QUANTIZATION OF POISSON-HOPF STACKS ASSOCIATED WITH GROUP LIE BIALGEBRAS

GILLES HALBOUT AND XIANG TANG

Abstract. Let G be a Poisson Lie group and $\mathfrak{g}$ its Lie bialgebra. Suppose that $\mathfrak{g}$ is a group Lie bialgebra. This means that there is an action of a discrete group Γ on G deforming the Poisson structure into coboundary equivalent ones. Starting from this we construct a non-trivial stack of Hopf-Poisson algebras and prove the existence of associated deformation quantizations. This non-trivial stack is a stack of functions on the formal Poisson group, dual of the starting Γ Poisson-Lie group. To quantize this non-trivial stack we use quantization of a Γ Lie bialgebra which is the infinitesimal of a Γ Poisson-Lie group (cf [MS] for simple Lie groups and Γ a covering of the Weyl group and [EH2] for quantization in the general case).

0. Introduction

In this paper, we study examples of Poisson Hopf stacks and their quantization. In [EH2], the first author and his author considered quantization of a Γ Lie bialgebra (**LBA**). As an outcome of this quantization, they constructed a functor from the category of Γ Lie bialgebra to the category of Γ quantized universal enveloping algebras (**QUE**). In this paper, we first study the dual of a Γ universal enveloping algebra. Similar to the duality between Lie bialgebras and Poisson-Lie groups, we discover a stack of Poisson formal series Hopf algebras (**PFSHA**), dual to a Γ Lie bialgebra. Then we study deformation quantization of this stack of Poisson formal series Hopf algebras. We construct the deformation quantization by applying the Drinfeld functor to a Γ quantized universal enveloping algebra, and obtain a stack of quantized formal series Hopf algebras (**QFSHA**). We summarize our results into the following commutative diagram.

$$\begin{array}{ccc} \Gamma\text{-}\mathbf{LBA} & \xrightarrow{\text{EH}} & \Gamma\text{-}\mathbf{QUE} \\ \approx \updownarrow & & \updownarrow \text{Dr} \\ \Gamma\text{-}\mathbf{PFSHA} & \xrightarrow{\text{Quant}} & \Gamma\text{-}\mathbf{QFSHA} \end{array}$$

Let Γ be a discrete group, G a Lie group and $\mathfrak{g}$ its Lie algebra. Suppose that $\mathfrak{g}$ is a Γ Lie bialgebra (or equivalently that G is a Γ Poisson group), i.e. a Lie algebra $(\mathfrak{g}, \mu_{\mathfrak{g}})$ together with a Lie cobracket δ_e, an action of Γ, $\theta : \Gamma \to \mathrm{Aut}(\mathfrak{g}, \mu_{\mathfrak{g}})$ and $f : \Gamma \to \wedge^2(\mathfrak{g})$ a map satisfying compatibility rules such that Γ acts on the double. Precise definitions and equivalent categories corresponding to these objects will be recalled in Section 1. Examples of Γ Lie bialgebras arise from the following situation: G is a Poisson-Lie group with Lie bialgebra $(\mathfrak{g}, \mu_{\mathfrak{g}}, \delta_{\mathfrak{g}})$, and $\Gamma \subset G$ is a discrete subgroup. Another example is when $\mathfrak{g}$ is a Kac-Moody Lie algebra $\mathfrak{g}$, and Γ is a covering of the Weyl group of $\mathfrak{g}$. In the latter case, a quantization was given ([MS]).

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Quantization of a general Γ Lie bialgebra was done in [EH2]. We will also recall this quantization result in Section 1.

It is then a natural question to ask what structure one gets on the corresponding dual groups. Considering the function algebra of a formal group, we get a trivial stack of Poisson Hopf algebras. In Section 3, we prove that we get a non-trivial stack of Poisson algebras of functions on the formal Poisson Lie group G^* dual to a Γ Poisson Lie group G. To do so, we will construct "lifts" of the elements $(f(\gamma))_{\gamma\in\Gamma}$ in the function algebra on G^*. In Section 2, we recall basic definitions of stacks and of their quantizations.

In Section 4, we construct quantization of these non-trivial Poisson-Hopf stacks. To do so we use quantization (cf [EH2]) of a Γ Lie bialgebra. To deduce from it a quantization of a non-trivial Poisson-Hopf stack we use the Drinfeld functor and prove that quantization of the elements $(f(\gamma))_{\gamma\in\Gamma}$ can be made "admissible" that is to say they will give quantizations of the corresponding "lifts". Definitions of the Drinfeld functor and admissibility will be recalled.

Finally, in Section 5, we give an explicit example corresponding to the case where G is a simple Lie group and Γ a covering of the corresponding Weyl group. In this case, quantization of Majid and Soilbelman [MS] will lead to an explicit quantization of the non-trivial Poisson-Hopf stack.

Our results in this paper fit very well in the Bressler-Gorokhovsky-Nest-Tsygan's framework [BGNT] of deformation quantization of gerbes. On one hand, our results provide interesting examples of quantization of stacks, on the other hand, the problems we are dealing with in this paper are more special and complicated because we need to treat Hopf algebra structure. In [KR] and [So] quantum Weyl groups are used to study R-matrices, and we hope that the results in this paper will shed a light on the general Γ R-matrices.

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1. Γ Lie bialgebras and equivalent categories

In this section, we recall some results of [EH2]

1.1. Γ **Lie algebras.** Define a group Lie algebra as a triple $(\Gamma, \mathfrak{g}, \theta_{\mathfrak{g}})$, where Γ is a group, $\mathfrak{g}$ is a Lie algebra and $\theta_{\mathfrak{g}} : \Gamma \to \mathrm{Aut}(\mathfrak{g})$ is a group morphism. It is the infinitesimal version of a Γ action on a group G. Group Lie algebras form a category.

If Γ is a discrete group, a Γ Lie algebra is a pair $(\mathfrak{g}, \theta_{\mathfrak{g}})$, such that $(\Gamma, \mathfrak{g}, \theta_{\mathfrak{g}})$ is a group Lie algebra. Γ Lie algebras form a subcategory of group Lie algebras. Such a Γ Lie algebra will be said to be the infinitesimal of a Γ group G.

Define a group cocommutative bialgebra as a triple (Γ, U, i), where Γ is a group, U is a cocommutative bialgebra, $U = \oplus_{\gamma\in\Gamma} U_\gamma$ is a decomposition of U, and $i : \mathbf{k}\Gamma \to U$ is a bialgebra morphism, such that $U_\gamma U_{\gamma'} \subset U_{\gamma\gamma'}$, $\Delta_U(U_\gamma) \subset U_\gamma^{\otimes 2}$, and i is compatible with the Γ grading.

We then define a Γ cocommutative bialgebra as a pair (U, i), such that (Γ, U, i) is a group cocommutative bialgebra. Γ cocommutative bialgebras form a category.

The category of group (resp., Γ) cocommutative bialgebras contains as a full subcategory the category of group (resp., Γ) universal enveloping algebras, where (U, Γ, i) satisfies the additional requirement that U_e is a universal enveloping algebra.

Define a group commutative bialgebra (in a symmetric monoidal category $\mathcal{S}$) as a triple $(\Gamma, \mathcal{O}, j)$, where Γ is a group, $\mathcal{O}$ is a commutative algebra (in $\mathcal{S}$) with a decomposition $\mathcal{O} = \oplus_{\gamma\in\Gamma}\mathcal{O}_\gamma$, such that $\mathcal{O}_\gamma\mathcal{O}_{\gamma'} = 0$ for $\gamma \neq \gamma'$, algebra morphisms $\Delta_{\gamma'\gamma''} : \mathcal{O}_{\gamma'\gamma''} \to \mathcal{O}_{\gamma'} \otimes \mathcal{O}_{\gamma''}$, $\eta : \mathbf{k} \to \mathcal{O}_e$ and $\varepsilon : \mathcal{O}_e \to \mathbf{k}$, satisfying axioms such that when Γ is finite, these morphisms add up to a bialgebra structure on $\mathcal{O}$; and $j : \mathcal{O} \to \mathbf{k}^\Gamma$ is a morphism of commutative algebras, compatible with the Γ gradings and the maps $\Delta_{\gamma'\gamma''}$ on both sides. We define Γ commutative bialgebras as above.

We define the category of group (resp., Γ) formal series Hopf (FSH) algebras as a full subcategory of the category of group (resp., Γ) commutative bialgebras in $\mathcal{S} = \{\text{pro-vector spaces}\}$ by the condition the $\mathcal{O}_e$ (or equivalently, each $\mathcal{O}_\gamma$) is a formal series algebra. Such FSH would correspond to functions on the formal dual group of a Γ group G.

Proposition 1.1. [EH2] *1) We have (anti)equivalences of categories $\{$group Lie algebras$\}$ $\leftrightarrow$ $\{$group universal enveloping algebras$\}$ $\leftrightarrow$ $\{$group FHS algebras$\}$ (the last map is an antiequivalence).*

2) If Γ is a group, these (anti)equivalences restrict to $\{\Gamma$-Lie algebras$\}$ $\leftrightarrow$ $\{\Gamma$-universal enveloping algebras$\}$ $\leftrightarrow$ $\{\Gamma$-FHS algebras$\}$.

If we denote the Γ universal enveloping algebra corresponding to a Γ Lie algebra $(\Gamma, \mathfrak{g}, \theta_\mathfrak{g})$ as $U(\mathfrak{g}) \rtimes \Gamma$. It is isomorphic to $U(\mathfrak{g}) \otimes \mathbf{k}\Gamma$ as a vector space; if we denote by $x \mapsto [x]$, $\gamma \mapsto [\gamma]$ the natural maps $\mathfrak{g} \to U(\mathfrak{g}) \rtimes \Gamma$, $\Gamma \to U(\mathfrak{g}) \rtimes \Gamma$, then the bialgebra structure of $U(\mathfrak{g}) \rtimes \Gamma$ is given by $[\gamma][x][\gamma^{-1}] = [\theta_\gamma(x)]$, $[\gamma][\gamma'] = [\gamma\gamma']$, $[e] = 1$, $[x][x'] - [x'][x] = [[x, x']]$, $\Delta([x]) = [x] \otimes 1 + 1 \otimes [x]$, $\Delta([\gamma]) = [\gamma] \otimes [\gamma]$.

When Γ is finite, the corresponding Γ FSH algebra is then $(U(\mathfrak{g}) \rtimes \mathbf{k}\Gamma)^*$, and in general, this is $\oplus_{\gamma\in\Gamma}(U(\mathfrak{g}) \otimes \mathbf{k}\gamma)^*$.

1.2. Γ Lie bialgebras.

