NONCOMMUTATIVE POISSON STRUCTURES ON ORBIFOLDS

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Abstract. In this paper, we compute the Gerstenhaber bracket on the Hochschild cohomology of $C^\infty(M)\rtimes\Gamma$. Using this computation, we classify all the noncommutative Poisson structures on $C^\infty(M)\rtimes\Gamma$ when M is a symplectic manifold. We provide examples of deformation quantizations of these noncommutative Poisson structures.

1. Introduction

It is well known [8] that the deformation theory of an associative algebra A is closely related to the Hochschild cohomology $HH^\bullet(A;A)$ of A. In particular, the infinitesimal deformation of A is governed by $HH^2(A;A)$. Furthermore, if we want the infinitesimal deformation to be integrable, we need to (necessarily but maybe not sufficiently) require that the two cocycle $\Pi\in C^2(A;A)$ associated to the infinitesimal deformation satisfies the equation $[\Pi,\Pi]_G=0$ in $HH^3(A;A)$, where $[\ ,\ ]_G$ is the Gerstenhaber bracket on $HH^\bullet(A;A)$.

When A is the algebra of smooth functions on a smooth manifold M, we know according to the Hochschild-Kostant-Rosenberg theorem that second Hochschild cohomology classes in $HH^\bullet(A;A)$ with the above integrability conditions are in one to one correspondence with Poisson structures on M. Inspired by this relation between the Poisson geometry and deformation theory, Block and Getzler [2] and Xu [18] independently introduced a notion of a noncommutative Poisson structure on an associative algebra in early 90's.

Definition 1.1. *A noncommutative Poisson structure on an associative algebra A is an element Π in the second Hochschild cohomology group $H^2(A,A)$ of A, whose Gerstenhaber bracket with itself vanishes, i.e.* $[\Pi,\Pi]_G=0$.

In this paper, we want to study noncommutative Poisson structures on orbifolds coming from global quotients. Let M be a compact smooth manifold, and Γ be a finite group acting on M. Our orbifold is the quotient space $X=M/\Gamma$. Because X is usually a topological space with quotient singularities, the algebra $C^\infty(M)^\Gamma$ of Γ-invariant smooth functions on M is not regular. As a replacement, we consider the crossed product algebra $C^\infty(M)\rtimes\Gamma$. Our main goal is to find out all noncommutative Poisson structures on $C^\infty(M)\rtimes\Gamma$ when M has a Γ-invariant symplectic structure.

The main difficulty in finding noncommutative Poisson structures on $C^\infty(M)\rtimes\Gamma$ is to compute the Gerstenhaber bracket on $HH^\bullet(C^\infty(M)\rtimes\Gamma;C^\infty(M)\rtimes\Gamma)$. In

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[13], the second author and the his coauthors computed the Hochschild cohomology of $C^\infty(M)\rtimes\Gamma$ as a vector space, i.e.

$$HH^\bullet(C^\infty(M)\rtimes\Gamma;C^\infty(M)\rtimes\Gamma)=\Gamma^\infty(\bigoplus_{\gamma\in\Gamma}\wedge^{\bullet-l(\gamma)}TM^\gamma\otimes\wedge^{l(\gamma)}N^\gamma)^\Gamma. \tag{1}$$

In the above equation, M^γ is the fixed point manifold of γ, and $l(\gamma)$ is the codimension of M^γ in M, and Γ acts on $\sqcup M^\gamma$ by conjugacy action, and N^γ is the normal bundle of the embedding of M^γ in M. We also remark that M^γ may have different components with different dimensions, and we take the disjoint union of all the components. (Following [13], in this paper we view $C^\infty(M)\rtimes\Gamma$ as a bornological algebra with the bornology defined by the Frechét topology. And $HH^\bullet$ is the continuous Hochschild cohomology of a bonological algebra. Accordingly, all the computation and constructions in this paper are local respect to the orbifold M/Γ. And we often work with a vector space (or a Γ-invariant open subset) with a linear Γ action, which we refer by "local" computation.) To compute the Gerstenhaber bracket on $HH^\bullet(C^\infty(M)\rtimes\Gamma;C^\infty(M)\rtimes\Gamma)$, we need to have quasi-isomorphisms from the Hochschild cochain complex $C^\bullet(C^\infty(M)\rtimes\Gamma;C^\infty(M)\rtimes\Gamma)$ to $\Gamma^\infty(\bigoplus_{\gamma\in\Gamma}\wedge^{\bullet-l(\gamma)}TM^\gamma\otimes\wedge^{l(\gamma)}N^\gamma)^\Gamma$ and vice versa. In [13], a map L is defined in the following direction, i.e.

$$L:C^\bullet(C^\infty(M)\rtimes\Gamma;C^\infty(M)\rtimes\Gamma)\longrightarrow\Gamma^\infty(\bigoplus_{\gamma\in\Gamma}\wedge^{\bullet-l(\gamma)}TM^\gamma\otimes\wedge^{l(\gamma)}N^\gamma)^\Gamma.$$

The quasi-isomorphism T map in the other direction is much harder to construct. It turns out we need to construct some nonlocal operators on $C^\infty(M)$, which we call twisted cocycles. These cocycles are closely related to the Lusztig-Demazure operator (*cf.* [14]). Using the maps T and L, we are able to compute the Gerstenhaber brackets on $HH^\bullet(C^\infty(M)\rtimes\Gamma;C^\infty(M)\rtimes\Gamma)$. Similar to the case of a manifold where the Gerstenhaber bracket corresponds to the Schouten-Nijenhuis bracket, the Gerstenhaber bracket on an orbifold is a generalization of the classical Schouten-Nijenhuis bracket. We call this bracket the twisted Schouten-Nijenhuis bracket on $\Gamma^\infty(\oplus_\gamma\wedge^{\bullet-l(\gamma)}TM^\gamma\otimes\wedge^{l(\gamma)}N^\gamma)^\Gamma$. Using the twisted Schouten-Nijenhuis bracket, we solve the equation $[\Pi,\Pi]=0$ on $HH^2(C^\infty(M)\rtimes\Gamma;C^\infty(M)\rtimes\Gamma)$, when M has a Γ invariant symplectic structure and. This leads to a full description of noncommutative Poisson structures on $C^\infty(M)\rtimes\Gamma$.

If we consider a complex symplectic vector space V with a symplectic Γ action, the cocycles used in the definition of symplectic reflection algebras [7] correspond to a special class of noncommutative Poisson structures on $\mathrm{Poly}(V)\rtimes\Gamma$, where $\mathrm{Poly}(V)$ is the algebra of polynomials on V. Furthermore, using the results from [7] we prove in this paper that all these cocycles can be extended to a formal deformation of the algebra $\mathrm{Poly}(V)\rtimes\Gamma$. As a generalization, we expect that all the noncommutative Poisson structures discovered in this paper can be extended to formal deformations, which will generalize the symplectic reflection algebras. This question is closely related to the following formality conjecture on orbifolds. We conjecture that the Hochschild complex of the algebra $C^\infty(M)\rtimes\Gamma$ is a formal differential graded Lie algebra. We plan to discuss this conjecture in future publications.

In the last part of this paper, we provide concrete new examples of noncommutative Poisson structures on $Poly(\mathbb{R}^4)\rtimes(\mathbb{Z}_n\times\mathbb{Z}_m)$ with $\mathbb{Z}_n=\mathbb{Z}/\mathbb{Z}_n$ and $\mathbb{Z}_m=\mathbb{Z}/m\mathbb{Z}$. These Poisson structures are not symplectic at all, and instead should be viewed as noncommutative quadratic Poisson structures. The connection between these "noncommutative quadratic Poisson structures" and quantum R matrices will be studied in the near future. In general, there are many interesting examples of noncommutative Poisson structures on orbifolds. We are going to study some of them together with Jean-Michel Oudom in detail in [11].

Besides the Gerstenhaber bracket, there is also a product structure on the Hochschild cohomology $HH^\bullet(C^\infty(M)\rtimes\Gamma; C^\infty(M)\rtimes\Gamma)$. In [15], with Pflaum, Posthuma and Tseng, the second author will study the product structure on the Hochschild cohomology of the deformed algebras of $C^\infty(M)\rtimes\Gamma$, which is closely related to the Chen-Ruan orbifold cohomology [3].

This paper is organized as follows. In Section 2, we will focus on the construction of twisted cocycles and a quasi-isomorphism T,

$$T:\Gamma^\infty(\bigoplus_{\gamma\in\Gamma}\wedge^{\bullet-l(\gamma)}TM^\gamma\otimes\wedge^{l(\gamma)}N^\gamma)^\Gamma\longrightarrow C^\bullet(C^\infty(M)\rtimes\Gamma; C^\infty(M)\rtimes\Gamma).$$

In Section 3, we will study the Gerstenhaber brackets on the Hochschild cohomology $HH^\bullet(C^\infty(M)\rtimes\Gamma; C^\infty(M)\rtimes\Gamma)$. And in Section 4, we will give a full description of noncommutative Poisson structures on $C^\infty(M)\rtimes\Gamma$, when M has a Γ invariant symplectic structure. And we will discuss the deformations of these Poisson structures. We construct a formal deformation of a special type of noncommutative Poisson structures and the second Poisson cohomology is computed in this case. We end this section by showing explicit two new families of noncommutative quadratic Poisson structures on $\mathbb{C}^2/\mathbb{Z}_n\times\mathbb{Z}_m$.

Remark 1.2. *Unless under specification, we work with the field $\mathbb{R}$, real vector space and real manifolds. Many results in this paper have analogs in the field $\mathbb{C}$, complex vector space and affine varieties.*

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2. Hochschild cohomology and Quasi-isomorphisms

In this section, we will construct a quasi-isomorphism T

$$T:\Gamma^\infty(\bigoplus_{\gamma\in\Gamma}\wedge^{\bullet-l(\gamma)}TM^\gamma\otimes\wedge^{l(\gamma)}N^\gamma)^\Gamma\longrightarrow C^\bullet(C^\infty(M)\rtimes\Gamma; C^\infty(M)\rtimes\Gamma).$$

In Section 2.1, we will recall the map L and the computation in [13]; in Section 2.2, we will construct the map T when M is a vector space; in Section 2.3, we will generalize the construction of T to a manifold M.

2.1. **Chain map L.** Let A be the algebra of polynomials on a vector space V, and Γ be a finite group acting linearly on V. We consider the crossed product algebra $A\rtimes\Gamma$, and construct a chain map L

$$L:C^\bullet(A\rtimes\Gamma, A\rtimes\Gamma)\longrightarrow\Gamma^\infty(\bigoplus_{\gamma\in\Gamma}\wedge^{\bullet-l(\gamma)}TV^\gamma\otimes\wedge^{l(\gamma)}N^\gamma)^\Gamma.$$

This map was constructed implicitly in the proof of Theorem 3.1, [13]. We make it explicit in the following.

The map L consists of compositions of three chain maps:

(1)
$$L_1 : C^\bullet(A \rtimes \Gamma, A \rtimes \Gamma) \longrightarrow C^\bullet(A, A \rtimes \Gamma),$$
where Γ acts on $C^\bullet(A, A \rtimes \Gamma)$ by
$$\gamma\Psi(a_1, \ldots, a_n) = U_{\gamma^{-1}} \cdot \Psi(\gamma(a_1), \ldots, \gamma(a_n)) \cdot U_\gamma.$$

Here U_γ denotes the élément γ seen in $A \rtimes \Gamma$. Given a Hochschild cocycle $\Psi \in C^k(A \rtimes \Gamma, A \rtimes \Gamma)$, $L_1(\Psi) \in C^k(A, A \rtimes \Gamma)$ is defined to be
$$L_1(\Psi)(f_1, \ldots, f_k) = \Psi(f_1, \ldots, f_k), \quad \forall f_1, \ldots, f_k \in A.$$

(2)
$$L_2 : C^\bullet(A, A \rtimes \Gamma) \longrightarrow \Big(\bigoplus_{\gamma\in\Gamma} \Gamma^\infty(\wedge^\bullet TV), \kappa_\gamma \wedge \ \Big).$$

As a A-A bimodule, $A \rtimes \Gamma$ has a natural splitting into a direct sum of submodules $\oplus_{\gamma\in\Gamma} A_\gamma$. Correspondingly, the Hochschild cochain complex $C^\bullet(A, A \rtimes \Gamma)$ has a natural splitting into $\oplus_{\gamma\in\Gamma} C^\bullet(A, A_\gamma)$. Therefore, we define L_2 to be the sum of the following maps L_2^γ for all $\gamma \in \Gamma$,
$$L_2^\gamma : C^\bullet(A, A_\gamma) \longrightarrow \big(\Gamma^\infty(\wedge^\bullet TV), \kappa_\gamma\big).$$

On V, we introduce the following vector field $X(x) = \sum_i x^i \frac{\partial}{\partial x^i}$. The vector field $\kappa_\gamma \in \Gamma^\infty(TV)$ is defined to be
$$\kappa_\gamma(x) = X(\gamma(x)) - X(x).$$

Given an element $\Psi \in C^k(A, A \rtimes \Gamma)$, we define $L_2^\gamma(\Psi) \in \Gamma^\infty(\wedge^k(TV))$, the usual projection to anti-symmetric linear operators by
$$L_2^\gamma(\Psi)(x) = \sum_{i_1,\ldots,i_k} \Psi((x_1^{i_1} - x^{i_1}) \otimes \cdots \otimes (x_k^{i_k} - x^{i_k})) \frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_k}},$$
where x_i is for the i-th function taken by the cochain Ψ and x is the output.

