Finally, using Kontsevich's quasi-isomorphism $\mathcal{K}$ we get an L_∞-module structure on $J^{poly}_*(\mathbb{R}^d_{formal})$ over $T^*_{poly}(\mathbb{R}^d_{formal})$. For this L_∞-module, we have the following theorem:

Theorem 1.13 (Shoikhet, [25])**.** *There exists a quasi-isomorphism $\mathcal{S}$ of L_∞-modules over $T^*_{poly}(\mathbb{R}^d_{formal})$ from $J^{poly}_*(\mathbb{R}^d_{formal})$ to $A^*(\mathbb{R}^d_{formal})$ such that*

(1) *The first structure map $\mathcal{S}_1$ is the quasi-isomorphism of Connes (1.33).*
(2) *The structure maps of $\mathcal{S}$ are $GL_d(\mathbb{R})$-equivariant.*
(3) *If $n>1$ then for any vector field $v \in T^0_{poly}(\mathbb{R}^d_{formal})$ linear in the coordinates, any polyvector fields $\chi_2,\dots,\chi_n \in T^*_{poly}(\mathbb{R}^d_{formal})$ and any chain $j \in J^{poly}_*(\mathbb{R}^d_{formal})$*

$$\mathcal{S}_n(v,\chi_2,\dots,\chi_n;j)=0$$

Remark 1. The third assertion of the above theorem is proved in [7] (see theorem 3).
Remark 2. Hopefully, one can prove the assertions of theorem 1.13 along the lines of Tamarkin and Tsygan [26, 27, 28].

2. The Fedosov resolutions

Let, as above, $E \to M$ be a C^∞ Lie algebroid with bracket $[,]$ on sections and the anchor ρ. Following [7] we introduce the formally completed symmetric algebra $\hat{S}(E^\vee)$ of the dual bundle $E^\vee$ and bundles $\mathcal{T}$, $\mathcal{D}$, $\mathcal{A}$, $\mathcal{J}$ naturally associated with $\hat{S}(E^\vee)$.

- $\hat{S}(E^\vee)$ is the formally completed symmetric algebra of the bundle $E^\vee$. Local sections are given by formal power series
$$\sum_{l=0}^{\infty} s_{i_1\dots i_l}(x) y^{i_1}\cdots y^{i_l}$$
where y^i are coordinates on the fibers of E and $s_{i_1\dots i_l}$ are components of a symmetric covariant E-tensor.
- $\mathcal{T}^* := \hat{S}(E^\vee)\otimes \wedge^{*+1}E$ is the graded bundle of formal fiberwise polyvector fields on E. Local homogeneous sections of degree k are of the form
$$\sum_{l=0}^{\infty} v^{j_0\dots j_k}_{i_1\dots i_l}(x) y^{i_1}\cdots y^{i_l} \frac{\partial}{\partial y^{j_0}}\wedge\cdots\wedge\frac{\partial}{\partial y^{j_k}}, \tag{2.1}$$
where $v^{j_0\dots j_k}_{i_1\dots i_l}$ are components of an E-tensor with symmetric covariant part (indices $i_1,\dots,i_l$) and antisymmetric contravariant part (indices $j_0,\dots,j_k$).
- $\mathcal{D}^* := \hat{S}(E^\vee)\otimes T^{*+1}(SE)$ is the graded bundle of formal fiberwise polydifferential operators on E with the shifted grading. A local homogeneous

section of degree k looks as follow

$$\sum_{l=0}^{\infty} P^{\alpha_0\dots\alpha_k}_{i_1\dots i_l}(x) y^{i_1}\cdots y^{i_l} \frac{\partial^{|\alpha_0|}}{\partial y^{\alpha_0}} \otimes \cdots \otimes \frac{\partial^{|\alpha_k|}}{\partial y^{\alpha_k}}, \tag{2.2}$$

where α_s are multi-indices, $P^{\alpha_0\dots\alpha_k}_{i_1\dots i_l}$ are components of an E-tensor with the obvious symmetry of the corresponding indices, and

$$\frac{\partial^{|\alpha_s|}}{\partial y^{\alpha_s}} = \frac{\partial}{\partial y^{j_1}} \cdots \frac{\partial}{\partial y^{j_{|\alpha_s|}}}$$

for $\alpha_s = (j_1 \dots j_{|\alpha_s|})$.

- $\mathcal{A}_* := \hat{S}(E^\vee) \otimes \wedge^{-*}(E^\vee)$ is the graded bundle of formal fiberwise differential forms on E with the reversed grading. Any local homogeneous section of degree $-k$ can be written as

$$\sum_{l=0}^{\infty} \omega_{i_1\dots i_l, j_1\dots j_k}(x) y^{i_1}\cdots y^{i_l} dy^{j_1}\wedge \cdots \wedge dy^{j_k}, \tag{2.3}$$

where $\omega_{i_1\dots i_l, j_1\dots j_k}$ are components of a covariant E-tensor symmetric in indices $i_1, \dots, i_l$ and antisymmetric in indices $j_1, \dots, j_k$.

- $\mathcal{J}_*$ is the bundle of Hochschild chains of $\hat{S}(E^\vee)$ over $\mathcal{O}_M$.

$$\mathcal{J} = \bigoplus_{k\ge 0} \mathcal{J}_k, \qquad \mathcal{J}_k := (\hat{S}E^\vee)^{\hat{\otimes}_{\mathcal{O}_M}(k+1)}, \tag{2.4}$$

where $\hat{\otimes}$ stands for the tensor product completed in the adic topology. Local sections of homogeneous degree k are formal power series

$$\sum_{\alpha_0,\dots,\alpha_k} a_{\alpha_0,\dots,\alpha_k}(x) y_0^{\alpha_0} y_1^{\alpha_1}\cdots y_k^{\alpha_k} \tag{2.5}$$

in $k{+}1$ copies $y_0, \dots, y_k$ of coordinates on the fibers of E. Here α_s are multi-indices, $a_{\alpha_0,\dots,\alpha_k}$ are components of a tensor with an obvious symmetry in the corresponding indices, and

$$y_m^{\alpha_m} = y_m^{j_1} \dots y_m^{j_{|\alpha_m|}}$$

for $\alpha_m = (j_1 \dots j_{|\alpha_m|})$.

For our purposes, we consider E-differential forms with values in the sheaves $\hat{S}(E^\vee)$, $\mathcal{T}$, $\mathcal{D}$, $\mathcal{A}$, $\mathcal{J}$. Below we list these sheaves of E-forms together with the algebraic structures they carry.

- ${}^E\Omega(\hat{S}(E^\vee))$ is a bundle of graded commutative algebras with grading given by the exterior degree of E-forms. ${}^E\Omega(\hat{S}(E^\vee))$ is also filtered by the degree of monomials in fiber coordinates y^i.
- ${}^E\Omega(\mathcal{T})$ is a sheaf of graded Lie algebras and ${}^E\Omega(\mathcal{A})$ is a sheaf of graded modules over ${}^E\Omega(\mathcal{T})$. These structures are induced by those of $T^*_{poly}(\mathbb{R}^d_{formal})$ and $A^*(\mathbb{R}^d_{formal})$, respectively and the grading is given by the sum of the exterior degree and the degree of an E-polyvector (resp. an E-form). $[,]_{SN}$ will denote the Lie bracket between sections of the sheaf ${}^E\Omega(\mathcal{T})$ and L_u (the Lie derivative) will denote the action of a fiberwise polyvector $u \in {}^E\Omega(\mathcal{T})$ on the sections of ${}^E\Omega(\mathcal{A})$. ${}^E\Omega(\mathcal{T})$ is also a sheaf of graded commutative algebras. The multiplication of sections in ${}^E\Omega(\mathcal{T})$ is given by the exterior product in the space $T^*_{poly}(\mathbb{R}^d_{formal})$ of E-polyvector fields on $\mathbb{R}^d_{formal}$.

The Lie bracket and the product in ${}^E\Omega(\mathcal{T})$ turn ${}^E\Omega(\mathcal{T})$ into a sheaf of Gerstenhaber algebras[3].

- ${}^E\Omega(\mathcal{D})$ is a sheaf of DGLAs and ${}^E\Omega(\mathcal{J})$ is a sheaf of DGLA modules over ${}^E\Omega(\mathcal{D})$. These structures are induced by those of $D^*_{poly}(\mathbb{R}^d_{formal})$ and $J^*_{poly}(\mathbb{R}^d_{formal})$, respectively and the grading is given by the sum of the exterior degree and the degree of a (co)chain. We denote by ∂ and $[,]_G$ respectively the differential and the Lie bracket on ${}^E\Omega(\mathcal{D})$, $\mathfrak{b}$ will stand for the differential on ${}^E\Omega(\mathcal{J})$ and $\mathcal{R}_P$ will denote the action of $P \in {}^E\Omega(\mathcal{D})$ on the sections of ${}^E\Omega(\mathcal{J})$. ${}^E\Omega(\mathcal{D})$ is also a sheaf of DGAs. The multiplication of sections is induced by the cup product in the space $D^*_{poly}(\mathbb{R}^d_{formal})$ of E-polydifferential operators on $\mathbb{R}^d_{formal}$.

Remark 1. Notice that $\mathcal{A}$ is a sheaf of exterior forms with values in $\hat{S}(E^\vee)$. However, we would like to distinguish $\mathcal{A}$ from ${}^E\Omega(\hat{S}(E^\vee))$. For this purpose we use two copies of a local basis of exterior forms. Those are $\{dy^i\}$ and $\{\xi^i\}$ for $\mathcal{A}$ and ${}^E\Omega(\hat{S}(E^\vee))$, respectively.

The following proposition shows that we have a distinguished sheaf of graded Lie algebras which acts on the sheaves ${}^E\Omega(\hat{S}(E^\vee))$, ${}^E\Omega(\mathcal{A})$, ${}^E\Omega(\mathcal{T})$, ${}^E\Omega(\mathcal{D})$, and ${}^E\Omega(\mathcal{J})$.

Proposition 2.1. *The sheaf ${}^E\Omega(\mathcal{T}^0)$ of E-forms with values in fiberwise vector fields is a sheaf of graded Lie algebras. The sheaves ${}^E\Omega(\hat{S}(E^\vee))$, ${}^E\Omega(\mathcal{A})$, ${}^E\Omega(\mathcal{T})$, ${}^E\Omega(\mathcal{D})$, and ${}^E\Omega(\mathcal{J})$ are sheaves of modules over ${}^E\Omega(\mathcal{T}^0)$ and the action of sections in ${}^E\Omega(\mathcal{T}^0)$ is compatible with the DG algebraic structures on ${}^E\Omega(\hat{S}(E^\vee))$, ${}^E\Omega(\mathcal{A})$, ${}^E\Omega(\mathcal{T})$, ${}^E\Omega(\mathcal{D})$, and ${}^E\Omega(\mathcal{J})$.*

Proof. Since the Schouten-Nijenhuis bracket (1.3), (1.4) has degree zero ${}^E\Omega(\mathcal{T}^0) \subset {}^E\Omega(\mathcal{T}) \subset {}^E\Omega(\mathcal{D})$ is a subsheaf of graded Lie algebras. While the action of ${}^E\Omega(\mathcal{T}^0)$ on the sections of ${}^E\Omega(\hat{S}(E^\vee))$ is obvious, the action on ${}^E\Omega(\mathcal{A})$ is given by the Lie derivative, the action on ${}^E\Omega(\mathcal{T})$ is the adjoint action corresponding to the Schouten-Nijenhuis bracket, the action on ${}^E\Omega(\mathcal{D})$ is given by the Gerstenhaber bracket and the action on ${}^E\Omega(\mathcal{J})$ is induced by the action of Hochschild cochains on Hochschild chains (see formula (3.4) in paper [7]). The compatibility of the action with the corresponding DGLA and DGLA module structures follows from the construction. The compatibility of the action with the product in ${}^E\Omega(\mathcal{T})$ follows from the axioms of the Gerstenhaber algebra [12] and the compatibility with the product in ${}^E\Omega(\mathcal{D})$ can be verified by a straightforward computation. □

Due to the above proposition the following 2-nilpotent derivation

$$\delta := \xi^i \frac{\partial}{\partial y^i} : {}^E\Omega^*(\hat{S}(E^\vee)) \to {}^E\Omega^{*+1}(\hat{S}(E^\vee)) \tag{2.6}$$

of the sheaf of algebras ${}^E\Omega(\hat{S}(E^\vee))$ obviously extends to 2-nilpotent differentials on ${}^E\Omega(\mathcal{T})$, ${}^E\Omega(\mathcal{D})$, ${}^E\Omega(\mathcal{A})$ and ${}^E\Omega(\mathcal{J})$. Furthermore, it follows from proposition 2.1 that δ is compatible with the DG algebraic structures on ${}^E\Omega(\mathcal{T})$, ${}^E\Omega(\mathcal{A})$, ${}^E\Omega(\mathcal{D})$, and ${}^E\Omega(\mathcal{J})$.

