

WEAK QUANTIZATION OF POISSON STRUCTURES

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ABSTRACT. In this paper we prove that any Poisson structure on a sheaf of Lie algebroids admits a weak deformation quantization, and give a sufficient condition for such a Poisson structure to admit an actual deformation quantization. We also answer the corresponding classification problems. In the complex symplectic case, we recover in particular some results of Nest-Tsygan and Polesello-Schapira.

We begin the paper with a recollection of known facts about deformation theory of cosimplicial differential graded Lie algebras.

CONTENTS

Introduction	1
Acknowledgements	2
1. Basic materials	2
1.1. Model categories and (co)simplicial methods	2
1.2. Homotopy theory of DG Lie algebras	5
2. Quantum type DG Lie algebras and 2-groupoids	6
2.1. The Deligne 2-groupoid of a quantum type DGLA	6
2.2. On the π_0 of the total space of a cosimplicial 2-groupoid	7
2.3. The Maurer-Cartan functor of a quantum type cosimplicial DGLA	8
2.4. The acyclic case	10
3. Applications to deformation quantization	10
3.1. (Weak) Poisson structures, deformations and equivalences	10
3.2. Existence and classification of weak quantizations	11
3.3. Classification in the complex symplectic case	12
3.4. Existence and classification of actual quantizations	17
References	18

INTRODUCTION

In this paper we prove a very general result concerning the deformation quantization problem for sheaves of Lie algebroids.

Following [11], any reasonable formal deformation problem can be described by a functor on differential graded (DG) artinian rings with values in simplicial sets, representable by some DG Lie (or perhaps L_∞) algebra. In this paper we deal with a deformation problem that is described by a *sheaf of* differential graded Lie algebras (DGLA). We solve this problem and take this opportunity to recall the construction of the deformation functor associated to cosimplicial DGLA.

Deformation quantization problem for a C^∞ Poisson manifold (M, π) has been solved by Kontsevich in [13]. Kontsevich first proves a formula in the local case ($M = \mathbb{R}^d$) and then apply an appropriate globalization procedure. Actually the existence of a deformation quantization is a part of a more general picture: Kontsevich proves in [13] that the DGLA of poly-differential operators is formal.

This formality theorem is generalized to a large class of sheaves of Lie algebroids in [5] (see also [2, 3, 4] for the particular cases of C^∞ and holomorphic Lie algebroids). In this paper, we prove that any Poisson structure admits a weak deformation quantization (Theorem 3.5). We also give a sufficient condition for such a Poisson structure to admit an actual deformation quantization. We also answer the corresponding classification problems. In the complex symplectic case, we recover in particular some results of Nest-Tsygan and Polesello-Schapira.

This paper can be seen as an attempt to understand some claims of [14] where this question is discussed in the context of algebraic geometry, and to give a correct formulation to some very unprecise remarks of [2] (remarks 3.25 and 3.26). We also want to emphasize the great importance of the extremely enlightening “homotopical point-of-view” [11] on deformation theory.

Throughout the paper k is a field with $\text{char}(k) = 0$.

Plan of the paper. In section 1 we review some basic materials concerning models for (cosimplicial) simplicial sets and (cosimplicial) DG Lie algebras. We also define the deformation functor associated to a cosimplicial DG Lie algebra.

In section 2 we recall the construction of the Deligne 2-groupoid associated to “quantum type” DG Lie algebras and give an explicit description of the deformation functor associated in this situation.

In section 3 we apply the previous constructions to some quantization problems. Namely, we first prove that any Poisson structure associated to a (locally free of finite rank) Lie algebroid admits a weak quantization. We then classify such weak quantizations and then give a sufficient condition for the existence of an usual quantization of a given Poisson structure. We compare our results with previous works [14, 18, 19, 17].

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1. BASIC MATERIALS

For the reader who wants to learn about model categories we refer to the very down-to-earth introduction [10] and references therein.

1.1. Model categories and (co)simplicial methods. Let us first recall that a *closed model category (CMC)* is a category equipped with three classes of morphisms

(called *fibrations*, *cofibrations*, and *weak equivalences*) satisfying the axioms (CM1)-(CM5) of [21].

1.1.1. *(Co)Simplicial objects.* Let \mathcal{C} be a category. Let us denote by $c\mathcal{C} = \mathcal{C}^{\Delta}$ (resp. $s\mathcal{C} = \mathcal{C}^{\Delta^{\text{op}}}$) the category of cosimplicial (resp. simplicial) objects in \mathcal{C} . Here Δ denotes the *ordinal number category* (or *simplicial category*), i.e. the category with objects ordered finite sets $[k] = \{0, \dots, k\}$ and morphisms (weakly) order preserving maps. In other words,

$$\Delta_k^l := \text{Hom}_{\Delta}([k], [l]) = \{(i_1, \dots, i_k) \mid 0 \leq i_1 \leq \dots \leq i_k \leq l\}.$$

Among all morphisms in Δ there are the following remarkable ones: for $i \in [k]$, $\delta_i = (0, \dots, i-1, i+1, \dots, k) : [k-1] \rightarrow [k]$ and $\sigma_i = (0, \dots, i, i, \dots, k) : [k+1] \rightarrow [k]$. Let C be a cosimplicial (resp. simplicial) object in \mathcal{C} . Moreover, any morphism in Δ is a composition of these two types of morphisms. The following easy lemma provides a full list of relations between them.

Lemma 1.1. *One has the following identities in Δ :*

- $\delta_i \delta_j = \delta_j \delta_{i-1}$ if $i > j$,
- $\sigma_i \sigma_j = \sigma_j \sigma_{i+1}$ if $i \geq j$,
- $\sigma_i \delta_j = \begin{cases} \delta_{j-1} \sigma_i & (\text{if } i < j-1) \\ \text{id} & (\text{if } i = j-1, j) \\ \delta_j \sigma_{i-1} & (\text{if } i > j) \end{cases}$

As a matter of notation, for cosimplicial (resp. simplicial) objects C , we will write $C^k := C([k])$ (resp. $C_k := C([k])$) for $k \geq 0$, and $f := C(f)$ (resp. $f^* := C(f)$) for any morphism f in Δ .

Example 1.2. For any $n \in \mathbb{N}$ we define the *geometric n -simplex* σ^n as the convex hull of the unit vectors in \mathbb{R}^{n+1} , and we identify $[n]$ with the vertices of σ^n . Then any map $\phi : [m] \rightarrow [n]$ can be extended to a PL map $\phi : \sigma^m \rightarrow \sigma^n$. Thus $(\sigma^n)_n$ is a cosimplicial PL manifold.

1.1.2. *The model category of simplicial sets.* Let us denote by **Sets** the category of sets. The category **sSets** of simplicial sets has a natural structure of a closed model category that we are going to describe.

For this we need to introduce some remarkable objects in **sSets**. The *standard n -simplex* is the simplicial set $\Delta^n = \text{Hom}_{\Delta}(-, [n]) : \Delta^{\text{op}} \rightarrow \mathbf{Sets}$. In other words, $\Delta_k^n = \Delta_k^n$. Moreover any $f \in \Delta_m^n$ induces a morphism of simplicial sets $\Delta^m \rightarrow \Delta^n$. Therefore one can also consider the *boundary* $\partial\Delta^n := \cup_{k=0}^n \delta_k \Delta^{n-1} \subset \Delta^n$ of Δ^n , and its *horns* $\Lambda^{i,n} := \cup_{k \neq i} \delta_k \Delta^{n-1} \subset \Delta^n$ ($i \in [n]$). In particular $\Delta_0^n = (\partial\Delta^n)_0 = [n]$ for $n \neq 0$.

For any simplicial set X_{\bullet} one can define its *realization* $|X|$ as a certain colimit in a category of topological spaces (see [10, Definition 1.19]; simply note that $|\Delta^n| = \sigma^n$). Then one can define the set of path components $\pi_0(X) := \pi_0(|X|)$ and homotopy groups $\pi_i(X, x) := \pi_i(|X|, x)$ ($i > 0$) for any $x \in \pi_0(X)$. Any morphism of simplicial sets $f : X \rightarrow Y$ induces morphisms $f_0 : \pi_0(X) \rightarrow \pi_0(Y)$ and $\pi_i(X, x) \rightarrow \pi_i(Y, f_0(x))$ for $i > 0$.

The CMC structure on **sSets** is such that the morphism $f : C \rightarrow D$ is a

- weak equivalence if it induces isomorphisms $\pi_0(X) \cong \pi_0(Y)$ and $\pi_n(X, x) \cong \pi_n(Y, f_0(x))$ for any $x \in \pi_0(X)$ and any $n > 0$,

- cofibration if f_n is injective for any $n \geq 1$.

A fibrant object X in $s\mathbf{Sets}$ is called a *weak ∞ -groupoid* (or *Kan complex*). It has the property that $\pi_0(X) = X_0 / \sim$, where $x \sim y$ if and only if there exists $\gamma \in X_1$ such that $\delta_1^* \gamma = x$ and $\delta_0^* \gamma = y$.

Moreover, a simplicial set C is fibrant if and only if the following condition is met:

every morphism $\Delta^{i,n} \rightarrow C$ can be extended to a morphism $\Delta^n \rightarrow C$.

We will also need the notion of *simplicial closed model category (SCMC)*: it is a CMC category \mathcal{M} enriched over simplicial sets (we denote by $\mathrm{Hom}_{\mathcal{M}}^{\Delta}(X, Y)$ the enriched Hom space) that satisfies Quillen's axiom (SM7) of [20]. The category of simplicial sets is naturally a SCMC, where the CMC structure is the previous one and the simplicial structure is given by

$$\mathrm{Hom}_{s\mathbf{Sets}}^{\Delta}(C, D)_n := \mathrm{Hom}_{s\mathbf{Sets}}(\Delta^n \times C, D).$$

1.1.3. *Reedy model categories.* The result of this paragraph first appeared in the unpublished paper [22]. Let \mathcal{M} be a CMC. We want to describe a model structure on $c\mathcal{M}$ (in his paper Reedy deals with $s\mathcal{M} = (c(\mathcal{M}^{\mathrm{op}}))^{\mathrm{op}}$).

