Bi-arrangements of hyperplanes and Orlik-Solomon bi-complexes

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Differential and combinatorial aspects of singularities
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1. Arrangements of hyperplanes and the Orlik-Solomon algebra

2. Periods of bi-arrangements of hyperplanes

3. Orlik-Solomon bi-complexes

The framework

Arnold's question '69

Let $\mathscr{A} = \{K_1, \ldots, K_k\}$ be a set of hyperplanes in $\mathbb{C}^n$, passing through the origin. Compute $H^\bullet(\mathbb{C}^n \setminus \mathscr{A})$ (cohomology algebra with rational coefficients).

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The generators e_i

$$K_i = \{f_i = 0\} \quad \rightsquigarrow \quad f_i : \mathbb{C}^n \setminus \mathscr{A} \to \mathbb{C} \setminus 0 \quad \rightsquigarrow \quad f_i^* : H^1(\mathbb{C} \setminus 0) \to H^1(\mathbb{C}^n \setminus \mathscr{A}).$$

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- Arnol'd conjectures that the classes e_i generate the cohomology algebra $H^\bullet(\mathbb{C}^n \setminus \mathscr{A})$.

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Definition

The *Orlik-Solomon algebra* of $\mathscr{A}$:

$$A_\bullet(\mathscr{A}) = \Lambda^\bullet(e_1,\ldots,e_k) \,/\, (d(e_{i_1}\wedge\cdots\wedge e_{i_r}), \{K_{i_1},\ldots,K_{i_r}\} \text{ dependent})$$

where $d(e_{i_1}\wedge\cdots\wedge e_{i_r}) = \sum_s (-1)^{s-1} e_{i_1}\wedge\cdots\wedge \widehat{e_{i_s}} \wedge\cdots\wedge e_{i_r}$.

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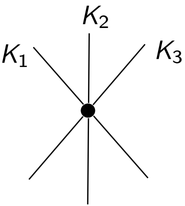
Example

$A_\bullet(\mathscr{A}) = \Lambda^\bullet(e_1, e_2, e_3) \,/\, (e_1 \wedge e_2 - e_1 \wedge e_3 + e_2 \wedge e_3)$.



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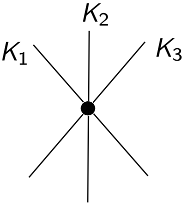

Theorem (Brieskorn '73, Orlik-Solomon '80)

We have an isomorphism of graded algebras $H^\bullet(\mathbb{C}^n \setminus \mathscr{A}) \cong A_\bullet(\mathscr{A})$.

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- $A_0(\mathscr{A}) = \mathbb{Q}$;
- for $\Sigma \in \mathscr{S}_r(\mathscr{A})$, $A_r^\Sigma(\mathscr{A})$ is defined as a kernel

$$0 \to A_r^\Sigma(\mathscr{A}) \xrightarrow{d} \bigoplus_{S \supset \Sigma} A_{r-1}^S(\mathscr{A}) \xrightarrow{d} \bigoplus_{T \supset \Sigma} A_{r-2}^T(\mathscr{A}) .$$

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Theorem (Looijenga '93)

In such a situation we have a spectral sequence

$$E_1^{-p,q} = \bigoplus_{S \in \mathscr{S}_p} H^{q-2p}(S)(-p) \otimes A_p^S(\mathscr{A}) \implies H^{-p+q}(X \setminus \mathscr{A}) \,.$$

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- D. '13 - application to rational homotopy theory.

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More interesting periods

(Multiple) zeta values : $\zeta(n) = \sum_{k \geqslant 1} \frac{1}{k^n}$, $\zeta(n_1, \ldots, n_r) = \sum_{1 \leqslant k_1 < \cdots < k_r} \frac{1}{k_1^{n_1} \cdots k_r^{n_r}}$.