Definition 1.2. A group Lie bialgebra is a 5-uple $(\Gamma, \mathfrak{g}, \theta_\mathfrak{g}, \delta_\mathfrak{g}, f)$ where $(\Gamma, \mathfrak{g}, \theta_\mathfrak{g})$ is a group Lie algebra, $\delta_\mathfrak{g} : \mathfrak{g} \to \wedge^2(\mathfrak{g})$ is[1] such that $(\mathfrak{g}, \delta_\mathfrak{g})$ is a Lie bialgebra, and $f : \Gamma \to \wedge^2(\mathfrak{g})$ is a map $\gamma \mapsto f_\gamma$, such that:

a) $\wedge^2(\theta_\gamma) \circ \delta \circ \theta_\gamma^{-1}(x) = \delta(x) + [f_\gamma, x \otimes 1 + 1 \otimes x]$ for any $x \in \mathfrak{g}$,

b) $f_{\gamma\gamma'} = f_\gamma + \wedge^2(\theta_\gamma)(f_{\gamma'})$,

c) $(\delta \otimes \mathrm{id})(f_\gamma) + [f_\gamma^{1,3}, f_\gamma^{2,3}]$ + cyclic permutations $= 0$.

Group Lie bialgebras form a category. When Γ is fixed, one defines the category of Γ Lie bialgebras as above.

A co-Poisson structure on a group cocommutative bialgebra (Γ, U, i) is a co-Poisson structure $\delta_U : A \to \wedge^2(U)$, such that $\delta_U(U_\gamma) \subset \wedge^2(U_\gamma)$. Co-Poisson group cocommutative bialgebras form a category

Co-Poisson group universal enveloping algebras form a full subcategory of the latter category. One defines the full subcategories of co-Poisson Γ cocommutative bialgebras and co-Poisson Γ enveloping algebras as above.

A Poisson structure on a group commutative bialgebra $(\Gamma, \mathcal{O}, j)$ is a Poisson bialgebra structure $\{-,-\} : \wedge^2(\mathcal{O}) \to \mathcal{O}$, such that $\{\mathcal{O}_\gamma, \mathcal{O}_\gamma\} \subset \mathcal{O}_\gamma$ and $\{\mathcal{O}_\gamma, \mathcal{O}_{\gamma'}\} = 0$ if $\gamma \neq \gamma'$. Poisson group bialgebras form a category, and Poisson group FSH algebras form a full subcategory when $\mathcal{S} = \{\text{pro-vector spaces}\}$. One defines the full subcategories of Poisson Γ bialgebras and Poisson Γ FSH algebras as above.

Example. Let G be a Poisson-Lie (e.g., algebraic) group, let $\Gamma \subset G$ be a subgroup (which we view as an abstract group). We define $\theta_\gamma := \mathrm{Ad}(\gamma)$, where $\mathrm{Ad} : G \to \mathrm{Aut}_{Lie}(\mathfrak{g})$ is the adjoint

[1] We view $\wedge^2(V)$ as a subspace of $V^{\otimes 2}$.

action. If $P : G \to \wedge^2(\mathfrak{g})$ is the Poisson bivector, satisfying $P(gg') = P(g') + \wedge^2(\mathrm{Ad}(g'))(P(g))$, then we set $f_\gamma := -P(\gamma)$. Then $(\mathfrak{g}, \Gamma, f)$ is a Γ Lie bialgebra.

Example. Assume that $(\mathfrak{g}, r_{\mathfrak{g}})$ is a quasitriangular Lie bialgebra and $\theta : \Gamma \to \mathrm{Aut}(\mathfrak{g}, t_{\mathfrak{g}})$ is an action of Γ on $\mathfrak{g}$ by Lie algebra automorphisms preserving $t_{\mathfrak{g}} := r_{\mathfrak{g}} + r_{\mathfrak{g}}^{2,1}$. If we set $f_\gamma := \theta_\gamma^{\otimes 2}(r) - r$, then $(\mathfrak{g}, \theta, f)$ is a Γ Lie bialgebra (we call this a quasitriangular Γ Lie bialgebra). For example, $\mathfrak{g}$ is a Kac-Moody Lie algebra, and $\Gamma = \widetilde{W}$ is a covering of the Weyl group of $\mathfrak{g}$ (cf [MS]).

Proposition 1.3. [EH2] *1) We have category (anti)equivalences {group bialgebras} $\leftrightarrow$ {co-Poisson group universal enveloping algebras} $\leftrightarrow$ {Poisson group FSH algebras}.*

2) These restrict to category (anti)equivalences {Γ-bialgebras} $\leftrightarrow$ {co-Poisson Γ universal enveloping algebras} $\leftrightarrow$ {Poisson Γ FSH algebras}.

If $(\mathfrak{g}, \theta_{\mathfrak{g}}, \delta_{\mathfrak{g}})$ is a Γ Lie bialgebra, then the co-Poisson structure on $U := U(\mathfrak{g}) \rtimes \Gamma$ is given by $\delta_U([x]) = [\delta_{\mathfrak{g}}(x)]$, and $\delta_U([\gamma]) = -[f_\gamma]([\gamma] \otimes [\gamma])$. (Here we also denote by $x \mapsto [x]$ the natural map $\wedge^2(\mathfrak{g}) \to \wedge^2(U(\mathfrak{g}) \rtimes \Gamma)$.)

1.3. **Quantization of Γ Lie bialgebras.** Let a Γ graded bialgebra (in a symmetric monoidal category $\mathcal{S}$) be a bialgebra A (in $\mathcal{S}$), equipped with a grading $A = \oplus_{\gamma\in\Gamma} A_\gamma$, such that $A_\gamma A_{\gamma'} \subset A_{\gamma\gamma'}$ and $\Delta_A(A_\gamma) \subset A_\gamma^{\otimes 2}$.

Assume that A is a Γ graded bialgebra in the category of topologically free $\mathbf{k}[[\hbar]]$-modules, quasicocommutative (in the sense that $A_0 := A/\hbar A$ is cocommutative). Then we get a co-Poisson structure on A_0. It is Γ graded, in the sense that $\delta_{A_0}((A_0)_\gamma) \subset \wedge^2((A_0)_\gamma)$. We therefore get a classical limit functor class : {Γ-graded quasicocommutative bialgebras} $\to$ {Γ-graded co-Poisson bialgebras}.

Definition 1.4. A quantization functor for Γ Lie bialgebras is a functor {co-Poisson Γ universal enveloping algebras} $\to$ {Γ-graded quasicocommutative bialgebras}, right inverse to class.

Assume that $(\mathfrak{g}, \theta, f)$ is a Γ Lie bialgebra. Let $(U_e, *, \Delta_e)$ be the (Etingof-Kazhdan) quantization of $(\mathfrak{g}, \delta)$(we will also denote the multiplication by m_e). We get from [EH2]:

Proposition 1.5. *There exist collections* $(\mathrm{F}_{\gamma,\gamma\gamma'})_{\gamma,\gamma'\in\Gamma}$ *of elements in* $U^{\otimes 2}$ *(with* $\mathrm{F}_{\gamma,\gamma\gamma'} = 1 + \hbar\,\mathrm{f}_1 + O(\hbar^2)$ *with* $\mathrm{Alt}(\mathrm{f}_1) = \wedge^2(\theta_\gamma)(f_{\gamma'})$*)* $(\mathrm{v}_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''})_{\gamma,\gamma',\gamma''\in\Gamma}$ *of elements in* $1 + \hbar^2 U$*,* $(U_\gamma, m_\gamma, \Delta_\gamma)_{\gamma\in\Gamma}$ *of bialgebras and* $(\mathrm{i}_{\gamma,\gamma\gamma'})_{\gamma,\gamma'\in\Gamma}$ *of algebra morphisms:* $(U_\gamma, m_\gamma) \to (U_{\gamma\gamma'}, m_{\gamma\gamma'})$ *such that*

- $\Delta_\gamma = \mathrm{i}_{e,\gamma}^{\otimes 2} \circ \mathrm{Ad}(\mathrm{F}_{e,\gamma}) \circ \Delta_e \circ \mathrm{i}_{e,\gamma}^{-1}$,
- $(\mathrm{F}_{e,\gamma} \otimes 1) * (\Delta_e \otimes \mathrm{id})(\mathrm{F}_{e,\gamma}) = (1 \otimes \mathrm{F}_{e,\gamma}) * (\mathrm{id} \otimes \Delta_e)(\mathrm{F}_{e,\gamma})$,
- $\mathrm{F}_{e,\gamma\gamma'} = \mathrm{v}_{e,\gamma,\gamma\gamma'}^{\otimes 2} * (\mathrm{i}_{e,\gamma}^{\otimes 2})^{-1}(\mathrm{F}_{\gamma,\gamma\gamma'}) * \mathrm{F}_{e,\gamma} * \Delta_e(\mathrm{v}_{e,\gamma,\gamma\gamma'})^{-1}$,
- $\mathrm{i}_{e,\gamma\gamma'} = \mathrm{i}_{\gamma,\gamma\gamma'} \circ \mathrm{i}_{e,\gamma} \circ \mathrm{Ad}(\mathrm{v}_{e,\gamma,\gamma\gamma'}^{-1})$,
- $\mathrm{v}_{e,\gamma\gamma',\gamma\gamma'\gamma''} * \mathrm{v}_{e,\gamma,\gamma\gamma'} = \mathrm{v}_{e,\gamma,\gamma\gamma'\gamma''} * \mathrm{i}_{e,\gamma}^{-1}(\mathrm{v}_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''})$.

Here e is the unit of the group to make the formulas shorter but could be any other element of the group and one would multiply γ, $\gamma\gamma'$ and $\gamma\gamma'\gamma''$ on the left by this elements in the formulas.

A quantization of the Γ Lie bialgebra is then obtained as follows: Set $U = S(\mathfrak{g}) \otimes \mathbf{k}\Gamma[[\hbar]]$ and $[x|\gamma] := x \otimes \gamma$, $[x \otimes x'|\gamma, \gamma'] := (x \otimes \gamma) \otimes (x' \otimes \gamma') \in U^{\otimes 2}$.

There are unique linear maps $m : U^{\otimes 2} \to U$ and $\Delta : U \to U^{\otimes 2}$, such that

$$m : [x|\gamma][x'|\gamma'] \mapsto [x * \mathrm{i}_{e,\gamma}^{-1}(\theta_\gamma(x')) * \mathrm{v}_{e,\gamma,\gamma\gamma'}^{-1} \,|\gamma\gamma']$$

$$\Delta : [x|\gamma] \mapsto [\Delta_e(x) * \mathrm{F}_{e,\gamma}^{-1} \,|\gamma, \gamma].$$

The unit for U is $[1|e]$, and the counit is the map $[x|\gamma] \mapsto \delta_{\gamma,e}\varepsilon(x)$.