First of all, for any object C in $c\mathcal{M}$ we define its *n -th matching object* in \mathcal{M} to be the colimit $M^n C := \varinjlim C^k$ taken over all surjections $[n] \rightarrow [k]$ in Δ_n^k with $k < n$. We have natural morphisms $C^n \rightarrow M^n C$.

Then $c\mathcal{M}$ has a CMC structure with a morphism $f : C \rightarrow D$ being a

- weak equivalence if $f^n : C^n \rightarrow D^n$ is a weak equivalence in \mathcal{M} for all $n \geq 0$,
- fibration if the induced map $C^n \rightarrow D^n \times_{M^n D} M^n C$ is a fibration in \mathcal{M} for all $n \geq 0$.

Remark 1.3. *One can of course define the n -th latching object of C to be the limit $L^n C := \varprojlim C^k$ taken over all injections $[k] \rightarrow [n]$ in Δ_n^k with $k < n$, and then cofibrations are characterized in a similar way as fibrations.*

A fibrant object C in $c\mathcal{M}$ is an object such that $C^n \rightarrow M^n C$ is a fibration for all n (this is an abstract way of describing descent condition). Moreover it is a standard fact that a fibrant object C in $c\mathcal{M}$ is termwise fibrant, that is to say C^n is fibrant for any $n \geq 0$ (the converse is obviously false).

1.1.4. *Cosimplicial simplicial sets.* It follows from the previous two paragraphs that the category $cs\mathbf{Sets}$ of cosimplicial simplicial sets has a natural structure of a CMC. Notice that there is a remarkable cofibrant object Δ in $cs\mathbf{Sets}$ defined as follows:

$$\begin{aligned} \Delta : \Delta &\longrightarrow s\mathbf{Sets} \\ [n] &\longmapsto \Delta^n = \mathrm{Hom}_{\Delta}(-, [n]) \\ f \in \Delta_m^n &\longmapsto (g \in \Delta_k^m \mapsto f \circ g \in \Delta_k^n). \end{aligned}$$

Using Δ we define the total space functor $\mathrm{Tot} : cs\mathbf{Sets} \longrightarrow s\mathbf{Sets}$ to be

$$C \longmapsto \mathrm{Tot}(C) := \mathrm{Hom}_{cs\mathbf{Sets}}^{\Delta}(\Delta, C).$$

The functor Tot preserves fibrations, trivial fibrations and weak equivalences between fibrant objects (as Δ is cofibrant).

The set $\mathrm{Tot}(C)_0$ of 0-simplices is given by sequences $(\alpha_0, \alpha_1, \dots, \alpha_n, \dots)$ such that $\alpha_n \in C_n^n$ and $s\alpha_i = s^* \alpha_j$ (in C_i^j) for any $s \in \Delta_i^j$ (this is $\mathrm{Hom}_{cs\mathbf{Sets}}(\Delta, C)$).

1.2. Homotopy theory of DG Lie algebras.

1.2.1. *The model category of DG Lie algebras.* Let us denote by **DG-Lie** the category of DG Lie algebras with morphisms being standard morphisms of DG Lie algebras. **DG-Lie** admits a closed model structure with a morphism $f : \mathfrak{g} \rightarrow \mathfrak{h}$ being a

- weak equivalence if $H(f)$ is an isomorphism¹,
- fibration if f is surjective.

In particular one can see that any object in **DG-Lie** is fibrant.

Moreover **DG-Lie** can be enriched over simplicial sets so that it becomes a SCMC. Namely,

$$\mathrm{Hom}_{\mathbf{DG-Lie}}^{\Delta}(\mathfrak{g}, \mathfrak{h})_n := \mathrm{Hom}_{\mathbf{DG-Lie}}(\mathfrak{g}, \Omega_n \otimes \mathfrak{h}),$$

with Ω_n being the DG commutative algebra of differential forms on the geometric n -simplex (in particular $(\Omega_n)_n$ is a simplicial DG commutative algebra).

1.2.2. *Hinich's deformation functor.* Let us denote by **Nilp** the category of pronilpotent commutative k -algebras, and by $s\mathbf{Sets}^{\mathbf{Nilp}}$ the SCMC of functors $\mathbf{Nilp} \rightarrow s\mathbf{Sets}$: weak equivalences (resp. (co)fibrations) in $s\mathbf{Sets}^{\mathbf{Nilp}}$ are termwise weak equivalences (resp. (co)fibrations).

Following [11] we define a functor

$$\mathbf{DG-Lie} \longrightarrow s\mathbf{Sets}^{\mathbf{Nilp}}; \mathfrak{g} \longmapsto \Sigma_{\mathfrak{g}}$$

that preserves fibrations and weak equivalence. Namely, for any DG Lie algebra (\mathfrak{g}, d, μ) and any pronilpotent commutative algebra \mathfrak{m} the set $\Sigma_{\mathfrak{g}}(\mathfrak{m})_n$ is the set of degree one elements Π in $\Omega_n \otimes \mathfrak{g} \otimes \mathfrak{m}$ such that

$$d\Pi + \frac{1}{2}\mu(\Pi, \Pi) = 0.$$

In other words, $\Sigma_{\mathfrak{g}}(\mathfrak{m})_n$ is the set of *Maurer-Cartan elements* of the DG Lie algebra $\Omega_n \otimes \mathfrak{g} \otimes \mathfrak{m}$.

1.2.3. *The Maurer-Cartan functor associated to a cosimplicial DGLA.* We consider the category $c\mathbf{DG-Lie}$ of cosimplicial DG Lie algebras with its Reedy model structure. Σ obviously extends to a functor $c\mathbf{DG-Lie} \rightarrow cs\mathbf{Sets}^{\mathbf{Nilp}}$ preserving fibrations and weak equivalences. Composing it with Tot we obtain a functor

$$Def : c\mathbf{DG-Lie} \longrightarrow s\mathbf{Sets}^{\mathbf{Nilp}}; Def_{\mathfrak{g}}(\mathfrak{m}) := \mathrm{Tot}(\Sigma_{\mathfrak{g}}(\mathfrak{m}))$$

that preserves fibrations, trivial fibrations and weak equivalences between fibrant objects. We finally define a functor

$$\underline{MC} := \pi_0 \circ Def : c\mathbf{DG-Lie} \longrightarrow \mathbf{Sets}^{\mathbf{Nilp}},$$

that sends weak equivalences between fibrant objects to equalities. We call $\underline{MC}_{\mathfrak{g}}$ the *Maurer-Cartan functor* of \mathfrak{g} , it is an homotopy invariant.

Remark 1.4. *Let X be a topological space and $X = \bigcup_{i \in I} U_i$ an open cover indexed by a totally order set I . Then for any presheaf of DG Lie algebras $U \mapsto \mathfrak{g}(U)$ one can construct a cosimplicial DG Lie algebra $\mathfrak{g}(\underline{U}^{\bullet})$ in an obvious way. Moreover if $U \mapsto \mathfrak{g}(U)$ is a sheaf then $\mathfrak{g}(\underline{U}^{\bullet})$ is fibrant (conversely, if $\mathfrak{g}(\underline{U}^{\bullet})$ is fibrant for ANY open cover then $U \mapsto \mathfrak{g}(U)$ is a sheaf).*

¹In other words, weak equivalences are quasi-isomorphisms.

1.2.4. *Sheaves of DG Lie algebras.* Given a site \mathcal{C} one can define a model structure on the category $\mathcal{P}_{\mathcal{C}}(\mathbf{DG-Lie})$ of *presheaves of DG Lie algebras* over \mathcal{C} (see [12]) in such a way that fibrant objects are precisely sheaves of DG Lie algebras. One could repeat the previous constructions in this context.

Nevertheless, if the site \mathcal{C} is not too bad (e.g. if it is the small site of a reasonable topological space) then this description is more or less equivalent to the cosimplicial one (thanks to remark 1.4).

2. QUANTUM TYPE DG LIE ALGEBRAS AND 2-GROUPOIDS

Following the terminology of [15], by a *quantum type DG Lie algebra* \mathfrak{g} we mean a DG Lie algebra such that $\mathfrak{g}^{[i]} = 0$ for $i < -1$. Getzler shows in [9] that if \mathfrak{g} is a quantum type DG Lie algebra then $\Sigma_{\mathfrak{g}}(\mathfrak{m})$ is weakly equivalent to the nerve of a strict 2-groupoid, called the *Deligne 2-groupoid* of $\mathfrak{g} \otimes \mathfrak{m}$ (see [8]). In this section we recall the construction of this 2-groupoid and then give an explicit description of the functor MC in the quantum type situation.

This description already appears in a slightly different formulation in [1, Section 3] (in particular one can recover Theorem 3.6 of [1] from our approach).