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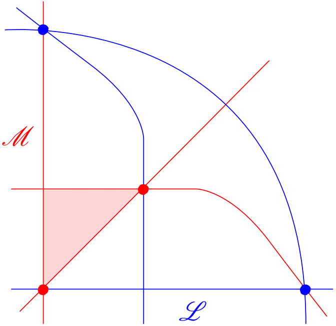

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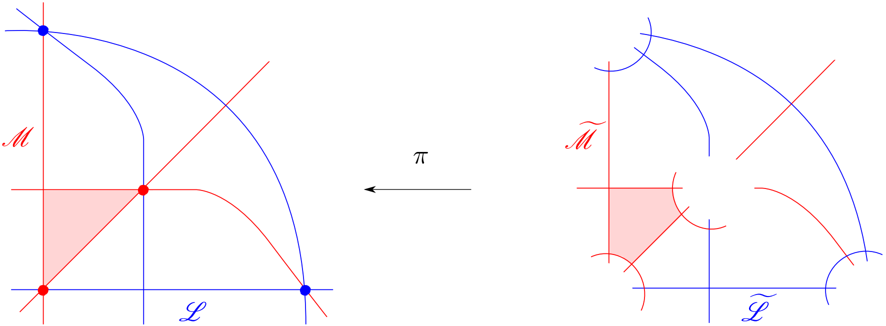

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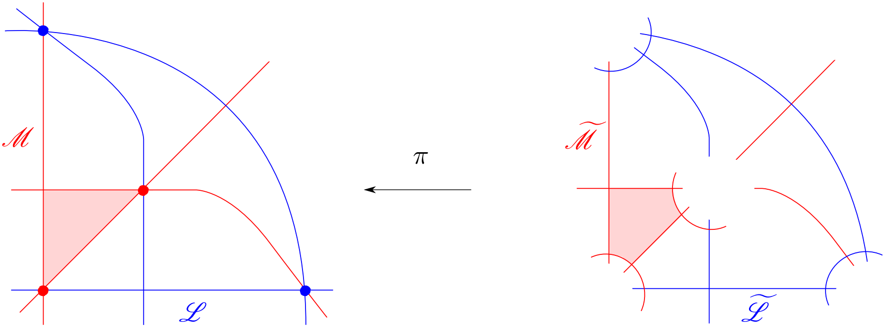

We are interested in the cohomology group $H = H^2(\widetilde{\mathbb{P}^2} \setminus \widetilde{\mathscr{L}}, \widetilde{\mathscr{M}} \setminus \widetilde{\mathscr{M}} \cap \widetilde{\mathscr{L}})$.

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- $\chi : \mathscr{S}(\mathscr{L} \cup \mathscr{M}) \to \{\lambda, \mu\}$ is a coloring function, satisfying

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Inspired by Aomoto '77 '82 and Beilinson-Goncharov-Schechtman-Varchenko '89.

Examples

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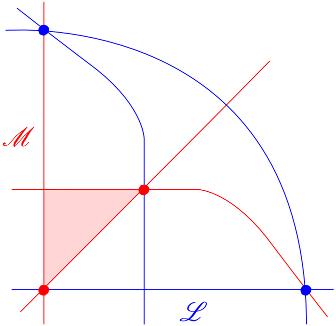

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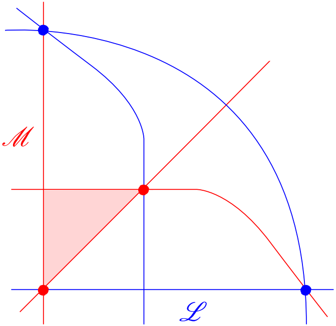

– The involution $\lambda \leftrightarrow \mu$ corresponds to Poincaré duality.

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We define $A_{i,j} = \displaystyle\bigoplus_{S \in \mathscr{S}_{i+j}} A^S_{i,j}$, by induction on the codimension $i+j$.

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– For a stratum Σ such that $\chi(\Sigma) = \lambda$, one defines $A^{\Sigma}_{i,j}$ as a *kernel* :

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Base step of the induction : $A_{0,0} = \mathbb{Q}$.