Proposition 1.6. [EH2] *This defines a bialgebra structure on U, quantizing the co-Poisson bialgebra structure induced by $(\mathfrak{g}, \theta, f)$.*

2. STACK

Let M be a smooth manifold.

Definition 2.1. A stack on M is the following data:

- an open cover of $M = \cup U_i$,
- a sheaf of rings A_i on every U_i,
- an isomorphism of sheaves of rings $G_{ij}\colon A_j|(U_i \cap U_j) \to A_i|(U_i \cap U_j)$ for every i, j,
- an invertible element $c_{ijk} \in A_i|(U_i \cap U_j \cap U_k)$ for every i, j, k satisfying
  - $G_{ij}G_{jk} = \mathrm{Ad}(c_{ijk})G_{ik}$
  - and for every i, j, k, l, $c_{ijk}c_{ikl} = G_{ij}(c_{jkl})c_{ijl}$.

If two such data $(U'_i, A'_i, G'_{ij}, c'_{ijk})$ and $(U''_i, A''_i, G''_{ij}, c''_{ijk})$ are given on M, an isomorphism between them is

- an open cover $M = \cup U_i$ refining both $\{U'_i\}$ and $\{U''_i\}$
- isomorphisms $H_i\colon A'_i \to A''_i$ on U_i
- and invertible elements b_{ij} of $A'_i|(U_i \cap U_j)$ such that
  - $G''_{ij} = H_i \,\mathrm{Ad}(b_{ij})G'_{ij}H_j^{-1}$
  - and $H_i^{-1}(c''_{ijk}) = b_{ij}G'_{ij}(b_{jk})c_{ijk}b_{ik}^{-1}$

In what follows, we will still call a stack a collection of rings A_i, group elements G_{ij} and elements c_{ijk} satisfying the conditions above that is to say we will work without considering the manifold M. More precisely, we will prove the existence of a stack of Poisson Hopf algebra corresponding to functions on the formal dual group G^*.

Theorem 2.2. *There exists a stack of Poisson Hopf algebras on G^*, i.e.:*

- *a collection $(\mathcal{O}_{G^*_\gamma})_{\gamma\in\Gamma}$ of Poisson Hopf algebras $(\mathcal{O}_{G^*}, m_0, \Delta_\gamma, \{-,-\}_\gamma)_{\gamma\in\Gamma}$,*
- *Poisson morphisms $j_{\gamma,\gamma\gamma'}\colon \mathcal{O}_{G^*_\gamma} \to \mathcal{O}_{G^*_{\gamma\gamma'}}$,*
- *elements $u_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''}$ of $\mathcal{O}_{G^*_\gamma}$ satisfying relations*
  - $j_{\gamma,\gamma\gamma'\gamma''} = j_{\gamma\gamma',\gamma\gamma'\gamma''} \circ j_{\gamma,\gamma\gamma'} \circ \mathrm{Ad}_{\star_\gamma}(u^{-1}_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''})$,
  - $u_{\gamma,\gamma\gamma'\gamma'',\gamma\gamma'\gamma''\gamma'''} \star_\gamma u_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''} = u_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''\gamma'''} \star_\gamma j^{-1}_{\gamma,\gamma\gamma'}(u_{\gamma\gamma',\gamma\gamma'\gamma'',\gamma\gamma'\gamma''\gamma'''})$.

The definition of the Baker-Campbell-Hausdorff product $\star_\gamma$ will be recalled in the next section.

Note that in this theorem (and the next one), one has to take inverses of maps $j_{\gamma,\gamma\gamma'}$ and of elements $u_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''}$ to get equations compatible with the ones of Definition 2.1

We will then prove the existence of a stack of algebras quantizing this stack of Poisson Hopf algebras:

Theorem 2.3. *There exists a stack of algebras:*

- *$(\mathbb{A}_\gamma, *_\gamma)_{\gamma\in\Gamma}$ quantizations*[2] *of the Poisson algebras $(\mathcal{O}_{G^*_\gamma}, \{-,-\}_\gamma)_{\gamma\in\Gamma}$,*
- *algebra morphisms $i_{\gamma,\gamma\gamma'}\colon \mathbb{A}_\gamma \to \mathbb{A}_{\gamma\gamma'}$,*
- *elements $v_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''}$ of $\mathbb{A}_\gamma$ such that elements $ev_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''} := \exp\left(\frac{v_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''}}{\hbar}\right)$ satisfy relations*
  - $i_{\gamma,\gamma\gamma'\gamma''} = i_{\gamma\gamma',\gamma\gamma'\gamma''} \circ i_{\gamma,\gamma\gamma'} \circ \mathrm{Ad}(ev^{-1}_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''})$,
  - $ev_{\gamma,\gamma\gamma'\gamma'',\gamma\gamma'\gamma''\gamma'''} *_\gamma ev_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''} = ev_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''\gamma'''} *_\gamma i^{-1}_{\gamma,\gamma\gamma'}(ev_{\gamma\gamma',\gamma\gamma'\gamma'',\gamma\gamma'\gamma''\gamma'''})$.

[2]By quantization, we mean deformation quantization, such that $\mathbb{A}_\gamma/\hbar\mathbb{A}_\gamma = \mathcal{O}_{G^*_\gamma}$, and $\frac{1}{\hbar}[\,,\,]_{*_\gamma} = \{\,,\,\}_\gamma + O(\hbar)$.

3. A Stack of Poisson bialgebras of functions on the formal group G^*

Let $(\mathfrak{g}, \theta_{\mathfrak{g}}, \delta_{\mathfrak{g}}, f)$ be a Γ Lie bialgebra. In this section we will construct a stack of Poisson bialgebras of functions on a formal Poisson group G^*.

3.1. **Notations.** Let $(\mathfrak{g}, \delta)$ be a Lie bialgebra and $(U(\mathfrak{g}), \Delta_0, \delta)$ its corresponding cocommutative coPoisson bialgebra. The latter can be seen as the dual of the function algebra of the formal Poisson Lie group G corresponding to $(\mathfrak{g}, \delta)$. In the same way, we will define $\mathcal{O}_{G^*}$ as the commutative Poisson Hopf algebra of functions of the formal Poisson Lie group G^* corresponding to the dual Lie bialgebra $\mathfrak{g}^*$. We define by $\mathfrak{m}_{G^*} \subset \mathcal{O}_{G^*}$ the maximal ideal of this ring. If k is an integer ≥ 1, we denote by $\mathcal{O}_{(G^*)^k}$ the ring of formal functions on $(G^*)^k$, by $\mathfrak{m}_{(G^*)^k}$ its maximal ideal and by $\mathfrak{m}^i_{(G^*)^k}$ the i-th power of this ideal.

If $f, g \in \mathfrak{m}^2_{(G^*)^k}$, then the series $f \star g = f + g + \frac{1}{2}\{f, g\} + \cdots + B_n(f, g) + \cdots$ is convergent, where $\sum_{i\geq 1} B_i(x, y)$ is the Baker-Campbell-Hausdorff (BCH) series specialized to the Poisson bracket of $\mathfrak{m}^2_{(G^*)^k}$. The product $\star$ defines a group structure on $\mathfrak{m}^2_{(G^*)^k}$.

Let us recall a useful technical lemma (see [EGH], p. 2477), proven for $m_{\mathfrak{g}^*}$ and still true for m_{G^*}:

Lemma 3.1. *For any $k \geq 1$ and $n \geq 2$, $f, h \in \mathfrak{m}^2_{(G^*)^k}$ and $g \in \mathfrak{m}^n_{(G^*)^k}$, one has*

$$f \star (h + g) = f \star h + g, \quad (f + g) \star h = f \star h + g \textit{ modulo } \mathfrak{m}^{n+1}_{(G^*)^k}.$$

If $f \in \mathcal{O}_{G^*}^{\hat{\otimes} n}$ and $P_1, \ldots, P_m$ are disjoint subsets of $\{1, \ldots, m\}$, one defines $f^{P_1,\ldots,P_n}$ using the coproduct of $\mathcal{O}_{G^*}$:

Definition 3.2. For $I_1, \ldots, I_m$ disjoint ordered subsets of $\{1, \ldots, n\}$, (U, Δ) a Hopf algebra and $a \in U^{\otimes m}$, we define

$$a^{I_1,\ldots,I_n} = \sigma_{I_1,\ldots,I_m} \circ (\Delta^{|I_1|} \otimes \cdots \otimes \Delta^{|I_n|})(a),$$

with $\Delta^{(1)} = \mathrm{id}$, $\Delta^{(2)} = \Delta$, $\Delta^{(n+1)} = (\mathrm{id}^{\otimes n-1} \otimes \Delta) \circ \Delta^{(n)}$, and $\sigma_{I_1,\ldots,I_m} : U^{\otimes \sum_i |I_i|} \to U^{\otimes n}$ is the morphism corresponding to the map $\{1, \ldots, \sum_i |I_i|\} \to \{1, \ldots, n\}$ taking $(1, \ldots, |I_1|)$ to I_1, $(|I_1| + 1, \ldots, |I_1| + |I_2|)$ to I_2, etc.

When U is cocommutative, this definition depends only on the underlying sets $I_1, \ldots, I_m$.

When $(\mathfrak{g}, \theta_{\mathfrak{g}}, \delta_{\mathfrak{g}}, f)$ is a Γ Lie bialgebra we thus get a collection of Lie bialgebras and so a collection $(\mathcal{O}_{G^*}, m_0, \Delta_\gamma, \{-,-\}_\gamma)_{\gamma\in\Gamma}$ of Poisson bialgebras. We will denote by $\star_\gamma$ the corresponding BCH products.

3.2. **"Lifts" and functional equations.** We will now construct "lifts" $\tilde{f}_{\gamma,\gamma\gamma'} \in \mathfrak{m}_{G^*}^{\hat{\otimes} 2}$ of the elements $\wedge^2(\theta_\gamma)(f_{\gamma'})$, $\gamma, \gamma' \in \Gamma$ that will satisfy similar relation as $\mathrm{F}_{\gamma,\gamma\gamma'}$ in Proposition 1.5.