Note that

$$\ker\delta \cap \hat{S}(E^\vee) \cong \mathcal{O}_M\,, \qquad \ker\delta \cap \mathcal{A}^* \cong {}^E A_* \tag{2.7}$$

[3]The definition of the Gerstenhaber algebra can be found in section 4.1 of the second part of [5] or in the original paper [12].

as sheaves of (graded) commutative algebras over $\mathcal{O}_M$. Similarly, $\ker \delta \cap \mathcal{T}$, (resp. $\ker \delta \cap \mathcal{D}$) is a sheaf of fiberwise polyvector fields (2.1) (resp. fiberwise polydifferential operators (2.2)) whose components do not depend on the fiber coordinates y^i. In other words,

$$\ker \delta \cap \mathcal{T}^* \cong \wedge^{*+1}(E) \tag{2.8}$$

as sheaves of graded commutative algebras and

$$\ker \delta \cap \mathcal{D}^* \cong \otimes^{*+1}(S(E)) \,, \tag{2.9}$$

as sheaves of DGAs over $\mathcal{O}_M$.

In fact, one can prove a more stronger statement:

Proposition 2.2. *For $\mathcal{B}$ being either of the sheaves $\hat{S}(E^\vee)$, $\mathcal{A}$, $\mathcal{T}$ or $\mathcal{D}$*

$$H^{\ge 1}({}^E\Omega(\mathcal{B}), \delta) = 0 \,.$$

Furthermore,

$$\begin{aligned} H^0({}^E\Omega(\hat{S}(E^\vee)), \delta) &\cong \mathcal{O}_M \,, \\ H^0({}^E\Omega(\mathcal{A}_*), \delta) &\cong {}^E A_* \,, \\ H^0({}^E\Omega(\mathcal{T}^*), \delta) &\cong \wedge^{*+1}(E) \end{aligned} \tag{2.10}$$

as sheaves of (graded) commutative algebras and

$$H^0({}^E\Omega(\mathcal{D}^*), \delta) \cong \otimes^{*+1}(S(E)) \tag{2.11}$$

as sheaves of DGAs over $\mathcal{O}_M$.

Proof. Due to equations (2.7), (2.8), and (2.9) the proposition will follow immediately if we construct an operator

$$\kappa : {}^E\Omega^*(\mathcal{B}) \to {}^E\Omega^{*-1}(\mathcal{B}) \tag{2.12}$$

such that for any section u of ${}^E\Omega(\mathcal{B})$

$$u = \delta\kappa(u) + \kappa\delta(u) + \mathcal{H}(u) \,, \tag{2.13}$$

where

$$\mathcal{H}(u) = u\Big|_{y^i=\xi^i=0} \,. \tag{2.14}$$

First, we define this operator on the sheaf ${}^E\Omega(\hat{S}(E^\vee))$

$$\kappa(a) = y^k \frac{\vec{\partial}}{\partial \xi^k} \int_0^1 a(x, ty, t\xi) \frac{dt}{t}, \qquad a \in {}^E\Omega^{>0}(\hat{S}(E^\vee)) \,, \qquad \kappa\Big|_{\hat{S}(E^\vee)} = 0 \,, \tag{2.15}$$

where the arrow over ∂ denotes the left derivative with respect to the anti-commuting variable ξ^i.

Next, we extend κ to sections of the sheaves ${}^E\Omega(\mathcal{A})$, ${}^E\Omega(\mathcal{T})$, ${}^E\Omega(\mathcal{D})$ in the componentwise manner. A direct computation shows that equation (2.13) holds and the proposition follows. □

Since our Lie algebroid E is a smooth bundle over M, it admits a global torsion free connection[4] ∂^E. Using this connection we define the following derivation of the DG sheaves ${}^E\Omega(\hat{S}(E^\vee))$, ${}^E\Omega(\mathcal{A})$, ${}^E\Omega(\mathcal{T})$, ${}^E\Omega(\mathcal{D})$, and ${}^E\Omega(\mathcal{J})$:

$$\nabla = {}^E d + \Gamma\cdot : {}^E\Omega^*(\mathcal{B}) \to {}^E\Omega^{*+1}(\mathcal{B}) , \qquad \Gamma = -\xi^i \Gamma^k_{ij} y^j \frac{\partial}{\partial y^k} , \tag{2.16}$$

where $\mathcal{B}$ is either of the sheaves $\hat{S}(E^\vee)$, $\mathcal{A}$, $\mathcal{T}$, $\mathcal{D}$, or $\mathcal{J}$, $\Gamma^k_{ij}(x)$ are Christoffel's symbols of the connection ∂^E and $\Gamma\cdot$ denotes the action of Γ on the sections of the sheaves ${}^E\Omega(\mathcal{B})$. Due to proposition 2.1 the operator ∇ (2.16) is compatible with the DG algebraic structures on ${}^E\Omega(\hat{S}(E^\vee))$, ${}^E\Omega(\mathcal{T})$, ${}^E\Omega(\mathcal{A})$, ${}^E\Omega(\mathcal{D})$, and ${}^E\Omega(\mathcal{J})$. Furthermore, the torsion freeness of the connection ∂^E implies that

$$\nabla\delta + \delta\nabla = 0 . \tag{2.17}$$

The standard curvature E-tensor $(R_{ij})^l_k(x)$ of the connection ∂^E provides us with the following fiberwise vector field:

$$R = -\frac{1}{2}\xi^i\xi^j (R_{ij})^l_k(x) y^k \frac{\partial}{\partial y^l} \in {}^E\Omega^2(\mathcal{T}^0) . \tag{2.18}$$

A direct computation shows that for $\mathcal{B}$ being any of the sheaves ${}^E S$, $\mathcal{A}$, $\mathcal{T}$, $\mathcal{D}$, or $\mathcal{J}$, we have

$$\nabla^2 = R\cdot : {}^E\Omega^*(\mathcal{B}) \to {}^E\Omega^{*+2}(\mathcal{B}) , \tag{2.19}$$

where $R\cdot$ denotes the action of the vector field R described in proposition 2.1.

Although ∇ is not flat the following theorem shows that the combination $\nabla - \delta$ can be extended to a flat connection on the sheaves ${}^E\Omega(\hat{S}(E^\vee))$, ${}^E\Omega(\mathcal{T})$, ${}^E\Omega(\mathcal{A})$, ${}^E\Omega(\mathcal{D})$, and ${}^E\Omega(\mathcal{J})$.

Theorem 2.3. *Let $\mathcal{B}$ be either of the sheaves ${}^E S$, $\mathcal{A}$, $\mathcal{T}$, $\mathcal{D}$, or $\mathcal{J}$. There exists a section*

$$A = \sum_{s=2}^{\infty} \xi^k A^j_{k,i_1\dots i_s}(x) y^{i_1}\cdots y^{i_s} \frac{\partial}{\partial y^j} \tag{2.20}$$

of the sheaf ${}^E\Omega^1(\mathcal{T}^0)$ such that the derivation

$$D := \nabla - \delta + A\cdot : {}^E\Omega^*(\mathcal{B}) \to {}^E\Omega^{*+1}(\mathcal{B}) \tag{2.21}$$

is 2-nilpotent

$$D^2 = 0 ,$$

and (2.21) is compatible with the DG algebraic structure on ${}^E\Omega(\mathcal{B})$.

Proof. The proof goes essentially along the lines of [6, theorem 2].

Thanks to equation (2.19) the condition $D^2 = 0$ is equivalent to the equation

$$R + \nabla A - \delta A + \frac{1}{2}[A, A]_{SN} = 0 . \tag{2.22}$$

We claim that a solution of (2.22) can be obtained by iterations of the following equation

$$A = \kappa R + \kappa(\nabla A + \frac{1}{2}[A, A]_{SN}) \tag{2.23}$$

[4]Recall that by the word "connection" we always mean an E-connection (1.6).

in degrees in the fiber coordinates y^i . Indeed, equation (2.13) implies that iterating (2.23) we get a solution of the equation

$$\kappa(R + \nabla A - \delta A + \frac{1}{2}[A, A]_{SN}) = 0\,.$$

We denote by C the left hand side of (2.22)

$$C = R + \nabla A - \delta A + \frac{1}{2}[A, A]_{SN}\,,$$

and mention that due to Bianchi's identities $\nabla R = \delta R = 0$

$$\nabla C - \delta C + [A, C] = 0\,. \tag{2.24}$$

Applying κ (2.15) to (2.24) and using the homotopy property (2.13) we get

$$C = \kappa(\nabla C + [A, C])\,.$$

The latter equation has the unique vanishing solution since the operator κ (2.15) raises the degree in the fiber coordinates y^i .

Proposition 2.1 implies that the differential (2.21) is compatible with the DG algebraic structures on ${}^E\Omega(\mathcal{B})$. Thus, the theorem is proved. □

In what follows we refer to the differential D (2.21) as *the Fedosov differential.*

The following theorem describes the cohomology of the Fedosov differential D for the sheaves ${}^E\Omega(\hat{S}(E^\vee))$, ${}^E\Omega(\mathcal{A})$, ${}^E\Omega(\mathcal{T})$, and ${}^E\Omega(\mathcal{D})$

Theorem 2.4. *For* $\mathbf{B}$ *being either of the sheaves* ${}^E\Omega(\hat{S}(E^\vee))$, ${}^E\Omega(\mathcal{A})$, ${}^E\Omega(\mathcal{T})$, *or* ${}^E\Omega(\mathcal{D})$

$$H^{\ge 1}(\mathbf{B}, D) = 0\,. \tag{2.25}$$

Furthermore,

$$H^0({}^E\Omega(\hat{S}(E^\vee)), D) \cong \mathcal{O}_M\,,$$
$$H^0({}^E\Omega(\mathcal{A}_*), D) \cong {}^E A_*\,, \tag{2.26}$$
$$H^0({}^E\Omega(\mathcal{T}^*), D) \cong \ker\delta \cap \mathcal{T}^*\,,$$

as sheaves of graded commutative algebras

$$H^0({}^E\Omega(\mathcal{D}^*), D) \cong \ker\delta \cap \mathcal{D}^* \tag{2.27}$$

as sheaves of DGAs over $\mathbb{R}$.

Proof. The first statement follows easily from the spectral sequence argument. Indeed, using the fiber coordinates y^i we introduce the decreasing filtration

$$\cdots \subset F^{p+1}\mathbf{B} \subset F^p\mathbf{B} \subset F^{p-1}\mathbf{B} \subset \cdots \subset F^0\mathbf{B} = \mathbf{B}\,,$$

where the components of the sections of the sheaf $F^p\mathbf{B}$ have degree in $y^i \ge p$.

Since $D(F^p\mathbf{B}) \subset F^{p-1}\mathbf{B}$ the corresponding spectral sequence starts with

$$E_{-1}^{p,q} = F^p\mathbf{B}^{p+q}\,.$$

It is easy to see that

$$d_{-1} = \delta\,.$$

Thus using proposition 2.2 we conclude that for any p, q satisfying the condition $p + q > 0$

$$E_0^{p,q} = E_1^{p,q} = \cdots = E_\infty^{p,q} = 0$$

and the first statement (2.25) follows.

Let $\mathcal{B}$ denote either of the bundles $\hat{S}(E^\vee)$, $\mathcal{A}$, $\mathcal{T}$, or $\mathcal{D}$. We claim that iterating the equation

$$\lambda(u) = u + \kappa(\nabla\lambda(u) + A\cdot\lambda(u))\,, \qquad u \in \Gamma(\mathcal{B}) \cap \ker\delta \tag{2.28}$$

we get a map of sheaves of graded vector spaces

$$\lambda : \mathcal{B} \cap \ker\delta \to \mathcal{B} \cap \ker D\,. \tag{2.29}$$

Here $A\cdot$ denotes the action of the fiberwise vector field A, defined in proposition 2.1. Indeed, let u be a section of $\mathcal{B}$. Then, due to formula (2.13) $\lambda(u)$ satisfies the following equation

$$\kappa(D(\lambda(u))) = 0\,. \tag{2.30}$$

Let us denote $D\lambda(u)$ by Y

$$Y = D\lambda(u)\,.$$

The equation $D^2 = 0$ implies that

$$DY = 0$$

which is equivalent to

$$\delta Y = \nabla Y + A\cdot Y \tag{2.31}$$

Applying (2.13) to Y and using equations (2.30), (2.31) we get

$$Y = \kappa(\nabla Y + A\cdot Y)\,.$$

The latter equation has the unique vanishing solution since the operator κ (2.15) raises the degree in the fiber coordinates y^i.

The map (2.29) is obviously injective. To prove that the map is surjective we notice that $\mathcal{H}$

$$\mathcal{H} \,:\, \mathcal{B} \to \mathcal{B} \cap \ker\delta$$

is a left inverse of the map (2.29). Thus it suffices to prove that if $a \in \Gamma(\mathcal{B}) \cap \ker D$ and

$$\mathcal{H}a = 0 \tag{2.32}$$

then a vanishes.

The condition $a \in \ker D$ is equivalent to the equation

$$\delta a = \nabla a + A\cdot a\,.$$

Hence, applying (2.13) to a and using (2.32) we get

$$a = \kappa(\nabla a + A\cdot a)\,.$$

The latter equation has the unique vanishing solution since the operator κ (2.15) raises the degree in the fiber coordinates y^i. Thus, the map (2.29) is bijective and the map $\mathcal{H}$

$$\mathcal{H} \,:\, \mathcal{B} \cap \ker D \to \mathcal{B} \cap \ker\delta \tag{2.33}$$

is the inverse of (2.29).