2.1. **The Deligne 2-groupoid of a quantum type DGLA.** We follow [8]. Let $(\mathfrak{g}, d, [,])$ be a DG Lie algebra and \mathfrak{m} a pronilpotent commutative k -algebra. We are going to define a 2-groupoid $Del_{\mathfrak{g}}(\mathfrak{m})$. Objects in $Del_{\mathfrak{g}}(\mathfrak{m})$ are given by the set of Maurer-Cartan elements, i.e. elements $\Pi \in (\mathfrak{g} \otimes \mathfrak{m})^{[1]}$ such that

$$d\Pi + \frac{1}{2}[\Pi, \Pi] = 0.$$

The pronilpotent group $\exp((\mathfrak{g} \otimes \mathfrak{m})^{[0]})$ acts on objects in the following way: for any $q \in (\mathfrak{g} \otimes \mathfrak{m})^{[0]}$ and any $\Pi \in (\mathfrak{g} \otimes \mathfrak{m})^{[1]}$

$$\exp(q) \cdot \Pi := \Pi - \sum_{n=0}^{\infty} \frac{\text{ad}(q)^n}{(n+1)!} d_{\Pi} q,$$

where $d_{\Pi}(q) = dq + [\Pi, q]$. The subset of Maurer-Cartan elements is obviously stable under this action². The translation groupoid associated to the group action $\exp((\mathfrak{g} \otimes \mathfrak{m})^{[0]}) \times MC(\mathfrak{g}, \mathfrak{m}) \rightarrow MC(\mathfrak{g}, \mathfrak{m})$ is the underlying 1-groupoid of Deligne 2-groupoid in the following sense: the set of morphisms between Π and $\exp(q) \cdot \Pi$ of the structure of a 1-groupoid, which is the translation groupoid associated to the group action

$$\exp((\mathfrak{g} \otimes \mathfrak{m})_{\Pi}^{[-1]}) \times \exp((\mathfrak{g} \otimes \mathfrak{m})^{[0]}) \rightarrow \exp((\mathfrak{g} \otimes \mathfrak{m})^{[0]}).$$

Here $(\mathfrak{g} \otimes \mathfrak{m})_{\Pi}^{[-1]}$ denotes the Lie algebra $((\mathfrak{g} \otimes \mathfrak{m})^{[-1]}, [-, -]_{\Pi})$, where $[u, v]_{\Pi} = [d_{\Pi}u, v]$, and the group action

$$\exp(u) \cdot \exp(q) := \exp(q) \exp(d_{\Pi}u).$$

comes from the Lie algebra morphism $d_{\Pi} : (\mathfrak{g} \otimes \mathfrak{m})_{\Pi}^{[-1]} \rightarrow (\mathfrak{g} \otimes \mathfrak{m})^{[0]}$.

For $\Pi \in MC(\mathfrak{g}, \mathfrak{m})$, $q \in (\mathfrak{g} \otimes \mathfrak{m})^{[0]}$ and $u, v \in (\mathfrak{g} \otimes \mathfrak{m})_{\Pi}^{[-1]}$, the *vertical* composition of 2-morphisms is given by the formula

$$(\exp(v), \exp(q) \exp(d_{\Pi}u), \Pi) \circ_v (\exp(u), \exp(q), \Pi) = (\exp(u) \exp(v), \exp(q), \Pi);$$

²This action is the exponentiation of the infinitesimal affine action $q \cdot \Pi = dq + [q, \Pi]$.

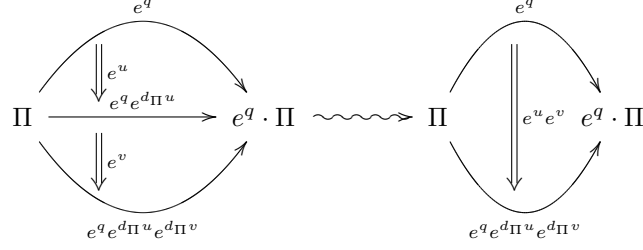


Figure 1. Vertical composition

and for $q' \in (\mathfrak{g} \otimes \mathfrak{m})^{[0]}$ and $u' \in (\mathfrak{g} \otimes \mathfrak{m})_{\exp(q) \cdot \Pi}^{[-1]}$ the *horizontal* composition of 2-morphisms is given by

$$\begin{aligned} & (\exp(u'), \exp(q'), \exp(q) \cdot \Pi) \circ_h (\exp(u), \exp(q), \Pi) \\ &= (\exp(e^{-ad(q)} u') \exp(u), \exp(q') \exp(q), \Pi). \end{aligned}$$

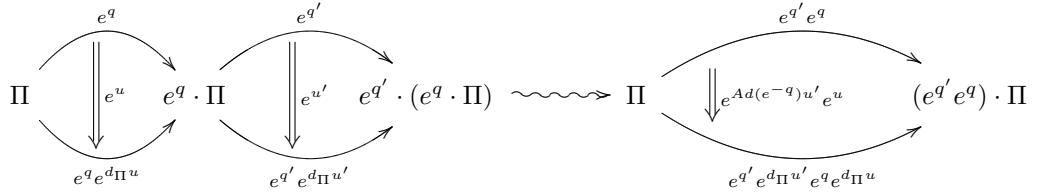


Figure 2. Horizontal composition

2.2. On the π_0 of the total space of a cosimplicial 2-groupoid.

2.2.1. *The nerve of a 2-groupoid.* In this paragraph we review the nerve construction for strict 2-groupoids (see [16]). Let \mathcal{G} be a 2-groupoid. The *nerve* of \mathcal{G} , denoted by $N\mathcal{G}$, is the simplicial set defined as follows: 0-simplices of $N\mathcal{G}$ are objects of \mathcal{G} , 1-simplices are 1-arrows in \mathcal{G} , 2-simplices are diagrams of the following form

(1)

$$\begin{array}{ccc} & \bullet & \\ a_{12} \nearrow & \Downarrow & \searrow a_{23} \\ \bullet & \xrightarrow{a_{13}} & \bullet \end{array},$$

3-simplices are commutative tetrahedra of the form

(2)

$$\begin{array}{ccc} & \bullet & \\ t_{14} \nearrow & \uparrow & \searrow t_{34} \\ & \bullet & \\ t_{12} \nearrow & \uparrow & \searrow t_{23} \\ & \bullet & \\ t_{13} \nearrow & \uparrow & \searrow t_{23} \\ & \bullet & \end{array},$$

and for $n \geq 3$ an n -simplex of $N\mathcal{G}$ is an n -simplex such that each of its sub-3-simplices is a commutative tetrahedron as above.

$N\mathcal{G}$ is fibrant and such that $\pi_0(N\mathcal{G})$ is the quotient set of \mathcal{G} , $\pi_1(N\mathcal{G}, x)$ is the group of 1-automorphisms of x , $\pi_2(N\mathcal{G}, x)$ is the group of 2-automorphisms of id_x , and $\pi_n(N\mathcal{G}, x) = 0$ for $n \geq 3$.

2.2.2. *Path components of the total space of a cosimplicial 2-groupoid.* Let \mathcal{G}^\bullet be a fibrant cosimplicial 2-groupoid, meaning that $N\mathcal{G}^\bullet$ is fibrant as a cosimplicial simplicial set. In this paragraph we describe the set $\pi_0(\text{Tot}(N\mathcal{G}^\bullet))$.

It follows from the nerve construction of the previous paragraph that $(\text{Tot}(N\mathcal{G}^\bullet))_0$ is the set of 4-tuples (m, g, a, t) , where

- m is an object in \mathcal{G}^0 ,
- g is a 1-arrow in \mathcal{G}^1 with source $\delta_2 m$ and target $\delta_1 m$,
- a is a diagram of the form (1) in \mathcal{G}^2 such that $a_{12} = \delta_3 g$, $a_{13} = \delta_2 g$ and $a_{23} = \delta_1 g$,
- t is a tetrahedron (2) in \mathcal{G}^3 such that $t_{123} = \delta_4 a$, $t_{134} = \delta_2 a$, $t_{124} = \delta_3 a$ and $t_{234} = \delta_1 a$.

Then $\pi_0(\text{Tot}(N\mathcal{G}^\bullet)) = (\text{Tot}(N\mathcal{G}^\bullet))_0 / \sim$, where two 0-simplices (m, g, a, t) and (m', g', a', t') are equivalent through \sim if there exists a triple (γ, α, τ) such that

- γ is a 1-arrow in \mathcal{G}^0 with source g and target g' ,
- α is a diagram of the form

$$\begin{array}{ccc} \bullet & \xrightarrow{g} & \bullet \\ \delta_2 \gamma \downarrow & & \downarrow \delta_1 \gamma \\ \bullet & \xrightarrow{g'} & \bullet \end{array},$$

- τ is a commutative diagram of the following form

$$\begin{array}{ccc} & \bullet & \\ & \nearrow & \searrow \\ \bullet & \xrightarrow{a} & \bullet \\ \tau_{12} \downarrow & & \downarrow \tau_{23} \\ \bullet & \xrightarrow{a'} & \bullet \\ \tau_{13} \downarrow & & \downarrow \tau_{23} \\ \bullet & \xrightarrow{\quad} & \bullet \end{array},$$

where $\tau_{12} = \delta_3 \gamma$, $\tau_{13} = \delta_2 \gamma$ and $\tau_{23} = \delta_1 \gamma$.

2.3. The Maurer-Cartan functor of a quantum type cosimplicial DGLA.

Let \mathfrak{g}^\bullet be a fibrant quantum type cosimplicial DGLA. Then for any pronilpotent commutative algebra \mathfrak{m} , we can now explicitly describe $\underline{MC}_{\mathfrak{g}^\bullet}(\mathfrak{m})$: it is the quotient of the set of *weak Maurer-Cartan elements* by *weak gauge equivalences*, that we define in the following two paragraphs.

2.3.1. *Weak Maurer-Cartan elements.* A weak Maurer-Cartan element is a triple (Π, g, a) such that

- Π is a standard Maurer-Cartan element in \mathfrak{g}^0 , that is to say $\Pi \in (\mathfrak{g}^0 \otimes \mathfrak{m})^{[1]}$ satisfies the Maurer-Cartan equation

$$d\Pi + \frac{1}{2}[\Pi, \Pi] = 0,$$

- $g \in \exp((\mathfrak{g}^1 \otimes \mathfrak{m})^{[0]})$ is such that

$$g \cdot (\delta_2 \Pi) = \delta_1 \Pi,$$

- $a \in \exp((\mathfrak{g}^2 \otimes \mathfrak{m})_{(\delta_3 \delta_2) \Pi}^{[-1]})$ is such that

$$\delta_1 g \delta_3 g = a^{-1} \cdot (\delta_2 g)$$

and satisfies

$$\delta_4 a \delta_2 a = \delta_4 \delta_3 g (\delta_1 a) \delta_3 a$$

Example 2.1 (Sheaves). If \mathfrak{g}^\bullet is the quantum cosimplicial DGLA naturally associated to a sheaf of quantum type DGLAs $\mathfrak{g}(-)$ on a topological space X and an open cover $X = \bigcup_{i \in I} U_i$ then a weak Maurer-Cartan element is a triple (Π, g, a) as follows: $\Pi = (\Pi_i)_{0 \leq i \leq m}$ with $\Pi_i \in (\mathfrak{g}(U_i) \otimes \mathfrak{m})^{[1]}$ satisfying the Maurer-Cartan equation for any i , $g = (g_{ij})_{0 \leq i < j \leq m}$ with $g_{ij} \in \exp((\mathfrak{g}(U_{ij}) \otimes \mathfrak{m})^{[0]})$ such that $g_{ij} \cdot \Pi_i = \Pi_j$ on U_{ij} for any $i < j$, $a = (a_{ijk})_{0 \leq i < j < k \leq m}$ with $a_{ijk} \in \exp((\mathfrak{g}(U_{ijk}) \otimes \mathfrak{m})_{\Pi_i}^{[-1]})$ such that

$$(3) \quad g_{jk} g_{ij} = a_{ijk}^{-1} \cdot g_{ik}$$

on U_{ijk} for any $i < j < k$, with the additional condition that

$$(4) \quad a_{ijk} a_{ikl} = g_{ij} (a_{jkl}) a_{ijl}$$

on U_{ijkl} for any $i < j < k < l$.