Proposition 3.3. *Let γ, γ' be in Γ. Then there exists $\tilde{f}_{\gamma,\gamma\gamma'}$ in $m_{G^*}^{\hat{\otimes} 2}$ the image of which in $\mathfrak{g}^{\otimes 2}$ under the square of the projection $\mathfrak{m}_{G^*} \to \mathfrak{m}_{G^*}/\mathfrak{m}^2_{G^*} = \mathfrak{g}$ equals $\wedge^2(\theta_\gamma)(f_{\gamma'})$, and such that*

$$(\tilde{f}_{\gamma,\gamma\gamma'} \otimes 1) \star_\gamma (\Delta_\gamma \otimes \mathrm{id})(\tilde{f}_{\gamma,\gamma\gamma'}) = (1 \otimes \tilde{f}_{\gamma,\gamma\gamma'}) \star_\gamma (\mathrm{id} \otimes \Delta_\gamma)(\tilde{f}_{\gamma,\gamma\gamma'}). \quad (1)$$

Such a $\tilde{f}_{\gamma,\gamma\gamma'}$ is unique up to the action of $\mathfrak{m}^2_{G^*}$ by $\lambda \cdot \tilde{f} = \lambda^1 \star_\gamma \lambda^2 \star_\gamma \tilde{f} \star_\gamma (-\lambda)^{12}$. We will call such a $\tilde{f}$ a twist for Δ_γ.

Proof. Let us construct $\tilde{f}_{\gamma,\gamma\gamma'}$ by induction: we will construct a convergent sequence $\tilde{f}_N \in \mathfrak{m}_{G^*}^{\hat{\otimes} 2}$ $(N \geq 2)$ satisfying (1) in $\mathfrak{m}_{G^*}^{\hat{\otimes} 3}/(\mathfrak{m}_{G^*}^{\hat{\otimes} 3} \cap \mathfrak{m}^N_{(G^*)^3})$, where $\mathfrak{m}^N_{(G^*)^3}$ is the N-th power of $\mathfrak{m}_{(G^*)^3}$. When $N = 3$, we take for $\tilde{f}_2$ any lift of $\wedge^2(\theta_\gamma)(f_{\gamma'})$ to $\mathfrak{m}_{G^*}^{\hat{\otimes} 2}$; then equation (1) is automatically satisfied.

To shorten the notation, we will write $\tilde f_{1,2}$ for $\tilde f_{\gamma,\gamma\gamma'}$, $\tilde f_{2,3}$ for $\tilde f_{\gamma\gamma',\gamma\gamma'\gamma''}$ and so on and the same thing for $\alpha_{-,-,-}$

Let N be an integer ≥ 3; assume that we have constructed $\tilde f_N$ in $\mathfrak{m}_{G^*}^{\widehat\otimes 2}$ satisfying equation (1) in $\mathfrak{m}_{G^*}^{\widehat\otimes 3}/(\mathfrak{m}_{G^*}^{\widehat\otimes 3}\cap\mathfrak{m}^N_{(G^*)^3})$. Set $\alpha^N_{1,2,3} := \tilde f^N_{1,2}\star_\gamma \tilde f^N_{12,3} - \tilde f^N_{2,3}\star_\gamma \tilde f^N_{1,23}$. Then $\alpha^N_{1,2,3}$ belongs to $\mathfrak{m}_{G^*}^{\widehat\otimes 3}\cap\mathfrak{m}^N_{(G^*)^3}$, and the following equalities hold in $\mathfrak{m}_{G^*}^{\widehat\otimes 4}/(\mathfrak{m}_{G^*}^{\widehat\otimes 4}\cap\mathfrak{m}^{N+1}_{(G^*)^4})$:

$$\begin{aligned}
\alpha^N_{12,3,4} = {} & \tilde f^N_{1,2}\star_\gamma \alpha^N_{12,3,4} = \tilde f^N_{1,2}\star_\gamma \tilde f^N_{12,3}\star_\gamma \tilde f^N_{123,4} - \tilde f^N_{1,2}\star_\gamma \tilde f^N_{3,4}\star_\gamma \tilde f^N_{12,34}\\
= {} & \alpha^N_{1,2,3} + \tilde f^N_{2,3}\star_\gamma \tilde f^N_{1,23}\star_\gamma \tilde f^N_{123,4} - \tilde f^N_{3,4}\star_\gamma \tilde f^N_{1,2}\star_\gamma \tilde f^N_{12,34}\\
& \text{(using Lemma 3.1)}\\
= {} & \alpha^N_{1,2,3} + \tilde f^N_{2,3}\star_\gamma \tilde f^N_{1,23}\star \tilde f^N_{123,4} - \tilde f^N_{3,4}\star(\tilde f^N_{2,34}\star_\gamma \tilde f^N_{1,234} + \alpha^N_{1,2,34})\\
& \text{(using Lemma 3.1 and the definition of } \alpha^N_{1,2,34})\\
= {} & \alpha^N_{1,2,3} + \tilde f^N_{2,3}\star_\gamma (\alpha^N_{1,23,4} + \tilde f^N_{23,4}\star_\gamma \tilde f^N_{1,234})\\
& -\alpha^N_{1,2,34} - \tilde f^N_{3,4}\star_\gamma \tilde f^N_{2,34}\star_\gamma \tilde f^N_{1,234}\\
& \text{(using the definition of } \alpha^N_{1,23,4} \text{ and Lemma 3.1)}\\
= {} & \alpha^N_{1,2,3} + \alpha^N_{1,23,4} + (\tilde f^N_{3,4}\star_\gamma \tilde f^N_{2,34} + \alpha^N_{2,3,4})\star_\gamma \tilde f^N_{1,234}\\
& -\alpha^N_{1,2,34} - \tilde f^N_{3,4}\star_\gamma \tilde f^N_{2,34}\star_\gamma \tilde f^N_{1,234}\\
& \text{(using the definition of } \alpha^N_{2,3,4} \text{ and Lemma 3.1)}\\
= {} & \alpha^N_{1,2,3} + \alpha^N_{1,23,4} - \alpha^N_{1,2,34} + \alpha^N_{2,3,4}\\
& \text{(using Lemma 3.1).}
\end{aligned}$$

Let us denote by $\overline{\alpha}^N$ the image of α^N in $(\mathfrak{m}_{\mathfrak{g}^*}^{\widehat\otimes 3}\cap\mathfrak{m}^N_{(\mathfrak{g}^*)^3})/(\mathfrak{m}_{\mathfrak{g}^*}^{\widehat\otimes 3}\cap\mathfrak{m}^{N+1}_{(\mathfrak{g}^*)^3}) = (S^{>0}(\mathfrak{g})^{\otimes 3})_N$, then we get

$$\overline{\alpha}^N_{12,3,4} + \overline{\alpha}^N_{1,2,34} = \overline{\alpha}^N_{1,2,3} + \overline{\alpha}^N_{1,23,4} + \overline{\alpha}^N_{2,3,4}.$$

This means that $\bar\alpha$ is a cocycle for the subcomplex $(S^{>0}(\mathfrak{g})^{\otimes\cdot}, d)$ of the co-Hochschild complex. Using [Dr2], Proposition 3.11, one proves that the k-th cohomology group of this subcomplex is $\wedge^k(\mathfrak{g})$, and that the antisymmetrization map coincides with the canonical projection from the space of cocycles to the cohomology group. For $N = 3$, the equations of Definition 1.2 implies $\mathrm{Alt}(\overline{\alpha}^3) = 0$, and hence $\overline{\alpha}^3$ is the coboundary of an element $\overline{\beta}_3 \in (S^{>0}(\mathfrak{g})^{\otimes 2})^3$. For $N > 3$, $\overline{\alpha}^N$ is the coboundary of an element $\overline{\beta}^N \in (S^{>0}(\mathfrak{g})^{\otimes 2})^N$, since the degree N part of the cohomology vanishes. We then set $\tilde f^{N+1} := \tilde f^N + \beta^N$, where $\beta^N \in \mathfrak{m}_{G^*}^{\widehat\otimes 2}\cap\mathfrak{m}^N_{(G^*)^2}$ is a representative of $\overline{\beta}^N$. Then $\tilde f^{N+1}$ satisfies (1) in $\mathfrak{m}_{G*}^{\widehat\otimes 3}/(\mathfrak{m}_{G^*}^{\widehat\otimes 3}\cap\mathfrak{m}^{N+1}_{(G^*)^3})$.

The sequence $(\tilde f_N)^{N\geq 2}$ has a limit $\tilde f$, which then satisfies (1).

The second part of the theorem can be proved in the same way or by analyzing the choices for $\overline{\beta}_N$ in the above proof. □

3.3. **Isomorphism of formal Poisson manifolds $G^*_\gamma \simeq G^*_{\gamma\gamma'}$.**

Proposition 3.4. *Let $\gamma, \gamma' \in \Gamma$ and let G^*_γ and $G^*_{\gamma\gamma'}$ be the formal Poisson-Lie groups associated to the corresponding Lie cobrackets. There exists an isomorphism of Poisson algebras $j_{\gamma,\gamma\gamma'}$: $\mathcal{O}_{G^*_\gamma} \simeq \mathcal{O}_{G^*_{\gamma\gamma'}}$.*

Proof. Let $P : \wedge^2(\mathcal{O}_{G^*_\gamma}) \to \mathcal{O}_{G^*_\gamma}$ be the Poisson bracket on $\mathcal{O}_{G^*_\gamma}$ corresponding to the Lie-Poisson Poisson structure on G^*_γ. Then $(\mathcal{O}_{G^*_\gamma}, m_0, P, \Delta_\gamma)$ is a Poisson formal series Hopf (PFSH) algebra; it corresponds to the formal Poisson-Lie group G^*_γ equipped with its Lie-Poisson structure.

Set ${}^{\tilde f_{\gamma,\gamma\gamma'}}\Delta_\gamma(a) = \tilde f_{\gamma,\gamma\gamma'} \star_\gamma \Delta_\gamma(a) \star_\gamma (-\tilde f_{\gamma,\gamma\gamma'})$ for any $a \in \mathcal{O}_{G^*_\gamma}$. It follows from the fact that $\tilde f_{\gamma,\gamma\gamma'}$ satisfies the equation (1) that $(\mathcal{O}_{G^*_\gamma}, m_0, P, {}^{\tilde f_{\gamma,\gamma\gamma'}}\Delta_\gamma)$ is a PFSH algebra.

Let us denote by **PFSHA** and **LBA** the categories of PSFH algebras and Lie bialgebras. We have a category equivalence $c : \mathbf{PFSHA} \to \mathbf{LBA}$, taking $(\mathcal{O}, m, P, \Delta)$ to the Lie bialgebra $(\mathfrak{c}, \mu, \delta)$, where $\mathfrak{c} := \mathfrak{m}/\mathfrak{m}^2$ ($\mathfrak{m} \subset \mathcal{O}$ is the maximal ideal), the Lie cobracket of $\mathfrak{c}$ is induced by $\Delta - \Delta^{2,1} : \mathfrak{m} \to \wedge^2(\mathfrak{m})$, and the Lie bracket of $\mathfrak{c}$ is induced by the Poisson bracket $P : \wedge^2(\mathfrak{m}) \to \mathfrak{m}$. The inverse of the functor c takes $(\mathfrak{c}, \mu, \delta)$ to $\mathcal{O} = \widehat{S}(\mathfrak{c})$ equipped with its usual product; Δ depends only on δ and P depends on (μ, δ).

