Bi-arrangements of hyperplanes and Orlik-Solomon bi-complexes

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Differential and combinatorial aspects of singularities
Kaiserslautern, Aug. 7, 2015

The framework

Arnold's question '69

Let $\mathscr{A} = \{K_1, \ldots, K_k\}$ be a set of hyperplanes in $\mathbb{C}^n$, passing through the origin. Compute $H^\bullet(\mathbb{C}^n \setminus \mathscr{A})$ (cohomology algebra with rational coefficients).

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- Arnol'd conjectures that the classes e_i generate the cohomology algebra $H^\bullet(\mathbb{C}^n \setminus \mathscr{A})$.

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Definition

The *Orlik-Solomon algebra* of $\mathscr{A}$:

$$A_\bullet(\mathscr{A}) = \Lambda^\bullet(e_1, \ldots, e_k) \,/\, (d(e_{i_1} \wedge \cdots \wedge e_{i_r}), \{K_{i_1}, \ldots, K_{i_r}\} \text{ dependent})$$

where $d(e_{i_1} \wedge \cdots \wedge e_{i_r}) = \sum_s (-1)^{s-1} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_s}} \wedge \cdots \wedge e_{i_r}$.

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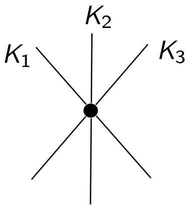
Example

$A_\bullet(\mathscr{A}) = \Lambda^\bullet(e_1,e_2,e_3) \,/\, (e_1\wedge e_2 - e_1\wedge e_3 + e_2\wedge e_3)$.



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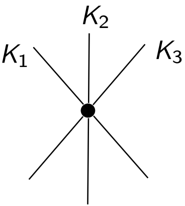

Theorem (Brieskorn '73, Orlik-Solomon '80)

We have an isomorphism of graded algebras $H^\bullet(\mathbb{C}^n\setminus\mathscr{A}) \cong A_\bullet(\mathscr{A})$.

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- $A_0(\mathscr{A}) = \mathbb{Q}$;
- for $\Sigma \in \mathscr{S}_r(\mathscr{A})$, $A_r^\Sigma(\mathscr{A})$ is defined as a kernel

$$0 \to A_r^\Sigma(\mathscr{A}) \xrightarrow{d} \bigoplus_{S \supset \Sigma} A_{r-1}^S(\mathscr{A}) \xrightarrow{d} \bigoplus_{T \supset \Sigma} A_{r-2}^T(\mathscr{A}) \; .$$

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Theorem (Looijenga '93)

In such a situation we have a spectral sequence

$$E_1^{-p,q} = \bigoplus_{S \in \mathscr{S}_p} H^{q-2p}(S)(-p) \otimes A_p^S(\mathscr{A}) \implies H^{-p+q}(X \setminus \mathscr{A}) .$$

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- D. '13 - application to rational homotopy theory.

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More interesting periods

(Multiple) zeta values : $\zeta(n) = \sum_{k \geqslant 1} \frac{1}{k^n}$, $\zeta(n_1, \ldots, n_r) = \sum_{1 \leqslant k_1 < \cdots < k_r} \frac{1}{k_1^{n_1} \cdots k_r^{n_r}}$.

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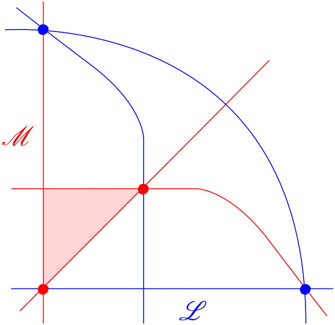

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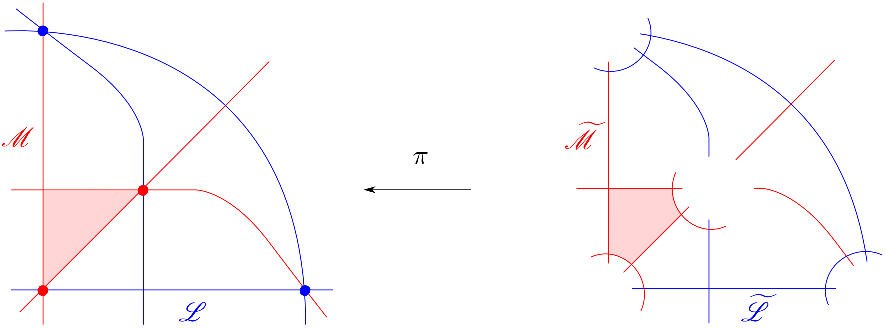

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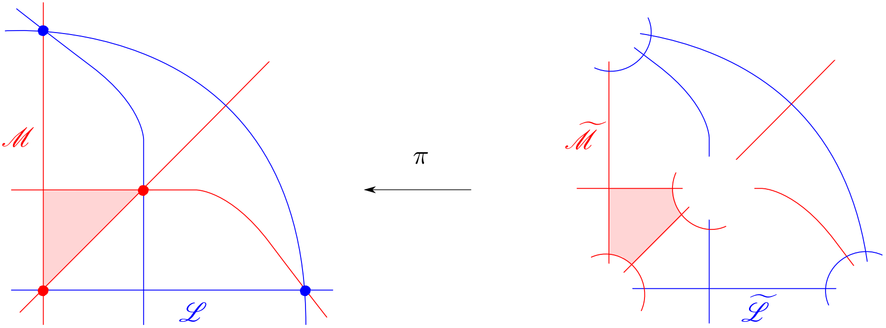

We are interested in the cohomology group $H = H^2(\widetilde{\mathbb{P}^2} \setminus \widetilde{\mathscr{L}}, \widetilde{\mathscr{M}} \setminus \widetilde{\mathscr{M}} \cap \widetilde{\mathscr{L}})$.

