Bi-arrangements of hyperplanes and Orlik-Solomon bi-complexes

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1. Arrangements of hyperplanes and the Orlik-Solomon algebra

2. Periods of bi-arrangements of hyperplanes

3. Orlik-Solomon bi-complexes

The framework

Arnold's question '69

Let $\mathscr{A} = \{K_1, \ldots, K_k\}$ be a set of hyperplanes in $\mathbb{C}^n$, passing through the origin. Compute $H^\bullet(\mathbb{C}^n \setminus \mathscr{A})$ (cohomology algebra with rational coefficients).

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Key example

$\mathscr{A}$ the braid arrangement, made of the hyperplanes $\{z_i = z_j\}$, $1 \leqslant i < j \leqslant n$.
$H^\bullet(\mathbb{C}^n \setminus \mathscr{A}) \cong H^\bullet(PB_n)$ the cohomology of the pure braid group on n strands.

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$$K_i = \{f_i = 0\} \quad \rightsquigarrow \quad f_i : \mathbb{C}^n \setminus \mathscr{A} \to \mathbb{C} \setminus 0 \quad \rightsquigarrow \quad f_i^* : H^1(\mathbb{C} \setminus 0) \to H^1(\mathbb{C}^n \setminus \mathscr{A}).$$

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- Let e be a generator of $H^1(\mathbb{C} \setminus 0)$, we have elements $e_i = f_i^*(e) \in H^1(\mathbb{C}^n \setminus \mathscr{A})$.
- Arnol'd conjectures that the classes e_i generate the cohomology algebra $H^\bullet(\mathbb{C}^n \setminus \mathscr{A})$.

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Definition

The *Orlik-Solomon algebra* of $\mathscr{A}$:

$$A_\bullet(\mathscr{A}) = \Lambda^\bullet(e_1,\ldots,e_k)\,/\,(d(e_{i_1}\wedge\cdots\wedge e_{i_r}), \{K_{i_1},\ldots,K_{i_r}\} \text{ dependent})$$

where $d(e_{i_1}\wedge\cdots\wedge e_{i_r}) = \sum_s (-1)^{s-1} e_{i_1}\wedge\cdots\wedge\widehat{e_{i_s}}\wedge\cdots\wedge e_{i_r}$.

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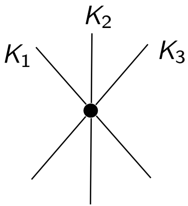
$A_\bullet(\mathscr{A}) = \Lambda^\bullet(e_1,e_2,e_3)\,/\,(e_1\wedge e_2 - e_1\wedge e_3 + e_2\wedge e_3)$.



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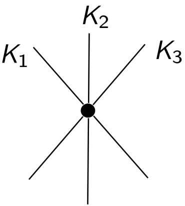

Theorem (Brieskorn '73, Orlik-Solomon '80)

We have an isomorphism of graded algebras $H^\bullet(\mathbb{C}^n\setminus\mathscr{A})\cong A_\bullet(\mathscr{A})$.

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$\mathscr{S}_r(\mathscr{A})$ the set of strata of $\mathscr{A}$ of codimension r, we have a decomposition

$$A_r(\mathscr{A}) = \bigoplus_{S \in \mathscr{S}_r(\mathscr{A})} A_r^S(\mathscr{A})$$

($A_r^S(\mathscr{A})$ spanned by the monomials $e_{i_1} \wedge \cdots \wedge e_{i_r}$ such that $K_{i_1} \cap \cdots \cap K_{i_r} = S$).

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- $A_0(\mathscr{A}) = \mathbb{Q}$;

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- for $\Sigma\in\mathscr{S}_r(\mathscr{A})$, $A_r^\Sigma(\mathscr{A})$ is defined as a kernel

$$0 \to A_r^\Sigma(\mathscr{A}) \xrightarrow{d} \bigoplus_{S\supset\Sigma} A_{r-1}^S(\mathscr{A}) \xrightarrow{d} \bigoplus_{T\supset\Sigma} A_{r-2}^T(\mathscr{A}) \ .$$

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More interesting periods

(Multiple) zeta values : $\zeta(n) = \sum_{k \geqslant 1} \frac{1}{k^n}$, $\zeta(n_1, \ldots, n_r) = \sum_{1 \leqslant k_1 < \cdots < k_r} \frac{1}{k_1^{n_1} \cdots k_r^{n_r}}$.