Then c restricts to a category equivalence $c_{\mathrm{fd}} : \mathbf{PFSHA}_{\mathrm{fd}} \to \mathbf{LBA}_{\mathrm{fd}}$ of subcategories of finite-dimensional objects (in the case of **PFSH**, we say that $\mathcal{O}$ is finite-dimensional if and only if $\mathfrak{m}/\mathfrak{m}^2$ is).

Let $\mathrm{dual} : \mathbf{LBA}_{\mathrm{fd}} \to \mathbf{LBA}_{\mathrm{fd}}$ be the duality functor. It is a category antiequivalence; we have $\mathrm{dual}(\mathfrak{g}, \mu, \delta) = (\mathfrak{g}^*, \delta^t, \mu^t)$. Then $\mathrm{dual} \circ c_{\mathrm{fd}} : \mathbf{PFSHA}_{\mathrm{fd}} \to \mathbf{LBA}_{\mathrm{fd}}$ is a category antiequivalence. Its inverse is the usual functor $\mathfrak{g} \mapsto U(\mathfrak{g})^*$. If G is the formal Poisson-Lie group with Lie bialgebra $\mathfrak{g}$, one sets $\mathcal{O}_G = U(\mathfrak{g})^*$.

Let us apply the functor c to $(\mathcal{O}_{G^*_\gamma}, m_0, P, {}^{\tilde f_{\gamma,\gamma\gamma'}}\Delta_\gamma)$. We obtain $\mathfrak{c} = \mathfrak{m}/\mathfrak{m}^2 = \mathfrak{g}$; the Lie bracket is unchanged with respect to the case $\tilde f_{\gamma,\gamma\gamma'} = 0$, so it is the Lie bracket of $\mathfrak{g}$; the Lie cobracket is given by $\delta_{\gamma\gamma'}(x) = \delta_\gamma + [\wedge^2(\theta_\gamma)(f_{\gamma'}), x \otimes 1 + 1 \otimes x]$ since the reduction of $\tilde f_{\gamma,\gamma\gamma'}$ modulo $(\mathfrak{m}_{G^*_\gamma})^2 \widehat{\otimes} \mathfrak{m}_{G^*_\gamma} + \mathfrak{m}_{G^*_\gamma} \widehat{\otimes} (\mathfrak{m}_{G^*_\gamma})^2$ is equal to $\wedge^2(\theta_\gamma)(f_{\gamma'})$.

Then applying $\mathrm{dual} \circ c_{\mathrm{fd}}$ to $(\mathcal{O}_{G^*_\gamma}, m_0, P, {}^{\tilde f_{\gamma,\gamma\gamma'}}\Delta_\gamma)$, we obtain the Lie bialgebra $(\mathfrak{g}^*, \delta_{\gamma\gamma'})$. So this PFSH algebra is isomorphic to the PFSH algebra of the formal Poisson-Lie group $G^*_{\gamma\gamma'}$. Let us call this PFSH algebra morphism $j_{\gamma,\gamma\gamma'}$.

In particular, the Poisson algebras $\mathcal{O}_{G^*_\gamma}$ and $\mathcal{O}_{G^*_{\gamma\gamma'}}$ are isomorphic. □

Remark 3.5. It is easy to check that the map $\mathfrak{g} = \mathfrak{m}_{G^*_\gamma}/\mathfrak{m}^2_{G^*_\gamma} \to \mathfrak{m}_{G^*_{\gamma\gamma'}}/\mathfrak{m}^2_{G^*_{\gamma\gamma'}} = \mathfrak{g}$ induced by the isomorphism $j_{\gamma,\gamma\gamma'}$ is the identity.

Remark 3.6. We have proven a stronger result than the existence of a Poisson algebra morphism $j_{\gamma,\gamma\gamma'}$: $\mathcal{O}_{G^*_\gamma} \simeq \mathcal{O}_{G^*_{\gamma\gamma'}}$. This morphism intertwines the coproducts in the following way:

$$\Delta_{\gamma\gamma'} = j^{\otimes 2}_{\gamma,\gamma\gamma'} \circ {}^{\tilde f_{\gamma,\gamma\gamma'}}\Delta_\gamma \circ j^{-1}_{\gamma,\gamma\gamma'}.$$

3.4. **Composition of equivalences.** Let us first prove the following lemma:

Lemma 3.7. *For γ, γ' in Γ, the element $(j^{\otimes 2}_{\gamma,\gamma\gamma'})^{-1}(\tilde f_{\gamma\gamma',\gamma\gamma'\gamma''}) \star_\gamma \tilde f_{\gamma,\gamma\gamma'}$ is a solution of the equation*

$$(\tilde f \otimes 1) \star_\gamma (\Delta_\gamma \otimes \mathrm{id})(\tilde f) = (1 \otimes \tilde f) \star_\gamma (\mathrm{id} \otimes \Delta_\gamma)(\tilde f). \tag{2}$$

Proof. One can check that directly or notice that $\tilde f_{\gamma\gamma',\gamma\gamma'\gamma''}$ is a twist for $\Delta_{\gamma\gamma'}$. Therefore $(j^{\otimes 2}_{\gamma,\gamma\gamma'})^{-1}(\tilde f_{\gamma\gamma',\gamma\gamma'\gamma''})$ is a twist for $(j^{\otimes 2}_{\gamma,\gamma\gamma'})^{-1} \circ \Delta_{\gamma\gamma'} \circ j_{\gamma,\gamma\gamma'} = {}^{\tilde f_{\gamma,\gamma\gamma'}}\Delta_\gamma$. Accordingly $(j^{\otimes 2}_{\gamma,\gamma\gamma'})^{-1}(\tilde f_{\gamma\gamma',\gamma\gamma'\gamma''}) \star_\gamma \tilde f_{\gamma,\gamma\gamma'}$ is a twist for Δ_γ. □

Let us then notice that the image of $(j^{\otimes 2}_{\gamma,\gamma\gamma'})^{-1}(\tilde{f}_{\gamma\gamma',\gamma\gamma'\gamma''})\star_\gamma \tilde{f}_{\gamma,\gamma\gamma'}$ under the square of the projection $\mathfrak{m}_{G^*} \to \mathfrak{m}_{G^*}/\mathfrak{m}^2_{G^*} = \mathfrak{g}$ equals $\wedge^2(\theta_\gamma)(f_{\gamma'})+\wedge^2(\theta_{\gamma\gamma'})(f_{\gamma''}) = \wedge^2(\theta_\gamma)(f_{\gamma'}+\wedge^2(\theta_{\gamma'})(f_{\gamma''})) = \wedge^2(\theta_\gamma)(f_{\gamma'\gamma''})$. Thanks to Proposition 3.3, there exists an element $u_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''}$ in $1+\mathfrak{m}^2_{G^*}$ such that

$$\tilde{f}_{\gamma,\gamma\gamma'\gamma''} = u^{\otimes 2}_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''} \star_\gamma (j^{\otimes 2}_{\gamma,\gamma\gamma'})^{-1}(\tilde{f}_{\gamma\gamma',\gamma\gamma'\gamma''}) \star_\gamma \tilde{f}_{\gamma,\gamma\gamma'} \star_\gamma \Delta_\gamma(u_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''})^{-1}.$$

Finally, from the previous section, we defined $j_{\gamma,\gamma\gamma'}$, $j_{\gamma\gamma',\gamma\gamma'\gamma''}$ and $j_{\gamma,\gamma\gamma'\gamma''}$ such that

(3)
$$\begin{aligned}
\Delta_{\gamma\gamma'\gamma''} &= j^{\otimes 2}_{\gamma,\gamma\gamma'\gamma''} \circ {}^{\tilde{f}_{\gamma,\gamma\gamma'\gamma''}}\Delta_\gamma \circ j^{-1}_{\gamma,\gamma\gamma'\gamma''}\\
&= j^{\otimes 2}_{\gamma,\gamma\gamma'\gamma''} \circ {}^{u^{\otimes 2}_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''}\star_\gamma (j^{\otimes 2}_{\gamma,\gamma\gamma'})^{-1}(\tilde{f}_{\gamma\gamma',\gamma\gamma'\gamma''})\star_\gamma \tilde{f}_{\gamma,\gamma\gamma'}\star_\gamma \Delta_\gamma(u_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''})^{-1}}\Delta_\gamma \circ j^{-1}_{\gamma,\gamma\gamma'\gamma''}\\
&= (j_{\gamma,\gamma\gamma'\gamma''} \circ \mathrm{Ad}_{\star_\gamma}(u_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''}))^{\otimes 2} \circ {}^{(j^{\otimes 2}_{\gamma,\gamma\gamma'})^{-1}(\tilde{f}_{\gamma\gamma',\gamma\gamma'\gamma''})\star_\gamma \tilde{f}_{\gamma,\gamma\gamma'}}\Delta_\gamma\\
&\quad \circ (j_{\gamma,\gamma\gamma'\gamma''} \circ \mathrm{Ad}_{\star_\gamma}(u_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''}))^{-1}\\
&= (j_{\gamma,\gamma\gamma'\gamma''} \circ \mathrm{Ad}_{\star_\gamma}(u_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''}) \circ j^{-1}_{\gamma,\gamma\gamma'} \circ j^{-1}_{\gamma\gamma',\gamma\gamma'\gamma''})^{\otimes 2} \circ \Delta_{\gamma\gamma'\gamma''}\\
&\quad \circ (j_{\gamma,\gamma\gamma'\gamma''} \circ \mathrm{Ad}_{\star_\gamma}(u_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''}) \circ j^{-1}_{\gamma,\gamma\gamma'} \circ j^{-1}_{\gamma\gamma',\gamma\gamma'\gamma''})^{-1}.
\end{aligned}$$

By the equivalence c_{fd} between the category $\mathbf{PFSHA}_{\mathrm{fd}}$ and $\mathbf{LBA}_{\mathrm{fd}}$ we get

$$j_{\gamma,\gamma\gamma'\gamma''} = j_{\gamma\gamma',\gamma\gamma'\gamma''} \circ j_{\gamma,\gamma\gamma'} \circ \mathrm{Ad}_{\star_\gamma}(u^{-1}_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''}).$$

3.5. **Cocycle relation for the $u_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''}$.** We will end this section by proving the following proposition that will prove Theorem 2.2:

Proposition 3.8. *For any $\gamma,\gamma',\gamma'',\gamma'''$ in Γ, we have*

$$u_{\gamma,\gamma\gamma'\gamma'',\gamma\gamma'\gamma''\gamma'''} \star_\gamma u_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''} = u_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''\gamma'''} \star_\gamma j^{-1}_{\gamma,\gamma\gamma'}(u_{\gamma\gamma',\gamma\gamma'\gamma'',\gamma\gamma'\gamma''\gamma'''}).$$

Proof. To shorten the notation, we will write $\tilde{f}_{1,2}$ for $\tilde{f}_{\gamma,\gamma\gamma'}$, $\tilde{f}_{2,3}$ for $\tilde{f}_{\gamma\gamma',\gamma\gamma'\gamma''}$ and so on and the same thing for the $j_{-,-}$ and the $u_{-,-,-}$. We will omit the BCH product $\star_\gamma$ and write $\star$ for the product $\star_{\gamma\gamma'}$, Δ_0 for the coproduct Δ_γ and Δ for the coproduct $\Delta_{\gamma\gamma'}$. We will also write $j(-)$ instead of $j^{\otimes 2}(-)$ when no confusion is possible.