It remains to prove that the map (2.29) is compatible with the multiplication of the sections of the sheaf $\mathcal{B}$, where $\mathcal{B}$ is either $\hat{S}(E^\vee)$, $\mathcal{A}$, $\mathcal{T}$, or $\mathcal{D}$. The latter follows immediately from the fact that the inverse map $\mathcal{H}$

$$\mathcal{H} \,:\, \mathcal{B} \to \mathcal{B} \cap \ker\delta \tag{2.34}$$

respects the corresponding algebra structures on $\hat{S}(E^\vee)$, $\mathcal{A}$, $\mathcal{T}$, and the DGA structure on $\mathcal{D}$. □

Let us now mention that since the Fedosov differential (2.21) is compatible with the graded algebraic structures on the sheaves ${}^E\Omega(\mathcal{T})$ and ${}^E\Omega(\mathcal{A})$ we conclude that $H^*({}^E\Omega(\mathcal{T}), D)$ is a sheaf of graded Lie algebras and $H^*({}^E\Omega(\mathcal{A}), D)$ is a sheaf of graded modules over $H^*({}^E\Omega(\mathcal{T}), D)$. On the other hand the above theorem tells us that

$$H^*({}^E\Omega(\mathcal{A}), D) = {}^E A_* \,,$$

and

$$H^*({}^E\Omega(\mathcal{T}), D) = \mathcal{T}^* \cap \ker \delta,$$

Furthermore, the sheaf $\mathcal{T}^* \cap \ker \delta$ in the right hand side of the latter equation can be canonically identified with ${}^E T^*_{poly} = \wedge^{*+1} E$ as a sheaf of vector spaces.

Thus, it is natural to ask whether the graded algebraic structures on the sheaves $\mathcal{T}^* \cap \ker \delta$ and ${}^E A_*$ coincide with the ones given by Lie bracket (1.3) (1.4) and the Lie derivative (1.7). A positive answer to this question is given by the following proposition:

Proposition 2.5. *The composition*

$$\mathcal{H}' = \nu \circ \mathcal{H} : \mathcal{T}^* \cap \ker D \to {}^E T^*_{poly} \tag{2.35}$$

of the map

$$\mathcal{H} \,:\, \mathcal{T}^* \cap \ker D \to \mathcal{T}^* \cap \ker \delta \tag{2.36}$$

with the identification of the sheaves $\mathcal{T}^* \cap \ker \delta$ *and* ${}^E T^*_{poly} \cong \wedge^{*+1} E$

$$\nu : \mathcal{T}^* \cap \ker \delta \xrightarrow{\sim} {}^E T^*_{poly} \tag{2.37}$$

induces an isomorphism of the sheaves of graded Lie algebras $H^*({}^E\Omega(\mathcal{T}), D) \cong {}^E T^*_{poly}$*. The map*

$$\mathcal{H} \,:\, \mathcal{A}_* \cap \ker D \to {}^E A_* \tag{2.38}$$

induces an isomorphism of the sheaves of graded modules $H^*({}^E\Omega(\mathcal{A}), D) \cong {}^E A_*$ *over the sheaf of graded Lie algebras* $H^*({}^E\Omega(\mathcal{T}), D) \cong {}^E T^*_{poly}$ *.*

Proof. The first part of the proposition is proved in [1] (see proposition 2.4). To prove the second part, we first remark that the maps $\mathcal{H}$ and ν are compatible with the cup products.

Next, we show that for any D-closed fiberwise differential form $\omega \in \Gamma(\mathcal{A})$ one has

$$\mathcal{H}(d^f \omega) = {}^E d \mathcal{H}(\omega) \,,$$

where $d^f = dy^i \frac{\partial}{\partial y^i}$ is the fiberwise De Rham differential on $\mathcal{A}$. Since

$$\mathcal{H} : \mathcal{A}_* \to {}^E A_*$$

is a morphism of graded commutative algebras, it is sufficient to prove it for functions and 1-forms:

- *First case.* Let f be a function on M and

$$\omega = \lambda(f) \,.$$

A direct computation shows that

$$\lambda(f) = f + y^i \rho(e_i) f \mod |y|^2 \,.$$

Therefore $d^f\omega = \rho(e_i) f dy^i \mod |y|$, and hence, $\mathcal{H}(d^f\omega) = {}^E df$.

- *Second case.* Let $\alpha = \alpha_i(x) dy^i$ be a E-1-form and

$$\omega = \lambda(\alpha)\,.$$

It is not hard to show that

$$\lambda(\alpha) = \alpha + y^i\big(\rho(e_i)\alpha_j - \Gamma^k_{ij}\alpha_k\big)dy^j \mod |y|^2\,.$$

Therefore,

$$d^f\omega = \big(\rho(e_i)\alpha_j - \Gamma^k_{ij}\alpha_k\big)dy^i \wedge dy^j \mod |y| = \big(\rho(e_i)\alpha_j - \frac{1}{2}c^k_{ij}\alpha_k\big)dy^i \wedge dy^j \mod |y|\,,$$

and hence,

$$\mathcal{H}(d^f\omega) = {}^E d\alpha\,.$$

To finish the proof we notice that for any fiberwise polyvector field $u \in \Gamma(\mathcal{T}^*)$ and any fiberwise differential form $\omega \in \Gamma(\mathcal{A})$, the equation

$$\mathcal{H}(\iota_u\omega) = \iota_{\mathcal{H}(u)} \circ \mathcal{H}(\omega)$$

is obviously satisfied. The latter implies that for any pair of D-closed sections $u \in \Gamma(\mathcal{T}^*)$, $\omega \in \Gamma(\mathcal{A}_*)$

$$\mathcal{H}(L_u\omega) = {}^E L_{\mathcal{H}(u)} \circ \mathcal{H}(\omega)\,,$$

and the proposition follows. □

Remark. Actually, we have proved a slightly stronger statement. Namely, we shown that the maps (2.38) and (2.35) induce an isomorphism of the sheaves of *calculi*.

$$(H^*({}^E\Omega(\mathcal{T}), D), H^*({}^E\Omega(\mathcal{A}), D)) \cong ({}^E T^*_{poly}, {}^E A_*)\,.$$

The precise definition of the calculi can be found in section 4.3 of the second part of [5].

Let us now recall that $\mathcal{T}^{-1} = \hat{S}(E^\vee)$, $\mathcal{T}^0$ is a sheaf of Lie-Rinehart algebras [24] over the sheaf of algebras $\mathcal{T}^{-1} = \hat{S}(E^\vee)$, and $\mathcal{D}^0$ is the universal enveloping algebroid of $\mathcal{T}^0$. Therefore, the inverse $(\mathcal{H}')^{-1}$ of the map (2.35) induces the morphism

$$\mu : \mathcal{U}E \to \mathcal{D}^0\,. \tag{2.39}$$

of the sheaves of bialgebras and for any $P \in \Gamma(\mathcal{U}E)$

$$D(\mu(P)) = 0\,. \tag{2.40}$$

We claim that

Proposition 2.6. *The map (2.39) gives the isomorphism*

$$\mu : \mathcal{U}E \to \mathcal{D}^0 \cap \ker D\,. \tag{2.41}$$

of the sheaves of bialgebras.

Proof. Notice that $\mathcal{U}E$ and $\mathcal{D}^0$ are both filtered sheaves of algebras. The filtration on $\mathcal{U}E$ is defined in (1.13) and the filtration on $\mathcal{D}^0$ is given by the degree of differential operators.

Thanks to the results of [23] and [24] we have the PBW theorem for Lie algebroids. This theorem says that the associated graded module of the filtration (1.13) on $\mathcal{U}E$ is

$$Gr(\mathcal{U}E) = S(E)$$

the symmetric algebra of the bundle E.

Furthermore, it is not hard to see that the map μ is compatible with the filtrations on $\mathcal{U}E$ and $\mathcal{D}^0$ and due to theorem 2.4 and proposition 2.2 μ induces the isomorphism

$$S(E) \cong \mathcal{D}^0 \cap \ker D$$

of the associated graded sheaves of vector spaces. Therefore, the snake lemma argument implies that the map (2.41) is also an isomorphism onto the sheaf $\mathcal{D}^0 \cap \ker D$ of D-flat sections of $\mathcal{D}^0$. □

Let us recall that ${}^E D^*_{poly}$ (resp. $\mathcal{D}^*$) is the tensor algebras of $\mathcal{U}E$ over $\mathcal{O}_M$ (resp. the tensor algebra of $\mathcal{D}^0$ over $\hat{S}(E^\vee)$). Using this fact we extend (2.39) to the morphism

$$\mu' : {}^E D^*_{poly} \to \mathcal{D}^* . \tag{2.42}$$

of sheaves of DGAs (over $\mathbb{R}$) by setting

$$\mu'\Big|_{{}^E D^0_{poly}} = \mu\,, \qquad \mu'\Big|_{\mathcal{O}_M} = \lambda\,,$$

where λ is defined in (2.29) .

Let us also observe that since the map (2.39) is an morphism of the sheaves of bialgebras the map (2.42) a morphism of the sheaves of DGLAs (over $\mathbb{R}$). Furthermore, theorem 2.4 implies that the sheaf of DGAs $\mathcal{D}^* \cap \ker D$ is generated by the sheaf $\mathcal{D}^0 \cap \ker D$ over the sheaf of commutative algebras $\hat{S}(E^\vee) \cap \ker D \cong \mathcal{O}_M$. Therefore using proposition 2.6 we get the following result:

Proposition 2.7 (proposition 2.5, [1]). *The map (2.42) gives an isomorphism of the sheaves of DGLAs*

$$\mu' : {}^E D^*_{poly} \xrightarrow{\sim} \mathcal{D}^* \cap \ker D\,. \tag{2.43}$$

This map is also compatible with the DGA structures on the sheaves ${}^E D^*_{poly}$ *and* $\mathcal{D}^* \cap \ker D$ *by construction.* □

Let us consider the map of sheaves of graded vector spaces

$$\gamma : \mathcal{J}_* \to {}^E J^{poly}_*\,, \qquad \gamma(j)(P) = (\mu'(P))(j)\Big|_{y^i=0}\,, \tag{2.44}$$

$$j \in \Gamma(\mathcal{J}_k)\,, \quad P \in \Gamma({}^E D^k_{poly})\,.$$

We claim that

Theorem 2.8. *For any* $q \geq 1$

$$H^q({}^E\Omega(\mathcal{J}), D) = 0\,, \tag{2.45}$$

and the map (2.44) gives an isomorphism of the sheaves of DG modules over the sheaf of DGLAs ${}^E D^*_{poly} \cong \mathcal{D}^* \cap \ker D$

$$\gamma : \mathcal{J}_* \xrightarrow{\sim} {}^E J^{poly}_*\,. \tag{2.46}$$

This isomorphism sends the Fedosov connection (2.21) on $\mathcal{J}^*$ *to the Grothendieck connection (1.17) on* ${}^E J^{poly}_*$.

Proof. The first statement (2.45) follows easily from the spectral sequence argument. Indeed, using the zeroth collection of the fiber coordinates y_0^i (2.5) we introduce the decreasing filtration on the sheaf ${}^E\Omega(\mathcal{J})$

$$\dots \subset F^{p+1}({}^E\Omega(\mathcal{J})) \subset F^{p}({}^E\Omega(\mathcal{J})) \subset F^{p-1}({}^E\Omega(\mathcal{J})) \subset \dots \subset F^{0}({}^E\Omega(\mathcal{J})) = {}^E\Omega(\mathcal{J}),$$

where the components of the sections (2.5) of the sheaf $F^p({}^E\Omega(\mathcal{J}))$ have degree in $y_0^i \ge p$.

Since $D(F^p({}^E\Omega(\mathcal{J}))) \subset F^{p-1}({}^E\Omega(\mathcal{J}))$ the corresponding spectral sequence starts with

$$E_{-1}^{p,q} = F^p({}^E\Omega(\mathcal{J})^{p+q}).$$

Next, we observe that

$$d_{-1} = \xi^i \frac{\partial}{\partial y_0^i},$$

and hence, due to the Poincaré lemma for the formal disk we have

$$E_0^{p,q} = E_1^{p,q} = \dots = E_\infty^{p,q} = 0$$

whenever $p+q>0$. Thus, the first statement (2.45) of the theorem follows.