Example 2.2 (Algebroid stack deformations of sheaves of algebras). Let $\mathcal{A}(-)$ be a sheaf of algebras over a topological space X . And let $\mathfrak{g}(-) = C(\mathcal{A}, \mathcal{A})$ be the sheaf of quantum type DGLA given by Hochschild cochains. Given a pronilpotent commutative algebra \mathfrak{m} and an open cover $X = \bigcup_i U_i$, the corresponding weak Maurer-Cartan elements are (according to the previous example) precisely the \mathfrak{m} -deformations of \mathcal{A} as an *algebroid stack* (see [14]).

2.3.2. *Weak gauge equivalences.* A weak (gauge) equivalence between two weak Maurer-Cartan elements (Π, g, a) and (Π', g', a') is a pair (γ, α) such that

- $\gamma \in \exp(\mathfrak{g}^0 \otimes \mathfrak{m})^{[0]}$ is a gauge equivalence between Π and Π' :

$$\Pi' = \gamma \cdot \Pi,$$

- $\alpha \in \exp(\mathfrak{g}^1 \otimes \mathfrak{m})_{\delta_2 \Pi}^{[-1]}$ is such that

$$\alpha \cdot (\delta_1 \gamma g) = g' \delta_2 \gamma$$

and satisfies

$$a \delta_2 \alpha = \delta_3 g (\delta_1 \alpha) \delta_3 \alpha \delta_3 \delta_2 \gamma (a')$$

Example 2.3 (Sheaves). Let us go back to example 2.1. In this context a weak equivalence is a pair (γ, α) as follows: $\gamma = (\gamma_i)_{0 \leq i \leq m}$ with $\gamma_i \in \exp(\mathfrak{g}(U_i) \otimes \mathfrak{m})^{[0]}$ being a gauge equivalence between Π_i and Π'_i , and $\alpha = (\alpha_{ij})_{0 \leq i < j \leq m}$ with $\alpha_{ij} \in \exp((\mathfrak{g}(U_{ij}) \otimes \mathfrak{m})^{[-1]})$ such that $\alpha_{ij} \cdot (\gamma_j g_{ij}) = g'_{ij} \gamma_i$ on U_{ij} for any $i < j$, with the additional condition that

$$a_{ijk} \alpha_{ik} = g_{ij} (\alpha_{jk}) \alpha_{ij} \gamma_i (a'_{ijk})$$

on U_{ijk} for any $i < j < k$.

2.4. The acyclic case. In this paragraph we assume that we are given a cosimplicial quantum type DGLA \mathfrak{g}^\bullet that is acyclic as a cosimplicial vector space. Namely $\check{H}^i(\mathfrak{g}^\bullet) = 0$ for any $i > 0$. We have the following:

Proposition 2.4. *$MC_{\mathfrak{g}^\bullet}(\mathfrak{m})$ is in one-to-one correspondance with the set of usual MC elements up to usual gauge equivalences in the DGLA $\check{H}^0(\mathfrak{g}^\bullet) \otimes \mathfrak{m}$.*

Proof. Let us first prove that any weak Maurer-Cartan element (Π, g, a) is weakly equivalent to a weak Maurer-Cartan element of the form $(\Pi', 1, 1)$. We do this in using two successive inductions:

- (1) Assume that $a = 1 \bmod \mathfrak{m}^i$. Then the tetrahedron equation together with $\check{H}^2(\mathfrak{g}^\bullet) = 0$ implies that $a = 1 + \check{d}b \bmod \mathfrak{m}^{i+1}$, with $b \in (\mathfrak{g}^2 \otimes \mathfrak{m}^i)^{[-1]}$. Therefore, applying the weak equivalence $(1, \exp(b))$ one obtains a weak Maurer-Cartan element (Π, g', a') with $a' = 1 \bmod \mathfrak{m}^{i+1}$. By induction (Π, g, a) is weakly equivalent to a Maurer-Cartan element of the form $(\Pi, g', 1)$.
- (2) Assume that $g' = 1 \bmod \mathfrak{m}^i$. Then the triangle equation together with $\check{H}^1(\mathfrak{g}^\bullet) = 0$ implies that $g = 1 + \check{d}h \bmod \mathfrak{m}^{i+1}$, with $h \in (\mathfrak{g}^1 \otimes \mathfrak{m}^i)^{[0]}$. Therefore, applying the weak equivalence $(\exp(h), 1)$ one obtains a weak Maurer-Cartan element $(\Pi', g'', 1)$ with $g'' = 1 \bmod \mathfrak{m}^{i+1}$. By induction $(\Pi, g', 1)$ is weakly equivalent to a Maurer-Cartan element of the form $(\Pi', 1, 1)$.

We then observe that if two weak Maurer-Cartan elements $(\Pi, 1, 1)$ and $(\Pi', 1, 1)$ are related by a weak equivalence (γ, α) then the weak equivalence $(\gamma, 1)$ also relates them.

Finally, weak Maurer-Cartan elements of the form $(\Pi, 1, 1)$ (resp. weak equivalences of the form $(\gamma, 1)$) are precisely usual Maurer-Cartan elements (resp. usual gauge equivalences) in $\check{H}^0(\mathfrak{g}^\bullet) \otimes \mathfrak{m}$. \square

3. APPLICATIONS TO DEFORMATION QUANTIZATION

Let (X, \mathcal{O}) be a topological space equipped with a sheaf of commutative (and associative) unital k -algebras and assume that \mathcal{L} is a sheaf of Lie algebroids³ over (X, \mathcal{O}) which is locally free and of constant rank $d \in \mathbb{N}^*$ as an \mathcal{O} -module.

We also assume that \mathcal{O} is locally acyclic (so that Čech resolution is relevant).

Recall from [2, 3, 4, 5] that there are two sheaves of quantum type DG-Lie algebras \mathfrak{g} and \mathfrak{h} associated to \mathcal{L} : \mathfrak{g} is the sheaf of \mathcal{L} -poly-vector fields equipped with zero differential and Schouten type bracket $[-, -]$, and \mathfrak{h} is the sheaf of \mathcal{L} -poly-differential operators (a-k-a Hochschild cochains associated to \mathcal{L}^4) equipped with Hochschild differential d_H and Gerstenhaber bracket $[-, -]_G$ (we refer to [2, 4] for details).

This is a well-known fact that \mathfrak{g} is the cohomology sheaf of (\mathfrak{h}, d_H) , and that the Lie bracket on \mathfrak{g} induced from $[-, -]_G$ is precisely $[-, -]$.

Let us chose an open cover $X = \bigcup_{i \in I} U_i$ of such that $\mathcal{O}|_{U_i}$ has trivial comology and $\mathcal{L}|_{U_i}$ is a free $\mathcal{O}|_{U_i}$ -module. We then denote by \mathfrak{g}^\bullet and \mathfrak{h}^\bullet the associated cosimplicial DGLAs.

3.1. (Weak) Poisson structures, deformations and equivalences.

³Lie algebroids are also called Lie-Rinehart algebras.

⁴Which are in fact *Cartier cochains* [7] for the counital \mathcal{O} -coalgebra $\mathcal{U}(\mathcal{L})$ with values in the trivial bicomodule.

3.1.1. *Definitions.* A *Poisson structure* (resp. *formal Poisson structure*) on \mathcal{L} is a usual Maurer-Cartan element in the graded Lie algebra $\check{H}^0(X, \mathfrak{g}^\bullet)$ (resp. in $\hbar\check{H}^0(X, \mathfrak{g}^\bullet)[[\hbar]]$).

A *formal weak Poisson structure* on \mathcal{L} is a weak Maurer-Cartan element (π, g, a) for \mathfrak{g}^\bullet and $\mathfrak{m} = \hbar k[[\hbar]]$. One can see that the first order term in the \hbar -series π is an actual Poisson structure.

A *weak formal deformation* (or simply, a *weak deformation*) is a weak Maurer-Cartan element for \mathfrak{h}^\bullet and $\mathfrak{m} = \hbar k[[\hbar]]$, and we denote its class modulo weak gauge equivalences by $\underline{\alpha} \in \underline{MC}_{\mathfrak{h}^\bullet}(\hbar k[[\hbar]])$.

An *actual deformation* is a weak Maurer-Cartan element for \mathfrak{h}^\bullet and $\mathfrak{m} = \hbar k[[\hbar]]$ of the form $(\Pi, g, 1)$. An *actual equivalence* between two actual deformations is a weak equivalence of the form $(\gamma, 1)$.

This terminology is justified by the example of complex manifolds below.

3.1.2. *Example: complex manifolds.* Let X be a complex manifold, $\mathcal{O} = \mathcal{O}_X$ be the sheaf of holomorphic functions on X , and \mathcal{L} be the sheaf of holomorphic vector fields on X . Then \mathfrak{g} (resp. \mathfrak{h}) is the DG Lie algebra of usual holomorphic poly-vector fields (resp. poly-differential operators) on X .

Weak deformations then correspond precisely to holomorphic formal deformations of \mathcal{O} as an algebroid stack (see the previous section), and NOT as a sheaf of algebras (the latest corresponding to actual deformations).

Actual deformations correspond to holomorphic formal deformations of \mathcal{O} as a sheaf of algebras.