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Inspired by Aomoto '77 '82 and Beilinson-Goncharov-Schechtman-Varchenko '89.

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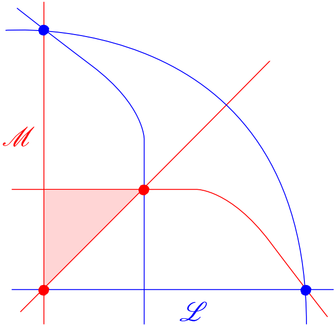

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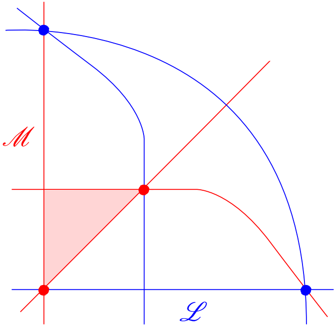

- The involution $\lambda \leftrightarrow \mu$ corresponds to Poincaré duality.

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\begin{tikzcd}
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We define $A_{i,j} = \bigoplus_{S \in \mathscr{S}_{i+j}} A^S_{i,j}$, by induction on the codimension $i+j$.

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– For a stratum Σ such that $\chi(\Sigma) = \lambda$, one defines $A_{i,j}^{\Sigma}$ as a *kernel* :

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Base step of the induction : $A_{0,0} = \mathbb{Q}$.

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Example

$\dim(A^{S}_{1,1}) =$ the number of hyperplanes $K \supset S$ such that $\chi(K) \neq \chi(S)$.

Exactness

Definition

A bi-arrangement of hyperplanes $(\mathscr{L}, \mathscr{M}, \chi)$ is *exact* if the above exact sequences can be continued to long exact sequences

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- Deletion and restriction formalism for exact bi-arrangements of hyperplanes.

The main theorem

Theorem (D. '14)

For an exact bi-arrangement of hyperplanes $(\mathscr{L},\mathscr{M},\chi)$ in $\mathbb{P}^n$, « the Orlik-Solomon bi-complex $A_{\bullet,\bullet}(\mathscr{L},\mathscr{M},\chi)$ computes the motive $H^{\bullet}(\mathscr{L},\mathscr{M},\chi)$ ».

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- *we consider the double complex* $A_{0 \leqslant \bullet \leqslant k, 0 \leqslant \bullet \leqslant n-k}$ *;*
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- The weight-graded quotients $\mathrm{gr}^W_{2k} H^\bullet(\mathscr{L}, \mathscr{M}, \chi)$ are combinatorial invariants, but not the whole motive $H^\bullet(\mathscr{L}, \mathscr{M}, \chi)$.

Explicit computations

Combinatorial notion of *tame* bi-arrangements of hyperplanes.

- Generic bi-arrangements are tame
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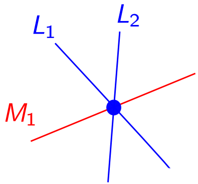
$A_{\bullet,\bullet}$ is the quotient of $\Lambda^\bullet(e_1, e_2) \otimes \Lambda^\bullet(f_1^\vee)$ by the relation $d(e_1 \wedge e_2) \otimes f_1^\vee = 0$.



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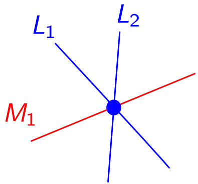

Example

One can define multiple zeta bi-arrangements $\mathscr{Z}(n_1,\ldots,n_r)$ that are tame.

The global context

Definition

A *bi-arrangement of hypersurfaces* in a complex manifold X is a triple $(\mathscr{L}, \mathscr{M}, \chi)$ where $\mathscr{L}$ and $\mathscr{M}$ are divisors in X and χ a coloring function, which locally looks like a bi-arrangement of hyperplanes.

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Theorem (D. '14)

In such a situation (for exact bi-arrangements) we have a spectral sequence

$$E_1^{-p,q} = \bigoplus_{\substack{i-j=p \\ S\in\mathscr{S}_{i+j}}} H^{q-2i}(S)(-i)\otimes A_{i,j}^S(\mathscr{L},\mathscr{M},\chi) \implies H^{-p+q}(\mathscr{L},\mathscr{M},\chi)\ .$$

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Idea of proof : by induction on the number of blow-ups in a resolution of singularities.

- True for normal crossing divisors (classical).
- For *one* blow-up, define a quasi-isomorphism $E_1 \xrightarrow{\sim} \widetilde{E}_1$ or $\widetilde{E}_1 \xrightarrow{\sim} E_1$.

Open problems

Geometric problems

(joint work with D. Juteau)

– Interpretation of the defect of exactness.

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- Orlik-Solomon bi-complexes as modules over Orlik-Solomon algebras.