We have by definition $\tilde{f}_{1,4}\Delta_0 u_{1,3,4} = u^{\otimes 2}_{1,3,4} j^{-1}_{1,3}(\tilde{f}_{3,4})\tilde{f}_{1,3}$. Multiplying this equality on the right by $\Delta_0 u_{1,2,3}$ and using the fact that $\tilde{f}_{1,3}\Delta_0 u_{1,2,3} = u^{\otimes 2}_{1,2,3} j^{-1}_{1,2}(\tilde{f}_{2,3})\tilde{f}_{1,2}$, we get

$$\tilde{f}_{1,4}\Delta_0 u_{1,3,4}\Delta_0 u_{1,2,3} = u^{\otimes 2}_{1,3,4} j^{-1}_{1,3}(\tilde{f}_{3,4}) u^{\otimes 2}_{1,2,3} j^{-1}_{1,2}(\tilde{f}_{2,3})\tilde{f}_{1,2}.$$

Using now that $j^{-1}_{1,3}(-)u_{1,2,3} = u_{1,2,3} j^{-1}_{1,2} \circ j^{-1}_{2,3}(-)$, we get

$$\tilde{f}_{1,4}\Delta_0 u = u^{\otimes 2} j^{-1}_{1,2} \circ j^{-1}_{2,3}(\tilde{f}_{3,4}) j^{-1}_{1,2}(\tilde{f}_{2,3})\tilde{f}_{1,2}, \tag{4}$$

where $u = u_{1,3,4}u_{1,2,3}$. On the other hand, we have $\tilde{f}_{2,4} \star \Delta u_{2,3,4} = u^{\otimes 2}_{2,3,4} \star j^{-1}_{2,3}(\tilde{f}_{3,4}) \star \tilde{f}_{2,3}$. Using the Poisson algebra morphism $j_{1,2}$ and the fact that $j^{-1}_{1,2} \circ \Delta = \tilde{f}_{1,2}\Delta_0(j^{-1}_{1,2}(-))\tilde{f}^{-1}_{1,2}$, we get

$$j^{-1}_{1,2}(\tilde{f}_{2,4})\tilde{f}_{1,2}\Delta_0(j^{-1}_{1,2}(u_{2,3,4}))\tilde{f}^{-1}_{1,2} = j^{-1}_{1,2}(u^{\otimes 2}_{2,3,4}) j^{-1}_{1,2} \circ j^{-1}_{2,3}(\tilde{f}_{3,4}) j^{-1}_{1,2}(\tilde{f}_{2,3}). \tag{5}$$

From $\tilde{f}_{1,4}\Delta_0 u_{1,2,4} = u^{\otimes 2}_{1,2,4} j^{-1}_{1,2}(\tilde{f}_{2,4})\tilde{f}_{1,2}$, using Equation (5), we get

$$\tilde{f}_{1,4}\Delta_0(u') = (u')^{\otimes 2} j^{-1}_{1,2} \circ j^{-1}_{2,3}(\tilde{f}_{3,4}) j^{-1}_{1,2}(\tilde{f}_{2,3})\tilde{f}_{1,2}, \tag{6}$$

where $u' = u_{1,2,4}j_{1,2}^{-1}(u_{2,3,4})$. Then Equations (4) and (6) imply that if $w = u(u')^{-1}$ then $\tilde{f}_{1,4}\Delta_0(w) = w\tilde{f}_{1,4}$, and so if $w' = j_{1,4}(w)$ then $\Delta_0(w') = w'$. Recall that by similar properties of $u_{i,j,k}$, $w' \in 1 + m^2_{G^*}$. Suppose that $w' \neq 1$ and set $i \geq 2$ the largest possible i such that $w' \in 1 + m^i_{G^*}$, but not in $1 + m^{i+1}_{G^*}$. Let $\bar{w}'$ be the projection of w' in $m^i_{G^*}/m^{i+1}_{G^*}$. Relation $\Delta_0(w') = w'$ implies that $\bar{w}'$ is in $\mathfrak{g}$ and so in $m^1_{G^*}$ which is a contradiction. Thus we have proved that $w = w' = 1$ and so that $u = u'$.

4. Quantization

4.1. **Duality of QUE and QFSH algebras.** In this subsection, we recall some facts from [Dr1] (proofs can be found in [Gav]). Let us denote by **QUE** the category of quantized universal enveloping (QUE) algebras and by **QFSH** the category of quantized formal series Hopf (QFSH) algebras. We denote by $\mathbf{QUE}_{\rm fd}$ and $\mathbf{QFSH}_{\rm fd}$ the subcategories corresponding to finite dimensional Lie bialgebras.

We have contravariant functors $\mathbf{QUE}_{\rm fd} \to \mathbf{QFSH}_{\rm fd}$, $U \mapsto U^*$ and $\mathbf{QFSH}_{\rm fd} \to \mathbf{QUE}_{\rm fd}$, $\mathcal{O} \mapsto \mathcal{O}^\circ$. These functors are inverse to each other. U^* is the full topological dual of U, i.e., the space of all continuous (for the $\hbar$-adic topology) $\mathbb{K}[[\hbar]]$-linear maps $U \to \mathbb{K}[[\hbar]]$. $\mathcal{O}^\circ$ the space of continuous $\mathbb{K}[[\hbar]]$-linear forms $\mathcal{O} \to \mathbb{K}[[\hbar]]$, where $\mathcal{O}$ is equipped with the $\mathfrak{m}$-adic topology (here $\mathfrak{m} \subset \mathcal{O}$ is the maximal ideal).

We also have covariant functors $\mathbf{QUE} \to \mathbf{QFSH}$, $U \mapsto U'$ and $\mathbf{QFSH} \to \mathbf{QUE}$, $\mathcal{O} \mapsto \mathcal{O}^\vee$. There functors are also inverse to each other. U' is a subalgebra of U, while $\mathcal{O}^\vee$ is the $\hbar$-adic completion of $\sum_{k\geq 0} \hbar^{-k}\mathfrak{m}^k \subset \mathcal{O}[1/\hbar]$.

We also have canonical isomorphisms $(U')^\circ \simeq (U^*)^\vee$ and $(\mathcal{O}^\vee)^* \simeq (\mathcal{O}^\circ)'$.

If $\mathfrak{a}$ is a finite dimensional Lie bialgebra and $U = U_\hbar(\mathfrak{a})$ is a QUE algebra quantizing $\mathfrak{a}$, then $U^* = \mathcal{O}_{A,\hbar}$ is a QFSH algebra quantizing the Poisson-Lie group A (with Lie bialgebra $\mathfrak{a}$), and $U' = \mathcal{O}_{A^*,\hbar}$ is a QFSH algebra quantizing the Poisson-Lie group A^* (with Lie bialgebra $\mathfrak{a}^*$). If now $\mathcal{O} = \mathcal{O}_{A,\hbar}$ is a QFSH algebra quantizing A, then $\mathcal{O}^\circ = U_\hbar(\mathfrak{a})$ is a QUE algebra quantizing $\mathfrak{a}$ and $\mathcal{O}^\vee = U_\hbar(\mathfrak{a}^*)$ is a QFSH algebra quantizing $\mathfrak{a}^*$.

We now compute these functors explicitly in the case of cocommutative QUE and commutative QFSH algebras. If $U = U(\mathfrak{a})[[\hbar]]$ with cocommutative coproduct (where $\mathfrak{a}$ is a Lie algebra), then U' is a completion of $U(\hbar\mathfrak{a}[[\hbar]])$; this is a flat deformation of $\widehat{S}(\mathfrak{a})$ equipped with its linear Lie-Poisson structure. If G is a formal group with function ring $\mathcal{O}_G$, then $\mathcal{O} := \mathcal{O}_G[[\hbar]]$ is a QFSH algebra, and $\mathcal{O}^\vee$ is a commutative QUE algebra; it is a quantization of ($S(\mathfrak{g}^*)$, commutative product, cocommutative coproduct, co-Poisson structure induced by the Lie bracket of $\mathfrak{g}$).

4.2. **Proof that "twists" can be made admissible.**

Definition 4.1. *An element x in a QUE algebra U is admissible if $x \in 1 + \hbar U$, and if $\hbar \log x$ is in $U' \subset U$.*

In this subsection, we will prove that for γ, γ' in Γ, the twist $\mathrm{F}_{\gamma,\gamma\gamma'}$ defined in Proposition 1.5 is twist equivalent to an admissible one. More precisely, we have

Proposition 4.2. *Let $\mathrm{F}_{\gamma,\gamma\gamma'}$ be as Proposition 1.5. Then there exists elements $\mathrm{b}_{\gamma,\gamma\gamma'}$ in U such that ${}^{\mathrm{b}_{\gamma,\gamma\gamma'}}\mathrm{F}_{\gamma,\gamma\gamma'} := \mathrm{b}^{\otimes 2}_{\gamma,\gamma\gamma'}\,\mathrm{F}_{\gamma,\gamma\gamma'}\,\Delta_\gamma(\mathrm{b}^{-1}_{\gamma,\gamma\gamma'})$ is admissible.*

Proof. Let us denote $\mathrm{F}_0 = \mathrm{F}_{\gamma,\gamma\gamma'}$. We will follow the proof of Proposition 5.2. in [EH3]: let us construct $\mathrm{b} = \mathrm{b}_{\gamma,\gamma\gamma'}$ as a product $\cdots \mathrm{b}_2\,\mathrm{b}_1$, where $\mathrm{b}_n \in 1 + \hbar^n U_0$, in such a way that if $\mathrm{F}_n := {}^{\mathrm{b}_n \cdots \mathrm{b}_1}\mathrm{F}_0$, then $\hbar\log(\mathrm{F}_n) \in U_0'^{\widehat{\otimes}2} + \hbar^{n+2}U_0^{\widehat{\otimes}2}$ (here U_0 denotes the augmentation ideal).