The example of $\zeta(2)$

(Goncharov-Manin '04)

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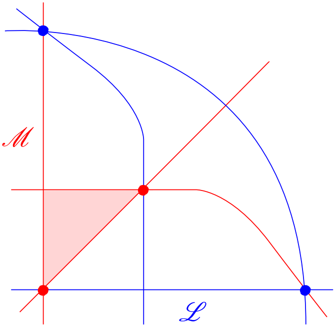

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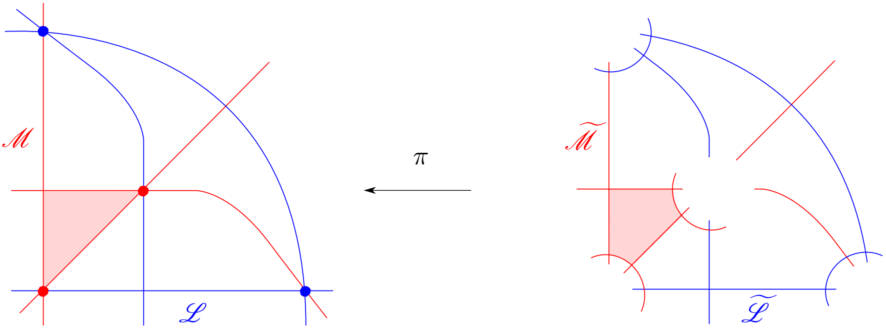

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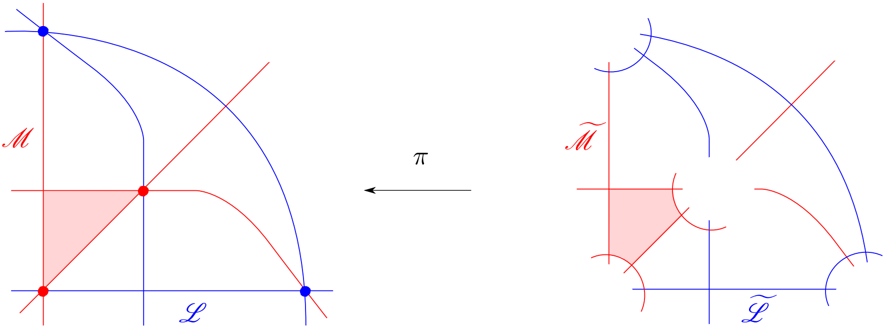

We are interested in the cohomology group $H = H^2(\widetilde{\mathbb{P}^2} \setminus \widetilde{\mathscr{L}}, \widetilde{\mathscr{M}} \setminus \widetilde{\mathscr{M}} \cap \widetilde{\mathscr{L}})$.

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- $\chi : \mathscr{S}(\mathscr{L} \cup \mathscr{M}) \to \{\lambda, \mu\}$ is a coloring function, satisfying

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Example

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- The involution $\lambda \leftrightarrow \mu$ corresponds to Poincaré duality.

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We define the *Orlik-Solomon bi-complex* $A_{\bullet,\bullet} = A_{\bullet,\bullet}(\mathscr{L},\mathscr{M},\chi)$:

$$
\begin{tikzcd}
{} \arrow[r, dotted] & A_{2,0} \arrow[r] \arrow[d, dotted] & A_{1,0} \arrow[r, "d'"] \arrow[d] & A_{0,0} \arrow[d, "d''"] \\
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We define $A_{i,j} = \displaystyle\bigoplus_{S \in \mathscr{S}_{i+j}} A^S_{i,j}$, by induction on the codimension $i + j$.

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– For a stratum Σ such that $\chi(\Sigma) = \lambda$, one defines $A^{\Sigma}_{i,j}$ as a *kernel* :

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– For a stratum Σ such that $\chi(\Sigma) = \mu$, one defines $A^{\Sigma}_{i,j}$ as a *cokernel* :

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A bi-arrangement of hyperplanes $(\mathscr{L}, \mathscr{M}, \chi)$ is *exact* if the above exact sequences can be continued to long exact sequences

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Remark

– All arrangements of hyperplanes $(\mathscr{A}, \varnothing, \lambda)$ are exact, $A_{\bullet,0}(\mathscr{A}, \varnothing, \lambda) = A_{\bullet}(\mathscr{A})$.