(3)
$$L_3 : \Big(\bigoplus_{\gamma\in\Gamma} \Gamma^\infty(\wedge^\bullet TV), \kappa \wedge \ \Big) \longrightarrow \bigoplus_{\gamma\in\Gamma} \big(\Gamma^\infty(\wedge^{\bullet - l(\gamma)} TV^\gamma \otimes \wedge^{l(\gamma)} N^\gamma), 0\ \big).$$

We fix a metric on V. Let V^γ be the subspace of V invariant under γ action, and N^γ be the subspace of V orthogonal to V^γ. Therefore, V can be written as $V^\gamma \oplus N^\gamma$.

We define L_3 to be the sum of L_3^γ, which is defined to be
$$L_3^\gamma(X) = pr^\gamma(X|_{V^\gamma}),$$
where pr^γ projects $\wedge^\bullet TV|_{V^\gamma}$ to $\wedge^{\bullet - l(\gamma)} TV^\gamma \otimes \wedge^{l(\gamma)} N^\gamma$.

It is proved in [13][Section 3] that L is a quasi-isomorphism of cochain complexes.

2.2. **Chain map T.** In this subsection, we construct an inverse map T to the above map L,
$$T : \bigoplus_{\gamma\in\Gamma} \Gamma^\infty(\wedge^{\bullet - l(\gamma)} TV^\gamma \otimes \wedge^{l(\gamma)} N^\gamma)^\Gamma \longrightarrow C^\bullet(A \rtimes \Gamma, A \rtimes \Gamma).$$

We divide this construction into two steps.

(1) **Step I: twisted cocycle**

We construct a “twisted cocycle” $\Omega_\gamma \in C^{l(\gamma)}(A, A_\gamma)$. We fix coordinates on V, such that $x^1, \dots, x^{\dim(V)-l\gamma} \in V^\gamma$ and $x^{\dim(V)-l(\gamma)+1}, \dots, x^{\dim(V)} \in N^\gamma$.

For any $\sigma \in S_{l(\gamma)}$, the permutation group of $l(\gamma)$ elements, we introduce the following vectors in N^γ. Let $(y^1, \dots, y^{l(\gamma)}) = (x^{\dim(V)-l(\gamma)+1}, \dots, x^{\dim(V)})$, and $(\tilde{y}^1, \dots, \tilde{y}^{l(\gamma)}) = \gamma(x^{\dim(v)-l(\gamma)+1}, \dots\ x^{\dim(V)})$. Define

$$\begin{array}{ll} z^0_\sigma = (y^1, \dots, y^{l(\gamma)}) & z^1_\sigma = (y^1, \dots, \tilde{y}^{\sigma(1)}, \dots) \\ z^2_\sigma = (y^1, \dots, \tilde{y}^{\sigma(1)}, \dots, \tilde{y}^{\sigma(2)}, \dots,) & \cdots \\ z^{l(\gamma)-1}_\sigma = (\tilde{y}^1, \dots, y^{\sigma(l(\gamma))}, \dots,) & z^{l(\gamma)}_\sigma = (\tilde{y}^1, \dots, \tilde{y}^{l(\gamma)}). \end{array}$$

Let Ω_γ be a $l(\gamma)-$cochain in $C^\bullet(A, A_\gamma)$ defined as follows:

$$\begin{aligned} &\Omega_\gamma(f_1, \dots, f_{l(\gamma)})(x, y) \\ &:= \tfrac{1}{l(\gamma)!} \sum_{\sigma \in S_{l(\gamma)}} \tfrac{(f_1(x, z^0_\sigma) - f_1(x, z^1_\sigma))(f_2(x, z^1_\sigma) - f_2(x, z^2_\sigma)) \cdots (f_{l(\gamma)}(x, z^{l(\gamma)-1}_\sigma) - f_{l(\gamma)}(x, z^{l(\gamma)}_\sigma))}{(y^1 - \tilde{y}^1) \cdots (y^{l(\gamma)} - \tilde{y}^{l(\gamma)})}. \end{aligned} \tag{2}$$

We remark that when σ is the identity permutation, then we have

$$\begin{array}{ll} z^0_{id} = (y^1, \dots, y^{l(\gamma)}) & z^1_{id} = (\tilde{y}^1, \dots, y^{l(\gamma)}) \\ & \cdots\cdots \\ z^{l(\gamma)-1}_{id} = (\tilde{y}^1, \dots, \tilde{y}^{l(\gamma)-1}, y^{l(\gamma)}) & z^{l(\gamma)}_{id} = (\tilde{y}^1, \dots, \tilde{y}^{l(\gamma)}). \end{array}$$

The corresponding contribution in the summation of expression (2) is

$$\frac{(f_1(x, z^0_{id}) - f_1(x, z^1_{id}))(f_2(x, z^1_{id}) - f_2(x, z^2_{id})) \cdots (f_{l(\gamma)}(x, z^{l(\gamma)-1}_{id}) - f_{l(\gamma)}(x, z^{l(\gamma)}_{id}))}{(y^1 - \tilde{y}^1) \cdots (y^{l(\gamma)} - \tilde{y}^{l(\gamma)})},$$

which converges to

$$\frac{\partial}{\partial y^1} f_1(x, 0) \cdots \frac{\partial}{\partial y^{l(\gamma)}} f_{l(\gamma)}(x, 0),$$

as $y^1, \cdots, y^{l(\gamma)}$ go to 0.

Therefore the identity component in Equation (2) can be viewed as a γ analog of the multi-differential operator

$$\frac{\partial}{\partial y^1} \otimes \cdots \otimes \frac{\partial}{\partial y^{l(\gamma)}}.$$

Summing over all permutations, Ω_γ can be viewed as a γ-analog of the multi-differential operator

$$\frac{\partial}{\partial y^1} \wedge \cdots \wedge \frac{\partial}{\partial y^{l(\gamma)}}.$$

It is straightforward to check that Ω_γ is a cocycle in $C^{l(\gamma)}(A, A_\gamma)$.

Example 2.1. *Let V be $\mathbb{R}$, and $\Gamma = \mathbb{Z}/2\mathbb{Z} = \{id, e\}$ act on $\mathbb{R}$ by $e : x \mapsto -x$. In this case, $\Omega_e \in C^1(A, A_e)$ is defined to be*

$$\Omega_e(f)(x) = \frac{f(x) - f(-x)}{2x}.$$

And $HH^\bullet(A, A_e)$ is computed to be

$$HH^\bullet(A, A_e) = \left\{ \begin{array}{ll} 0 & \bullet \neq 1 \\ \mathbb{R} & \bullet = 1 \end{array} \right. ,$$

where $HH^1(A, A_e)$ is generated by Ω_e.

(2) **Step II: Inverse map of L**

We use the twisted cocycle constructed in the previous step to obtain an inverse map of L

$$T : \Gamma^\infty(\bigoplus_{\gamma\in\Gamma} \wedge^{\bullet - l(\gamma)} TV^\gamma \otimes \wedge^{l(\gamma)} N^\gamma)^\Gamma \longrightarrow C^\bullet(A \rtimes \Gamma, A \rtimes \Gamma).$$

We write T as a composition of two maps T_1 and T_2. T_1 is map

$$T_1 : \Gamma^\infty(\bigoplus_{\gamma\in\Gamma} \wedge^{\bullet - l(\gamma)} TV^\gamma \otimes \wedge^{l(\gamma)} N^\gamma)^\Gamma \longrightarrow C^\bullet(A, A \rtimes \Gamma)^\Gamma.$$

And T_2 is the standard map constructed in the proof of the Eilenberg-Zilber theorem.

$$T_2 : C^\bullet(A, A \rtimes \Gamma)^\Gamma \subset C^0(\Gamma, C^\bullet(A, A \rtimes \Gamma)) \longrightarrow C^\bullet(A \rtimes \Gamma, A \rtimes \Gamma).$$

First, the map T_1 is a sum of the following maps

$$T_1^\gamma : \Gamma^\infty(\wedge^{\bullet - l(\gamma)} TV^\gamma \otimes \wedge^{l(\gamma)} N^\gamma) \longrightarrow C^\bullet(A, A_\gamma),$$

which is defined as follows.

Given $\xi \in \Gamma^\infty(\wedge^{\bullet - l(\gamma)} TV^\gamma \otimes \wedge^{l(\gamma)} N^\gamma)$, we write ξ to be $X \otimes \Lambda_\gamma$, where $X \in \Gamma^\infty(\wedge^{\bullet - l(\gamma)} TV^\gamma)$ and $\Lambda_\gamma \in \Gamma^\infty(\wedge^{l(\gamma)} N^\gamma)$ is defined be

$$\frac{\partial}{\partial x^{\dim(V) - l(\gamma)+1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\dim(V)}}. \tag{3}$$

Define

$$T_1^\gamma(\xi) = X \sharp \Omega_\gamma, \quad \forall \xi \in \Gamma^\infty(\wedge^{\bullet - l(\gamma)} TV^\gamma \otimes \wedge^{l(\gamma)} N^\gamma),$$

where $X\sharp\Omega_\gamma(f_1, \ldots, f_k)$ is equal to

$$X(f_1, \ldots, f_{k-l(\gamma)})\Omega_\gamma(f_{k-l(\gamma)+1}, \ldots, f_k).$$

It is straightforward to check that T_1^γ is a chain morphism and compatible with the Γ-action, i.e.

$$T_1^\gamma(\alpha(\xi)) = \alpha(T_1^\gamma(\xi)), \quad \forall \alpha \in \Gamma,$$

And we define T_1 to be the sum of T_1^γ. The restriction of T_1 to the Γ-invariant sections gives the desired map

$$T_1 : \Gamma^\infty(\bigoplus_{\gamma\in\Gamma} \wedge^{\bullet - l(\gamma)} TV^\gamma \otimes \wedge^{l(\gamma)} N^\gamma)^\Gamma \longrightarrow C^\bullet(A, A \rtimes \Gamma)^\Gamma.$$

Secondly, we explain the construction of T_2, which is standard. Given $\Phi \in C^k(A, A \rtimes \Gamma)$,

$$T_2(\Phi)(a_1 U_{\gamma_1}, \ldots, a_k U_{\gamma_k}) = \Phi(a_1, \ldots, \gamma_1 \cdots \gamma_{k-1}(a_k))U_{\gamma_1 \cdots \gamma_k}.$$

Lemma 2.2. *The twisted cocycle Ω_γ satisfies the following properties:*

(a)

$$\Omega_\gamma(x^1 \otimes \cdots \otimes x^{l(\gamma)}) = 1;$$

(b)

$$\Omega_\gamma(x^1 \otimes \cdots \otimes c \otimes \cdots \otimes x^{l(\gamma)}) = 0,$$

when c is a constant function.

Generally, Ω_γ and Λ_γ have the same values on linear functions.

Proof. A straightforward check. □

By this Lemma 2.2, we have the following proposition for the map L_2.