We have already $\hbar\log(\mathrm{F}_0) \in \hbar^2 U_0^{\widehat{\otimes}2}$.

Expand $\mathrm{F}_0 = 1^{\otimes 2} + \hbar\,\mathrm{f}_1 + \cdots$, then $\mathrm{Alt}(\mathrm{f}_1) = r$. Moreover, the coefficient of $\hbar$ in $\mathrm{F}_0^{1,2}\,\mathrm{F}_0^{12,3} = \mathrm{F}_0^{2,3}\,\mathrm{F}_0^{1,23}$ yields $d(\mathrm{f}_1) = 0$, where $d : U(\mathfrak{g})_0^{\otimes 2} \to U(\mathfrak{g})_0^{\otimes 3}$ is the co-Hochschild differential. It follows that for some $a_1 \in U(\mathfrak{g})_0$, we have $\mathrm{f}_1 = r + d(a_1)$. Then if we set $\mathrm{b}_1 := \exp(\hbar a_1)$ and $\mathrm{F}_1 = {}^{\mathrm{b}_1}\,\mathrm{F}_0$, we get $\mathrm{F}_1 \in 1^{\otimes 2} + \hbar r + \hbar^2 U_0^{\widehat{\otimes} 2}$. Then $\hbar\log(\mathrm{F}_1) \in \hbar^2 r + \hbar^3 U_0^{\widehat{\otimes} 2} \subset U_0'^{\widehat{\otimes} 2} + \hbar^3 U_0^{\widehat{\otimes} 3}$.

Assume that for $n \geq 2$, we have constructed $\mathrm{b}_1, \ldots, \mathrm{b}_{n-1}$ such that $\alpha_{n-1} := \hbar\log(\mathrm{F}_{n-1}) \in U_0'^{\widehat{\otimes} 2} + \hbar^{n+1} U_0^{\widehat{\otimes} 2}$.

Let us recall to technical lemmas from [EH3]:

Lemma 4.3. *The quotient $(U' + \hbar^n U)/(U' + \hbar^{n+1}U)$ identifies with $U(\mathfrak{g})/U(\mathfrak{g})_{\leq n}$. In the same way, the quotient $(U_0'^{\widehat{\otimes} k} + \hbar^n U_0^{\widehat{\otimes} k})/(U_0'^{\widehat{\otimes} k} + \hbar^{n+1} U_0^{\widehat{\otimes} k})$ identifies with $U(\mathfrak{g})_0^{\otimes k}/(U(\mathfrak{g})_0^{\otimes k})_{\leq n}$ and the quotient of $\mathfrak{g}$-invariant subspaces $(U_0'^{\widehat{\otimes} k} + \hbar^n U_0^{\widehat{\otimes} k})^{\mathfrak{g}}/(U_0'^{\widehat{\otimes} k} + \hbar^{n+1} U_0^{\widehat{\otimes} k})^{\mathfrak{g}}$ identifies with $(U(\mathfrak{g})_0^{\otimes k})^{\mathfrak{g}}/(U(\mathfrak{g})_0^{\otimes k})^{\mathfrak{g}}_{\leq n}$.*

Lemma 4.4. *Assume that $n \geq 2$. If $f_1, f_2 \in (U_0')^2 + \hbar^{n+1}U_0$ and $g, h \in \hbar^n U_0$, then $(f_1 + g) \star_\hbar (f_2 + h) = g + h$ modulo $(U_0')^2 + \hbar^{n+1}U_0$, where $\star_\hbar$ is the CBH product for the Lie bracket $[a,b]_\hbar = [a,b]/\hbar$.*

Let us denote by $\bar{\alpha}$ the image of the class of α_{n-1} in $U(\mathfrak{g})_0^{\otimes 2}/(U(\mathfrak{g})_0^{\otimes 2})_{\leq n+1}$ under the isomorphism of this space with $(U_0'^{\widehat{\otimes} 2} + \hbar^{n+1} U_0^{\widehat{\otimes} 2})/(U_0'^{\widehat{\otimes} 2} + \hbar^{n+2} U_0^{\widehat{\otimes} 2})$ (see Lemma 4.3). Let $\alpha \in U(\mathfrak{g})_0^{\otimes 2}$ be a representative of $\bar{\alpha}$, then $\alpha_{n-1} = \alpha' + \hbar^{n+1}\alpha$, where $\alpha' \in U_0'^{\widehat{\otimes} 2} + \hbar^{n+2} U_0^{\widehat{\otimes} 2}$. Then the twist equation gives

$$(7)\qquad (-\alpha' - \hbar^{n+1}\alpha)^{1,23} \star_\hbar (-\alpha' - \hbar^{n+1}\alpha)^{2,3} \star_\hbar (\alpha' + \hbar^{n+1}\alpha)^{1,2} \star_\hbar (\alpha' + \hbar^{n+1}\alpha)^{12,3} = 0.$$

According to Lemma 4.4, the image of equality (7) in $(U^{\widehat{\otimes} 3} + \hbar^{n+1} U'^{\widehat{\otimes} 3})/(U^{\widehat{\otimes} 3} + \hbar^{n+2} U'^{\widehat{\otimes} 3}) \simeq U(\mathfrak{g})^{\otimes 3}/(U(\mathfrak{g})^{\otimes 3})_{\leq n+1}$ is $d(\bar{\alpha}) = 0$, where d is the co-Hochschild differential on the quotient $U(\mathfrak{g})_0^{\otimes \bullet}/(U(\mathfrak{g})_0^{\otimes \bullet})_{\leq n+1}$. Since $n \geq 2$, the relevant cohomology group vanishes, so $\bar{\alpha} = d(\bar{\beta})$, where $\bar{\beta} \in U(\mathfrak{g})_0/(U(\mathfrak{g})_0)_{\leq n+1}$. Let $\beta \in U(\mathfrak{g})_0$ be a representative of $\bar{\beta}$ and set $\mathrm{b}_n := \exp(\hbar^n\beta)$, $\mathrm{F}_n := {}^{\mathrm{b}_n}\,\mathrm{F}_{n-1}$, $\alpha_n := \hbar\log(\mathrm{F}_n)$. Then

$$\alpha_n = (\hbar^{n+1}\beta)^1 \star_\hbar (\hbar^{n+1}\beta)^2 \star_\hbar \alpha_{n-1} \star_\hbar (-\hbar^{n+1}\beta)^{12}.$$

According to Lemma 4.4, the image of α_n in

$$(U_0^{\widehat{\otimes} 2} + \hbar^{n+1} U_0'^{\widehat{\otimes} 2})/(U_0^{\widehat{\otimes} 2} + \hbar^{n+2} U_0'^{\widehat{\otimes} 2}) \simeq U(\mathfrak{g})_0^{\otimes 2}/(U(\mathfrak{g})_0^{\otimes 2})_{\leq n+1}$$

is $\bar{\alpha} - d(\bar{\beta}) = 0$. So α_n belongs to $U_0^{\widehat{\otimes} 2} + \hbar^{n+2} U_0'^{\widehat{\otimes} 2}$, as required. This proves the induction step. □

4.3. **Proof of Theorem 2.3.** Thanks to the previous subsection, we now know that there exists an element $\mathrm{b}_{\gamma,\gamma\gamma'}$ in U such that ${}^{\mathrm{b}_{\gamma,\gamma\gamma'}}\,\mathrm{F}_{\gamma,\gamma\gamma'} := \mathrm{b}_{\gamma,\gamma\gamma'}^{\otimes 2}\,\mathrm{F}_{\gamma,\gamma\gamma'}\,\Delta_\gamma(\mathrm{b}_{\gamma,\gamma\gamma'}^{-1})$ is admissible. Let us define

$$\mathrm{F}'_{\gamma,\gamma\gamma'} = {}^{\mathrm{b}_{\gamma,\gamma\gamma'}}\,\mathrm{F}_{\gamma,\gamma\gamma'}, \qquad \mathrm{i}'_{\gamma,\gamma\gamma'} = \mathrm{i}_{\gamma,\gamma\gamma'} \circ \mathrm{Ad}(\mathrm{b}_{\gamma,\gamma\gamma'}^{-1})$$

and

$$\mathrm{v}'_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''} = \mathrm{b}_{\gamma,\gamma\gamma'\gamma''}\,\mathrm{v}_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''}\,\mathrm{i}_{\gamma,\gamma\gamma'}^{-1}(\mathrm{b}_{\gamma\gamma',\gamma\gamma'\gamma''}^{-1})\,\mathrm{b}_{\gamma,\gamma\gamma'}^{-1}.$$

Then it is clear that $\mathrm{F}'_{\gamma,\gamma\gamma'}$, $\mathrm{i}'_{\gamma,\gamma\gamma'}$ and $\mathrm{v}'_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''}$ still satisfy the conclusion of Theorem 1.5.