Since (2.39) is a morphism of sheaves of bialgebras the map μ' is compatible with the operation $\bullet$ (1.15)

$$\mu'(P \bullet Q) = \mu'(P) \bullet \mu'(Q), \qquad P, Q \in \Gamma({}^E D^*_{poly}).$$

Furthermore, μ' is obviously compatible with cyclic permutations

$$t\, \mu'(P_0 \otimes P_1 \otimes \dots \otimes P_l) = \mu'(P_1 \otimes P_2 \otimes \dots \otimes P_l \otimes P_0), \qquad P_i \in \Gamma(\mathcal{U}E).$$

Hence, for any $P \in \Gamma({}^E D^*_{poly})$ and any $a \in \Gamma(\mathcal{J}_*)$

$$ {}^E S_P(\gamma(a)) = \gamma(\mathcal{R}_{\mu'(P)}(a)). \tag{2.47}$$

Since $\mathcal{J}_*$ is dual to $\mathcal{D}^* \cap \ker \delta$ and $\mathcal{D}^* \cap \ker\delta \cong \mathcal{D}^* \cap \ker D \cong {}^E D^*_{poly}$ the map (2.46) is an isomorphism. It remains to prove that the map (2.46) sends the Fedosov connection (2.21) to the Grothendieck connection (1.17). This statement is proved by the following line of equations:

$$\begin{aligned}
\gamma(D_u j)(P) &= (\mu'(P))(D_u j)\Big|_{y^i=0} = (D_u[\mu'(P)(j)])\Big|_{y^i=0} \\
&= \rho(u)[\mu'(P)(j)]\Big|_{y^i=0} - (\iota_u \delta \bullet [\mu'(P)(j)])\Big|_{y^i=0} \\
&= \rho(u)[\mu'(P)(j)]\Big|_{y^i=0} - (\mu'(u) \bullet \mu'(P))(j)\Big|_{y^i=0} \\
&= \rho(u)[\mu'(P)(j)]\Big|_{y^i=0} - \mu'(u \bullet P)(j)\Big|_{y^i=0} \\
&= \rho(u)(\gamma(j))(P) - (\gamma(j))(u \bullet P) = (\nabla^G_u \gamma(j))(P),
\end{aligned}$$

where $u \in \Gamma(E)$, $j \in \Gamma(\mathcal{J}_k)$, $P \in \Gamma({}^E D^k_{poly})$, ι denotes the contraction of an E-vector field with E-differential forms, ρ is the anchor map, the operation $\bullet$ is defined in (1.15), and u is viewed both as a section of E and an E-differential operator.

□

3. Proof of the formality theorem for E-chains and its applications

Let us denote by

- $\lambda_A : {}^E A_* \to {}^E\Omega(\mathcal{A}_*)$ the map λ (2.29) defined in the proof of Theorem 2.4 for $\mathcal{B} = {}^E\Omega(\mathcal{A}_*)$,
- $\lambda_T : {}^E T^*_{poly} \to {}^E\Omega(\mathcal{T})$, the inverse of the map $\mathcal{H}'$ (2.35),
- $\lambda_D : {}^E D^*_{poly} \to {}^E\Omega(\mathcal{D})$, the map μ' (2.42) and
- $\lambda_C : {}^E C^{poly}_* \to {}^E\Omega(\mathcal{J})$, the composition $\gamma^{-1} \circ \varrho$ of the inverse of the map γ (2.44) with the map ϱ (1.28).

The results of the previous section can be represented in the form of the following commutative diagrams of sheaves of DGLAs, DGLA modules, and their honest (not L_∞-) morphisms

$$
\begin{array}{ccc}
({}^E T^*_{poly}, 0, [,]_{SN}) & \stackrel{\lambda_T}{\succ\!\!\longrightarrow} & ({}^E\Omega(\mathcal{T}), D, [,]_{SN}) \\
\downarrow^{E}L_{mod} & & \downarrow L_{mod} \\
({}^E A_*, 0) & \stackrel{\lambda_A}{\succ\!\succ\!\!\longrightarrow} & ({}^E\Omega(\mathcal{A}), D),
\end{array}
\tag{3.1}
$$

$$
\begin{array}{ccc}
({}^E\Omega(\mathcal{D}), D + \partial, [,]_G) & \stackrel{\lambda_D}{\longleftarrow\!\!\prec} & ({}^E D^*_{poly}, \partial, [,]_G) \\
\downarrow \mathcal{R}_{mod} & & \downarrow^{E}R_{mod} \\
({}^E\Omega(\mathcal{J}), D + \mathfrak{b}) & \stackrel{\lambda_C}{\longleftarrow\!\!\prec\!\prec} & ({}^E C^{poly}_*, \mathfrak{b}),
\end{array}
$$

where the horizontal arrows correspond to embeddings of the sheaves of DGLAs (resp. of DGLA modules) constructed in the previous section. These embeddings are quasi-isomorphisms by theorems 2.4, 2.8 and propositions 2.5, 2.7 .

Next, due to claims *1* and *2* in theorem 1.12 we have a fiberwise quasi-isomorphism

$$
\mathcal{K} : ({}^E\Omega(\mathcal{T}), 0, [,]_{SN}) \succ\!\!\longrightarrow ({}^E\Omega(\mathcal{D}), \partial, [,]_G)
\tag{3.2}
$$

from the sheaf of DGLAs $({}^E\Omega(\mathcal{T}), 0, [,]_{SN})$ to the sheaf of DGLAs $({}^E\Omega(\mathcal{D}), \partial, [,]_G)$. Composing quasi-isomorphism (3.2) with the action of ${}^E\Omega(\mathcal{D})$ on ${}^E\Omega(\mathcal{J})$ we get an L_∞-module structure on ${}^E\Omega(\mathcal{J})$ over ${}^E\Omega(\mathcal{T})$.

Due to claims *1* and *2* in theorem 1.13 we have a fiberwise quasi-isomorphism

$$
\mathcal{S} : ({}^E\Omega(\mathcal{J}), \mathfrak{b}) \succ\!\succ\!\!\longrightarrow ({}^E\Omega(\mathcal{A}), 0)
\tag{3.3}
$$

from the sheaf of L_∞-modules ${}^E\Omega(\mathcal{J})$ to the sheaf of DGLA modules ${}^E\Omega(\mathcal{A})$ over ${}^E\Omega(\mathcal{T})$.

Thus we get the following commutative diagram

$$
\begin{array}{ccc}
({}^E\Omega(\mathcal{T}), 0, [,]_{SN}) & \stackrel{\mathcal{K}}{\succ\!\!\longrightarrow} & ({}^E\Omega(\mathcal{D}), \partial, [,]_G) \\
\downarrow L_{mod} & & \downarrow \mathcal{R}_{mod} \\
({}^E\Omega(\mathcal{A}), 0) & \stackrel{\mathcal{S}}{\longleftarrow\!\!\prec\!\prec} & ({}^E\Omega(\mathcal{J}), \mathfrak{b}),
\end{array}
\tag{3.4}
$$

where by commutativity we mean that $\mathcal{S}$ is a morphism of the sheaves of L_∞-modules $({}^E\Omega(\mathcal{J}), \mathfrak{b})$ and $({}^E\Omega, 0)$ over the sheaf of DGLAs $({}^E\Omega(\mathcal{T}), 0, [,]_{SN})$ and the L_∞-module structure on $({}^E\Omega(\mathcal{J}), \mathfrak{b})$ over $({}^E\Omega(\mathcal{T}), 0, [,]_{SN})$ is obtained by composing the quasi-isomorphism $\mathcal{K}$ with the action $\mathcal{R}$ (see (3.4) in [7]) of $({}^E\Omega(\mathcal{D}), \partial, [,]_G)$ on $({}^E\Omega(\mathcal{J}), \mathfrak{b})$.

Let us now restrict ourselves to an open subset $V \subset M$ such that $E\Big|_V$ is trivial. Over any such subset the E-de Rham differential (1.5) is well defined for either of the sheaves ${}^E\Omega(\mathcal{A})$, ${}^E\Omega(\mathcal{T})$, ${}^E\Omega(\mathcal{J})$, and ${}^E\Omega(\mathcal{D})$. Furthermore, since the quasi-isomorphisms (3.2) and (3.3) are fiberwise we can add to all the differentials in diagram (3.4) the E-de Rham differential (1.5). Thus we get a new commutative diagram

$$
\begin{array}{ccc}
({}^E\Omega(\mathcal{T})\Big|_V, {}^Ed, [,]_{SN}) & \overset{\mathcal{K}}{\succ\!\!\rightarrow} & ({}^E\Omega(\mathcal{D})\Big|_V, {}^Ed+\partial, [,]_G) \\
\downarrow^{L}_{mod} & & \downarrow^{\mathcal{R}}_{mod} \\
({}^E\Omega(\mathcal{A})\Big|_V, {}^Ed) & \overset{\mathcal{S}}{\leftarrow\!\!\prec\!\!\prec} & ({}^E\Omega(\mathcal{J})\Big|_V, {}^Ed+\mathfrak{b})
\end{array}
\tag{3.5}
$$

of the L_∞-morphism $\mathcal{K}$ and the morphism of L_∞-modules $\mathcal{S}$.

We claim that

Proposition 3.1. *The L_∞-morphism $\mathcal{K}$ and the morphism of L_∞-modules $\mathcal{S}$ in (3.5) are quasi-isomorphisms.*

Proof. This statement follows easily from the standard argument of the spectral sequence. Indeed, we can naturally regard ${}^E\Omega(\mathcal{T})$ and ${}^E\Omega(\mathcal{D})$ (resp. ${}^E\Omega(\mathcal{J})$ and ${}^E\Omega(\mathcal{A})$) as sheaves of double complexes and the exterior degree provides us with the following descending filtration

$$F^p({}^E\Omega^*(\mathcal{B})) = \bigoplus_{k\geq p} {}^E\Omega^k(\mathcal{B}),$$

where $\mathcal{B}$ is either $\mathcal{T}$ or $\mathcal{D}$ (resp. $\mathcal{J}$ or $\mathcal{A}$).

The corresponding versions of Vey's [31] and Hochschild-Kostant-Rosenberg-Connes-Teleman [3], [15], [29] theorems for $\mathbb{R}^d_{formal}$ imply that $\mathcal{K}$ (resp. $\mathcal{S}$) induces a quasi-isomorphism on the level of E_0. Hence, $\mathcal{K}$ (resp. $\mathcal{S}$) induces a quasi-isomorphism on the level of E_∞. The standard snake lemma argument of homological algebra implies that $\mathcal{K}$ (resp. $\mathcal{S}$) in (3.5) is a quasi-isomorphism. □

On the open subset V we can represent the Fedosov differential (2.21) in the following (non-covariant) form

$$D = {}^Ed + B\,\cdot\,, \tag{3.6}$$

$$B = \sum_{p=0}^{\infty} \xi^i B^k_{i;j_1\dots j_p}(x) y^{j_1}\dots y^{j_p}\frac{\partial}{\partial y^k}.$$

If we regard B as a section of ${}^E\Omega^1(\mathcal{T}^0)\Big|_V$ then the nilpotency condition $D^2=0$ says that B is a Maurer-Cartan section of the sheaf of DGLAs $({}^E\Omega(\mathcal{T})\Big|_V, {}^Ed, [,]_{SN})$. In the terminology of section 2 in [7] this means that the sheaf of DGLAs $({}^E\Omega(\mathcal{T})\Big|_V,$

$D, [,]_{SN})$ is obtained from $({}^E\Omega(\mathcal{T})\Big|_V, {}^E d, [,]_{SN})$ via the twisting procedure by the Maurer-Cartan element B.

According to proposition 1 in section 2 of [7] the element

$$B_D = \sum_{k=1}^{\infty} \frac{1}{k!} \mathcal{K}_k(B, \dots, B)$$

is a Maurer-Cartan section of $({}^E\Omega(\mathcal{D})\Big|_V, {}^E d + \partial, [,]_G)$. Moreover, due to claim *3* in theorem 1.12

$$B_D = B\,,$$

where B is viewed as a section of the sheaf ${}^E\Omega^1(\mathcal{D}^0)\Big|_V$.

Thus twisting the quasi-isomorphism $\mathcal{K}$ in (3.5) by the Maurer-Cartan element B we get the quasi-isomorphism

$$\mathcal{K}^{tw} \,:\, ({}^E\Omega(\mathcal{T})\Big|_V, D, [,]_{SN}) \succ\!\!\rightarrow ({}^E\Omega(\mathcal{D})\Big|_V, D + \partial, [,]_G)\,.$$

Since the DGLA module structure on ${}^E\Omega(\mathcal{A})$ over ${}^E\Omega(\mathcal{T})$ (resp. on ${}^E\Omega(\mathcal{J})$ over ${}^E\Omega(\mathcal{D})$) is honest the twist by the Maurer-Cartan element described in section 2 of [7] does not change these structures. Hence, by virtue of propositions 3 and 4 in [7] the twisting procedure turns diagram (3.5) into the commutative diagram

$$\begin{array}{ccc} ({}^E\Omega(\mathcal{T})\Big|_V, D, [,]_{SN}) & \stackrel{\mathcal{K}^{tw}}{\succ\!\!\rightarrow} & ({}^E\Omega(\mathcal{D})\Big|_V, D + \partial, [,]_G) \\ \downarrow^{L}_{mod} & & \downarrow^{\mathcal{R}}_{mod} \\ ({}^E\Omega(\mathcal{A})\Big|_V, D) & \stackrel{\mathcal{S}^{tw}}{\leftarrow\!\!\prec\!\!\prec} & ({}^E\Omega(\mathcal{J})\Big|_V, D + \mathfrak{b}), \end{array} \tag{3.7}$$

where $\mathcal{S}^{tw}$ is a quasi-isomorphism obtained from $\mathcal{S}$ by twisting via the Maurer-Cartan section B of the sheaf of DGLAs $({}^E\Omega(\mathcal{T})\Big|_V, {}^E d, [,]_{SN})$.