Proposition 2.3. *Given $\xi \in \Gamma^\infty(\oplus_\gamma \wedge^{k-l(\gamma)} TV^\gamma \otimes \wedge^{l(\gamma)} N^\gamma)^\Gamma$, we write $\xi = \sum_\gamma X_\gamma \otimes \Lambda_\gamma$, where Λ_γ is defined the same as Equation (3).*

The composition map $L_2 \circ T_1$ satisfies

$$L_2(T_1(\xi)) = \sum_\gamma X_\gamma \otimes \Lambda_\gamma = \xi.$$

Proof. We compute $L_2(T_1(\xi))(x)$ as follows: $L_2(T_1(\xi))(x) =$

$$\begin{aligned}
=& \sum_{i_1,\dots,i_k} T_1(\xi)((x_1-x)^{i_1}\otimes\cdots\otimes(x_k-x)^{i_k})\frac{\partial}{\partial x^{i_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{i_k}}\\
=& \sum_\gamma \sum_{i_1,\dots,i_k} T_1(X_\gamma\otimes\Lambda_\gamma)((x_1-x)^{i_1}\otimes\cdots\otimes(x_k-x)^{i_k})\frac{\partial}{\partial x^{i_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{i_k}}\\
=& \sum_\gamma \sum_{i_1,\dots,i_k} X_\gamma((x_1-x)^{i_1},\dots,(x_{k-l(\gamma)}-x)^{i_{k-l(\gamma)}})\\
&\times \Omega_\gamma((x_{k-l(\gamma)+1}-x)^{i_{k-l(\gamma)+1}},\dots,(x_k-x)^{i_k})\frac{\partial}{\partial x^{i_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{i_k}}\\
=& \sum_\gamma \sum_{i_1,\dots,i_k} X_\gamma((x_1-x)^{i_1},\dots,(x_{k-l(\gamma)}-x)^{i_{k-l(\gamma)}})\frac{\partial}{\partial x^{i_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{i_{k-l(\gamma)}}}\\
&\otimes \Omega_\gamma((x_{k-l(\gamma)+1}-x)^{i_{k-l(\gamma)+1}},\dots,(x_k-x)^{i_k})\frac{\partial}{\partial x^{i_{k-l(\gamma)+1}}}\wedge\cdots\wedge\frac{\partial}{\partial x^{i_k}}\\
=& \sum_\gamma X_\gamma\otimes\Lambda_\gamma.
\end{aligned}$$

□

We define $T = T_2 \circ T_1$, and have the following theorem.

Theorem 2.4. *The map T is a quasi-isomorphism. In particular, $L \circ T = id$.*

Proof. We notice that $L_1(T_2) = id$ on $C^\bullet(A; A \rtimes \Gamma)^\Gamma$, and therefore have

$$L \circ T(\xi) = L_3(L_2(L_1(T_2(T_1(\xi))))) = L_3(L_2(T_1(\xi))),$$

which is equal to ξ by Proposition 2.3 □

2.3. **The case of a smooth manifold.** We generalize the maps L and T to the case of a finite group action on a smooth manifold. The detail study of these generalizations will be discussed in [15]. Here we introduce them briefly.

For the map L, L_1 and L_3 can be generalized to the manifold case directly. To generalize L_2, we use Connes' map [4][Lemma 44] from $C^\infty(M)$'s Koszul resolution to its Bar resolution. This map induces a map L_2^γ from the Hochschild cochain complex $C^\bullet(C^\infty(M), C^\infty(M)_\gamma)$ to $\wedge^\bullet TM$.

For the map T, T_2 generalizes to the manifold case directly because it is purely algebraic. To define T_1, we consider a γ-invariant tubular neighborhood $\mathcal{M}^\gamma$ of M^γ. $\mathcal{M}^\gamma$ is a fiber bundle over M^γ, and we fix a Γ-invariant Ehresmann connection on $\mathcal{M}^\gamma$. Furthermore, we choose a cut-off function ρ_γ on $\mathcal{M}^\gamma$ which is equal to 1 on a γ-invariant neighborhood of M^γ and vanishes outside $\mathcal{M}^\gamma$. Given a section

$\xi_\gamma = X_\gamma \otimes \Lambda_\gamma$ of $\wedge^{\bullet - l(\gamma)} TM^\gamma \otimes \wedge^{l(\gamma)} N^\gamma$ to $\mathcal{M}^\gamma$, we use the Ehresmann connection to extend X_γ to a multi-vector field $\tilde{X_\gamma}$ on $\mathcal{M}^\gamma$, and define Ω_γ a linear map on $C^\infty(\mathcal{M}^\gamma)$ by the same formula as Equation (2).

We define $T_1 : \Gamma^\infty(\oplus_\gamma \wedge^{k-l(\gamma)} TM^\gamma \otimes \wedge^{l(\gamma)} N^\gamma)^\Gamma \longrightarrow C^k(C^\infty(M); C^\infty(M) \rtimes \Gamma)^\Gamma$ by

$$T_1(\xi)(f_1, \dots, f_k) = \sum_\gamma \rho_\gamma \tilde{X}_\gamma(f_1, \dots, f_{k-l(\gamma)}) \Omega_\gamma(f_{k-l(\gamma)+1}, \dots, f_k) U_\gamma,$$

for $\xi = \sum_\gamma X_\gamma \otimes \Lambda_\gamma$.

Again, we know that $L \circ T = id$. Then therefore, since L is a quasi-isomorphism proved in [13], T is also a quasi-isomorphism.

Remark 2.5. *Apparently, the definition of T_1 depends on a choice of the normal bundle N^γ, the Γ-invariant metric, and the cut-off function. Therefore, T is not a canonical map. However, we notice that at any $x \in M^\gamma$, inside T_xM, there is a canonical complementary subspace to $T_xM^\gamma \subset T_xM$ determined by the representation of $< \gamma >$ on T_xM independent of the choices of the metrics. Therefore, it is easy to check (c.f. Proposition 2.3) that when restricted to the $\infty-$jets of M^γ, T is independent of all the choices.*

3. Gerstenhaber bracket

We recall the definition of the Gerstenhaber bracket on the Hochschild cohomology of an algebra A. We define a pre-Lie product $\circ$ on $C^\bullet(A; A)$. For $\phi \in C^k(A; A)$, $\psi \in C^k(A; A)$, $\phi \circ \psi \in C^{k+l-1}(A; A)$ is defined by

$$\begin{aligned} &\phi \circ \psi(a_1, \dots, a_{k+l-1}) \\ =& \sum_{i=1}^k (-1)^{(i-1)(l-1)} \phi(a_1, \dots, a_{i-1}, \psi(a_i, \dots, a_{i+l-1}), a_{i+l}, \dots, a_{k+l-1}), \end{aligned}$$

for $a_i \in A$, $i = 1, \dots k+l-1$. The Gerstenhaber bracket $[\ ,\ ]$ on $C^\bullet(A; A)$ is defined to be the commutator of the pre-Lie product, i.e.

$$[\phi, \psi] = \phi \circ \psi - (-1)^{(k-1)(l-1)} \psi \circ \phi.$$

In this section, we compute the Gerstenhaber bracket on the Hochschild cohomology of the algebra $C^\infty(M) \rtimes \Gamma$. Because the Gerstenhaber bracket is the commutator of the pre-Lie product, we will mainly work on the pre-Lie product, and state the results for the Gerstenhaber brackets. Since all the computation and constructions are local respect to the orbifold M/Γ, it is sufficient to work out everything locally on a vector space.

Let $\xi \in \Gamma^\infty(\oplus_{\gamma \in \Gamma} \wedge^{k-l(\gamma)} TV^\gamma \otimes \wedge^{l(\gamma)} N^\gamma)^\Gamma$, and $\eta \in \Gamma^\infty(\oplus_{\beta \in \Gamma} \wedge^{l-l(\beta)} TV^\beta \otimes \wedge^{l(\beta)} N^\beta)^\Gamma$. We compute the pre-Lie product between ξ and η by $L(T(\xi) \circ T(\eta))$. We write ξ as the sum of $\xi_\gamma = X_\gamma \otimes \Lambda_\gamma$, and η as the sum of $\eta_\beta = Y_\beta \otimes \Lambda_\beta$, with $X_\gamma \in \Gamma^\infty(\wedge^{k-l(\gamma)} TV^\gamma)$, $Y_\beta \in \Gamma^\infty(\wedge^{l-l(\beta)} TV^\beta)$, $\Lambda_\gamma \in \Gamma^\infty(\wedge^{l(\gamma)} N^\gamma)$, $\Lambda_\beta \in \Gamma^\infty(\wedge^{l(\beta)} N^\beta)$.

We compute $L_1(T(\xi)\circ T(\eta)) \in C^{k+l-1}(A, A\rtimes\Gamma)$ first.

$$\begin{aligned}
&L_1(T(X)\circ T(Y))(f_1,\ldots,f_{k+l-1})\\
=&\sum_s(-1)^{(s-1)(l-1)}T(\xi)(f_1,\ldots,f_s,T(\eta)(f_{s+1},\ldots,f_{s+l}),f_{s+l+1},\ldots,f_{k+l-1})\\
=&\sum_s(-1)^{(s-1)(l-1)}T(\sum_\alpha\xi_\alpha)(f_1,\ldots,f_s,T(\sum_\beta\eta_\beta)(f_{s+1},\ldots,f_{s+l}),f_{s+l+1},f_{k+l-1})\\
=&\sum_\alpha\sum_\beta\sum_s(-1)^{(s-1)(l-1)}T(\xi_\alpha)(f_1,\ldots,f_s,T(\eta_\beta)(f_{s+1},\ldots,f_{s+l}),\\
&f_{s+l+1},\ldots,f_{k+l-1})\\
=&\sum_\gamma\sum_{\alpha\beta=\gamma}\sum_s(-1)^{(s-1)(l-1)}T_1^\alpha(\xi_\alpha)(f_1,\ldots,f_s,T_1^\beta(\eta_\beta)(f_{s+1},\ldots,f_{s+l}),\\
&\beta(f_{s+l+1}),\ldots,\beta(f_{k+l-1}))U_\gamma
\end{aligned}$$

Therefore, $L_2^\gamma(L_1(T(\xi)\circ T(\eta)))(x)$ is computed by

$$\begin{aligned}
&\sum_{i_1,\ldots,i_{k+l-1}}\sum_{\alpha\beta=\gamma}\sum_s(-1)^{(s-1)(l-1)}T_1^\alpha(\xi_\alpha)((x_1-x)^{i_1},\ldots,(x_s-x)^{i_s},\\
&T_1^\beta(\eta_\beta)((x_{s+1}-x)^{i_{s+1}},\ldots,(x_{s+l}-x)^{i_{x+l}}),\beta((x_{s+l+1}-x)^{i_{s+l+1}}),\ldots,\\
&\beta((x_{k+l-1}-x)^{i_{k+l-1}}))\frac{\partial}{\partial x^{i_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{i_{k+l-1}}}.
\end{aligned}\tag{4}$$

We look at the term $T_1^\beta(\eta_\beta)((x_{s+1}-x)^{i_{s+1}},\ldots,(x_{s+l}-x)^{i_{x+l}})$. Using the expression $\eta_\beta = X_\beta\otimes\Lambda_\beta$, we have

$$\begin{aligned}
&T_1^\beta(\eta_\beta)((x_{s+1}-x)^{i_{s+1}},\ldots,(x_{s+l}-x)^{i_{x+l}}))\\
=&X_\beta\sharp\Omega_\beta((x_{s+1}-x)^{i_{s+1}},\ldots,(x_{s+l}-x)^{i_{x+l}}))\\
=&X_\beta((x_{s+1}-x)^{i_{s+1}},\ldots,(x_{s+l-l(\beta)}-x)^{i_{s+l-l(\beta)}})\\
&\times\Lambda_\beta((x_{s+l-l(\beta)}-x)^{i_{s+l-l(\beta)+1}},\ldots,(x_{s+l}-x)^{i_{x+l}}))\\
=&\eta_\beta((x_{s+1}-x)^{i_{s+1}},\ldots(x_{s+l}-x)^{i_{s+l}}).
\end{aligned}$$

In the second equality of the above equation, we have used Lemma 2.2 that Ω_β and Λ_β agree on linear functions.