3.1.3. *Quantization problems.* To any actual deformation (Π_{\hbar}, g_{\hbar}) on \mathcal{L} one can associate canonically a Poisson structure by taking the skew-symmetrization of the first order term in the \hbar -series Π_{\hbar} . The *strong quantization problem* is then as follows:

Problem 3.1. *Let π be a Poisson structure on \mathcal{L} . Does there exist an actual deformation with associated Poisson structure being π ?*

If it exists, such an actual deformation is called an *actual quantization* of π .

In full generality the answer to this problem is NO. This leads us to formulate a weaker version of this problem. As above, to any weak deformation $(\Pi_{\hbar}, g_{\hbar}, a_{\hbar})$ on \mathcal{L} one can associate canonically a Poisson structure. The *weak quantization problem* is then as follows:

Problem 3.2. *Let π be a Poisson structure on \mathcal{L} . Does there exist a weak deformation with associated Poisson structure being π ?*

If it exists, such a weak deformation is called a *weak quantization* of π .

3.2. Existence and classification of weak quantizations.

3.2.1. *A formality theorem.* There exist several results [24, 4, 23, 5] about the extension of Kontsevich formality theorem for algebraic varieties and/or complex manifolds. The one we will use in the paper is taken from [5] (Theorem 6.4.1) that we translate as follows in the language we are using here:

Theorem 3.3 ([5]). *The sheaves of DG-Lie algebras \mathfrak{g} and \mathfrak{h} are weakly equivalent.*

Remark 3.4. *The graded vector subspace $\tilde{\mathfrak{h}} \subset \mathfrak{h}$ defined by*

$$\tilde{\mathfrak{h}}^{[k]} := (\ker(\epsilon : \mathcal{U}(\mathcal{L}) \rightarrow \mathcal{O}))^{\otimes_{\mathcal{O}} k+1} \subset \mathcal{U}(\mathcal{L})^{\otimes_{\mathcal{O}} k+1} = \mathfrak{h}^{[k]}$$

actually is a DG Lie subalgebra of \mathfrak{h} , and the inclusion is a (objectwise) quasi-isomorphism : as a counital \mathcal{O} -coalgebra $\mathcal{U}(\mathcal{L})$ is the cofree counital cocommutative and coassociative \mathcal{O} -coalgebra (this is PBW Theorem for Lie algebroids), for which this result is a standard fact (see e.g. [6], Ch. IX). Thus for any open $U \subset X$, $\text{Del}_{\tilde{\mathfrak{h}}(U)}$ and $\text{Del}_{\mathfrak{h}(U)}$ are weakly equivalent, and then we can work with $\tilde{\mathfrak{h}}$ instead of \mathfrak{h} . This is more convenient since cochains of positive degree in $\tilde{\mathfrak{h}}$ are vanishing when acting on $k \subset \mathcal{O}$ in (at least) one argument. ⁵

3.2.2. Main result. As a direct consequence of Theorem 3.3 and of discussion in the previous section we have the following result, which in particular gives a positive answer to Problem 3.2:

Theorem 3.5. *1) Any Poisson structure π on \mathcal{L} admits a weak quantization.*

2) For any Poisson structure π on \mathcal{L} there is a one-to-one correspondence

$$\frac{\{w.P.s. (\pi_{\hbar}, g_{\hbar}, a_{\hbar}) \text{ s.t. } \pi_{\hbar} = \hbar\pi + o(\hbar)\}}{\text{weak equivalences}} \xleftrightarrow{\quad} \frac{\{\text{weak quantizations of } \pi\}}{\text{weak equivalences}}.$$

This result has first been conjectured (and proved ?) by Leitner and Yekutieli (see [15]) when $\mathcal{O} = \mathcal{O}_X$ and $\mathcal{L} = \mathcal{T}_X$ are respectively the structure and tangent sheaf of a smooth algebraic variety (they speak about *twisted* things while we write *weak*).

3.3. Classification in the complex symplectic case. In this paragraph X is (the underlying topological space of) a complex manifold and $\mathcal{O} = \mathcal{O}_X$ is the sheaf of holomorphic functions on it. Thanks to the $\bar{\partial}$ -Poincaré lemma the natural inclusion $\mathfrak{g}^{\bullet} \hookrightarrow (\mathcal{A}^{0,*}(\mathfrak{g}^{\bullet}), \bar{\partial})$ is a weak equivalence. Moreover, as $\mathcal{A}^{0,*}(\mathfrak{g})$ is the sheaf of sections of a C^{∞} -bundle it is acyclic; therefore $\underline{MC}_{\mathfrak{g}^{\bullet}}(\hbar k[[\hbar]])$ is described by the set of usual Maurer-Cartan elements up to usual gauge equivalences in the DGLA of global sections $\hbar \mathcal{A}^{0,*}(\mathfrak{g})(X)[[\hbar]]$ with $\bar{\partial}$ as differential. We write $\mathcal{G} = (\mathcal{A}^{0,*}(\mathfrak{g})(X), \bar{\partial})$ and introduce the following bigrading on it: $\mathcal{G}^{k,l} = \mathcal{A}^{0,l}(\mathfrak{g}^{[k]})(X)$.

Let us suppose that we are given a Poisson structure π on a holomorphic Lie algebroid \mathcal{L} over (X, \mathcal{O}) which is *symplectic*. Namely, viewed as an \mathcal{O} -linear map $\pi^{\sharp} : \mathcal{L}^{\vee} = \text{Hom}_{\mathcal{O}}(\mathcal{L}, \mathcal{O}) \rightarrow \mathcal{L}$ it is invertible. Consider the bundle $\mathcal{L}\Omega^* = \wedge^* \mathcal{L}^{\vee}$ of \mathcal{L} -differential forms and recall (see [4]) that sections of it (\mathcal{L} -forms for short) are endowed with the following \mathcal{L} -de Rham differential: for any \mathcal{L} - k -form η and \mathcal{L} -vector fields $\sigma_0, \dots, \sigma_k$,

$$(5) \quad \mathcal{L}d\eta(\sigma_0, \dots, \sigma_k) := \sum_i (-1)^i \rho(\sigma_i) \eta(\sigma_0, \dots, \hat{\sigma}_i, \dots, \sigma_k) \\ + \sum_{i < j} (-1)^{i+j} \eta([\sigma_i, \sigma_j], \sigma_0, \dots, \hat{\sigma}_i, \dots, \hat{\sigma}_j, \dots, \sigma_k).$$

Let us denote $J : \mathfrak{g} \rightarrow \wedge^{*+1} \mathcal{L}^{\vee}$ the inverse map of π^{\sharp} extended by taking iterated exterior products. Let us denote ω the image of π through the map J . By direct computation one gets that J sends the differential $[-, \pi]$ onto the \mathcal{L} -de Rham differential $\mathcal{L}d$. Recall also that if u is a \mathcal{L} -polyvector field then one can define

⁵In the case when $\mathcal{O} = \mathcal{O}_X$ and $\mathcal{L} = \mathcal{T}_X$ are the structure and tangent sheaf of a smooth algebraic variety X , elements of \mathfrak{h} are called *normalized* poly-differential operators in [24].

contraction ι_u with u and \mathcal{L} -Lie derivative $\mathcal{L}L_u$ by u . These operations are related by the following formulas: for u and v of homogeneous degree k and l one has

$$\begin{aligned}\mathcal{L}L_u &= \mathcal{L}d \circ \iota_u + (-1)^k \iota_u \circ \mathcal{L}d, \\ \mathcal{L}L_u \circ \mathcal{L}L_v - (-1)^{kl} \mathcal{L}L_v \circ \mathcal{L}L_u &= \mathcal{L}L_{[u,v]},\end{aligned}$$

and

$$\mathcal{L}L_u \circ \iota_v - (-1)^{k(l+1)} \iota_v \circ \mathcal{L}L_u = (-1)^k \iota_{[u,v]}.$$

Let us introduce the bicomplex $\mathcal{L}\mathcal{A}^{*,*}$: $\mathcal{L}\mathcal{A}^{k,l} = \mathcal{A}^{0,l}(\wedge^k \mathcal{L}^\vee)$ with differentials $\mathcal{L}d$ and $\bar{\partial}$. It is naturally equipped with a descending filtration:

$$F^p(\mathcal{L}\mathcal{A}) = \bigoplus_{k \geq p} \mathcal{L}\mathcal{A}^{k,*}.$$

Theorem 3.6. *There is a one-to-one correspondence*

$$\frac{\{\text{weak quantizations of } \pi\}}{\text{weak equivalences}} \longleftrightarrow \frac{1}{\hbar} \omega + H_{\text{tot}}^2(F^1(\mathcal{L}\mathcal{A})) + \hbar H_{\text{tot}}^2(\mathcal{L}\mathcal{A})[[\hbar]].$$

Proof. We know from Theorem 3.5 that the set of weak quantizations of π up to weak equivalences is in bijection with the set of formal weak Poisson structures $(\pi_{\hbar}, g_{\hbar}, a_{\hbar})$ such that $\pi_{\hbar} = \hbar\pi + o(\hbar)$ up to weak equivalences. And we have just seen that this set is itself in bijection with the set of usual Maurer-Cartan elements $\tilde{\pi}_{\hbar}$ such that $\tilde{\pi}_{\hbar}^{1,0} = \hbar\pi + o(\hbar)$ up to usual gauge equivalences in $\hbar\mathcal{G}[[\hbar]]$.