We claim that the morphism $\mathcal{K}^{tw}$ (resp. $\mathcal{S}^{tw}$) does not depend on the choice of the trivialization of E over V and hence is a well-defined L_∞-morphism of sheaves of DGLAs (resp. sheaves of DGLA modules). Indeed, the term in B that depends on the choice of the trivialization of E is linear in the fiber coordinates y^i. But due to claim 4 in theorem 1.12 and claim 3 in theorem 1.13 this term contribute neither to $\mathcal{K}^{tw}$ nor to $\mathcal{S}^{tw}$.

Thus the quasi-isomorphisms $\mathcal{K}^{tw}$ and $\mathcal{S}^{tw}$ are well defined and we arrive at the following commutative diagram

$$\begin{array}{ccc} ({}^E\Omega(\mathcal{T}), D, [,]_{SN}) & \stackrel{\mathcal{K}^{tw}}{\succ\!\!\rightarrow} & ({}^E\Omega(\mathcal{D}), D + \partial, [,]_G) \\ \downarrow^{L}_{mod} & & \downarrow^{\mathcal{R}}_{mod} \\ ({}^E\Omega(\mathcal{A}), D) & \stackrel{\mathcal{S}^{tw}}{\leftarrow\!\!\prec\!\!\prec} & ({}^E\Omega(\mathcal{J}), D + \mathfrak{b}). \end{array} \tag{3.8}$$

Assembling diagrams (3.1) and (3.8) we get the desired chain (1.34) of quasi-isomorphisms between the sheaves of DGLA modules $({}^E T^*_{poly}, {}^E A_*)$ and $({}^E D^*_{poly}, {}^E C^{poly}_*)$. It is obvious from the construction that the terms and the quasi-isomorphisms of the resulting diagram (1.34) are functorial in the pair (E, ∂^E), where ∂^E is a torsion-free connection on E. Thus, theorem 1.11 is proved. □

The obvious applications of the formality theorem for E-chains are related to the deformations associated with Poisson Lie algebroids. Namely, theorem 1.11 allows us to get an elegant description of the Hochschild homology and the traces of these deformations.

First, we recall that

Definition 3.2. *A Lie algebroid* (E, M, ρ) *equipped with an* E*-bivector* $\pi \in \Gamma(M, {}^{E}T^{1}_{poly})$ *satisfying the Jacobi identity*

$$[\pi, \pi]_{SN} = 0 \tag{3.9}$$

is called a Poisson Lie algebroid.

Following [22] a quantization of a Poisson Lie algebroid is a construction of an element

$$\Pi \in \Gamma(M, {}^{E}D^{1}_{poly})[[\hbar]] \tag{3.10}$$

satisfying the condition of the classical limit

$$\Pi = I \otimes I \bmod \hbar\,, \qquad \Pi - t(\Pi) = \hbar\pi \bmod \hbar\,, \tag{3.11}$$

and the "associativity" condition

$$[\Pi, \Pi]_{G} = 0\,. \tag{3.12}$$

Here $\hbar$ is an auxiliary variable and t denotes the (cyclic) permutation of components of $\Pi \in \Gamma(M, {}^{E}D^{1}_{poly})[[\hbar]] = \Gamma(M, \mathcal{U}E \otimes \mathcal{U}E)[[\hbar]]$.

Furthermore, two deformations Π and Π' of (E, M, ρ, π) are called *equivalent* if there exists a formal power series

$$\Psi = I + \hbar\Psi_1 + \hbar^2\Psi_2 + \ldots \in \Gamma(M, \mathcal{U}E)[[\hbar]]$$

such that

$$\Pi'(\Delta\Psi) = (\Psi \otimes \Psi)\,\Pi\,, \tag{3.13}$$

where Δ is the coproduct (1.14) in $\mathcal{U}E$.

Thanks to the formality theorem for the sheaf of DGLAs ${}^{E}D^{*}_{poly}$ proved in [1] and theorem 2.2.2 in [11] we have a bijective correspondence between the moduli space of Maurer-Cartan elements of the DGLA $\Gamma(M, {}^{E}T^{*}_{poly})[[\hbar]]$ of E-polyvector fields and the moduli space of Maurer-Cartan elements of the DGLA $\Gamma(M, {}^{E}D^{*}_{poly})[[\hbar]]$ of E-polydifferential operators (tensored with $\mathbb{R}[[\hbar]]$). In other words, if we consider the cone

$$\pi_\hbar = \hbar\pi + \hbar^2\pi_1 + \hbar^3\pi_2 + \ldots\,,$$

$$[\pi_\hbar, \pi_\hbar]_{SN} = 0\,, \tag{3.14}$$

$$\pi_i \in \Gamma(M, {}^{E}T^{1}_{poly})$$

of formal power series in $\hbar$ acted upon by the Lie algebra $\hbar\Gamma(M, E)[[\hbar]]$

$$\pi_\hbar \to [u, \pi_\hbar]\,, \qquad u \in \hbar\Gamma(M, E)[[\hbar]]\,, \tag{3.15}$$

then

Corollary 3.3. *The deformations (3.10) associated with a Poisson Lie algebroid* (E, M, ρ, π) *modulo the relation (3.13) are in a bijective correspondence with the points of the cone (3.14) modulo the action (3.15) of the prounipotent group corresponding to the Lie algebra* $\hbar\Gamma(M, E)[[\hbar]]$. □

An orbit $[\pi_\hbar]$ on the cone (3.14) corresponding to a deformation Π (3.10) is called *the class of the deformation* and any point $\pi_\hbar$ of this orbit is called *a representative* of the class.

Given a deformation Π (3.10) associated with a Poisson Lie algebroid (E, M, ρ, π) one can define Hochschild chain complex of this deformation as the graded vector space

$$\Gamma(M, {}^E C^{poly}_*)[[\hbar]] \tag{3.16}$$

equipped with the differential

$${}^E R_\Pi : {}^E C^{poly}_* \to {}^E C^{poly}_{*+1} \,.$$

Furthermore, one defines the Hochschild cochain complex of the deformation Π as the graded vector space

$$\Gamma(M, {}^E D^*_{poly})[[\hbar]] \tag{3.17}$$

equipped with the differential

$$[\Pi, \,] : {}^E D^*_{poly} \to {}^E D^{*+1}_{poly} \,.$$

Due to claim 5 of proposition 2 in [7], claim 5 of proposition 3 in [7], and theorem 1.11 we get the following result:

Corollary 3.4. *Let Π be a deformation associated with a Poisson Lie algebroid (E, M, ρ, π) and let $\pi_\hbar$ be a representative of the class of this deformation. Then the complex of Hochschild cohomology (3.17) of the deformation Π is quasi-isomorphic to the complex of E-polyvector fields*

$$(\Gamma(M, {}^E T^*_{poly})[[\hbar]], [\pi_\hbar, \,]) \tag{3.18}$$

with the differential $[\pi_\hbar, \,]$. The complex of Hochschild homology (3.16) of the deformation Π is quasi-isomorphic to the complex of E-forms

$$({}^E A_*(M)[[\hbar]], {}^E L_{\pi_\hbar}) \tag{3.19}$$

with the differential ${}^E L_{\pi_\hbar}$. □

Given a deformation Π (3.10) associated with a Poisson Lie algebroid (E, M, ρ, π) one can define *a trace* of this deformation as an $\mathbb{R}[[\hbar]]$-linear functional

$$tr : \mathcal{O}(M)[[\hbar]] \to \mathbb{R}[[\hbar]] \tag{3.20}$$

satisfying the following condition

$$tr(j(\Pi) - j(t(\Pi))) = 0 \,, \qquad \forall\, j \in \Gamma(M, {}^E J^{poly}_1) \cap \ker \nabla^G \,. \tag{3.21}$$

It is not hard to see that corollary 3.4 implies the following statement:

Corollary 3.5. *Let Π be a deformation associated with a Poisson Lie algebroid (E, M, ρ, π) and let $\pi_\hbar$ be a representative of the class of this deformation. Then the vector space of traces of the deformation Π is isomorphic to the vector space of continuous $\mathbb{R}[[\hbar]]$-linear $\mathbb{R}[[\hbar]]$-valued functionals on $\mathcal{O}(M)[[\hbar]]$ vanishing on all functions $f \in \mathcal{O}(M)[[\hbar]]$ of the following form*

$$f = j(\pi_\hbar) \,, \qquad j \in \Gamma(M, {}^E J^{poly}_1) \cap \ker \nabla^G \,,$$

where $\pi_\hbar$ is viewed as a series E-bidifferential operators. □

4. Formality theorems for holomorphic Lie algebroids

Let now M be a complex manifold. Let us write $TM = T^{1,0}M \oplus T^{0,1}M$ for the decomposition of the tangent bundle into the sum of the holomorphic tangent bundle and anti-holomorphic tangent bundle. We denote by $\mathcal{O}_M$ the structure sheaf of holomorphic functions on M and by z^α local coordinates on M. We have to adapt the definition of holomorphic Lie algebroids:

Definition 4.1. *A holomorphic Lie algebroid over a complex manifold M is a holomorphic vector bundle E of finite rank whose sheaf of sections is a sheaf of Lie algebras equipped with a holomorphic map of sheaves of Lie algebras*

$$\rho : E \to T^{1,0} .$$

satisfying the same condition described (for the smooth case) in formula (1.1).

Let E be a holomorphic Lie algebroid. As in section 1, one can define the following sheaves (which are also holomorphic vector bundles):

- ${}^E T^*_{poly}$ is the sheaf of E-polyvector fields. We regard ${}^E T^*_{poly}$ as a sheaf of DGLAs with the vanishing differential and with the Lie bracket $[,]_{SN}$ defined as in (1.3), (1.4).
- ${}^E A_*$ is the sheaf of E-differential forms with converted grading:

$$(4.1) \qquad {}^E A_* = \wedge^{-*} E^\vee , \qquad {}^E A_0 = \mathcal{O}_M .$$

  We regard ${}^E A_*$ as a sheaf of DGLA modules over ${}^E T^*_{poly}$ with the vanishing differential and with the action ${}^E L$ defined as in (1.7). For sections

$$(4.2) \qquad a = \sum_{m \ge 0} a_{i_1 \dots i_m}(z) dy^{i_1} \dots dy^{i_m}$$

  of the sheaf ${}^E A_*$ we reserve the basis of local E-1-forms $\{dy^i\}$, where y^i are fiber coordinates on E.
- ${}^E D^*_{poly}$ is a sheaf of E-polydifferential operators. We regard ${}^E D^*_{poly}$ as a sheaf of DGLAs with the bracket $[,]_G$ and the differential ∂ defined as in (1.16) and (1.26). Notice that the tensor product of sections (over $\mathcal{O}_M$) of ${}^E D^*_{poly}$ turns ${}^E D^*_{poly}$ into a sheaf of DGAs.
- ${}^E J^{poly}_*$ is the sheaf of E-polyjets

$${}^E J^{poly} = \bigoplus_{k \ge 0} {}^E J^{poly}_k , \quad {}^E J^{poly}_* := Hom_{\mathcal{O}_M}({}^E D^*_{poly}, \mathcal{O}_M) ,$$

  which we regard as a sheaf of DGLA modules over ${}^E D^*_{poly}$ with the action ${}^E S$ and the differential $\mathfrak{b}$ defined as in (1.22) and (1.26). The sheaf ${}^E J^{poly}_*$ is also equipped with the Grothendieck connection

$$(4.3) \qquad \nabla^G : T^{1,0} \otimes {}^E J^{poly}_* \mapsto {}^E J^{poly}_* , \qquad \nabla^G_u(j)(P) := \rho(u)(j(P)) - j(u \bullet P) ,$$

  where $u \in \Gamma(T^{1,0})$ is a holomorphic vector field, $P \in \Gamma({}^E D^k_{poly})$, $j \in \Gamma({}^E J^{poly}_k)$ and the operation $\bullet$ is defined in (1.15). The connection (4.3) is compatible the DGLA module structure on ${}^E J^{poly}_*$.
- ${}^E C^{poly}_*$ is the graded sheaf of ∇^G-flat E-polyjets with converted grading

$$(4.4) \qquad {}^E C^{poly}_* := \ker \nabla^G \cap {}^E J^{poly}_{-*} .$$

Due to the compatibility of the Grothendieck connection (4.3) with the DGLA module structure on E-polyjets ${}^E C^{poly}_*$ can be viewed as a sheaf of DG modules over sheaf of DGLAs ${}^E D^*_{poly}$. We refer to ${}^E C^{poly}_*$ as a sheaf of Hochschild E-chains or E-chains for short.

The main result of this section can be formulated as follows:

Theorem 4.2. *For any holomorphic Lie algebroid E over a complex manifold M the sheaves of DGLA modules $({}^E T^*_{poly}, {}^E A_*)$ and $({}^E D^*_{poly}, {}^E C^{poly}_*)$ are quasi-isomorphic.*

Omitting the sheaves ${}^E A_*$ and ${}^E C^{poly}_*$ in the above theorem we get the following corollary:

Corollary 4.3. *For any holomorphic Lie algebroid E over a complex manifold M the sheaves of DGLAs ${}^E T^*_{poly}$ and ${}^E D^*_{poly}$ are quasi-isomorphic.*

We would like to mention that this corollary is parallel to the result of A. Yekutieli [34], who proved this statement for the tangent Lie algebroid $TX \to X$ of any smooth algebraic variety X over a field $\mathbb{K}$ for which $\mathbb{R} \subset \mathbb{K}$.