Substituting the above expression of the $T_1^\beta(\eta_\beta)$ into Equation (4), we have that

$$\begin{aligned}
&L_2^\gamma(T(\xi_\alpha)\circ T(\eta_\beta))\\
=&\sum_{i_1,\ldots,i_{k+l-1}}\sum_s(-1)^{(s-1)(l-1)}T_1^\alpha(\xi_\alpha)((x_1-x)^{i_1},\ldots,(x_s-x)^{i_s},\\
&\eta_\beta((x_{s+1}-x)^{i_{s+1}},\ldots,(x_{s+l}-x)^{i_{s+l}}),\beta((x_{s+l+1}-x)^{i_{s+l+1}}),\ldots,\\
&\beta((x_{k+l-1}-x)^{i_{k+l-1}}))\frac{\partial}{\partial x^{i_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{i_{k+l-1}}}.
\end{aligned}$$

Lemma 3.1. *The restriction of $L(T(\xi_\alpha)\circ T(\eta_\beta))$ to $V^\alpha \cap V^\beta$ is*

$$\begin{aligned}&\sum_{i_1,\ldots,i_{k+l-1}}\sum_s(-1)^{(s-1)(l-1)}\xi_\alpha((x_1-x)^{i_1},\ldots,(x_s-x)^{i_s},\\ &\eta_\beta((x_{s+1}-x)^{i_{s+1}},\ldots,(x_{s+l}-x)^{i_{s+l}}),\beta((x_{s+l+1}-x)^{i_{s+l+1}}),\ldots,\\ &\beta((x_{k+l-1}-x)^{i_{k+l-1}}))\frac{\partial}{\partial x^{i_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{i_{k+l-1}}}.\end{aligned}$$

Proof: Because $T_1(\xi_\alpha)\in C^k(A;A_\alpha)$, $T_1(\eta_\beta)\in C^l(A;A_\beta)$, $T_1(\xi_\alpha)\circ T_1(\eta_\beta)$ is in $C^{k+l-1}(A;A_{\alpha\beta})$. Therefore, the restriction of $L(T(\xi_\alpha)\circ T(\eta_\beta))$ to $V^\alpha\cap V^\beta$ is same to the restriction of $L_2^{\alpha\beta}(T_1(\xi_\alpha)\circ T_1(\eta_\beta)))$ to $V^\alpha\cap V^\beta$.

We observe that the restriction of $\Omega_\alpha(f_1,\ldots f_{l(\alpha)})$ to V^α agrees with the restriction of $\Lambda_\alpha(f_1,\ldots,\ f_{l(\alpha)})$. This is because we are setting the variables in the normal direction of V^α equal to 0. Therefore, we have that the restriction of $\Omega_\alpha(f_1,\ldots,f_{l(\alpha)})$ to $V^\alpha\cap V^\beta$ is same to $\Lambda_\alpha(f_1,\ldots,f_{l(\alpha)})$. Using this observation, we have that the restriction of

$$\begin{aligned}&T_1(\xi_\alpha)((x_1-x)^{i_1},\ldots,(x_s-x)^{i_s},\eta_\beta((x_{s+1}-x)^{i_{s+1}},\ldots(x_{s+l}-x)^{i_{s+l}}),\\ &\beta((x_{s+l+1}-x)^{i_{s+l+1}}),\ldots,\beta((x_{k+l-1}-x)^{i_{k+l-1}}))\end{aligned}$$

to $V^\alpha\cap V^\beta$ is same to

$$\begin{aligned}&\xi_\alpha((x_1-x)^{i_1},\ldots,(x_s-x)^{i_s},\eta_\beta((x_{s+1}-x)^{i_{s+1}},\ldots,(x_{s+l}-x)^{i_{s+l}}),\\ &\beta((x_{s+l+1}-x)^{i_{s+l+1}}),\ldots,\beta((x_{k+l-1}-x)^{i_{k+l-1}})).\qquad\square\end{aligned}$$

Inspired by the results of Lemma 3.1, we introduce the following definition.

Definition 3.2. *Let M be a smooth manifold with a diffeomorphism γ. For all $\xi\in\wedge^k TM$, $\eta\in\wedge^l TM$. The γ-twisted pre-Lie product $\xi\circ_\gamma\eta$ is defined to be*

$$\begin{aligned}&\sum_{i_1,\ldots,i_{k+l-1}}\sum_s(-1)^{(s-1)(l-1)}\xi((x_1-x)^{i_1},\ldots,(x_s-x)^{i_s},\\ &\eta(((x_{s+1}-x)^{i_{s+1}},\ldots(x_{s+l}-x)^{i_{s+l}}),\gamma((x_{s+l+1}-x)^{i_{s+l+1}}),\ldots,\\ &\gamma((x_{k+l-1}-x)^{i_{k+l-1}})))\frac{\partial}{\partial x^{i_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{i_{k+l-1}}}.\end{aligned}$$

We notice that $(\sqcup_{\gamma\in\Gamma}M^\gamma)/\Gamma=\sqcup_{(\gamma):\gamma\in\Gamma}M^\gamma/C(\gamma)$, where (γ) is the conjugacy class of γ in Γ, and $C(\gamma)$ is the centralizer of γ in Γ. Therefore, $\Gamma^\infty(\oplus_\gamma\wedge^\bullet TM^\gamma\otimes\wedge^{l(\gamma)})^\Gamma$ is isomorphic to $\oplus_{(\gamma)}\Gamma^\infty(\wedge^\bullet TM^\gamma\otimes\wedge^{l(\gamma)}N^\gamma)^{C(\gamma)}$. We use the this identification in the following lemma.

Lemma 3.3. *Let*

$$\xi_{(\alpha)}=\sum_{\alpha\in(\alpha)}X_\alpha\otimes\Lambda_\alpha\in\Gamma^\infty(\wedge^{k-l(\alpha)}TV^\alpha\otimes\wedge^{l(\alpha)}N^\alpha)^{C(\alpha)}$$

and

$$\eta_{(\beta)}=\sum_{\beta\in(\beta)}Y_\beta\otimes\Lambda_\beta\in\Gamma^\infty(\wedge^{l-l(\beta)}TV^\beta\otimes\wedge^{l(\beta)}N^\beta)^{C(\beta)},$$

if $V^\alpha+V^\beta\neq V$ for all $\alpha\in(\alpha),\beta\in(\beta)$, then

$$L(T(\xi_{(\alpha)})\circ T(\eta_{(\beta)}))=0.$$

Proof. Following the same computation as to Equation (4), we have

$$\sum_{i_1,\ldots,i_{k+l-1}}\sum_{\alpha\beta=\gamma}\sum_{s}(-1)^{(s-1)(l-1)}T_1^{\alpha}(\xi_\alpha)((x_1-x)^{i_1},\ldots,(x_s-x)^{i_s},$$
$$T_1^{\beta}(\eta_\beta)((x_{s+1}-x)^{i_{s+1}},\ldots,(x_{s+l}-x)^{i_{x+l}}),$$
$$\beta((x_{s+l+1}-x)^{i_{s+l+1}}),\ldots,\beta((x_{k+l-1}-x)^{i_{k+l-1}}))\frac{\partial}{\partial x^{i_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{i_{k+l-1}}}.$$

Since $V^\alpha+V^\beta\neq V$, then its normal directions $N^\perp=N^\alpha\cap N^\beta\neq 0$. We observe that $T(\xi_\alpha)$ contains all the derivations[1] along $N^\perp$, and $T(\eta_\beta)$ contains all the derivations along N^β. As η_β is a section of $\wedge^{l-l(\beta)}TV^\beta\otimes\wedge^{l(\beta)}N^\beta$, $T_1^\beta(\eta_\beta)((x_{s+1}-x)^{i_{s+1}},\ldots,(x_{s+l}-x)^{i_{s+l}}))$ is independent of $N^\perp$. Therefore, we see that to have a nonzero contribution in $L(T(\xi_\alpha)\circ T(\eta_\beta))$, we need that $x^{i_1},\cdots,x^{i_{s+l}},\beta(x^{i_{s+l+1}}),\cdots,\beta(x^{i_{k+l-1}})$ contains two copies of the variables along $N^\perp$ and one copies of the variables along $N^\beta/N^\perp$. However, because $i_1,\ldots,i_{k+l-1}$ are distinguished and N^β is β invariant, $x^{i_1},\ldots,x^{s+k-1}$, $\beta(x^{s+k}),\ldots,\beta(x^{i_{k+l-1}})$ has at most one copy of the variables along the N^β direction. There are no enough variables along the $N^\perp$ direction. This implies that $L(T(\xi_{(\alpha)})\circ T(\eta_{(\beta)}))$ vanishes. □

Lemma 3.4. *The condition $V^\alpha+V^\beta=V$ is equivalent to the equality that $l(\alpha)+l(\beta)=l(\alpha\beta)$*

Proof. As $\dim(V^\alpha)+\dim(V^\beta)=\dim(V^\alpha+V^\beta)+\dim(V^\alpha\cap V^\beta)$, we have that

$$\begin{aligned}&l(\alpha)+l(\beta)\\ =&\dim(V)-\dim(V^\alpha)+\dim(V)-\dim(V^\beta)\\ =&\dim(V)-\dim(V^\alpha+V^\beta)+\dim(V)-\dim(V^\alpha\cap V^\beta)\\ \geq&\dim(V)-\dim(V^\alpha+V^\beta)+\dim(V)-\dim(V^{\alpha\beta})\\ =&\dim(V)-\dim(V^\alpha+V^\beta)+l(\alpha\beta),\end{aligned}$$

where we have used the fact that $V^\alpha\cap V^\beta\subset V^{\alpha\beta}$. Therefore, $l(\alpha)+l(\beta)=l(\alpha\beta)$ implies that $V=V^\alpha+V^\beta$.

On the other hand, assume that $V^\alpha+V^\beta=V$ and let $<\ ,\ >$ be a Γ invariant metric on V. For any $v\in V^{\alpha\beta}$, we have $\alpha\beta(v)=v$, and accordingly $\beta(v)=\alpha^{-1}(v)$, and $\beta(v)-v=\alpha^{-1}(v)-v$. Furthermore, as the metric $<\ ,\ >$ is Γ invariant, we see that $\beta(v)-v$ is orthogonal to V^β with respect to the metric $<\ ,\ >$ and $\alpha^{-1}(v)-v$ is orthogonal to V^α. Therefore we have that $\beta(v)-v=\alpha^{-1}(v)-v$ is orthogonal to $V^\alpha+V^\beta$, which is equal to V by the assumption. This implies that v has to belong to $V^\alpha\cap V^\beta$, and we have $V^\alpha\cap V^\beta=V^{\alpha\beta}$. This together with the above equations implies that

$$l(\alpha)+l(\beta)=\dim(V)-\dim(V^\alpha+V^\beta)+\dim(V)-\dim(V^{\alpha\beta})=l(\alpha\beta).$$

□

We summarize the above computation into the following theorem[2]

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[1] Rigorously speaking, $\Omega_\alpha,\Omega_\beta$ are not derivations. Here, we use the word "derivation" loosely, because they behave like derivations on linear functions.

[2] A similar content of the following theorem was stated in the first version of [1], but the proof there contained a crucial gap.