Let us write $\pi_{\hbar} = \tilde{\pi}_{\hbar}^{1,0}$, $q_{\hbar} = \tilde{\pi}_{\hbar}^{0,1}$ and $r_{\hbar} = \tilde{\pi}_{\hbar}^{-1,2}$. The Maurer-Cartan equation reads

$$\begin{aligned}(a) \quad & [\pi_{\hbar}, \pi_{\hbar}] = 0 & (b) \quad & \bar{\partial}(\pi_{\hbar}) + [q_{\hbar}, \pi_{\hbar}] = 0 \\ (c) \quad & \bar{\partial}(q_{\hbar}) + [\pi_{\hbar}, r_{\hbar}] + \frac{1}{2}[q_{\hbar}, q_{\hbar}] = 0 & (d) \quad & \bar{\partial}(r_{\hbar}) = 0.\end{aligned}$$

We now define a bundle isomorphism

$$\begin{aligned}T_{\mathbb{C}}^{\vee}M &\xrightarrow{\sim} T^{1,0}M \oplus T^{\vee 0,1}M \\ \xi + \bar{\xi} &\longmapsto \pi_{\hbar}^{\sharp}(\xi) - \iota_{q_{\hbar}}(\bar{\xi}) + \bar{\xi}\end{aligned}$$

whose inverse extends to a graded (but NOT bigraded) algebra isomorphism

$$\tilde{J}_{\hbar} : \mathcal{G}[-1] \xrightarrow{\sim} \mathcal{L}\mathcal{A}.$$

Then let $\tilde{\omega}_{\hbar} = \frac{1}{\hbar}\omega + O(1)$ be the image of $\tilde{\pi}_{\hbar}$ under \tilde{J}_{\hbar} and write $\omega_{\hbar} = \tilde{\omega}_{\hbar}^{2,0}$, $v_{\hbar} = \tilde{\omega}_{\hbar}^{1,1}$ and $u_{\hbar} = \tilde{\omega}_{\hbar}^{0,2}$. One has $\omega_{\hbar} = \frac{1}{\hbar}\omega + O(1)$, $v_{\hbar} = O(1)$ and $u_{\hbar} = O(\hbar)$.

Lemma 3.7. 1. $\omega_{\hbar} = J_{\hbar}(\pi_{\hbar})$ (where J_{\hbar} is defined as J , with π replaced by π_{\hbar}).

2. $q_{\hbar} = -\pi_{\hbar}^{\sharp}(v_{\hbar})$.

3. $r_{\hbar} = u_{\hbar} - \frac{1}{2}(\iota_{q_{\hbar}}^2)(\omega_{\hbar}) = u_{\hbar} - \frac{1}{2}\iota_{q_{\hbar}}v_{\hbar}$.

Proof of the lemma. The first part is well-known.

Before proving the second and the third parts observe that for any $q \in \mathcal{G}^{0,1}$ one has $(\iota_q \otimes \text{id})(\omega) = \frac{1}{2}\iota_q(\omega) = (\text{id} \otimes \iota_q)(\omega)$ and $\iota_q(\omega) = -J(q)$.

Therefore $q_{\hbar} = \pi_{\hbar}^{\sharp}(v_{\hbar}) - (\iota_{q_{\hbar}} \otimes \pi_{\hbar}^{\sharp} + \pi_{\hbar}^{\sharp} \otimes \iota_{q_{\hbar}})(\omega_{\hbar}) = \pi_{\hbar}^{\sharp}(v_{\hbar}) + 2q_{\hbar}$, which proves part 2. And finally $r_{\hbar} = u_{\hbar} - \iota_{q_{\hbar}}(v_{\hbar}) + (\iota_{q_{\hbar}} \otimes \iota_{q_{\hbar}})(\omega_{\hbar}) = u_{\hbar} - \frac{1}{2}\iota_{q_{\hbar}}^2(\omega_{\hbar})$. \square

We now prove

Proposition 3.8. $(\mathcal{L}d + \bar{\partial})(\tilde{\omega}_{\hbar}) = 0$ if and only if $\tilde{\pi}_{\hbar}$ is a Maurer-Cartan element.

Proof of the proposition. First of all, by applying J_{\hbar} to (a) one sees that it is equivalent to $\mathcal{L}d(\omega_{\hbar}) = 0$.

Then $J_{\hbar}(\bar{\partial}(\pi_{\hbar}) + [\pi_{\hbar}, q_{\hbar}]) = -\bar{\partial}(\omega_{\hbar}) + \mathcal{L}d(J_{\hbar}(q_{\hbar})) = -\bar{\partial}(\omega_{\hbar}) - \mathcal{L}d(v_{\hbar})$. Therefore (b) is equivalent to $\bar{\partial}(\omega_{\hbar}) + \mathcal{L}d(v_{\hbar}) = 0$.

Now assume that (a) and (b) are satisfied. The following lemma tells us that, under this assumption, (c) is equivalent to $\bar{\partial}(v_{\hbar}) + \mathcal{L}d(u_{\hbar}) = 0$.

Lemma 3.9. *The l.h.s. of (c) is equal to the (0, 2)-part of $\tilde{J}_{\hbar}^{-1}(\mathcal{L}d + \bar{\partial})(\tilde{\omega}_{\hbar})$.*

Proof of the lemma. The (0, 2)-part of $\tilde{J}_{\hbar}^{-1}(\mathcal{L}d + \bar{\partial})(\tilde{\omega}_{\hbar})$ is

$$(6) \quad \begin{aligned} & \pi_{\hbar}^{\sharp}(\bar{\partial}v_{\hbar} + \mathcal{L}du_{\hbar}) - (\pi_{\hbar}^{\sharp} \otimes \iota_{q_{\hbar}} + \iota_{q_{\hbar}} \otimes \pi_{\hbar}^{\sharp})(\bar{\partial}\omega_{\hbar} + \mathcal{L}dv_{\hbar}) \\ & + (\pi_{\hbar}^{\sharp} \otimes \iota_{q_{\hbar}} \otimes \iota_{q_{\hbar}} + \iota_{q_{\hbar}} \otimes \pi_{\hbar}^{\sharp} \otimes \iota_{q_{\hbar}} + \iota_{q_{\hbar}} \otimes \iota_{q_{\hbar}} \otimes \pi_{\hbar}^{\sharp})(\mathcal{L}d\omega_{\hbar}) \\ & = \pi_{\hbar}^{\sharp}(\bar{\partial}v_{\hbar} + \mathcal{L}du_{\hbar}) - \pi_{\hbar}^{\sharp}\iota_{q_{\hbar}}(\bar{\partial}\omega_{\hbar} + \mathcal{L}dv_{\hbar}) + \frac{1}{2}\pi_{\hbar}^{\sharp}\iota_{q_{\hbar}}^2(\mathcal{L}d\omega_{\hbar}). \end{aligned}$$

We now compute each term separately. First of all

$$\pi_{\hbar}^{\sharp}(\bar{\partial}v_{\hbar} + \mathcal{L}du_{\hbar}) = [\pi_{\hbar}, u_{\hbar}] + \pi_{\hbar}^{\sharp}(\bar{\partial}v_{\hbar}).$$

Then

$$\begin{aligned} \pi_{\hbar}^{\sharp}\iota_{q_{\hbar}}(\bar{\partial}\omega_{\hbar} + \mathcal{L}dv_{\hbar}) &= \pi_{\hbar}^{\sharp}(\mathcal{L}d\iota_{q_{\hbar}}v_{\hbar} - \mathcal{L}L_{q_{\hbar}}v_{\hbar} + \bar{\partial}\iota_{q_{\hbar}}\omega_{\hbar} + \iota_{\bar{\partial}q_{\hbar}}\omega_{\hbar}) \\ &= [\pi_{\hbar}, \iota_{q_{\hbar}}^2\omega_{\hbar}] - \pi_{\hbar}^{\sharp}(\mathcal{L}L_{q_{\hbar}}v_{\hbar}) + \pi_{\hbar}^{\sharp}(\bar{\partial}v_{\hbar}) - \bar{\partial}(q_{\hbar}). \end{aligned}$$

Finally

$$\begin{aligned} \pi_{\hbar}^{\sharp}\iota_{q_{\hbar}}^2(\mathcal{L}d\omega_{\hbar}) &= \pi_{\hbar}^{\sharp}\iota_{q_{\hbar}}(\mathcal{L}d\iota_{q_{\hbar}} - \mathcal{L}L_{q_{\hbar}})(\omega_{\hbar}) = \pi_{\hbar}^{\sharp}(\mathcal{L}d\iota_{q_{\hbar}}^2 - 2\mathcal{L}L_{q_{\hbar}}\iota_{q_{\hbar}} - \iota_{[q_{\hbar}, q_{\hbar}]})(\omega_{\hbar}) \\ &= [\pi_{\hbar}, \iota_{q_{\hbar}}^2\omega_{\hbar}] - 2\pi_{\hbar}^{\sharp}(\mathcal{L}L_{q_{\hbar}}v_{\hbar}) + [q_{\hbar}, q_{\hbar}]. \end{aligned}$$

Therefore the r.h.s. of (6) gives

$$\bar{\partial}(q_{\hbar}) + [\pi_{\hbar}, u_{\hbar} - \frac{1}{2}\iota_{q_{\hbar}}^2\omega_{\hbar}] + \frac{1}{2}[q_{\hbar}, q_{\hbar}] = \bar{\partial}(q_{\hbar}) + [\pi_{\hbar}, r_{\hbar}] + \frac{1}{2}[q_{\hbar}, q_{\hbar}],$$

that is precisely the l.h.s. of (c). The lemma is proved. \square

We assume finally that (a) (b) and (c) are satisfied. The next lemma implies that, under this assumption, (d) is equivalent to $\bar{\partial}(u_{\hbar}) = 0$.