Notice that applying theorem 4.2 to the tangent algebroid $T^{1,0}M \to M$ we prove the following version of Tsygan's formality conjecture for complex manifolds:

Theorem 4.4. *For any complex manifold M the sheaf of DGLA modules $C^{poly}(M)$ of Hochschild chains over the sheaf $D_{poly}(M)$ of (holomorphic) polydifferential operators is formal.* $\square$

The proof of theorem 4.2 occupies the rest of the section.

First, we observe that any holomorphic Lie algebroid E can be viewed as a smooth Lie algebroid in the sense of definition 1.1, where the anchor map is naturally extended to the smooth sections of E. It is clear that the sheaf of Lie algebras $T^{0,1}$ acts on E and that this action commutes with ρ as ρ is holomorphic. Thus we get

Proposition 4.5. *Let F be the smooth vector bundle $F = E \oplus T^{0,1}$. Then F is a smooth Lie algebroid over M with the anchor map $\rho_F : F \to T^{1,0} \oplus T^{0,1}$ given by $\rho_F \Big|_E = \rho$ and $\rho_F \Big|_{T^{0,1}} = \mathrm{id} : T^{0,1} \to T^{0,1}$.* $\square$

For a holomorphic vector bundle $\mathcal{B}$ over M we consider the sheaf of smooth F-differential forms with values in $\mathcal{B}$:

$$
{}^F\Omega(\mathcal{B}) = \bigoplus_{p,q} {}^F\Omega^{p,q}(\mathcal{B}) \,, \tag{4.5}
$$

$$
{}^F\Omega^{p,q}(\mathcal{B}) = \wedge^p E^\vee \otimes \wedge^q T^{*0,1} M \otimes \mathcal{B} \,.
$$

We reserve the local basis $\{\xi^i\}$ of anti-commuting fiber coordinates on E and the local basis $\{d\bar{z}^\alpha\}$ of antiholomorphic exterior forms on M for sections of ${}^F\Omega(\mathcal{B})$:

$$
a = \sum_{p,q} a_{i_1 \dots i_p \,;\, \alpha_1, \dots, \alpha_q}(z, \bar{z}) \xi^{i_1} \dots \xi^{i_p} d\bar{z}^{\alpha_1} \dots d\bar{z}^{\alpha_q} \,, \tag{4.6}
$$

$$
a_{i_1 \dots i_p \,;\, \alpha_1, \dots, \alpha_q}(z, \bar{z}) \in \Gamma^{\text{smooth}}(\mathcal{B}) \,.
$$

We denote by $\bar{d}$ the Dolbeault differential

$$
\bar{d} = d\bar{z}^\alpha \partial_{\bar{z}^\alpha} : {}^F\Omega^{p,*}(\mathcal{B}) \mapsto {}^F\Omega^{p,*+1}(\mathcal{B}) \,. \tag{4.7}
$$

It is obvious that the (DG) algebraic structures on the sheaves ${}^E T^*_{poly}$, ${}^E A_*$, ${}^E D^*_{poly}$, and ${}^E J^{poly}_*$, can be naturally extended to the sheaves ${}^F\Omega^{0,*}({}^E T^*_{poly})$, ${}^F\Omega^{0,*}({}^E A_*)$, ${}^F\Omega^{0,*}({}^E D^*_{poly})$, and ${}^F\Omega^{0,*}({}^E J^{poly}_*)$. Similarly, the Grothendieck connection (4.3) on ${}^E J^{poly}_*$ extends to the operator

$$\nabla^G : T^{1,0} \otimes {}^F\Omega^{0,*}({}^E J^{poly}_*) \mapsto {}^F\Omega^{0,*}({}^E J^{poly}_*)\,, \tag{4.8}$$

which is compatible with the action ${}^E S$ of ${}^F\Omega^{0,*}({}^E D^*_{poly})$ on ${}^F\Omega^{0,*}({}^E J^{poly}_*)$ and with the differential $\mathfrak{b}$ on ${}^F\Omega^{0,*}({}^E J^{poly}_*)$.

Since ${}^E T^*_{poly}$, ${}^E A_*$, ${}^E D^*_{poly}$, and ${}^E J^{poly}_*$ are holomorphic vector bundles it makes sense to speak about the Dolbeault differential (4.7)

$$\bar{d} : {}^F\Omega^{0,*}(\mathcal{B}) \mapsto {}^F\Omega^{0,*+1}(\mathcal{B})\,, \tag{4.9}$$

for $\mathcal{B}$ being either ${}^E T^*_{poly}$, ${}^E A_*$, ${}^E D^*_{poly}$, or ${}^E J^{poly}_*$. It is obvious that $\bar{d}$ is compatible with the (DG) algebraic structures on ${}^F\Omega^{0,*}(\mathcal{B})$ and with the Grothendieck connection (4.8) on ${}^F\Omega^{0,*}({}^E J^{poly}_*)$.

Furthermore, due to the $\bar{d}$-Poincaré lemma we have

Proposition 4.6. *If $\mathcal{B}$ is either ${}^E T^*_{poly}$, ${}^E A_*$, ${}^E D^*_{poly}$, or ${}^E J^{poly}_*$ then the canonical inclusion of sheaves*

$$\text{inc} : \mathcal{B} \hookrightarrow {}^F\Omega^{0,*}(\mathcal{B}) \tag{4.10}$$

is a quasi-isomorphism of complexes of sheaves $(\mathcal{B}, 0)$ and $({}^F\Omega^{0,*}(\mathcal{B}), \bar{d})$. The inclusion inc *is compatible with the (DG) algebraic structures on $\mathcal{B}$, and ${}^F\Omega^{0,*}(\mathcal{B})$, and with the Grothendieck connection (4.3), (4.8).* □

Due to this proposition it suffices to prove that the sheaves of DGLA modules $({}^F\Omega^{0,*}({}^E T^*_{poly}), {}^F\Omega^{0,*}({}^E A_*))$, and $({}^F\Omega^{0,*}({}^E D^*_{poly}), {}^F\Omega^{0,*}({}^E J^{poly}_*))$ are quasi-isomorphic. To show this we follow the lines of section 2 and introduce the formally completed symmetric algebra $\hat{S}(E^\vee)$ of the dual bundle $E^\vee$ and (holomorphic) bundles $\mathcal{T}$, $\mathcal{D}$, $\mathcal{A}$, $\mathcal{J}$ associated with $\hat{S}(E^\vee)$ (see page 11). As in section 2, $\mathcal{T}$ and $\mathcal{D}$ are sheaves of DGLAs while $\mathcal{A}$ and $\mathcal{J}$ are sheaves of DGLA modules over $\mathcal{T}$ and $\mathcal{D}$, respectively. $\mathcal{D}$ is also a sheaf of DGA's.

Next, we consider sheaves of smooth F-differential forms with values in the bundles $\hat{S}(E^\vee)$ $\mathcal{T}$, $\mathcal{D}$, $\mathcal{A}$, and $\mathcal{J}$. It is clear that the sheaves ${}^F\Omega(\hat{S}(E^\vee))$, ${}^F\Omega(\mathcal{A})$, ${}^F\Omega(\mathcal{T})$, ${}^F\Omega(\mathcal{D})$, and ${}^F\Omega(\mathcal{J})$ acquire the corresponding (DG) algebraic structures and the Dolbeault differential (4.7) is obviously compatible with these structures.

Furthermore, we have the following obvious analogue of proposition 2.1

Proposition 4.7. *The sheaf ${}^F\Omega(\mathcal{T}^0)$ of F-forms with values in fiberwise vector fields is a sheaf of graded Lie algebras. The sheaves ${}^F\Omega(\hat{S}(E^\vee))$, ${}^F\Omega(\mathcal{A})$, ${}^F\Omega(\mathcal{T})$, ${}^F\Omega(\mathcal{D})$, and ${}^F\Omega(\mathcal{J})$ are sheaves of modules over ${}^F\Omega(\mathcal{T}^0)$ and the action of ${}^F\Omega(\mathcal{T}^0)$ is compatible with the DG algebraic structures on ${}^F\Omega(\hat{S}(E^\vee))$, ${}^F\Omega(\mathcal{A})$, ${}^F\Omega(\mathcal{T})$, ${}^F\Omega(\mathcal{D})$, ${}^F\Omega(\mathcal{J})$ and with the Dolbeault differential (4.7).* □

Due to this proposition one can extend the following differential

$$\delta := \xi^i \frac{\partial}{\partial y^i} : {}^F\Omega^{*,q}(\hat{S}(E^\vee)) \to {}^F\Omega^{*+1,q}(\hat{S}(E^\vee))$$

of the sheaf of algebras ${}^F\Omega(\hat{S}(E^\vee))$ to the sheaves ${}^F\Omega(\mathcal{T})$, ${}^F\Omega(\mathcal{D})$, ${}^F\Omega(\mathcal{A})$ and ${}^F\Omega(\mathcal{J})$ so that δ is compatible with the (DG) algebraic structures on ${}^F\Omega(\mathcal{T})$, ${}^F\Omega(\mathcal{A})$,

${}^F\Omega(\mathcal{D})$, and ${}^F\Omega(\mathcal{J})$, and with the differential $\bar{d}$ (4.7). Here $\{y^i\}$ (resp. $\{\xi^i\}$) denote commuting (resp. anticommuting) fiber coordinates of the bundle E.

We now have an analogue of proposition 2.2

Proposition 4.8. *For $\mathcal{B}$ being either of the sheaves $\hat{S}(E^\vee)$, $\mathcal{A}$, $\mathcal{T}$ or $\mathcal{D}$ and $q \geq 0$,*

$$H^{\geq 1}({}^F\Omega^{*,q}(\mathcal{B}), \delta) = 0\,.$$

Furthermore,

$$\begin{aligned} H^0({}^F\Omega^{*,q}(\hat{S}(E^\vee)), \delta) &\cong {}^F\Omega^{0,q}(M, \mathcal{O}_M)\,, \\ H^0({}^F\Omega^{*,q}(\mathcal{A}_*), \delta) &\cong {}^F\Omega^{0,q}(M, {}^E A_*)\,, \\ H^0({}^F\Omega^{*,q}(\mathcal{T}^*), \delta) &\cong {}^F\Omega^{0,q}(M, \wedge^{*+1}(E)) \end{aligned} \tag{4.11}$$

as sheaves of (graded) commutative algebras and

$$H^0({}^F\Omega^{*,q}(\mathcal{D}^*), \delta) \cong {}^F\Omega^{0,q}(M, \otimes^{*+1}(S(E))) \tag{4.12}$$

as sheaves of DGAs over $\mathcal{O}_M$.

Proof. As in proposition 2.2 it suffices to construct an operator ($q \geq 0$)

$$\kappa : {}^F\Omega^{*,q}(\mathcal{B}) \to {}^F\Omega^{*-1,q}(\mathcal{B}) \tag{4.13}$$

such that for any section u of ${}^F\Omega(\mathcal{B})$ equation

$$u = \delta\kappa(u) + \kappa\delta(u) + \mathcal{H}(u)\,, \tag{4.14}$$

is still true, where now

$$\mathcal{H}(u) = u\Big|_{y^i=\xi^i=0}\,. \tag{4.15}$$

and y^i are as above fiber coordinates on E. As in the proof of proposition 2.2 we define κ on ${}^F\Omega(\hat{S}(E^\vee))$ by equation (2.15) and then extend it to ${}^F\Omega(\mathcal{T})$, ${}^F\Omega(\mathcal{A})$, and ${}^F\Omega(\mathcal{D})$ in the componentwise manner. □

Let us choose a connection ∂^E on E which is compatible with the complex structure on E

$$\partial^E = {}^E d + \bar{d} + \xi^i \Gamma_i : {}^F\Omega^*(E) \to {}^F\Omega^{*+1}(E)\,, \tag{4.16}$$

where $\xi^i\Gamma_i$ is locally a section of the sheaf ${}^E\Omega^1(End(E))$ and ${}^E d : {}^F\Omega_M^{*,q} \to {}^F\Omega_M^{*+1,q}$ is defined in (1.5).

It is not hard to show that such a connection always exists, and moreover, one can always choose ∂^E to be torsion free.