Theorem 3.5. *Consider*

$$\xi_{(\alpha)} \in \Gamma^\infty(\wedge^{k-l(\alpha)}TM^\alpha \otimes \wedge^{l(\alpha)}N^\alpha)^{C(\alpha)},$$
$$\eta_{(\beta)} \in (\Gamma^\infty(\wedge^{l-l(\beta)}TM^\beta \otimes \wedge^{l(\beta)}N^\beta)^{C(\beta)}.$$

The $(\gamma) \subset \Gamma$ *component of* $L(T(\xi_{(\alpha)}) \circ T(\eta_{(\beta)}))$ *is*

$$pr^\gamma(\sum_{\substack{\gamma=\lambda\mu\\ \lambda\in(\alpha),\mu\in(\beta)\\ l(\gamma)=l(\lambda)+l(\mu)}} \tilde{\xi}_\lambda \circ_\mu \tilde{\eta}_\mu),$$

where $\tilde{\xi}_\lambda$ *and* $\tilde{\eta}_\mu$ *are extensions of* ξ_λ *and* η_μ *to* M *as explained in Section 2.3.*

And the (γ) *component of the Gerstenhaber bracket* $L([T(\xi_{(\alpha)}), T(\eta_{(\beta)})])$ *is*

$$pr^\gamma(\sum_{\substack{\gamma=\lambda\mu\\ \lambda\in(\alpha),\mu\in(\beta)\\ l(\gamma)=l(\lambda)+l(\mu)}} \tilde{\xi}_\lambda \circ_\mu \tilde{\eta}_\mu - \tilde{\eta}_{\lambda\mu\lambda^{-1}} \circ_\mu \tilde{\xi}_\lambda).$$

We call the above defined bracket on $\Gamma^\infty(\wedge^{\bullet-l(\gamma)}TM^\gamma \otimes \wedge^{l(\gamma)}N^\gamma)^\Gamma$ *the "twisted Schouten-Nijenhuis bracket" on the orbifold* M/Γ*.*

Proof. Straight forward from Lemma 3.1 and 3.3. □

Remark 3.6. *In the statement of Theorem 3.5, it seems that the pre-Lie bracket between* ξ *and* η *depends on their extensions to* M*. However, according to Remark 2.5, when restricted to* $M^\alpha \cap M^\beta$*, the extensions* $\tilde{\xi}_\alpha, \tilde{\eta}_\beta$ *are both canonical. Therefore, the Gerstenhaber bracket computed above is independent of extensions.*

Theorem 3.5 gives a full description of the pre-Lie product and the Gerstenhaber bracket. In the following, we discuss a special case when all elements in (α) commute with all elements in (β). Under this assumption, we have a more explicit description of the twisted Schouten-Nijenhuis bracket.

Definition 3.7. *Let* Γ *be a group with conjugacy classes* (α) *and* (β)*. We say* (α) *commutes with* (β) *when all elements in* (α) *commute with all elements in* (β)*.*

Remark 3.8. *Our following results with the assumption of commuting conjugacy classes can be weakened to the assumption that the corresponding actions of the conjugacy classes commute.*

Because of Lemma 3.3, we are reduced to look at Gerstenhaber brackets in the case when $V^\alpha + V^\beta = V$, which implies that $V^\alpha \cap V^\beta = V^{\alpha\beta}$. We write V as a direct sum of $V^{\alpha\beta} \oplus N^\alpha \oplus N^\beta$, such that $V^\alpha = V^{\alpha\beta} \oplus N^\beta$ and $V^\beta = V^{\alpha\beta} \oplus N^\alpha$. We remark that because (α) commutes with (β), N^α and N^β are both α and β invariant subspaces.

Lemma 3.9. *Assume that* (α) *commutes with* (β)*. If* $X_\alpha \in \Gamma^\infty(\wedge^{k-l(\alpha)}TN^\beta)^{C(\alpha)}$ *with* $k - l(\alpha) > 0$*, then*

$$L(T(\xi_{(\alpha)}) \circ T(\eta_{(\beta)})) = 0.$$

Proof. We look at the term $T(\eta_\beta)(\cdots)$ in $T(X_\alpha \otimes \Lambda_\alpha) \circ T(\eta_\beta)$ at $\gamma = \alpha\beta$. It contains derivations along all the directions of N^β. Now if $X_\alpha \in \Gamma^\infty(\wedge^{k-l(\alpha)}TN^\beta)$, then $\beta(X_\alpha)$ also belongs to $\Gamma^\infty(\wedge^{k-l(\alpha)}TN^\beta)$ again. Therefore $T(X_\alpha \otimes \Lambda_\alpha) \circ T(Y_\beta \otimes \Lambda_\beta)$ contains too many derivations along N^β as $T(Y_\beta \otimes \Lambda_\beta)(\cdots)$ is constant along the direction of N^β. We conclude that

$$L(T(X_\alpha \otimes \Lambda_\alpha) \circ T(Y_\beta \otimes \Lambda_\beta)) = 0,$$

and

$$L(T(\xi_{(\alpha)})\circ T(\eta_{(\beta)}))=0.$$

$\square$

Lemma 3.10. *Under the same assumption as Lemma 3.9, if* $l-l(\beta)\geq 2$*, the for* $Y_\beta\in\Gamma^\infty(\wedge^{l-l(\beta)}TN^\alpha)^{C(\beta)}$*,*

$$L(T(\xi_{(\alpha)})\circ T(\eta_{(\beta)}))=0.$$

Proof. Because of Lemma 3.9, we can drop the β twist Equation (4). At the component $\gamma=\alpha\beta$, we look at the number of derivations along the direction of N^α. $T(\xi_\alpha)$ contributes $l(\alpha)$ and $T(Y_\beta\otimes\Lambda_\beta)$ contributes $l-l(\beta)$. Therefore, $T(\xi_\alpha)\circ T(Y_\beta\otimes\Lambda_\beta)$ contains at least the following number of derivations along N^α,

$$l(\alpha)+(l-l(\beta))-1\geq l(\alpha)+2-1\geq l(\alpha)+1.$$

This implies the statement of this lemma, because $dim(N^\alpha)=l(\alpha)$. $\square$

We summarize the above computations into the following theorem.

Theorem 3.11. *Assume that* (α) *and* (β) *commute. Let*

$$\xi_{(\alpha)}=\sum_{\alpha\in(\alpha)}X_\alpha\otimes\Lambda_\alpha\in\Gamma^\infty(\wedge^{k-l(\alpha)}TV^\alpha\otimes\wedge^{l(\alpha)}N^\alpha)^{C(\alpha)},$$

and

$$\eta_{(\beta)}=\sum_{\beta\in(\beta)}Y_\beta\otimes\Lambda_\beta\in\Gamma^\infty(\wedge^{l-l(\beta)}TV^\beta\otimes\wedge^{l(\beta)}N^\beta)^{C(\beta)}.$$

Then the component of $L(T(X_{(\alpha)})\circ T(Y_{(\beta)}))$ *in*

$$\Gamma^\infty(\wedge^{k+l-l(\alpha)-1}TV^{\alpha\beta}\otimes\wedge^{l(\alpha)}N^{\alpha\beta})^{C(\alpha\beta)}$$

is computed as follows.

1. *When* $V^\alpha+V^\beta\neq V$ *for all* $\alpha\in(\alpha),\beta\in(\beta)$, $L(T(X_{(\alpha)})\circ T(Y_{(\beta)}))=0$.
2. *When* $V^\alpha+V^\beta=V$*, we write* $V=V^{\alpha\beta}\oplus N^\alpha\oplus N^\beta$*, where* $V^{\alpha\beta}$ *is the invariant subspace of* $\alpha\beta$*,* N^α *is the subspace orthogonal to* V^α*, and* N^β *is the subspace orthogonal to* V^β*. In this case,* $V^\alpha=V^{\alpha\beta}\oplus N^\beta$*, and* $V^\beta=V^{\alpha\beta}\oplus N^\alpha$*. According to this decomposition, we write*

$$X_\alpha=\sum X_\alpha^{i_1\cdots i_s,p_1\cdots p_{k-l(\alpha)-s}}\frac{\partial}{\partial x^{i_1}}\wedge\cdots\frac{\partial}{\partial x^{i_s}}\wedge\frac{\partial}{\partial x^{p_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{p_{k-l(\alpha)-s}}}$$
$$\in\Gamma^\infty\big(\wedge^s\,TV^{\alpha\beta}\big)\otimes\Gamma^\infty\big(\wedge^{k-l(\alpha)-s}\,TN^\beta\big)$$
$$Y_\beta=\sum Y_\beta^{j_1\cdots j_t,q_1\cdots q_{l-l(\beta)-t}}\frac{\partial}{\partial x^{j_1}}\wedge\cdots\frac{\partial}{\partial x^{j_t}}\wedge\frac{\partial}{\partial x^{q_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{q_{l-l(\beta)-t}}}$$
$$\in\Gamma^\infty\big(\wedge^t\,TV^{\alpha\beta}\big)\otimes\Gamma^\infty\big(\wedge^{k-l(\beta)-t}\,TN^\alpha\big).$$

The component of $L(T(\xi_{(\alpha)})\circ T(\eta_{(\beta)}))$ *in*

$$\Gamma^\infty(\wedge^{k+l-l(\gamma)-1}TV^\gamma\otimes\wedge^{l(\gamma)}(N^\alpha\oplus N^\beta))^{C(\gamma)}$$

is computed to be

$$\sum_{\substack{\gamma=\lambda\mu\\ \lambda\in(\alpha),\,\mu\in(\beta)\\ l(\gamma)=l(\lambda)+l(\mu)}} \sum_{i_1,\ldots,i_{k-l(\lambda)},j_1,\ldots,j_{l-l(\mu)}} (-1)^{(z-1)(l-1)+(k-z)l+(l-l(\mu))l(\lambda)} X_\lambda^{i_1\cdots\widehat{i_z}\cdots i_{k-l(\lambda)}}$$

$$\frac{\partial}{\partial x^{i_z}} Y_\mu^{j_1\cdots j_{l-l(\mu)}} \frac{\partial}{\partial x^{i_1}}\wedge\cdots\widehat{\frac{\partial}{\partial x^{i_z}}}\cdots\wedge\frac{\partial}{\partial x^{i_{k-l(\lambda)}}}\wedge\frac{\partial}{\partial x^{j_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{j_{l-l(\mu)}}}\otimes\Lambda_\lambda\wedge\Lambda_\mu$$

$$+\sum_{i_1,\ldots,i_{k-l(\lambda)},j_1,\ldots,j_{l-l(\mu)-1},q_z} (-1)^{(k-l(\lambda))(l-1)+(k-l(\mu))(l(\lambda)-1)} X_\lambda^{i_1\cdots i_{k-l(\lambda)}}$$

$$\frac{\partial}{\partial x^{q_z}} Y_\mu^{j_1,\ldots,j_{l-l(\mu)-1},q_z} \frac{\partial}{\partial x^{i_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{i_{k-l(\lambda)}}}\wedge\frac{\partial}{\partial x^{j_1}}\wedge\cdots\frac{\partial}{\partial x^{j_{l-l(\mu)-1}}}\otimes\Lambda_\lambda\wedge\Lambda_\mu.$$

Proof. The first statement is a corollary of Lemma 3.3. We are left to show the second statement.

By Lemma 3.9, we conclude that X_α must be from $\Gamma^\infty(\wedge^{k-l(\alpha)}TV^{\alpha\beta})$ to have nontrivial contribution in $L([T(\xi_{(\alpha)}),T(\eta_{(\beta)})])$.

Similarly, by Lemma 3.10, we conclude that to have nontrivial contribution in $L([T(\xi_{(\alpha)}),T(\eta_{(\beta)})])$, Y_β has to be from one of the following space

$$(i)\ Y_\beta\in\Gamma^\infty(\wedge^{l-l(\beta)}TV^{\alpha\beta})\qquad (ii)\ Y_\beta\in\Gamma^\infty(\wedge^{l-l(\beta)-1}TV^{\alpha\beta})\otimes\Gamma^\infty(TN^\alpha).$$

We can apply Theorem 3.5 to compute the pre-Lie product. Because both $V^{\alpha\beta}$ and N^α are subspaces of V^β which is the β fixed point set, we can drop the β twist of the pre-Lie product. Therefore, we are left with the stand pre-Lie product.