Exactness

Definition

A bi-arrangement of hyperplanes $(\mathscr{L},\mathscr{M},\chi)$ is *exact* if the above exact sequences can be continued to long exact sequences

$$0 \to A_{i,j}^{\Sigma} \xrightarrow{d'} \bigoplus_{S\supset\Sigma} A_{i-1,j}^{S} \xrightarrow{d'} \bigoplus_{T\supset\Sigma} A_{i-2,j}^{T} \xrightarrow{d'} \cdots$$

or

$$0 \leftarrow A_{i,j}^{\Sigma} \xleftarrow{d''} \bigoplus_{S\supset\Sigma} A_{i,j-1}^{S} \xleftarrow{d''} \bigoplus_{T\supset\Sigma} A_{i,j-2}^{T} \xleftarrow{d''} \cdots$$

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- All arrangements of hyperplanes $(\mathscr{A},\varnothing,\lambda)$ are exact, $A_{\bullet,0}(\mathscr{A},\varnothing,\lambda)=A_{\bullet}(\mathscr{A})$.
- Deletion and restriction formalism for exact bi-arrangements of hyperplanes.

The main theorem

Theorem (D. '14)

For an exact bi-arrangement of hyperplanes $(\mathscr{L}, \mathscr{M}, \chi)$ *in* $\mathbb{P}^n$*, « the Orlik-Solomon bi-complex* $A_{\bullet,\bullet}(\mathscr{L}, \mathscr{M}, \chi)$ *computes the motive* $H^\bullet(\mathscr{L}, \mathscr{M}, \chi)$ *».*

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More precisely, for each $k = 0, \ldots, n$ *:*

- *we consider the double complex* $A_{0 \leqslant \bullet \leqslant k, 0 \leqslant \bullet \leqslant n-k}$ *;*
- *we let* ${}^{(k)}A_\bullet$ *be its total complex ;*
- *then* $\mathrm{gr}^W_{2k} H^r(\mathscr{L}, \mathscr{M}, \chi) \cong H_{2k-r}({}^{(k)}A_\bullet)$
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Remark

- For arrangements of hyperplanes, we recover the (projective) Brieskorn-Orlik-Solomon theorem, with only weight $\mathrm{gr}_{2k} H^k$.
- The weight-graded quotients $\mathrm{gr}^W_{2k} H^\bullet(\mathscr{L}, \mathscr{M}, \chi)$ are combinatorial invariants, but not the whole motive $H^\bullet(\mathscr{L}, \mathscr{M}, \chi)$.

Explicit computations

Combinatorial notion of *tame* bi-arrangements of hyperplanes.

- Generic bi-arrangements are tame
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For a tame bi-arrangement of hyperplanes $(\mathscr{L}, \mathscr{M}, \chi)$*, the Orlik-Solomon bi-complex* $A_{\bullet,\bullet}(\mathscr{L}, \mathscr{M}, \chi)$ *is an explicit sub-quotient of* $A_\bullet(\mathscr{L}) \otimes A_\bullet(\mathscr{M})^\vee$*.*

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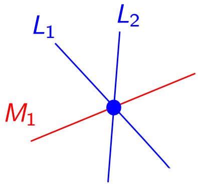
Example

$A_{\bullet,\bullet}$ is the quotient of $\Lambda^\bullet(e_1,e_2)\otimes\Lambda^\bullet(f_1^\vee)$ by the relation $d(e_1\wedge e_2)\otimes f_1^\vee=0$.



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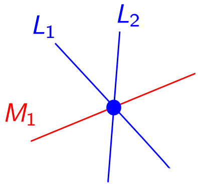

Example

One can define multiple zeta bi-arrangements $\mathscr{Z}(n_1, \ldots, n_r)$ that are tame.

The global context

Definition

A *bi-arrangement of hypersurfaces* in a complex manifold X is a triple $(\mathscr{L}, \mathscr{M}, \chi)$ where $\mathscr{L}$ and $\mathscr{M}$ are divisors in X and χ a coloring function, which locally looks like a bi-arrangement of hyperplanes.

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Theorem (D. '14)

In such a situation (for exact bi-arrangements) we have a spectral sequence

$$E_1^{-p,q} = \bigoplus_{\substack{i-j=p \\ S\in\mathscr{S}_{i+j}}} H^{q-2i}(S)(-i) \otimes A^S_{i,j}(\mathscr{L}, \mathscr{M}, \chi) \implies H^{-p+q}(\mathscr{L}, \mathscr{M}, \chi) \, .$$

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Example

Toric arrangements $\leftrightarrow$ integrals in cubical coordinates : $\zeta(2) = \displaystyle\iint_{[0,1]^2} \frac{ds\, dt}{1-st}$.

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