As in (2.16) we extend (4.16) to a derivation of the DG sheaves ${}^F\Omega(\hat{S}(E^\vee))$, ${}^F\Omega(\mathcal{A})$, ${}^F\Omega(\mathcal{T})$, ${}^F\Omega(\mathcal{D})$, and ${}^F\Omega(\mathcal{J})$:

$$\nabla = {}^E d + \Gamma \cdot + \bar{d} : {}^F\Omega^*(\mathcal{B}) \to {}^F\Omega^{*+1}(\mathcal{B})\,, \tag{4.17}$$

where $\mathcal{B}$ is either of the sheaves $\hat{S}(E^\vee)$, $\mathcal{A}$, $\mathcal{T}$, $\mathcal{D}$, or $\mathcal{J}$, $\Gamma = -\xi^i \Gamma_{ij}^k y^j \frac{\partial}{\partial y^k}$, $\Gamma_{ij}^k(x)$ are Christoffel's symbols of the connection ∂^E (4.16) and $\Gamma\cdot$ denotes the action of Γ on the sections of the sheaves ${}^F\Omega(\mathcal{B})$. Due to proposition 4.7 the operator ∇ (4.17) is compatible with the DG algebraic structures on ${}^F\Omega(\hat{S}(E^\vee))$, ${}^F\Omega(\mathcal{T})$, ${}^F\Omega(\mathcal{A})$, ${}^F\Omega(\mathcal{D})$, and ${}^F\Omega(\mathcal{J})$, and since ∇ is torsion free

$$\nabla\delta + \delta\nabla = 0\,. \tag{4.18}$$

Regarding (4.17) as a connection on $\mathcal{B}$ one can see that the curvature of (4.17) has the components of type $(2,0)$ and $(1,1)$

$$\nabla^2 = R^{2,0} + R^{1,1}\,, \qquad R^{2,0} = ({}^E d + \Gamma)^2, \qquad R^{1,1} = \bar{d}\Gamma\,. \tag{4.19}$$

We now prove the existence of a complex Fedosov differential D:

Theorem 4.9. *Let $\mathcal{B}$ be either of the sheaves $\hat{S}(E^\vee)$, $\mathcal{A}$, $\mathcal{T}$, $\mathcal{D}$, or $\mathcal{J}$. There exists a section*

$$A = \sum_{s=2}^{\infty} \xi^k A^j_{k,i_1\dots i_s}(z,\bar{z}) y^{i_1}\cdots y^{i_s} \frac{\partial}{\partial y^j} \tag{4.20}$$

of the sheaf ${}^F\Omega^{1,0}(\mathcal{T}^0)$ and a section

$$\bar{A} = \sum_{s=2}^{\infty} d\bar{z}^\alpha \bar{A}^j_{\alpha,i_1\dots i_s}(z,\bar{z}) y^{i_1}\cdots y^{i_s} \frac{\partial}{\partial y^j} \tag{4.21}$$

of the sheaf ${}^F\Omega^{0,1}(\mathcal{T}^0)$ such that the derivation

$$D := \nabla - \delta + A\,\cdot + \bar{A}\cdot\, : {}^F\Omega^*(\mathcal{B}) \to {}^F\Omega^{*+1}(\mathcal{B}) \tag{4.22}$$

is 2-nilpotent ($D^2 = 0$) and compatible with the DG algebraic structure on ${}^F\Omega(\mathcal{B})$.

Proof. Let us rewrite $D = D^{1,0} + D^{0,1}$ with

$$D^{1,0} = {}^E d + \Gamma\,\cdot\, - \delta + A\cdot\,, \qquad D^{0,1} = \bar{d} + \bar{A}\cdot$$

and mimic the proof of theorem 2.3.

Due to (4.18) and (4.19) the condition $(D^{1,0})^2 = 0$ is equivalent to the equation

$$R^{2,0} + ({}^E d + \Gamma\,\cdot) A - \delta A + \frac{1}{2}[A,A]_{SN} = 0\,.$$

This equation has a solution obtained by iterations of the following equation (with respect to the degrees in fiber coordinates y^i's)

$$A = \kappa R^{2,0} + \kappa(({}^E d + \Gamma\,\cdot) A + \frac{1}{2}[A,A]_{SN})$$

(the proof is the same as for theorem 2.3).

Using (4.19) once again we observe that the condition $D^{1,0}D^{0,1} + D^{0,1}D^{1,0} = 0$ is equivalent to

$$R^{1,1} + \bar{d}A + ({}^E d + \Gamma\,\cdot)\bar{A} - \delta\bar{A} + [A,\bar{A}]_{SN} = 0\,,$$

which, using the same arguments, has a solution obtained by iterations of the equation

$$\bar{A} = \kappa(R^{1,1} + \bar{d}A + ({}^E d + \Gamma\,\cdot)\bar{A} + [A,\bar{A}]_{SN}).$$

Indeed, denoting

$$C^{1,1} = R^{1,1} + \bar{d}A + ({}^E d + \Gamma\,\cdot)\bar{A} - \delta\bar{A} + [A,\bar{A}]_{SN}\,,$$

and using that $\delta A = R^{2,0} + ({}^E d + \Gamma\,\cdot)A + \frac{1}{2}[A,A]_{SN}$ ($(D^{1,0})^2 = 0$), $\bar{d}R^{2,0} = 0$ and $\delta R^{1,1} = 0$ (Bianchi's identities for ∇) we get

$$({}^E d + \Gamma\,\cdot)C^{1,1} - \delta C^{1,1} + [A, C^{1,1}] = 0.$$

We have $\kappa C^{1,1} = 0$ by construction of $\bar{A}$ and so, by the "Hodge-de Rham" decomposition (4.14), we have

$$C^{1,1} = \kappa(({}^E d + \Gamma\,\cdot)C^{1,1} + [A, C^{1,1}]).$$

The latter equation has the unique vanishing solution, which gives the result.

Let us now check the condition $(D^{0,1})^2 = 0$. This will be true if the section

$$C^{0,2} = \bar{d}\bar{A} + \frac{1}{2}[\bar{A}, \bar{A}] \in {}^F\Omega^{0,2}(\mathcal{T}^0)$$

is zero. One has again $D^{1,0}C^{0,2} = 0$ and $\kappa C^{0,2} = 0$ because it does not have ξ's. As before, one can conclude that $C^{0,2} = 0$.

The compatibility of (4.22) with the corresponding DG algebraic structures follows from proposition 4.7. □

We now describe the cohomology of the Fedosov differential D for the sheaves ${}^F\Omega(\hat{S}(E^\vee))$, ${}^F\Omega(\mathcal{A})$, ${}^F\Omega(\mathcal{T})$, and ${}^F\Omega(\mathcal{D})$

Theorem 4.10. *Let $\mathcal{B}$ be either of the sheaves $\hat{S}(E^\vee)$, $\mathcal{A}$, $\mathcal{T}$, or $\mathcal{D}$. Then*

$$H({}^F\Omega^*(\mathcal{B}), D) \cong H({}^F\Omega^{0,*}(M, \mathcal{B}) \cap \ker \delta, \bar{d})\,.$$

as sheaves of (differential) graded (commutative) algebras.

Proof. Let us consider the double complex $({}^F\Omega^{*,*}(\mathcal{B}), D^{1,0}+D^{0,1})$. Using the degree in the fiber coordinates y^i we introduce on this complex a decreasing filtration. Applying the spectral sequence argument (as in the proof of theorem 2.4) and using proposition 4.8 we conclude that for any $i \geq 0$, the cohomology of the complex $({}^F\Omega^{*,i}(\mathcal{B}), D^{1,0})$ is concentrated in degree $* = 0$. Therefore,

$$H({}^F\Omega^*(\mathcal{B}), D) = H({}^F\Omega^{0,*}(\mathcal{B}) \cap \ker D^{1,0}, D^{0,1})\,. \tag{4.23}$$

Following the lines of the proof of theorem 2.4 it is not hard to show that iterating the equation

$$\lambda(u) = u + \kappa(\nabla\lambda(u) + A \cdot \lambda(u) + \bar{A} \cdot \lambda(u))\,, \qquad u \in {}^F\Omega^{0,q}(\mathcal{B}) \cap \ker \delta \tag{4.24}$$

we get an isomorphism of sheaves (of graded vector spaces)

$$\lambda : {}^F\Omega^{0,q}(\mathcal{B}) \cap \ker \delta \to {}^F\Omega^{0,q}(\mathcal{B}) \cap \ker D^{1,0}\,, \tag{4.25}$$

and moreover, the map λ (4.25) has a natural inverse given by the map $\mathcal{H}$ (4.15).

We claim that λ gives a quasi-isomorphism of complexes

$$\lambda : ({}^F\Omega^{0,*}(\mathcal{B}) \cap \ker \delta, \bar{d}) \to ({}^F\Omega^*(\mathcal{B}), D).$$

Indeed, due to (4.23) it suffices to show that for any $u \in {}^F\Omega^{0,q}(M, \mathcal{B}) \cap \ker \delta$, one has

$$\lambda(\bar{d}(u)) = D^{0,1}\lambda(u).$$

The term $\lambda(\bar{d}(u))$ is the only element in ${}^F\Omega^{0,q}(\mathcal{B})$ such that $\mathcal{H}(\lambda(\bar{d}(u))) = \bar{d}(u)$ and $D^{1,0}\lambda(\bar{d}(u)) = 0$. It is clear that $\mathcal{H}(D^{0,1}\lambda(u)) = \bar{d}(u)$ and one has

$$D^{1,0}D^{0,1}\lambda(u) = -D^{0,1}D^{1,0}\lambda(u) = 0,$$

since map λ (4.24) lands in $\ker D^{1,0}$.

The map λ (4.25) is compatible with the corresponding multiplications in $\hat{S}(E^\vee)$, $\mathcal{A}$, $\mathcal{T}$, or $\mathcal{D}$ since so is the map $\mathcal{H}$ (4.15). The theorem is proved. □

It is not hard to prove the following analogue of proposition 2.5 :

Proposition 4.11. *The composition*

$$\mathcal{H}' = \nu \circ \mathcal{H} : {}^F\Omega^{0,*}(\mathcal{T}) \cap \ker D^{1,0} \to {}^F\Omega^{0,*}({}^E T^*_{poly}) \tag{4.26}$$

of the map

$$\mathcal{H} : {}^F\Omega^{0,*}(\mathcal{T}) \cap \ker D^{1,0} \to {}^F\Omega^{0,*}(\mathcal{T}) \cap \ker \delta \tag{4.27}$$

with the identification of the sheaves $\mathcal{T}^* \cap \ker \delta$ *and* ${}^E T^*_{poly} \cong \wedge^{*+1} E$

$$\nu : \mathcal{T}^* \cap \ker \delta \xrightarrow{\sim} {}^E T^*_{poly} \tag{4.28}$$

is an isomorphism of the sheaves of DGLAs

$$({}^F\Omega^{0,*}(\mathcal{T}) \cap \ker D^{1,0}, D^{0,1}, [,]_{SN}) \cong ({}^F\Omega^{0,*}({}^E T^*_{poly}), \bar{d}, [,]_{SN}) \tag{4.29}$$

The map

$$\mathcal{H} : {}^F\Omega^{0,*}(\mathcal{A}_*) \cap \ker D^{1,0} \to {}^F\Omega^{0,*}({}^E A_*) \tag{4.30}$$

is an isomorphism of the sheaves of DGLA modules

$$({}^F\Omega^{0,*}(\mathcal{A}_*) \cap \ker D^{1,0}, D^{0,1}) \cong ({}^F\Omega^{0,*}({}^E A_*), \bar{d})$$

over the sheaf of DGLAs (4.29). □

Thanks to equation (4.23) this proposition implies that the map $\mathcal{H}'$ gives a quasi-isomorphism of the sheaves of DGLAs $({}^F\Omega^*(\mathcal{T}), D, [,]_{SN})$ and $({}^F\Omega^{0,*}({}^E T^*_{poly}), \bar{d}, [,]_{SN})$.

Playing with the PBW theorem for the Lie algebroids (as we did in the proof of proposition 2.6) and with the cup product in the sheaves $\mathcal{D}$ and ${}^E D^*_{poly}$ (see equation (2.42)) one can prove the following analogue of proposition 2.7

Proposition 4.12. *There exists an isomorphism of the sheaves of DGLAs*

$$\mu' : ({}^F\Omega^{0,*}({}^E D^*_{poly}), \bar{d}, [,]_G) \xrightarrow{\sim} ({}^F\Omega^{0,*}(\mathcal{D}) \cap \ker D^{1,0}, D^{0,1}, [,]_G), \tag{4.31}$$

which is compatible with the DGA structures on the sheaves ${}^F\Omega^{0,*}({}^E D^*_{poly})$ *and* ${}^F\Omega^{0,*}(\mathcal{D})$. □

Thanks to equation (4.23) this proposition implies that the map μ' (4.31) gives a quasi-isomorphism of the sheaves of DGLAs $({}^F\Omega^*(\mathcal{D}), D, [,]_G)$ and $({}^F\Omega^{0,*}({}^E D^*_{poly}), \bar{d}, [,]_G)$.

Let us consider the map

$$\gamma : {}^F\Omega^{0,q}(\mathcal{J}_*) \to {}^F\Omega^{0,q}({}^E J^{poly}_*), \qquad \gamma(j)(P) = (\mu'(P))(j)\Big|_{y^i=0}, \tag{4.32}$$

where $j \in {}^F\Omega^{0,q}(\mathcal{J}_k)$ and P is a holomorphic section of ${}^E D^k_{poly}$.