When $Y_\beta\in\Gamma^\infty(\wedge^{l-l(\beta)}TV^{\alpha\beta})$, we have $L([T(\xi_\alpha),T(\eta_\beta)])=$

$$\sum_{i_1,\ldots,i_{k-l(\alpha)},j_1,\ldots,j_{l-l(\beta)}} (-1)^{(z-1)(l-1)+(k-z)l+(l-l(\beta))l(\alpha)} X_\alpha^{i_1\cdots\widehat{i_z}\cdots i_{k-l(\alpha)}}$$

$$\frac{\partial}{\partial x^{i_z}} Y_\beta^{j_1\cdots j_{l-l(\beta)}} \frac{\partial}{\partial x^{i_1}}\wedge\cdots\widehat{\frac{\partial}{\partial x^{i_z}}}\cdots\wedge\frac{\partial}{\partial x^{i_{k-l(\alpha)}}}\wedge\frac{\partial}{\partial x^{j_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{j_{l-l(\beta)}}}.$$

When $Y_\beta\in\Gamma^\infty(\wedge^{l-l(\beta)-1}TV^{\alpha\beta})\otimes\Gamma^\infty(TN^\alpha)$, we compute $L([T(\xi_\alpha),T(\eta_\beta)])=$

$$\sum_{i_1,\ldots,i_{k-l(\alpha)},j_1,\ldots,j_{l-l(\beta)-1},q_z} (-1)^{(k-l(\alpha))(l-1)+(k-l(\beta))(l(\alpha)-1)} X_\alpha^{i_1\cdots i_{k-l(\alpha)}}$$

$$\frac{\partial}{\partial q_z} Y_\beta^{j_1,\ldots,j_{l-l(\beta)-1},q_z} \frac{\partial}{\partial x^{i_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{i_{k-l(\alpha)}}}\wedge\frac{\partial}{\partial x^{j_1}}\wedge\cdots\frac{\partial}{\partial x^{j_{l-l(\beta)-1}}}.$$

□

Corollary 3.12. *Assume that* (α) *commutes with* (β). *Let* $\xi_{(\alpha)}\in\Gamma^\infty(\wedge^{k-l(\alpha)}TM^\alpha\otimes\wedge^{l(\alpha)}N^\alpha)^{C(\alpha)}$, *and* $\eta_{(\beta)}\in\Gamma^\infty(\Gamma^\infty(\wedge^{l-l(\gamma)}TM^\beta\otimes\wedge^{l(\beta)}N^\beta)^{C(\beta)})$. *Then the* (γ) *component of* $\xi_{(\alpha)}\circ\eta_{(\beta)}$ *in*

$$\Gamma^\infty(\wedge^{k+l-1-l(\gamma)}TM^\gamma\otimes\wedge^{l(\gamma)}N^\gamma)^{C(\gamma)},$$

is computed by

$$pr^\gamma(\sum_{\substack{\gamma=\lambda\mu\\ \lambda\in(\alpha),\,\mu\in(\beta)\\ l(\gamma)=l(\lambda)+l(\mu)}}\tilde{\xi}_\lambda\circ\tilde{\eta}_\mu|_{M^{\lambda\mu}}),$$

where $\circ$ *is the standard pre-Lie product by setting* $\gamma = id$ *in Definition 3.2, and* $\tilde{\xi}_\lambda, \tilde{\eta}_\mu$ *are extensions*[3] *of* ξ_λ, η_μ *to* M *as explained in Section 2.3.*

And the (γ) *component of the twisted Schouten-Nijenhuis bracket is*

$$L([T(\xi_{(\alpha)}), T(\eta_{(\beta)})]) = pr^\gamma(\sum_{\substack{\gamma = \lambda\mu \\ \lambda \in (\alpha), \mu \in (\beta) \\ l(\gamma) = l(\lambda) + l(\mu)}} [\tilde{\xi}_\lambda, \tilde{\eta}_\mu]|_{M^{\lambda\mu}}).$$

Remark 3.13. *Lemma 3.9, 3.10 and Theorem 3.11 have a natural extension when we drop the assumption of the commutativity between* (α) *and* (β)*. Again the twisted Schouten-Nijenhuis bracket is simplified but more complicated than Corollary 3.12. We will not discuss the formulas here.*

4. Noncommutative Poisson structure and symplectic reflection algebras

In this section, we want to find all possible noncommutative Poisson structures on $C^\infty(M) \rtimes \Gamma$ when M has a Γ-invariant symplectic form.

4.1. **Noncommutative Poisson structure.** We assume that the manifold M has a Γ-invariant symplectic structure. Because the group Γ is finite, there always exists a Γ-invariant compatible almost complex structure J on M. Accordingly, for any γ in Γ, the fixed point manifold M^γ is again a symplectic manifold with a compatible almost complex structure [9]. Therefore, all the fixed point manifolds $M^\gamma, \gamma \in \Gamma$ are of even dimension. And furthermore, the restriction of the $l(\gamma)$-th wedge power of the corresponding Poisson structure defines a global section on $\wedge^{l(\gamma)} N^\gamma$. Therefore, we can choose Λ_γ to be constant for any $\gamma \in \Gamma$.

Since all the fixed point manifolds have at least codimension equal to 2, there is no contribution of $\Gamma^\infty(\oplus_{\gamma; l(\gamma)=1} TM^\gamma \otimes N^\gamma)^\Gamma$ in the Hochschild cohomology of $C^\infty(M) \rtimes \Gamma$. Hence, we consider the set S of elements $\gamma \in \Gamma$ such that the fixed point subspace of γ is of codimension 2. It is easy to see that S is closed under the conjugacy action of Γ. By (1) and $\wedge^0 TV^\gamma \otimes \wedge^2 N^\gamma = \wedge^2 N^\gamma$, the second Hochschild cohomology of $A \rtimes \Gamma$ is isomorphic to

$$\Gamma^\infty(\wedge^2(TV))^\Gamma \bigoplus (\oplus_{\gamma \in S} \wedge^2 N^\gamma))^\Gamma,$$

where Γ action on the second component is the conjugacy action.

Theorem 4.1. *Assume that* M *is a real symplectic manifold, with a* Γ *symplectic action. Let* π *be an element in* $(\Gamma^\infty(\wedge^2(TM)))^\Gamma$*, and* $\sum_{\gamma \in S} \Lambda_\gamma$ *be an element in* $(\oplus_{\gamma \in S} \wedge^2 N^\gamma)^\Gamma$*. Then* $\pi + \sum_\gamma \Lambda_\gamma$ *is noncommutative Poisson structure on* $C^\infty(M) \rtimes \Gamma$ *if and only if*

1. *On* M*,* $[\pi, \pi] = 0$*;*
2. *For any* $\gamma \in S$*,* $pr^\gamma([\pi, \widetilde{\Lambda_\gamma}]|_{M^\gamma}) = 0$*.*

Proof. We compute $L([T(\pi + \sum_{\gamma \in S} \Lambda_\gamma), T(\pi + \sum_{\gamma \in S} \Lambda_\gamma)])$. It decomposes into the sum of four terms

$$i)\ L([T(\pi), T(\pi)]), \qquad ii)\ \sum_{\alpha \in S} L([T(\pi), T(\Lambda_\alpha)]),$$

$$iii)\ \sum_{\alpha \in S} L([T(\Lambda_\alpha), T(\pi)]), \qquad iv)\ \sum_{\alpha, \beta \in S} L([T(\Lambda_\alpha), T(\Lambda_\beta)]).$$

---

[3] Remark 2.5 on extensions still applies to this case here.

We compute the above i)-iv) terms separately.

1. $L([T(\pi),T(\pi)])$. On the identity component the Gerstenhaber bracket corresponds to the standard Schouten-Nijenhuis bracket. Therefore, we have

   $$L([T(\pi),T(\pi)])=[\pi,\pi],$$

   which is again on the identity component.
2. $\sum_{\alpha\in S}L([T(\pi),T(\Lambda_\alpha)])$. π is from the identity component and Λ_α is from the α component. Noticing that $(id)=\{id\}$ commutes with (α), we apply Corollary 3.12 to compute these terms,

   $$\sum_{\alpha\in S}(L([T(\pi),T(\Lambda_\alpha)])=\sum_{\alpha\in S}pr^\alpha([\pi,\widetilde{\Lambda}_\alpha]|_{M^\alpha}),$$

   where $[\pi,\widetilde{\Lambda}_\alpha]|_{M^\alpha}$ is on the α component.
3. $\sum_{\alpha\in S}L([T(\Lambda_\alpha),T(\pi)]))$. This is similar to the previous case. We apply Corollary 3.12 to compute the terms,

   $$\sum_{\alpha\in S}(L([T(\pi),T(\Lambda_\alpha)])=\sum_{\alpha\in S}pr^\alpha([\widetilde{\Lambda}_\alpha,\pi]_{M^\alpha}),$$

   where $[\widetilde{\Lambda}_\alpha,\pi]_{M^\alpha}$ is on the α component.
4. $\sum_{\alpha,\beta\in S}L([T(\Lambda_\alpha),T(\Lambda_\beta)])$. We decompose the computation into two subcases. We fix a Γ-invariant almost complex structure on M compatible with the symplectic form, which is always achievable.
   (a) $M^\alpha\cap M^\beta\neq M^\alpha$ or M^β.
   In this case, at each point $x\in M^\alpha\cap M^\beta$, the sum of T_xM^α and T_xM^β is equal to T_xM because T_xM^α and T_xM^β are both of complex codimension 1 and do not agree. (Otherwise $T_xM^\alpha=T_xM^\beta$ and $M^\alpha\cap M^\alpha$ will be either M^α or M^β near x.) Therefore, $T_xM=T_xM^\alpha\cap T_xM^\beta\oplus N^\alpha_x\oplus N^\beta_x$. We prove that at each point $x\in M^\alpha\cap M^\beta$, $T_xM^{\alpha\beta}=T_xM^\alpha\cap T_xM^\beta$.
   Let $v\in T_xM^{\alpha\beta}$. We have that $\alpha\beta(v)=v$. Therefore, $\beta(v)=\alpha^{-1}(v)$ and $\beta(v)-v=\alpha^{-1}(v)-v$. It is easy to check that $\beta(v)-v$ is orthogonal to T_xM^β, $\alpha^{-1}(x)-x$ is orthogonal to T_xM^α. By $T_xM^\alpha+T_xM^\beta=T_xM$, we see that $\beta(v)-v=\alpha^{-1}(v)-v=0$. This implies that $v\in T_xM^\alpha\cap T_xM^\beta$. Therefore, $M^{\alpha\beta}$ is equal to $M^\alpha\cap M^\beta$ with codimension 4.
   We conclude that $L([T(\Lambda_\alpha),T(\Lambda_\beta)])$ is supported on the component of $M^{\alpha\beta}=M^\alpha\cap M^\beta$ which is of codimension 4. By Equation (1), there is no nontrivial 3-Hochschild cocycle supported on a codimension 4 component of $\alpha\beta$. Therefore, the bracket has to be equal to 0.
   (b) $M^\alpha\cap M^\beta=M^\alpha($ and $M^\beta)$. In this case, N^α agrees with N^β. Λ_α is $C^\infty(M^\alpha)$ proportional to Λ_β. Both α and β action preserves $N^\beta=N^\alpha$ direction. Using Equation (4), it is easy to see that $L([T(\Lambda_\alpha),T(\Lambda_\beta)])$ is supported along the $N^\alpha=N^\beta$ direction, as both α and β preserve $N^\alpha=N^\beta$. This implies that $L([T(\Lambda_\alpha),T(\Lambda_\beta)])$ has to be equal to 0 because $\dim(N^\alpha=N^\beta)=2$ while $L([T(\Lambda_\alpha),T(\Lambda_\beta)])$ is a tri-vector field.

In summary, we have that

$$\begin{aligned}&L([T(\pi+\sum_{\gamma\in S}\Lambda_\gamma),T(\pi+\sum_{\gamma}\Lambda_\gamma)])\\ =&[\pi,\pi]+\sum_{\alpha\in S}pr^\alpha([\pi,\tilde{\Lambda}_\alpha]|_{M^\alpha})+\sum_{\alpha\in S}pr^\alpha([\tilde{\Lambda}_\alpha,\pi]|_{M^\alpha})\\ =&[\pi,\pi]+2\sum_{\alpha\in S}pr^\alpha([\pi,\tilde{\Lambda}_\alpha]|_{M^\alpha}).\end{aligned}$$

This concludes the Theorem.

□

Corollary 4.2. *Let V be a real symplectic vector space with a Γ invariant linear symplectic form ω. Let π be the corresponding Poisson structure of ω. Then $\kappa = \pi+\sum_{\alpha\in S}\Lambda_\alpha$ is a noncommutative Poisson structure on $A\rtimes\Gamma$ if and only if Λ_α is constant on V^α.*

Proof. Under the assumption of the corollary, the restriction of the symplectic form ω to N^α for $\alpha\in S$ is a symplectic two form. We denote the corresponding Poisson structure on N^α by π_α. Accordingly, we can write $\Lambda_\alpha = f_\alpha\pi_\alpha$, where f_α is a polynomial function on V^α.

By Theorem 4.1, $\pi+\sum_{\alpha\in S}f_\alpha\pi_\alpha$ is a noncommutative Poisson structure if and only if

(1) $[\pi,\pi]=0$,
(2) $[\pi,f_\alpha]|_{V^\alpha}=0$.