Lemma 3.10. *The l.h.s. of (d) is equal to the (-1, 3)-part of $\tilde{J}_{\hbar}^{-1}(\mathcal{L}d + \bar{\partial})(\omega_{\hbar})$.*

Proof of the lemma. The (-1, 3)-part of $\tilde{J}_{\hbar}^{-1}(\mathcal{L}d + \bar{\partial})(\omega_{\hbar})$ is

$$\begin{aligned} & -\iota_{q_{\hbar}}^{\otimes 3}(\mathcal{L}d\omega_{\hbar}) + \iota_{q_{\hbar}}^{\otimes 2}(\bar{\partial}\omega_{\hbar} + \mathcal{L}dv_{\hbar}) - \iota_{q_{\hbar}}(\bar{\partial}v_{\hbar} + \mathcal{L}du_{\hbar}) + \bar{\partial}u_{\hbar} \\ & = -\frac{1}{6}\iota_{q_{\hbar}}^3(\mathcal{L}d\omega_{\hbar}) + \frac{1}{2}\iota_{q_{\hbar}}^2(\bar{\partial}\omega_{\hbar} + \mathcal{L}dv_{\hbar}) - \iota_{q_{\hbar}}(\bar{\partial}v_{\hbar} + \mathcal{L}du_{\hbar}) + \bar{\partial}u_{\hbar}. \end{aligned}$$

As for the previous lemma one computes each term separately. For the reader's convenience we give the results without computations:

- $\iota_{q_{\hbar}}^3(\mathcal{L}d\omega_{\hbar}) = -3(\mathcal{L}L_{q_{\hbar}}\iota_{q_{\hbar}} + \iota_{[q_{\hbar}, q_{\hbar}]})\iota_{q_{\hbar}}\omega_{\hbar}$,
- $\iota_{q_{\hbar}}^2(\bar{\partial}\omega_{\hbar} + \mathcal{L}dv_{\hbar}) = (\bar{\partial}\iota_{q_{\hbar}} + 2\iota_{\bar{\partial}q_{\hbar}} - 2\mathcal{L}L_{q_{\hbar}}\iota_{q_{\hbar}} - \iota_{[q_{\hbar}, q_{\hbar}]})\iota_{q_{\hbar}}\omega_{\hbar}$,
- $\iota_{q_{\hbar}}(\bar{\partial}v_{\hbar} + \mathcal{L}du_{\hbar}) = (\bar{\partial}\iota_{q_{\hbar}} + \iota_{\bar{\partial}q_{\hbar}})\iota_{q_{\hbar}}\omega_{\hbar} - \mathcal{L}L_{q_{\hbar}}u_{\hbar}$.

Therefore the $(-1, 3)$ -part of $\tilde{J}_\hbar^{-1}(\mathcal{L}d + \bar{\partial})(\omega_\hbar)$ is

$$\bar{\partial}(u_\hbar - \frac{1}{2}\iota_{q_\hbar}^2 \omega_\hbar) + \mathcal{L}L_{q_\hbar}(u_\hbar - \frac{1}{2}\iota_{q_\hbar}^2 \omega_\hbar) = \bar{\partial}(r_\hbar) + [q_\hbar, r_\hbar],$$

i.e. the l.h.s. of (d). The lemma is proved. \square

This ends the proof of the proposition. \square

Therefore we obtain a map from the set of Maurer-Cartan elements $\tilde{\pi}_\hbar$ in $\hbar\mathcal{G}[[\hbar]]$ such that $\tilde{\pi}_\hbar^{1,0} = \hbar\pi + o(\hbar)$ to the set of 2-cocycles $\tilde{\omega}_\hbar = \frac{1}{\hbar}\omega + O(1)$ in $(\mathcal{L}\mathcal{A}^{*,*}, \mathcal{L}d + \bar{\partial})$ with $(0, 2)$ -part being zero mod \hbar . This map is obviously bijective. Let us now assume that we have another Maurer-Cartan element $\tilde{\pi}'_\hbar$ in $\hbar\mathcal{G}[[\hbar]]$ such that $(\tilde{\pi}'_\hbar)^{1,0} = \hbar\pi + o(\hbar)$, with image under the above map denoted $\tilde{\omega}'_\hbar$. It remains to prove the following:

Proposition 3.11. *$\tilde{\pi}_\hbar$ and $\tilde{\pi}'_\hbar$ are gauge equivalent if and only if $\tilde{\omega}'_\hbar = \tilde{\omega}_\hbar + (\mathcal{L}d + \bar{\partial})(\tilde{\theta}_\hbar)$ with $\tilde{\theta}_\hbar$ a 1-cochain who's $(0, 1)$ -part is zero mod \hbar .*

Proof of the proposition. The gauge equivalence between $\tilde{\pi}_\hbar$ and $\tilde{\pi}'_\hbar$ can be reformulated as follows: $\tilde{\pi}'_\hbar = \tilde{\pi}_{t|t=1}$ where $\tilde{\pi}_t$ is the solution of the differential equation

$$(7) \quad \frac{d\tilde{\pi}_t}{dt} - (\bar{\partial}\tilde{\alpha}_t + [\tilde{\pi}_t, \tilde{\alpha}_t]) = 0$$

with initial condition $\tilde{\pi}_{t|t=0} = \tilde{\pi}_\hbar$ and $\tilde{\alpha}_t \in \exp(\hbar\mathcal{G}^0[[\hbar]])$. As above we write $\pi_t = \tilde{\pi}_t^{1,0}$, $q_t = \tilde{\pi}_t^{0,1}$ and $r_t = \tilde{\pi}_t^{-1,2}$, and we define \tilde{J}_t in the same way as \tilde{J}_\hbar . We also write $\tilde{\omega}_t = \tilde{J}_t(\tilde{\pi}_t)$, $\omega_t = \tilde{\omega}_t^{2,0}$, $v_t = \tilde{\omega}_t^{1,1}$ and $u_t = \tilde{\omega}_t^{0,2}$. We finally write $\alpha_t = \tilde{\alpha}_t^{0,0}$ and $\beta_t = \tilde{\alpha}_t^{-1,1}$, and define $\tilde{\theta}_t = \tilde{J}_t(\tilde{\alpha}_t) = \theta_t + \gamma_t$ ($\theta_t = \tilde{\theta}_t^{1,0}$ and $\gamma_t = \tilde{\theta}_t^{0,1}$). So we have $\alpha_t = \pi_t^\sharp(\theta_t)$ and $\beta_t = \gamma_t - \iota_{q_t}(\theta_t)$. It suffices to prove that the differential equation (7) is equivalent to

$$(8) \quad \frac{d\tilde{\omega}_t}{dt} - (\mathcal{L}d + \bar{\partial})(\tilde{\theta}_t) = 0.$$

First of all the $(1, 0)$ -part of (7) is equivalent to the $(2, 0)$ -part of (8), i.e. one has

$$\dot{\pi}_t - [\pi_t, \alpha_t] = 0 \iff \dot{\omega}_t - \mathcal{L}d(\theta_t) = 0,$$

which directly follows from the fact that $J_t(\dot{\pi}_t) = -\dot{\omega}_t + \mathcal{L}d(\theta_t)$.

Let us now assume that the $\dot{\omega}_t - \mathcal{L}d(\theta_t) = 0$.

Lemma 3.12. *Under the previous assumption the $(0, 1)$ -part of (7) is equivalent to the $(1, 1)$ -part of (8).*

Proof of the lemma. Since we assumed that $\dot{\omega}_t - \mathcal{L}d(\theta_t) = 0$ then it suffices to prove that the $(0, 1)$ -part of $\tilde{J}_t^{-1}(\tilde{\omega}_t - (\mathcal{L}d + \bar{\partial})(\tilde{\theta}_t))$ is equal to the opposite of the $(0, 1)$ -part of the l.h.s. of (7). The $(0, 1)$ -part of $\tilde{J}_t^{-1}(\tilde{\omega}_t - (\mathcal{L}d + \bar{\partial})(\tilde{\theta}_t))$ is

$$\begin{aligned} & \pi_t^\sharp(\dot{v}_t - \bar{\partial}\theta_t - \mathcal{L}d\gamma_t) - (\pi_t^\sharp \otimes \iota_{q_t} + \iota_{q_t} \otimes \pi_t^\sharp)(\dot{\omega}_t - \mathcal{L}d\theta_t) \\ &= \pi_t^\sharp(\dot{v}_t - \bar{\partial}\theta_t - \mathcal{L}d\gamma_t) - \pi_t^\sharp \iota_{q_t}(\dot{\omega}_t - \mathcal{L}d\theta_t). \end{aligned}$$

Let us compute the first term of this sum:

$$\begin{aligned} \pi_t^\sharp(\dot{v}_t - \bar{\partial}\theta_t - \mathcal{L}d\gamma_t) &= \overbrace{\pi_t^\sharp v_t} - \dot{\pi}_t^\sharp(v_t) + \bar{\partial}(\pi_t^\sharp \theta_t) - (\bar{\partial}\pi_t^\sharp)(\theta_t) + [\pi_t, \gamma_t] \\ &= -\dot{q}_t - \dot{\pi}_t^\sharp(v_t) + \bar{\partial}(\alpha_t) + ([q_t, \pi_t]^\sharp)(\theta_t) + [\pi_t, \gamma_t]. \end{aligned}$$

Then the second term is

$$\begin{aligned} \pi_t^\# \iota_{q_t} (\dot{\omega}_t - \mathcal{L}d\theta_t) &= \overbrace{\pi_t^\# \iota_{q_t} \omega_t - \pi_t^\# \iota_{\dot{q}_t} \omega_t - \pi_t^\# \iota_{q_t} \omega_t - \pi_t^\# (\mathcal{L}d\iota_{q_t} \theta_t - \mathcal{L}L_{q_t} \theta_t)} \\ &= -\dot{q}_t + \dot{q}_t - \pi_t^\# v_t + [\pi_t, \iota_{q_t} \theta_t] + \pi_t^\# \mathcal{L}L_{q_t} \theta_t \\ &= -\pi_t^\# v_t + [\pi_t, \iota_{q_t} \theta_t] - [q_t, \alpha_t] + ([q_t, \pi_t]^\#) \theta_t. \end{aligned}$$

Therefore the $(0, 1)$ -part of $\tilde{J}_t^{-1}(\dot{\omega}_t - (\mathcal{L}d + \bar{\partial})(\tilde{\theta}_t))$ is

$$-\dot{q}_t + \bar{\partial}(\alpha_t) + [\pi_t, \gamma_t - \iota_{q_t} \theta_t] + [q_t, \alpha_t] = -\dot{q}_t + \bar{\partial}(\alpha_t) + [\pi_t, \beta_t] + [q_t, \alpha_t],$$

which is minus the $(0, 1)$ -part of the l.h.s. of (7). \square

To end the proof, assuming (7) is true for degrees $(1, 0)$ and $(0, 1)$ (and (8) is true for degrees $(2, 0)$ and $(1, 1)$), we have to prove the following lemma:

