CHERN SIMONS THEORY IN DIMENSION THREE

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Abstract. These notes are both an introduction to and a survey on Chern-Simons theory for 3-manifolds. In particular, we describe the rigidity properties and the critical points of the Chern-Simons action (the flat connections) and we construct the Chern-Simons line bundle over the moduli spaces of flat connections over closed surfaces. In the case of gauge group $PSL(2,\mathbb{C})$ we develop the applications to the symplectic geometry of these moduli spaces, three-dimensional hyperbolic geometry, and volumes of holonomy representations of 3-manifolds. By recalling the derivation of the Chern-Simons 3-forms from Chern-Weil theory, we also describe explicit simplicial formulas of the Chern-Simons action of flat connections which were obtained recently. We conclude with the quantization of Chern-Simons theory for $PSL(2,\mathbb{C})$: we discuss the status of the Reshetikhin-Turaev TQFT on this problem, and, by using elementary geometric tools and known results of Teichmüller theory, we reconstruct both the full classical Chern-Simons theory developed previously (thus providing a kind of axiomatic definition for it), and its quantizations based on the non restricted quantum group $U_\varepsilon sl(2,\mathbb{C})$ at roots of unity.

Contents

1. Foreword 2
2. Preliminaries: bundles and connections 3
3. The Chern-Simons action 9
4. Rigidity and volumes of representations 19
5. The Chern-Simons line bundle, I 27
6. The Chern-Simons line bundle, II 38
6.1. A very brief review on character varieties 38
6.2. Moduli spaces and character varieties 42
6.3. Chern-Simons invariants of marked cobordisms 44
7. Variation formulas and applications 47
7.1. The Chern-Simons line over a torus 48
7.2. Applications to cusped hyperbolic manifolds 50
7.3. 1-dimensional representations of mapping classes of surfaces 51
8. Chern-Weil theory and secondary characteristic classes 51
8.1. Construction of Cheeger-Chern-Simons classes 51
8.2. The universal canonical cochain on $BPSL(2,\mathbb{C})^\delta$ 57
9. Simplicial formulas 58
10. Quantum Chern-Simons theories 58
References 58

1. Foreword

These notes grew out of lectures given at the Scuola di dottorato Galileo Galilei, Universita' di Pisa, Italy, in February 2009. The goal was to provide a self-contained treatment of various aspects of three dimensional Chern-Simons theory.

Loosely speaking, three dimensional Chern-Simons theory is the theory of an integral, called *Chern-Simons action*, of some "characteristic" differential form defined over the spaces of connections on 3-manifolds with values in a fixed Lie algebra. This integral was introduced (in any dimension) in the seminal paper of S-S. Chern and J. Simons [CS]. It produces very powerful invariants of 3-manifolds which have many applications to their geometry and topology, and is also at the heart of the geometric understanding of the three dimensional quantum field theories based on the combinatorics of representations of quantum groups, which describe fundamental *algebraic structures* specific to knots, surfaces and 3-manifolds.

To fullfill the aim of this note would require at least to develop, on one hand, the gauge-theoretic aspects of the Chern-Simons action and of the Chern-Simons line bundle, which is defined over the moduli spaces of flat connections on closed surfaces, and, on another hand, the homological interpretation of the Chern-Simons action as a secondary characteristic class for flat connections. We have tried to do both, clarifying as far as possible their relationships as well as providing the tools for understanding (if not working with) the more recent results. We hope this has been successful in particular regarding the simplicial formulas of the Chern-Simons action for flat $(P)SL(2,\mathbb{C})$-connections and their quantization via quantum groups. We provide new results in this area. We do not consider the gauge theoretic aspects of Chern-Simons theory for 3-manifolds with corners, which relies on the Wess-Zumino-Witten $(1+1)$-dimensional conformal field theory for surfaces with boundary and is developed in [Fr], because we will be able to recover it by using the simplicial formulas.

In Section 3 and 5 we define the Chern-Simons action and the Chern-Simons line bundle via gauge theory, following T.R. Ramadas - I.M. Singer - J. Weitsman [RSW], S. Freed [Fr], and P. Kirk and E. Klassen [KK] approach. We give applications, in particular to hyperbolic geometry, and develop the relationships with the volumes of holonomy representations of 3-manifolds in Section 4 and 7. We pay a particular attention to variation formulas for cusped hyperbolic manifolds. Chern-Simons theory gives a very powerful unifying framework to deal with all these objects. In this direction, perhaps the most fundamental results are due to T. Yoshida [Yo], J.L. Dupont [Du1], C.D. Hogdson [Ho] and W. Neumann [Ne0, Ne1]. We describe also in detail the complex symplectic structure on the moduli space of flat $sl(2,\mathbb{C})$-connections on surfaces in terms of the Chern-Simons connection. This symplectic structure extends the one defined by M. Atiyah and R. Bott [AB] in the case of flat $su(2)$-connections, as well as the Weil-Petersson symplectic structure on Teichmüller space [Go1, Go2].
A purely topological reformulation of the Chern-Simons line bundle is given in Section 6. It serves mainly as an introduction to both the applications given in Section 7, and to the results of the following sections.

In Section 8 and 9 we change our way, and construct the simplicial formulas of Dupont and Neumann for the Chern-Simons action of flat $sl(2,\mathbb{C})$-connections on closed 3-manifolds. In order to make their results accessible we have included some elements of Chern-Weil theory and Cheeger-Chern-Simons secondary characteristic classes. Our main tools are [CS], [Du0]; the reader may also consult the general references [MiSt] and [Mo1, Mo2]. Then we present the original approach of Dupont and Neumann, which relies on the study of the cohomology of the classifying space $BPSL(2,\mathbb{C})^{\delta}$, where $PSL(2,\mathbb{C})$ is given the discrete topology ([Du1], [Ne1]). We also give a new direct derivation of these simplicial formulas, based on a careful analysis of the integral of the Chern-Simons 3-form over 3-simplices. This new derivation is interesting on its own, as it may be regarded as the basic building block for quantization, and allows to understand the Chern-Simons action of flat connections on manifolds with boundary in the spirit of combinatorial field theories.