Equation (1) is automatically satisfied because π is Poisson. Because $[\pi,\pi_\alpha]=0$, Equation (2) is reduced to

$$pr^\alpha([\pi,f\pi_\alpha]|_{V^\alpha})=pr^\alpha([\pi,f]\wedge\pi_\alpha|_{V^\alpha})=[\pi,f]|_{V^\alpha}\wedge\pi_\alpha=0.$$

Therefore, $\pi+\sum_{\alpha\in S}f_\alpha\pi_\alpha$ is a noncommutative Poisson structure if and only if $[\pi,f_\alpha]|_{V^\alpha}=0$, for all $\alpha\in S$. Because f is a polynomial with only variables from V^α and π is linear, $[\pi,f]=[\pi,f]|_{V^\alpha}$. To have a noncommutative Poisson structure, f_α has to satisfy $[\pi,f_\alpha]=0$ for all $\alpha\in S$. Because ω is symplectic, $[\pi,f_\alpha]=0$ enforces f_α to be a constant. Therefore, Λ_α is also a constant on V^α for all $\alpha\in S$. □

Remark 4.3. *The complex analogs of Theorem 4.1 and Corollary 4.2 hold when M(or V) has a Γ-invariant compatible almost quaternionic structure. The proofs generalize directly.*

4.2. **Remarks on deformation quantizations.** As is known that a noncommutative Poisson structure on an algebra A is in one to one correspondence to infinitesimal deformation of A. It is natural to ask whether one can integrate the infinitesimal deformation to a real one. This is also related to the idea of deformation quantization in mathematical physics. In [16], the second author introduced a notion of deformation quantization of a noncommutative Poisson structure, which we recall in the following.

Definition 4.4. *A deformation quantization of a noncommutative Poisson structure Π on an associative algebra A is an associative product $\star_\hbar$ on $A[[\hbar]]$, such that $f\star_\hbar g=\sum_i \hbar^i C_i(f,g)$ for $f,g\in A$ satisfying*

(1) $C_0(f,g)=fg$;
(2) *The Hochschild cohomology class* $[C_1]$ *is equal to* Π.

It is natural to ask whether all the noncommutative Poisson structures defined in Theorem 4.1 can be deformation quantized. One special case is already known and well studied, when there is only an Γ-invariant Poisson structure on M. The deformation quantization of these type of noncommutative Poisson structures on $C^\infty(M)\rtimes\Gamma$ is studied in [16], [17], [5], [13], etc. The other special case is the following proposition, which is essentially due to Etingof and Ginzburg [7][Theorem 1.3].

Proposition 4.5. *The noncommutative Poisson structure on a symplectic vector space obtained in Corollary 4.2 can be deformation quantized.*

Proof. In the following, we work with the field $\mathbb{C}$, because we will use the construction of the symplectic reflection algebras in [7] and Theorem 1.3 therein. Everything extends to the field $\mathbb{R}$, because Theorem 1.3 in [7] still holds in the real case. (The real group algebra of a finite group is semisimple.)

In [7], a symplectic reflection algebra $H_{t,c}$ is introduced as

$$TV\rtimes\Gamma/I<x\otimes y-y\otimes x-\kappa(x,y)\in T^2V\oplus\mathbb{C}\Gamma>_{x,y\in V},$$

where (V,ω) is a finite dimensional complex symplectic vector space over $\mathbb{C}$, TV its tensor algebra, and κ is defined to be

$$\kappa(x,y)=t\pi(x,y)+\sum_{\alpha\in S}c_\alpha\pi_\alpha U_\alpha,$$

a Γ-invariant section of $\wedge^2V+\oplus_{\alpha\in S}\wedge^2N^\alpha$.

We assign V degree 1, and $\mathbb{C}\Gamma$ degree 0. This defines an increasing filtration $F_\bullet$ on $H_{t,c}$. It was proved by in [7][Theorem 1.3] that $H_{t,c}$ satisfies Poincaré-Birkhoff-Witt property, i.e. the tautological embedding $V\hookrightarrow gr(H_{t,c})$ extends to an isomorphism $Q:\mathrm{Poly}(V)\rtimes\Gamma\to gr(H_{t,c})$. We define gr_i to be the projection from $gr(H_{t,c})$ to its $i-$th degree component.

We define a formal deformation quantization of $\mathrm{Poly}(V)\rtimes\Gamma$ as follows. For $fU_\alpha, gU_\beta\in\mathrm{Poly}(V)\rtimes\Gamma$,

$$fU_\alpha\star gU_\beta=\sum_{i,j,k=0}^{\infty}\hbar^{i+j-k}Q^{-1}(gr_k(gr_i(Q(fU_\alpha))gr_j(Q(gU_\beta)))).$$

In particular, $C_i(fU_\alpha,gU_\beta)$ is defined to be

$$C_i(fU_\alpha,gU_\beta)=\sum_{p+q-r=i}Q^{-1}(gr_r(gr_p(Q(fU_\alpha))gr_q(Q(gU_\beta)))).$$

Because $Q(fU_\alpha)$ and $Q(gU_\beta)$ are of finite degrees, p,q in the summation are both finite. Therefore, the sum in the definition of C_i is finite and C_i is well defined.

We check that $\star$ is associative. $(fU_\alpha \star gU_\beta) \star hU_\gamma =$

$$
\begin{aligned}
&= \sum_{i,j,k=0}^{\infty} \hbar^{i+j-k} Q^{-1}(gr_k(gr_i(Q(fU_\alpha))gr_j(Q(gU_\beta)))) \star hU_\gamma \\
&= \sum_{i,j,k,p,q,r=0}^{\infty} \hbar^{i+j-k}\hbar^{p+q-r} Q^{-1}(gr_r(gr_p(Q(Q^{-1}(gr_k(gr_i(Q(fU_\alpha)gr_j(Q(gU_\beta)))))))) \\
&\quad gr_q(Q(hU_\gamma)))) \\
&= \sum_{i,j,k=0}^{\infty}\sum_{p,q,r=0}^{\infty} \hbar^{i+j+p+q-k-r} Q^{-1}(gr_r(gr_p(gr_k(gr_i(Q(fU_\alpha))gr_j(Q(gU_\beta))))gr_q(Q(hU_\gamma)))) \\
&= \sum_{i,j,k=p,q,r}^{\infty} \hbar^{i+j+q-r} Q^{-1}(gr_r(gr_k(gr_i(Q(fU_\alpha))gr_j(Q(gU_\beta)))gr_q(Q(hU_\gamma)))) \\
&= \sum_{i,j,q=0}^{\infty} \hbar^{i+j+q} \sum_{r=0}^{\infty} \hbar^{i+j+q-r} Q^{-1}(gr_r(\sum_{k=0}^{\infty} gr_k(gr_i(Q(fU_\alpha))gr_j(Q(gU_\beta)))gr_q(Q(hU_\gamma)))) \\
&= \sum_{i,j,k,r=0}^{\infty} \hbar^{i+j+q-r} Q^{-1}(gr_r(gr_i(Q(fU_\alpha))gr_j(Q(gU_\beta))gr_q(Q(hU_\gamma)))),
\end{aligned}
$$

which by the similar computation is equal to

$$fU_\alpha \star (gU_\beta \star hU_\gamma).$$

We look at $C_1(fU_\alpha, gU_\beta) = \sum_{i+j-k=1} Q^{-1}(gr_k(gr_i(Q(fU_\alpha))gr_j(Q(gU_\beta))))$. To check that C_1 is cohomologus to κ. We compute $L(C_1)$ as follows using the definition.

$$
\begin{aligned}
&=L_3(\sum_{i_1,i_2} C_1(x_1^{i_1} - x^{i_1}, x_2^{i_2} - x^{i_2})\frac{\partial}{\partial x^{i_1}} \wedge \frac{\partial}{\partial x^{i_2}}) \\
&=L_3(\sum_{i_1,i_2} gr_0((x_1 - x)^{i_1}(x_2 - x)^{i_2})\frac{\partial}{\partial x^{i_1}} \wedge \frac{\partial}{\partial x^{i_2}}) \\
&=L_3(\sum_{i_2<i_1} (x^{i_1}x^{i_2} - x^{i_2}x^{i_1}))\frac{\partial}{\partial x^{i_1}} \wedge \frac{\partial}{\partial x^{i_2}} \\
&= \sum_{i_2<i_1} (\omega(x^{i_1}, x^{i_2}) + \sum_{\alpha\in S} c_\alpha \omega_\alpha(x^{i_1}, x^{i_2})U_\alpha)\frac{\partial}{\partial x^{i_1}} \wedge \frac{\partial}{\partial x^{i_2}} \\
&=\frac{1}{2}\sum_{i_1,i_2}(\omega(x^{i_1}, x^{i_2}) + \sum_{\alpha\in S} c_\alpha \omega_\alpha(x^{i_1}, x^{i_2})U_\alpha)\frac{\partial}{\partial x^{i_1}} \wedge \frac{\partial}{\partial x^{i_2}}.
\end{aligned}
$$

In the third equality of the above equation, we have used the definition of C_1 and the product structure in $H_{t,c}$. When $i_1 < i_2$, $x^{i_1}x^{i_2}$ has no degree 0 term. When $i_1 > i_2$, degree 0 term of $x^{i_1}x^{i_2}$ is $x^{i_1}x^{i_2} - x^{i_2}x^{i_1}$.

In conclusion, $\star$ is a deformation quantization of $A \rtimes \Gamma$ with the noncommutative Poisson structure equal to $\frac{1}{2}\kappa$. □

Remark 4.6. *The generalization of the deformation quantization defined in Proposition 4.5 to affine varieties was studied by Etingof* [6].

It is natural to ask whether the deformation quantization constructed in Theorem 4.5 is unique up to isomorphisms. From the knowledge in Poisson geometry, we know that the isomorphic classes of deformation quantization of a Poisson structure is determined by its second Poisson cohomology. We denote the Poisson cohomology group of $A \rtimes \Gamma$ associated to κ by $H^\bullet_\kappa(A\rtimes\Gamma)$. In following we compute $H^2_\kappa(A\rtimes\Gamma)$.

Proposition 4.7. *Let (V,ω) be a real symplectic*[4] *vector space with a finite group Γ symplectic action. Let κ be defined as in Corollary 4.2. Then the second Poisson cohomology of κ is isomorphic to*

$$\{\sum_{\gamma\in S} c_\gamma\pi_\gamma \in \Gamma^\infty(\bigoplus_{\gamma\in S}\wedge^2 N^\gamma)^\Gamma |\ \textit{for all } \gamma\in S, c_\gamma \textit{ is a constant on } V^\gamma.\}.$$

Proof. According to Equation (1) and the fact that the codimensions of all V^α are even, the second Hochschild cohomology of $A\rtimes\Gamma$ consists of

$$\Gamma^\infty(\wedge^2 TV)^\Gamma \bigoplus (\sum_{\gamma\in S}\wedge^0 V^\gamma)^\Gamma.$$

Let $\Xi+\sum_{\gamma\in S} f_\gamma\pi_\gamma$ be from $\Gamma^\infty(\wedge^2TV)^\Gamma\bigoplus(\sum_{\gamma\in S}\wedge^0V^\gamma)^\Gamma$. We compute $L([T(\Xi+\sum_{\gamma\in S} f_\gamma\pi_\gamma), T(\kappa)])$ as follows. The computation is similar to the proof of Theorem 4.1. We deal with the four terms separately.

$$1)\ L([T(\Xi),T(\pi)]),\qquad 2)\ L([T(\Xi),T(\sum_{\gamma\in S}c_\gamma\pi_\gamma)])$$

$$3)\ L([T(\sum_{\gamma\in S}f_\gamma\pi_\gamma),T(\pi)]),\quad 4)\ L([T(\sum_{\gamma\in S}f_\gamma\pi_\gamma),T(\sum_{\gamma\in S}c_\gamma\pi_\gamma)]).$$

The following computation follows exactly the same reasons as in the proof of Theorem 4.1.

$$1) L([T(\Xi),T(\pi)]) = [\Xi,\pi];$$

$$2) L([T(\Xi),T(\sum_{\gamma\in S}c_\gamma\pi_\gamma)]) = \sum_{\gamma\in S} pr^\gamma(c_\gamma[\Xi,\pi_\gamma]|_{V^\gamma});$$

$$3) L([T(\sum_{\gamma\in S}f_\gamma\pi_\gamma),T(\pi)]) = \sum_{\gamma\in S} pr^\gamma([f_\gamma\pi_\gamma,\pi]|_{V^\gamma}) = \sum_{\gamma\in S} pr^\gamma([f_\gamma,\pi]\wedge\pi_\gamma)|_{V^\gamma};$$

$$4) L([T(\sum_{\gamma\in S}f_\gamma\pi_\gamma),T(\sum_{\gamma\in S}c_\gamma\pi_\gamma)]) = 0.$$

According to the above computation, $\Xi+\sum_{\gamma\in S} f_\gamma\pi_\gamma$ is κ closed if and only if it satisfies the following equations.

$$\begin{array}{ll} 1) & [\Xi,\pi]=0,\\ 2) & pr^\gamma(\{c_\gamma[\Xi,\pi_\gamma]+[f_\gamma,\pi]\wedge\pi_\gamma\}|_{V^\gamma})=0, \text{ for all}\gamma\in\Gamma.\end{array} \tag{5}$$

We denote the space of solutions to the above equations by Z^2_κ.