Lemma 3.13. *Under the preceding assumptions the $(-1, 2)$ -part of (7) is equivalent to the $(0, 2)$ -part of (8).*

Proof of the lemma. Thanks to the assumptions we made, it is sufficient to prove that the $(-1, 2)$ -part of $\tilde{J}_t^{-1}(\dot{\omega}_t - (\mathcal{L}d + \bar{\partial})(\tilde{\theta}_t))$ is equal to the $(-1, 2)$ -part of the l.h.s. of (7). The $(-1, 2)$ -part of $\tilde{J}_t^{-1}(\dot{\omega}_t - (\mathcal{L}d + \bar{\partial})(\tilde{\theta}_t))$ is

$$\begin{aligned} &\iota_{q_t}^{\otimes 2} (\dot{\omega}_t - \mathcal{L}d\theta_t) - \iota_{q_t} (\dot{v}_t - \bar{\partial}\theta_t - \mathcal{L}d\gamma_t) + \dot{u}_t - \bar{\partial}\gamma_t \\ &= \frac{1}{2} \iota_{q_t}^2 (\dot{\omega}_t - \mathcal{L}d\theta_t) - \iota_{q_t} (\dot{v}_t - \bar{\partial}\theta_t - \mathcal{L}d\gamma_t) + \dot{u}_t - \bar{\partial}\gamma_t. \end{aligned}$$

\square

Let us compute the first term of this sum:

$$\begin{aligned} \frac{1}{2} \iota_{q_t}^2 (\dot{\omega}_t - \mathcal{L}d\theta_t) &= \frac{1}{2} \overbrace{\iota_{q_t}^2 \omega_t - \iota_{\dot{q}_t} \iota_{q_t} \omega_t} - \frac{1}{2} \iota_{q_t} (\mathcal{L}d\iota_{q_t} - \mathcal{L}L_{q_t}) \theta_t \\ &= \frac{1}{2} \overbrace{\iota_{q_t} v_t - \iota_{\dot{q}_t} v_t} + (\mathcal{L}L_{q_t} \iota_{q_t} + \frac{1}{2} \iota_{[q_t, q_t]}) \theta_t. \end{aligned}$$

The second term is

$$\iota_{q_t} (\dot{v}_t - \bar{\partial}\theta_t - \mathcal{L}d\gamma_t) = \overbrace{\iota_{q_t} v_t} - \iota_{\dot{q}_t} v_t - \bar{\partial}(\iota_{q_t} \theta_t) - \iota_{\bar{\partial}q_t} \theta_t + \mathcal{L}L_{q_t} \gamma_t.$$

Therefore the $(-1, 2)$ -part of $\tilde{J}_t^{-1}(\dot{\omega}_t - (\mathcal{L}d + \bar{\partial})(\tilde{\theta}_t))$ is equal to:

$$\begin{aligned} &-\frac{1}{2} \overbrace{\iota_{q_t} v_t} + (\mathcal{L}L_{q_t} \iota_{q_t} + \frac{1}{2} \iota_{[q_t, q_t]}) \theta_t + \bar{\partial}(\iota_{q_t} \theta_t) + \iota_{\bar{\partial}q_t} \theta_t - \mathcal{L}L_{q_t} \gamma_t + \dot{u}_t - \bar{\partial}\gamma_t \\ &= \dot{r}_t - \mathcal{L}L_{q_t} \beta_t - \bar{\partial}\beta_t + \iota_{\bar{\partial}q_t + \frac{1}{2}[q_t, q_t]} \theta_t = \dot{r}_t - [q_t, \beta_t] - \bar{\partial}\beta_t - \iota_{[\pi_t, r_t]} \theta_t \\ &= \dot{r}_t - [q_t, \beta_t] - \bar{\partial}\beta_t - [\pi_t^\#(\theta_t), r_t] = \dot{r}_t - [q_t, \beta_t] - \bar{\partial}\beta_t - [\alpha_t, r_t], \end{aligned}$$

which is the $(-1, 2)$ -part of the l.h.s. of (7). \square

The theorem is proved. \square

In the case of a complex symplectic manifold (like in paragraph 3.1.2) the differential $\mathcal{L}d$ is the usual holomorphic differential ∂ . Therefore (using the $\bar{\partial}$ -Poincare lemma) one obtains the following

Corollary 3.14. *There is a one-to-one correspondence*

$$\frac{\{w.P.s. \Pi_{\hbar} \text{ s.t. } [\Pi_{\hbar}] = \hbar[\Pi] + o(\hbar)\}}{\text{weak equivalences}} \longleftrightarrow \frac{1}{\hbar}\omega + F^1\check{H}^2(X, \mathbb{C}) + \hbar\check{H}^2(X, \mathbb{C})[[\hbar]].$$

This classification result is similar to the one in [18, 19].

3.4. Existence and classification of actual quantizations. Recall that a weak deformation is an actual one if and only if the a_{ijk} 's (see Example 2.1) are exactly 1. Let us denote by $\mathcal{O}^{\mathcal{L}} \subset \mathcal{O}$ the sheaf of subalgebras of \mathcal{L} -invariants.

3.4.1. *A sufficient condition for the existence.*

Proposition 3.15. *Assume that the map $\check{H}^2(X, \mathcal{O}^{\mathcal{L}}) \rightarrow \check{H}^2(X, \mathcal{O})$ (which is given by $\mathcal{O}^{\mathcal{L}} \hookrightarrow \mathcal{O}$) is surjective. Then any Poisson structure admits an actual quantization.*

Proof. We proceed by induction: we prove that for any $n \geq 1$ one can build a weak quantization with $a = 1 + O(\hbar^n)$. It is obvious for $n = 1$, and for $n = 2$ it follows from the fact that the starting Poisson structure is not a weak one. Assume now that we have the result for $n \geq 2$ and let us write $a = 1 + \hbar^n a_n + O(\hbar^{n+1})$. Taking the coefficient of \hbar^n in equation (4) we get that $\check{d}(a_n) = 0$. By assumption $a_n = \tilde{a}_n + \check{d}(\alpha_n)$, where \tilde{a}_n is a Čech 2-cocycle with values in $\mathcal{O}^{\mathcal{L}}$. Using the gauge transformation $\exp(\hbar^n \alpha_n)$ we get that the quantization is weakly equivalent to a one with $a = 1 + \hbar^n \tilde{a}_n + O(\hbar^{n+1})$. It is an immediate check that replacing \tilde{a}_n with 0, equations of Example 2.1 are still satisfied. So we get the result for $n + 1$. \square

Remark 3.16. *Observe that the operation of replacing an invariant element with 0 is not a weak equivalence. Therefore we have not proved that, under the hypothesis of the proposition, any weak quantization is weakly equivalent to an actual one.*

Let us come back to the example of §3.1.2 and assume that we are given a holomorphic symplectic 2-form. In this case, as $\mathcal{O}^{\mathcal{L}} = \mathbb{C}$, our condition for the existence of an actual (i.e., in this case, holomorphic) quantization is the same as in [17]: surjectivity of $\check{H}^2(X, \mathbb{C}) \rightarrow \check{H}^2(X, \mathcal{O}_X)$. Indeed, Nest and Tsygan prove that this condition is sufficient for the existence of a Fedosov connection $\nabla = \bar{\partial} + \nabla_0 + \text{ad}A + \text{ad}B$, where A and B are respectively $(0, 1)$ - and $(1, 0)$ -forms with values in the Weyl algebra bundle of X (cf. [17], Theorem 5.9: in that theorem only surjectivity of $\check{H}^2(X, \mathbb{C}) \rightarrow \check{H}^2(X, \mathcal{O}_X)$ is used to prove the existence of the connection). Then they prove (Theorem 5.6 and following remarks) that taking flat sections of a Fedosov connection one gets the desired quantization.

3.4.2. *A partial classification result.*

Proposition 3.17. *Assume that $\check{H}^2(X, \mathcal{O}) = 0$ and that the map $\check{H}^1(X, \mathcal{O}^{\mathcal{L}}) \rightarrow \check{H}^1(X, \mathcal{O})$ (which is given by $\mathcal{O}^{\mathcal{L}} \hookrightarrow \mathcal{O}$) is surjective. Then any Poisson structure admits an actual quantization and we have a one-to-one correspondence between equivalence classes of weak quantizations and equivalence classes of actual quantizations*

Remark 3.18. *We saw that the condition $\check{H}^2(X, \mathcal{O}) = 0$ implies that any weak deformation is weakly equivalent to an actual one. In the same way one can prove that any formal weak Poisson structure is weakly equivalent to an actual one.*

Proof. Thanks to the previous remark, we only need to prove that a weak equivalence can be replaced by a strong one. Following the same proof as in Proposition 3.15 and the fact that $\check{H}^1(M, \mathbb{C}) \rightarrow \check{H}^1(M, \mathcal{O}_M)$ is surjective, one can replace the functions involved in the definition of the weak equivalence isomorphisms with constants ones and so 0 functions. \square

Let us again study the example of paragraph 3.1.2 ($\mathcal{O}^{\mathcal{L}} = \mathbb{C}$) and assume that we are given a holomorphic symplectic 2-form ω . In this case, existence and classification of actual (i.e. holomorphic) quantizations is the same as in [17]: thanks to the corollary of Proposition 3.6 and the assumption $\check{H}^2(X, \mathcal{O}_X) = 0$, equivalence classes of holomorphic quantizations are in one-to-one correspondence with $\frac{1}{\hbar}\omega + \check{H}^2(X, \mathbb{C})[[\hbar]]$ (or $\frac{1}{\hbar}\omega + H^2(F^1\mathcal{A}^{*,*}(X))[[\hbar]]$). Note that in Nest and Tsygan's construction one really needs the condition $\check{H}^2(X, \mathcal{O}_X) = 0$ otherwise the set of preimages of the surjective map $\check{H}^2(X, \mathbb{C}) \rightarrow \check{H}^2(X, \mathcal{O}_X)$ would be an affine space (not a vectorial space). So choices of preimages of this map in the construction of a Fedosov cannot be done canonically by choosing element in the space $H^2(F^1\Omega^{*,*}(X))$.

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