For this map we have the following analogue of theorem 2.8

Theorem 4.13. *For any* $q \geq 0$

$$H^q({}^F\Omega^*(\mathcal{J}), D) = H^q({}^F\Omega^{0,*}(\mathcal{J}) \cap \ker D^{1,0}, D^{0,1}). \tag{4.33}$$

and the map γ *(4.32) provides us with an isomorphism of the sheaves of DGLA modules*

$$\gamma : {}^F\Omega^{0,*}(\mathcal{J}_*) \xrightarrow{\sim} {}^F\Omega^{0,*}({}^E J^{poly}_*) \tag{4.34}$$

over the sheaf of DGLAs

$$({}^F\Omega^{0,*}(\mathcal{D}) \cap \ker D^{1,0}, D^{0,1}, [,]_G) \cong ({}^F\Omega^{0,*}({}^E D^*_{poly}), \bar{d}, [,]_G).$$

The map γ *sends the component* $D^{1,0}$ *to the Grothendieck connection (4.8) and the component* $D^{0,1}$ *to the Dolbeault differential* $\bar{d}$ *(4.7).*

Proof. The main part of the proof of this statement can be read off from the proof of theorem 2.8. We only have to prove that the component $D^{0,1}$ of the Fedosov differential (4.22) gets sent to the Dolbeault differential $\bar{d}$. In other words, we have to prove that

$$\gamma(D^{0,1}j)(P) = (\bar{d}\gamma(j))(P)\,, \tag{4.35}$$

where $j \in {}^F\Omega^{0,*}(\mathcal{J}_k)$ and P is a holomorphic section of ${}^E D^k_{poly}$.

Due to proposition 4.12 $\mu'(P)$ commutes with $D^{0,1}$ since P is holomorphic. Thus, by definition of the map γ (4.32) we get

$$\gamma(D^{0,1}j)(P) = (\mu'(P))(D^{0,1}j)\Big|_{y^i=0} = (D^{0,1}[\mu'(P)(j)])\Big|_{y^i=0} = (\bar{d}\gamma(j))(P)\,.$$

Hence, equation (4.35) holds and the theorem is proved. □

We have constructed the following honest (not L_∞) quasi-isomorphisms of the sheaves of DGLA modules

- $\lambda_T : ({}^F\Omega^{0,*}(M, {}^E T^*_{poly}), \bar{d}, [,]_{SN}) \to ({}^F\Omega(\mathcal{T}), D, [,]_{SN})$,
- $\lambda_A : ({}^F\Omega^{0,*}(M, {}^E A_*), \bar{d}) \to ({}^F\Omega(\mathcal{A}_*), D)$,
- $\lambda_D : ({}^F\Omega^{0,*}(M, {}^E D^*_{poly}), \bar{d}, [,]_G) \to ({}^F\Omega(\mathcal{D}), D, [,]_G)$, and
- $\lambda_C : ({}^F\Omega^{0,*}(M, {}^E C^{poly}_*), \bar{d}) \to ({}^F\Omega(\mathcal{J}), D)$.

Namely, the map λ_T is the inverse of $\mathcal{H}'$ (4.26) the map λ_A is the inverse of $\mathcal{H}$ (4.30) $\lambda_D = \mu'$ (4.31), and λ_C is composition of the identification (4.4) and the inverse of γ (4.32).

Our results can be summarized in the following commutative diagrams

$$\begin{array}{ccc}
({}^F\Omega^{0,*}(M, {}^E T^*_{poly}), \bar{d}, [,]_{SN}) & \overset{\lambda_T}{\rightarrowtail} & ({}^F\Omega(\mathcal{T}), D, [,]_{SN}) \\
\downarrow {}^E L_{mod} & & \downarrow L_{mod} \\
({}^F\Omega^{0,*}(M, {}^E A_*), \bar{d}) & \overset{\lambda_A}{\rightarrowtail} & ({}^F\Omega(\mathcal{A}), D), \\
 & & \\
({}^F\Omega(\mathcal{D}), D+\partial, [,]_G) & \overset{\lambda_D}{\leftarrowtail} & ({}^F\Omega^{0,*}(M, {}^E D^*_{poly}), \bar{d}+\partial, [,]_G) \\
\downarrow \mathcal{R}_{mod} & & \downarrow {}^E R_{mod} \\
({}^F\Omega(\mathcal{J}), D+\mathfrak{b}) & \overset{\lambda_C}{\leftarrowtail} & ({}^F\Omega^{0,*}(M, {}^E C^{poly}_*), \bar{d}+\mathfrak{b}),
\end{array} \tag{4.36}$$

where the action ${}^E R$ is obtained from the action ${}^E S$ of ${}^F\Omega^{0,*}(M, {}^E D^*_{poly})$ on ${}^F\Omega^{0,*}(M, {}^E J^{poly}_*)$ via the identification (4.4).

Due to claims *1* and *2* in theorem 1.12 and claims *1* and *2* in theorem 1.13 we get the following commutative diagram

$$
\begin{array}{ccc}
({}^F\Omega(\mathcal{T}), 0, [,]_{SN}) & \stackrel{\mathcal{K}}{\succ\!\!\longrightarrow} & ({}^F\Omega(\mathcal{D}), \partial, [,]_G) \\
\downarrow^{L}_{mod} & & \downarrow^{\mathcal{R}}_{mod} \\
({}^F\Omega(\mathcal{A}), 0) & \stackrel{\mathcal{S}}{\longleftarrow\!\!\prec\!\!\prec} & ({}^F\Omega(\mathcal{J}), \mathfrak{b}),
\end{array}
\tag{4.37}
$$

where by commutativity we mean that $\mathcal{S}$ is a morphism of the sheaves of L_∞-modules $({}^F\Omega(\mathcal{J}), \mathfrak{b})$ and $({}^F\Omega, 0)$ over the sheaf of DGLAs $({}^F\Omega(\mathcal{T}), 0, [,]_{SN})$ and the L_∞-module structure on $({}^F\Omega(\mathcal{J}), \mathfrak{b})$ over $({}^F\Omega(\mathcal{T}), 0, [,]_{SN})$ is obtained by composing the quasi-isomorphism $\mathcal{K}$ with the action $\mathcal{R}$ (see (3.4) in [7]) of $({}^F\Omega(\mathcal{D}), \partial, [,]_G)$ on $({}^F\Omega(\mathcal{J}), \mathfrak{b})$.

Let us now restrict ourselves to an open subset $V \subset M$ such that $E\Big|_V$ is trivial. Over any such subset the E-de Rham differential (1.5) is well defined for either of the sheaves ${}^F\Omega(\mathcal{A})$, ${}^F\Omega(\mathcal{T})$, ${}^F\Omega(\mathcal{J})$, and ${}^F\Omega(\mathcal{D})$. So again, we get a new commutative diagram

$$
\begin{array}{ccc}
({}^F\Omega(\mathcal{T})\Big|_V, {}^E d + \bar{d}, [,]_{SN}) & \stackrel{\mathcal{K}}{\succ\!\!\longrightarrow} & ({}^F\Omega(\mathcal{D})\Big|_V, {}^E d + \bar{d} + \partial, [,]_G) \\
\downarrow^{L}_{mod} & & \downarrow^{\mathcal{R}}_{mod} \\
({}^F\Omega(\mathcal{A})\Big|_V, {}^E d + \bar{d}) & \stackrel{\mathcal{S}}{\longleftarrow\!\!\prec\!\!\prec} & ({}^F\Omega(\mathcal{J})\Big|_V, {}^E d + \bar{d} + \mathfrak{b})
\end{array}
\tag{4.38}
$$

in which the L_∞-morphism $\mathcal{K}$ and the morphism of L_∞-modules $\mathcal{S}$ are quasi-isomorphisms.

On the open subset V we can represent the Fedosov differential (2.21) in the following (non-covariant) form

$$
D = {}^E d + \bar{d} + B\,\cdot + \bar{B}\,\cdot\,, \tag{4.39}
$$

$$
B = \sum_{p=0}^{\infty} \xi^i B^k_{i;j_1\dots j_p}(z,\bar{z}) y^{j_1}\dots y^{j_p} \frac{\partial}{\partial y^k}\,,
$$

and

$$
\bar{B} = \sum_{p=0}^{\infty} d\bar{z}^\alpha \bar{B}^k_{\alpha;j_1\dots j_p}(z,\bar{z}) y^{j_1}\dots y^{j_p} \frac{\partial}{\partial y^k}\,,
$$

where the z^α are local coordinates on M. If we regard $B + \bar{B}$ as a section of ${}^F\Omega^1(\mathcal{T}^0)\Big|_V$ then the nilpotency condition $D^2 = 0$ says that $B + \bar{B}$ is a Maurer-Cartan section of the sheaf of DGLAs $({}^F\Omega(\mathcal{T})\Big|_V, {}^E d + \bar{d}, [,]_{SN})$.

Thus applying the twisting procedures developed in section 2 of [7] and using claim 3 of theorem 1.12 we get the following commutative diagram

$$
\begin{array}{ccc}
({}^F\Omega(\mathcal{T})\Big|_V, D, [,]_{SN}) & \stackrel{\mathcal{K}^{tw}}{\succ\!\!\longrightarrow} & ({}^F\Omega(\mathcal{D})\Big|_V, D + \partial, [,]_G) \\
\downarrow^{L}_{mod} & & \downarrow^{\mathcal{R}}_{mod} \\
({}^F\Omega(\mathcal{A})\Big|_V, D) & \stackrel{\mathcal{S}^{tw}}{\longleftarrow\!\!\prec\!\!\prec} & ({}^F\Omega(\mathcal{J})\Big|_V, D + \mathfrak{b}),
\end{array}
\tag{4.40}
$$

in which $\mathcal{K}^{tw}$ is a quasi-isomorphism of the sheaves of DGLAs and $\mathcal{S}^{tw}$ is a quasi-isomorphism of the sheaves of DGLA modules.

Due to claim 4 in theorem 1.12 and claim 3 in theorem 1.13 the quasi-isomorphisms do not depend on the trivialization of E over V.

Thus we constructed the following commutative diagram of sheaves of DGLAs, DGLA modules and their (L_∞-) quasi-isomorphisms:

$$\begin{array}{ccc} ({}^F\Omega(\mathcal{T}), D, [,]_{SN}) & \overset{\mathcal{K}^{tw}}{\succ\!\!\rightarrow} & ({}^F\Omega(\mathcal{D}), D+\partial, [,]_G) \\ \downarrow^{L}_{mod} & & \downarrow^{\mathcal{R}}_{mod} \\ ({}^F\Omega(\mathcal{A}), D) & \overset{\mathcal{S}^{tw}}{\leftarrow\!\!\prec\!\prec} & ({}^F\Omega(\mathcal{J}), D+\mathfrak{b}), \end{array} \tag{4.41}$$

Combining the diagrams in (4.36), (4.41) together with proposition 4.6 we see that the sheaves of DGLA modules $({}^ET^*_{poly}, {}^EA_*)$ and $({}^ED^*_{poly}, {}^EC^{poly}_*)$ are connected by a chain of quasi-isomorphisms. Thus, theorem 4.2 in proved. □

5. Concluding remarks

We would like to mention that the functoriality of the chain of quasi-isomorphisms (1.34) between the pair of sheaves of DGLA modules implies the following interesting results

Corollary 5.1. *Let (E, M, ρ) be a C^∞ Lie algebroid equipped with a smooth action of a group G. If one can construct a G-invariant connection ∂^E on E then there exists a chain of G-equivariant quasi-isomorphisms between the sheaves of DGLA modules $({}^ET^*_{poly}, {}^EA_*)$ and $({}^ED^*_{poly}, {}^EC^{poly}_*)$.* □

In particular,

Corollary 5.2. *If (E, M, ρ) is a C^∞ Lie algebroid equipped with a smooth action of a finite or compact group G then the DGLA modules $(\Gamma(M, {}^ET^*_{poly})^G, \Gamma(M, {}^EA_*)^G)$ and $(\Gamma(M, {}^ED^*_{poly})^G, \Gamma(M, {}^EC^{poly}_*)^G)$ are quasi-isomorphic.* □

It would be interesting to prove the corresponding version of the algebraic index theorem [22], [27], which should relate a cyclic chain in the complex associated with a deformation Π (3.10) to its principal part and characteristic classes of the Lie algebroid (E, M, ρ). It would be also interesting to investigate how other characteristic classes [4], [10], [18] of Lie algebroids could enter this picture.

Corollary 4.2 does not in general give a chain of quasi-isomorphisms between the DGLAs $\Gamma({}^ET^*_{poly})$ and $\Gamma({}^ED^*_{poly})$ of global sections. However, since the sheaves of smooth forms ${}^F\Omega^{0,*}({}^ET^*_{poly})$ and ${}^F\Omega^{0,*}({}^ED^*_{poly})$ are soft one could speculate about the deformations associated with E as about the Maurer-Cartan elements of the DGLA ${}^F\Omega^{0,*}(M, {}^ED^*_{poly})[[\hbar]]$. Using the correspondence between the Dolbeault and Cech pictures one could relate these speculations to Kontsevich's algebroid picture of deformation quantization of algebraic varieties [17].

Finally, we think that the technique of mixed resolutions proposed by A. Yekutieli [34] could help us to prove Tsygan's formality conjecture for Hochschild chains of the structure sheaf of a smooth algebraic varieties over an arbitrary field of characteristic 0.

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