Thanks to the first subsection of this section, applying the functor $\mathbf{QUE} \to \mathbf{QFSH}$ to the algebras $(U_{\gamma,\gamma}, \Delta_\gamma)$ we get algebras $(U'_\gamma, *_\gamma, \Delta_\gamma)$ which are quantizations of the Poisson algebras $(\mathcal{O}_{G^*_\gamma}, \{-,-\}_\gamma)$. Since the twists $\mathrm{F}'_{\gamma,\gamma\gamma'}$ are admissible, the algebra morphisms $\mathrm{i}'_{\gamma,\gamma\gamma'}$ restrict to the QFSH algebras U'_γ. Then to end the proof of Theorem 2.3, one has to prove:

Proposition 4.5. *The elements $\mathrm{v}'_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''}$ are admissible.*

Proof. Let us denote $\mathrm{v} = \mathrm{v}'_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''}$. Suppose v is not admissible and let n be the bigger i such that $\alpha_0 := \hbar\log(\mathrm{v}) \in U_0 + \hbar^{n+1}U_0$. By the assumption on v, we know that $n \geq 2$. Let us denote by $\bar{\alpha}$ the image of the class of α_0 in $U(\mathfrak{g})_0/(U(\mathfrak{g})_0)_{\leq n+1}$ under the isomorphism of this space with $(U_0 + \hbar^{n+1}U_0)/(U_0 + \hbar^{n+2}U_0)$ (see Lemma 4.3). Let $\alpha \in U(\mathfrak{g})_0$ be a representative of $\bar{\alpha}$, then $\alpha_0 = \alpha' + \hbar^{n+1}\alpha$, where $\alpha' \in U_0 + \hbar^{n+2}U_0$. Let f, f' and f'' be respectively the $\hbar\log$ of $\mathrm{F}'_{\gamma,\gamma\gamma'}$, $\mathrm{F}'_{\gamma\gamma',\gamma\gamma'\gamma''}$ and $\mathrm{F}_{\gamma,\gamma\gamma'\gamma''}$. Then the compatibility equation for composition of twists gives

$$f'' = (\alpha' + \hbar^{n+1}\alpha)^{\otimes 2} \star_\hbar \mathrm{i}^{-1}_{\gamma,\gamma\gamma'}(f') \star_\hbar f \star_\hbar (-\alpha' - \hbar^{n+1}\alpha)^{12} = 0. \tag{8}$$

According to Lemma 4.4, the image of equality (8) in $(U^{\widehat{\otimes}2} + \hbar^{n+1}U'^{\widehat{\otimes}2})/(U^{\widehat{\otimes}2} + \hbar^{n+2}U'^{\widehat{\otimes}2}) \simeq U(\mathfrak{g})^{\otimes 2}/(U(\mathfrak{g})^{\otimes 2})_{\leq n+1}$ is $d(\bar{\alpha}) = 0$. So $\bar{\alpha} \in \mathfrak{g}$ which is a contradiction with $n \geq 2$. □

5. Example of simple group with action of the Weyl group

5.1. **Quantization of Majid and Soibelman** [MS]**.** We start with briefly recalling the Majid and Soibelman's approach to quantum Weyl group. Let $\mathfrak{g}$ be a complex simple Lie algebra, $U_\hbar(\mathfrak{g})$ be the natural deformation of the universal enveloping algebra $U(\mathfrak{g})$. Lustig [Lu] and Soibelman [So] first independently noticed that a simple reflection w in the Weyl group W of $\mathfrak{g}$ defines an automorphism α_w on $U_\hbar(\mathfrak{g})$. Then one can extend $U_\hbar(\mathfrak{g})$ by elements $\bar{w}$ with $\alpha_w(g) = \bar{w}g\bar{w}^{-1}$ for all simple reflections in W. The extended algebra is called by "quantum Weyl group" and denoted by $\widetilde{U_\hbar(\mathfrak{g})}$. In [KR] and [So], $\widetilde{U_\hbar(\mathfrak{g})}$ is used to construct explicit formulas for solutions to the Yang-Baxter equation.

In [MS], Majid and Soibelman discovered the bicrossed product structure on $\widetilde{U_\hbar(\mathfrak{g})}$. Let w_i, $1 \leq i \leq \mathrm{rank}\,(\mathfrak{g})$ be simple reflections in W and t_j, $1 \leq j \leq \mathrm{rank}\,(\mathfrak{g})$ be elements in the maximal torus corresponding to $\phi_j \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ with $\phi_j : sl_2 \hookrightarrow \mathfrak{g}$ embedding to the j-th vertex of the Dynkin diagram. Define $\widetilde{W}$ be the group generated by w_i and t_j, which is a covering of the Weyl group W with the kernel isomorphic to the direct sum of k-copies of $\mathbb{Z}_2$ ($k = \mathrm{rank}(\mathfrak{g})$). The quantum Weyl group $\widetilde{U_\hbar(\mathfrak{g})}$ is proved in [MS][Corollary 3.4] to isomorphic to the bicrossed product

$$k\widetilde{W}^\psi \bowtie_{\alpha,\chi} U_\hbar(\mathfrak{g}),$$

where $\alpha : U_\hbar(\mathfrak{g}) \otimes k\widetilde{W} \to U_q(g)$, $\chi : k\widetilde{W} \otimes k\widetilde{W} \to U_\hbar(\mathfrak{g})$, and $\psi : k\widetilde{W} \to U_\hbar(\mathfrak{g}) \otimes U_\hbar(\mathfrak{g})$ are linear maps defined by

$$\alpha(a \otimes wt) = t^{-1}\alpha_w(a)t, \qquad \chi(w_1t_1, w_2t_2) = x^{-1}, \qquad \psi(wt) = (\bar{w}^{-1} \otimes \bar{w}^{-1})\Delta\bar{w}.$$

In the above equation of χ, x is defined to be an element in $U_\hbar(\mathfrak{g})$ such that $\alpha_{w_1w_2(\alpha_{w_1}(t_1)t_2)} = \alpha_{w_1t_1}\alpha_{w_2t_2}Ad_{x^{-1}}$ with $x \in U_\hbar(\mathfrak{g})$.

Proposition 5.1. *The quantum Weyl group $\widetilde{U_\hbar(\mathfrak{g})}$ is a quantization of the $\Gamma = \widetilde{W}$ Lie bialgebra $(\mathfrak{g}, [\ ,\ ], \delta)$, where $(\mathfrak{g}, [\ ,\ ], \delta)$ is the Lie bialgebra structure on $\mathfrak{g}$ corresponding to the deformation $U_\hbar(\mathfrak{g})$, and $\widetilde{W}$ acts on $\mathfrak{g}$ as the Weyl group (t acts on $\mathfrak{g}$ by adjoint action), and $f_\gamma = \wedge^2(\gamma) \circ \delta \circ \gamma^{-1} - \delta$ for $\gamma \in \widetilde{W}$.*

Proof. Inspired by the above bicrossed product structure on $\widetilde{U_\hbar(\mathfrak{g})}$, we introduce the following Γ quantized universal enveloping algebras for $\Gamma = \widetilde{W}$ generated by the following data.

- $(U_\hbar(\mathfrak{g})_\gamma, m_\gamma, \Delta_\gamma) = (U_\hbar(\mathfrak{g}), m, \Delta_\gamma)$, where m is the canonical multiplication on $U_\hbar(\mathfrak{g})$ and $\Delta_\gamma = \alpha(-,\gamma)^{\otimes 2} \circ Ad(\psi(\gamma)) \circ \Delta \circ \alpha^{-1}(-,\gamma)$ with Δ the canonical coproduct on $U_\hbar(\mathfrak{g})$.

- $i_{\gamma,\gamma\gamma'} : (U_\hbar(\mathfrak{g}), m_\gamma) \to (U_\hbar(\mathfrak{g}), m_{\gamma\gamma'})$ by $i_{e,\gamma} = \alpha(- \otimes \gamma) : U_\hbar(\mathfrak{g}) \to U_\hbar(\mathfrak{g})$ and $i_{\gamma,\gamma\gamma'} = i_{e,\gamma'}$.
- $F_{e,\gamma} \in U_\hbar(\mathfrak{g})^{\otimes 2}$ is set to equal to $\psi(\gamma)$ and $F_{\gamma,\gamma\gamma'} = F_{e,\gamma'}$. According to [MS][Lemma 3.3], for any reflection $w_i \in W$, $F_{e,w_it} = \psi(w_i) = e^{\frac{1}{2}\hbar H_i \otimes H_i/(\alpha_i,\alpha_i)}(\mathcal{R}_i)_{12}^{-1} = 1 + \hbar f_1 + O(\hbar^2)$. (Here (H_i, X_i^+, X_i^-) corresponds to the embedding $\phi_i : sl_2 \hookrightarrow \mathfrak{g}$ for the i-th root α_i with normal (α_i, α_i).) Because the part of $e^{\frac{1}{2}\hbar H_i \otimes H_i/(\alpha_i,\alpha_i)}$ is symmetric, the antisymmetrization of f_1 is equal to the antisymmetrization of the first order term of $(\mathcal{R}_i)_{21}^{-1}$, which is equal to the definition of f_{w_i} by the asymptotic expansion of $\mathcal{R}_i$. This result extends to an arbitrary element γ simply because w_i generates W.
- $v_{e,\gamma,\gamma\gamma'} = \chi(\gamma, \gamma\gamma') \in U_\hbar(\mathfrak{g})^{\otimes 2}$. According to the definition of $\chi(\gamma, \gamma\gamma')$ we see that v can be chosen be an element in $1 + \hbar^2 U_\hbar(\mathfrak{g})$ because the α action is associative up to the $\hbar$-linear terms by [KR][Formula (13)] and [KS][Prop 1.4.10].

It is straight forward to check that the cocycle conditions for α, χ, ψ, and their compatibilities are equivalent to the conditions for $(U_\hbar, m, \Delta_\gamma, i_{\gamma,\gamma\gamma'}, F_{\gamma,\gamma\gamma}, v_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''})$ to be a $\Gamma = \widetilde{W}$ quantized universal enveloping algebra. Therefore, the corresponding Γ quantized universal enveloping algebra is isomorphic to $\widetilde{U_\hbar(\mathfrak{g})}$. □

5.2. **Admissibility of the twists.**

Corollary 5.2. *The twists $F_{\gamma,\gamma\gamma'}$ and $v_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''}$ defined in Proposition 5.1 are admissible. Therefore, the quantum Weyl group defines a stack of formal series Hopf algebras quantizing the corresponding stack of Poisson Hopf algebras dual to $(\widetilde{W}, \mathfrak{g}, [\ ,\ ], \delta, f_\gamma)$.*

Proof. We look at the formulas for $F_{e,wt}$. According to ψ's formula, if w_i is a simple reflection, then $F_{e,w_it} = e^{\frac{1}{2}\hbar H_i \otimes H_i/(\alpha_i,\alpha_i)}(\mathcal{R}_i)_{12}^{-1}$. Taking $\hbar \log$ on $F_{e,w}$, we have

$$\hbar^2 \frac{1}{2} H_i \otimes H_i/(\alpha_i, \alpha_i) + \hbar \log((\mathcal{R}_i)_{12}^{-1}).$$

The first term is primitive as H_i is primitive. And the second term $\hbar \log((\mathcal{R}_i)_{12}^{-1})$ is primitive because $\hbar \log(\mathcal{R}_i)$ is primitive which was proved in [EH1][Theorem 0.1]. Therefore, we conclude that F_{e,w_it} is admissible when w is a simple reflection. And this property extends to a general element γ directly by products.

By Proposition 4.5, we also know that v is admissible because F is admissible.

We conclude the corollary by Theorem 2.3. □

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Institut de Mathématiques et de Modélisation de Montpellier, Université de Montpellier 2, CC5149, Place Eugène Bataillon, F-34095 Montpellier CEDEX 5, France
E-mail address: halbout@@math.univ-montp2.fr

Department of Mathematics, Washington University, St. Louis, Missouri, USA, 63130
E-mail address: xtang@@math.wustl.edu