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Example

$\dim(A^{S}_{1,1})$ = the number of hyperplanes $K \supset S$ such that $\chi(K) \neq \chi(S)$.

Exactness

Definition

A bi-arrangement of hyperplanes $(\mathscr{L}, \mathscr{M}, \chi)$ is *exact* if the above exact sequences can be continued to long exact sequences

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– Deletion and restriction formalism for exact bi-arrangements of hyperplanes.

The main theorem

Theorem (D. '14)

For an exact bi-arrangement of hyperplanes $(\mathscr{L},\mathscr{M},\chi)$ *in* $\mathbb{P}^n$*, « the Orlik-Solomon bi-complex* $A_{\bullet,\bullet}(\mathscr{L},\mathscr{M},\chi)$ *computes the motive* $H^\bullet(\mathscr{L},\mathscr{M},\chi)$ *».*

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More precisely, for each $k=0,\ldots,n$ *:*

- *we consider the double complex* $A_{0\leqslant\bullet\leqslant k,0\leqslant\bullet\leqslant n-k}$ *;*
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- The weight-graded quotients $\mathrm{gr}^W_{2k}H^\bullet(\mathscr{L},\mathscr{M},\chi)$ are combinatorial invariants, but not the whole motive $H^\bullet(\mathscr{L},\mathscr{M},\chi)$.

Explicit computations

Combinatorial notion of *tame* bi-arrangements of hyperplanes.

– Generic bi-arrangements are tame

– tame $\Longrightarrow$ exact.

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For a tame bi-arrangement of hyperplanes $(\mathscr{L}, \mathscr{M}, \chi)$*, the Orlik-Solomon bi-complex* $A_{\bullet,\bullet}(\mathscr{L}, \mathscr{M}, \chi)$ *is an explicit sub-quotient of* $A_\bullet(\mathscr{L}) \otimes A_\bullet(\mathscr{M})^\vee$*.*

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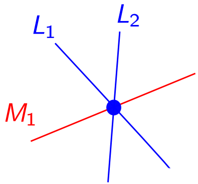
$A_{\bullet,\bullet}$ is the quotient of $\Lambda^\bullet(e_1, e_2) \otimes \Lambda^\bullet(f_1^\vee)$ by the relation $d(e_1 \wedge e_2) \otimes f_1^\vee = 0$.



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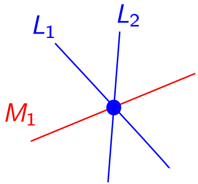

Example

One can define multiple zeta bi-arrangements $\mathscr{Z}(n_1, \ldots, n_r)$ that are tame.

The global context

Definition

A *bi-arrangement of hypersurfaces* in a complex manifold X is a triple $(\mathscr{L}, \mathscr{M}, \chi)$ where $\mathscr{L}$ and $\mathscr{M}$ are divisors in X and χ a coloring function, which locally looks like a bi-arrangement of hyperplanes.

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Theorem (D. '14)

In such a situation (for exact bi-arrangements) we have a spectral sequence

$$E_1^{-p,q} = \bigoplus_{\substack{i-j=p \\ S\in\mathscr{S}_{i+j}}} H^{q-2i}(S)(-i)\otimes A^S_{i,j}(\mathscr{L},\mathscr{M},\chi) \implies H^{-p+q}(\mathscr{L},\mathscr{M},\chi)\ .$$

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Idea of proof : by induction on the number of blow-ups in a resolution of singularities.

- True for normal crossing divisors (classical).
- For *one* blow-up, define a quasi-isomorphism $E_1 \xrightarrow{\sim} \widetilde{E}_1$ or $\widetilde{E}_1 \xrightarrow{\sim} E_1$.

Open problems

Geometric problems

(joint work with D. Juteau)

– Interpretation of the defect of exactness.

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- Orlik-Solomon bi-complexes as modules over Orlik-Solomon algebras.