Next we compute the Poisson coboundary in $(\Gamma^\infty(\wedge^2TV)\bigoplus_{\alpha\in S}\wedge^2N^\gamma)^\Gamma$.

According to Equation (1), $HH^1(A\rtimes\Gamma, A\rtimes\Gamma)$ consists of Γ invariant vector field on V because all V^γs for $\alpha\neq id$ have at least codimension 2. Let $X\in\Gamma^\infty(TV)^\Gamma$. We compute $L([T(\kappa),T(X)])$ as follows. Because X is on (id)-component and $c_\gamma\pi_\gamma$ is on (α)-component which commutes with (id), we can apply Corollary 3.12.

$$\begin{aligned}&L([T(\kappa),T(X)])\\ =&[\pi,X]+\sum_{\gamma\in s}pr^\gamma(c_\gamma[\pi_\gamma,X]|_{V^\gamma}).\end{aligned}$$

---

[4] When V is a complex vector space, we need to assume that V has a Γ-invariant compatible almost quaterionic structure.

We denote the space of elements in $\Gamma^\infty(\wedge^2 TV)^\Gamma \bigoplus (\sum_{\gamma\in S}\wedge^0 V^\gamma)^\Gamma$ of the above form by B^2_κ.

We want to find out the quotient Z^2_κ/B^2_κ. Given $\Xi\in\Gamma^\infty(\wedge^2 TV)^\Gamma$ with $[\Xi,\pi]=0$, because π is from a symplectic form, we can find a Γ-invariant vector field $X\in\Gamma^\infty(TV)$ such that $[\pi,X]=\Xi$. Because we know that $L([T(\kappa),T(X)])=[\pi,X]+\sum_{\gamma\in S} g_\gamma\pi_\gamma=[\pi,X]+\sum_{\gamma\in S}c_\gamma pr^\gamma([\pi_\gamma,X]|_{V^\gamma})$ is κ-closed. We conclude that

$$\sum_{\gamma\in S}(g_\gamma-f_\gamma)\pi_\gamma$$

is also κ closed. Substituting this expression into second equation of (5), we have that for all $\gamma\in S$

$$pr^\gamma([\pi,(g_\gamma-f_\gamma)]\wedge\pi_\gamma|_{V^\gamma})=[\pi,(g_\gamma-f_\gamma)]|_{V^\gamma}\wedge\pi_\gamma|_{V^\gamma}=0.$$

Since $g_\gamma-f_\gamma$ is supported on V_γ and π is linear, we have

$$[\pi,g_\gamma-f_\gamma]|_{V^\gamma}=0\Leftrightarrow[\pi,g_\gamma-f_\gamma]=0.$$

Because π is symplectic, this implies that $g_\gamma-f_\gamma$ has to be a constant. It is obvious that $X+\sum_{\gamma\in S}(g_\gamma+c_\gamma)\pi_\gamma$ is κ closed.

We are left to show that any nonzero element like $\sum_{\gamma\in S}a_\gamma\pi_\gamma$ is not a coboundary for any constant a_γ, $\gamma\in S$. If $X\in\Gamma^\infty(TV)^\Gamma$ such that $L(T(\kappa),T(X))=\sum_{\gamma\in S}a_\gamma\pi_\gamma$, then by the similar computation as above we have that

$$[\pi,X]=0,\qquad pr^\gamma(c_\gamma[\pi_\gamma,X]|_{V^\gamma})=a_\gamma\pi_\gamma.$$

for any $\gamma\in S$.

As π is from a symplectic form, $[\pi,X]=0$ implies that there is a function f such that $X=[\pi,f]$. Therefore, $[\pi_\gamma,X]=[\pi_\gamma,[\pi,f]]$ contains no component proportional to π_γ. Accordingly, $pr^\gamma(c_\gamma[\pi_\gamma,X])=0=a_\gamma\pi_\gamma$.

In conclusion, we see that the quotient space Z^2_κ/B^2_κ is

$$\{\sum_{\gamma\in S}c_\gamma\pi_\gamma\in\Gamma^\infty(\bigoplus_{\gamma\in S}\wedge^2 N^\gamma)^\Gamma|\text{ for all }\gamma\in S, c_\gamma\text{ is a constant on }V^\gamma.\}$$

□

Remark 4.8. *Proposition 4.7 shows that the dimension of all the infinitesimal deformation of the noncommutative Poisson structure κ is equal to the size of the set S. Furthermore, it is easy to check that the all the infinitesimal deformation actually corresponds to Poisson structures. This gives another explanation of Corollary 4.2.*

Proposition 4.5, 4.7 inspires a series of interesting questions. The cocycle κ is a very special type of noncommutative Poisson structure on $A\rtimes\Gamma$ defined in Theorem 4.1. Can all of the noncommutative Poisson structures defined in Theorem 4.1 be deformation quantized? If their deformation quantizations exist, how many are they? All these problems have a general version on $C^\infty(M)\rtimes\Gamma$. It is closely related to the Conjecture 1, [5] by Dolgushev and Etingof. We plan to address these questions in the future publication.

4.3. **Noncommutative quadratic Poisson structures.** In this subsection, we provide some new examples of noncommutative Poisson structures other than those in Corollary 4.2. We can easily see that these Poisson structures are not symplectic at all, and they should be viewed as generalized quadratic Poisson structures.

We consider the space of $\mathbb{R}^4 = \mathbb{C}\times\mathbb{C}$ with the following $\mathbb{Z}_n\times\mathbb{Z}_m$ action, where $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$. Let (z_1, z_2) be holomorphic coordinates on $\mathbb{C}^2$, and $(k, l)\in\mathbb{Z}_n\times\mathbb{Z}_m$. Define

$$\begin{aligned}(k,l) :&(z_1,z_2) \longrightarrow (\exp(\frac{2k\pi i}{n})z_1, \exp(\frac{2l\pi i}{m})z_2)\\ &(\bar z_1,\bar z_2) \longrightarrow (\exp(-\frac{2k\pi i}{n})\bar z_1, \exp(-\frac{2l\pi i}{m})\bar z_2).\end{aligned}$$

The fixed point subspace of $(k, l)\in\mathbb{Z}_n\times\mathbb{Z}_m$ can be described explicitly.

1. if $k\neq 0, l\neq 0$, the fixed point set of (k, l) consists of only one point, the origin;
2. if $k = 0, l\neq 0$, $(0, l)$'s fixed point set is $\mathbb{C}\times\{0\}\subset\mathbb{C}\times\mathbb{C}$;
3. if $k\neq 0, l = 0$, $(k, 0)$'s fixed point set is $\{0\}\times\mathbb{C}\subset\mathbb{C}\times\mathbb{C}$;
4. if $k = l = 0$, $(0, 0)$'s fixed point space is $\mathbb{C}\times\mathbb{C}$.

To look for noncommutative Poisson structures on $Poly(\mathbb{R}^4)\rtimes(\mathbb{Z}_n\times\mathbb{Z}_m)$, we only need to consider the fixed point space of the identity, which is $\mathbb{C}^2$, and those codimension 2 fixed point subspaces, which are $(k, 0)$'s fixed point subspace $\{0\}\times\mathbb{C}$, and $(0, l)$'s fixed point subspace $\mathbb{C}\times\{0\}$. We consider the following collection of bivector fields where $\alpha, \beta, \lambda_k, \mu_l$ are real constants.

1. on $(0, 0)$'s fixed point subspace, we consider $\Pi^\alpha_{0,0} = i\alpha|z_2|^2\frac{\partial}{\partial z_1}\wedge\frac{\bar\partial}{\partial\bar z_1}$. We notice that $\Pi_{0,0}$ is $\mathbb{Z}_n\times\mathbb{Z}_m$ invariant, and satisfies $[\Pi_{0,0}, \Pi_{0,0}] = 0$;
2. on $(k, 0)$'s fixed point subspace, we consider $\Pi_{k,0} = i\lambda_k|z_2|^2\frac{\partial}{\partial z_1}\wedge\frac{\bar\partial}{\partial\bar z_1}$, which is a smooth section of the determinant bundle of the normal bundle over $\{0\}\times\mathbb{C}\subset\mathbb{C}\times\mathbb{C}$. Again we notice that $\Pi_{k,0}$ is $\mathbb{Z}_n\times\mathbb{Z}_m$ invariant, and $[\Pi_{0,0}, \Pi_{k,0}] = 0$;
3. on $(0, l)$'s fixed point subspace, we consider $\Pi_{0,l} = i\mu_l|z_1|^2\frac{\partial}{\partial z_2}\wedge\frac{\bar\partial}{\partial\bar z_2}$, which is a smooth section of the determinant bundle of the normal bundle over $\mathbb{C}\times\{0\}\subset\mathbb{C}\times\mathbb{C}$. We notice that $\Pi_{0,l}$ is $\mathbb{Z}_n\times\mathbb{Z}_m$ invariant, but
$$\begin{aligned}[\Pi_{0,0}, \Pi_{0,l}] = -\alpha\mu_l\Big[&|z_2|^2(z_1\frac{\partial}{\partial z_1} - \bar z_1\frac{\bar\partial}{\partial\bar z_1})\wedge\frac{\partial}{\partial z_2}\wedge\frac{\bar\partial}{\partial\bar z_2}\\ &+ |z_1|^2(z_2\frac{\partial}{\partial z_2} - \bar z_2\frac{\bar\partial}{\partial\bar z_2})\wedge\frac{\partial}{\partial z_1}\wedge\frac{\bar\partial}{\partial\bar z_1}\Big]\neq 0.\end{aligned}$$
However, if we look at the restriction of $[\Pi_{0,0}, \Pi_{0,l}]$ to $(0, l)$'s fixed point subspace which is $\mathbb{C}\times\{0\}$, it does vanish. We have that
$$pr^{(0,l)}([\Pi_{0,0}, \Pi_{0,l}]|_{\mathbb{C}\times\{0\}}) = 0.$$

By Theorem 4.1, we conclude that the collection of $(\Pi^\alpha_{0,0}, \Pi_{k,0}, \Pi_{0,l})$ defines a family of noncommutative Poisson structures on $Poly(\mathbb{R}^4)\rtimes(\mathbb{Z}_n\times\mathbb{Z}_m)$. One can also easily check that if we replace $\Pi^\alpha_{0,0}$ by $\Pi^\beta_{0,0} = i\beta|z_1|^2\frac{\partial}{\partial z_2}\wedge\frac{\bar\partial}{\partial\bar z_2}$, $(\Pi^\beta_{0,0}, \Pi_{k,0}, \Pi_{0,l})$ defines another family of noncommutative Poisson structures on $Poly(\mathbb{R}^4)\rtimes(\mathbb{Z}_n\times\mathbb{Z}_m)$.

There are various ways to generalize the above families of examples to higher dimensions. For example, one can consider $\mathbb{Z}_n\times\mathbb{Z}_m$ acts on the first two components of $\mathbb{C}^k$ as same as the above, but acts trivially on the left $\mathbb{C}^{k-2}$ component. Then the above two families of noncommutative Poisson structures naturally extend to $Poly(\mathbb{R}^{2k}) \rtimes (\mathbb{Z}_n \times \mathbb{Z}_m)$. We will leave the more nontrivial generalizations to the future [11].

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