In the last section we consider quantum Chern-Simons theories. After a brief presentation of what has been done on the side of *global* geometric quantization, we present a certain number of results which we believe have notably clarified some of the relationships between the "classical" Chern-Simons theory and the combinatorially defined quantum field theories for $SL(2,\mathbb{C})$.
First we consider the deformation quantization of $SL(2,\mathbb{C})$-characters via skein theory and the Kauffman bracket.
Then we define the quantization of the simplicial formulas of the Chern-Simons action, based on the representation theory of the quantum group $U_\varepsilon sl(2,\mathbb{C})$ at roots of unity, and developed by the author with R. Benedetti in [BB] and subsequent papers.

Finally we show how essentially the same constructions can be applied to reconstruct completely the classical Chern-Simons theory described in the previous sections by using only "elementary" geometric means.

There is an extensive litterature on Chern-Simons theory. Good introductions to the subject and other developpments can be found eg. in the books of E. Guadagnini [Ga] and T. Kohno [Ko].

2. Preliminaries: bundles and connections

For the material in this section, see eg. [Du0], [Na] or [Mo1].

Principal bundles. All manifolds, maps, actions, etc are assumed to be smooth. Let G be a Lie group, and P, X manifolds. A *principal G-bundle* is a map $\pi : P \to X$, where G acts freely on P on the right, and the quotient P/G is diffeomorphic to X. We denote the action of g by $p \mapsto p \cdot g$.
For all $x \in X$ and $p \in P_x = \pi^{-1}(x)$, we have a diffeomorphism commuting with the right G actions

$$(1) \qquad \begin{array}{rccc} \Phi_p : & G & \longrightarrow & P_x \\ & g & \longmapsto & p \cdot g. \end{array}$$

If $p' = p \cdot g'$, then $\Phi_p^{-1} \circ \Phi_{p'} = L'_g$, the left translation on G by g'. Hence the collection of maps Φ_p identifies T_pP_x with the set of vectors fields on G invariant under left translation, that is, the Lie algebra $\mathfrak{g}$ of G. In fact

$$\begin{array}{rccc} d\Phi_p : & \mathfrak{g} & \longrightarrow & T_pP_x \\ & a & \longmapsto & (X_a)_p \end{array} \tag{2}$$

where for all $f : P \to \mathbb{R}$ the vector field X_a is given by

$$df_p(X_a) = \frac{d}{dt}\left(f(p \cdot e^{ta})\right)_{t=0}$$

Since the projection map $\pi : P \to X$ is a submersion, a principal G bundle is locally trivial: for any sufficiently small open set U_i in X we can construct local sections $s_i : U_i \to P$ by lifting linearly independendent vector fields on U_i and considering their flows. Then, for any such a section we have a trivialization

$$\begin{array}{rccc} \tau_i : & U_i \times G & \longrightarrow & \pi^{-1}(U_i) \\ & (x, g) & \longmapsto & s_i(x) \cdot g \end{array} \tag{3}$$

If $U_i \cap U_j \neq \emptyset$ we get

$$\tau_j^{-1} \circ \tau_i : (x, g) \longmapsto (x, g_{i,j}(x)g) \tag{4}$$

The map $g_{i,j} : U_i \cap U_j \to G$ is called a *transition function* of P. Note that transition functions act on the fiber on the left and commute with the right G-action. Clearly, the bundle π can be reconstructed from the transitions functions associated to any open covering of X by open sets U_i with sections s_i as above.

Connections. A connection on P is a subbundle HP of TP which at any point $p \in P$ is a complement in T_pP to the tangent space T_pP_x of the fiber, and is equivariant in the sense that

$$R_{g*}H_pP = H_{p \cdot g}P. \tag{5}$$

We call H_pP the horizontal subspace of T_pP, and vectors in T_pP_x are called vertical. We say that a path γ in P is horizontal if $\gamma'(t) \in H_{\gamma(t)}P$ for all t.
A connection can be equivalently defined as a one-form $w \in T^*P \otimes \mathfrak{g}$ such that

$$w(X_a) = a \tag{6}$$

$$R_g^* w = \mathrm{Ad}_{g^{-1}} w \tag{7}$$

where the vertical vector field X_a is defined in (2), and Ad is the adjoint action of G, which applies here to the values in $\mathfrak{g}$ of w. Then $H_pP = \mathrm{Ker}(w_p)$, and w_p is the parallel projection on T_pP_x followed by the identification with $\mathfrak{g}$.
The equation (6) means that the restriction $i_x^* w$ of w to the fibers, where $i_x : P_x \hookrightarrow P$ is the inclusion, can be identified with the Maurer-Cartan form θ on G, which is defined by:

$$\begin{array}{rccc} \theta_g : & T_gG & \longrightarrow & \mathfrak{g} \\ & X & \longmapsto & L_{g^{-1}*}X. \end{array} \tag{8}$$

For a matrix group G the Maurer-Cartan form can be written as follows:

$$\theta = g^{-1}dg \tag{9}$$

where g means the matrix of coordinate functions on G.
In fact, if we fix an isomorphism Φ_p as in (1), for any $a \in \mathfrak{g} = T_eG$ we have $(\Phi_p^* i_x^* w)_e(a) = a = \theta(a)$. Since Φ_p commutes with the right G action, by (7) and

the obvious $R_g^*\theta = \mathrm{Ad}_{g^{-1}}\theta$, this equality extends to any tangent vector of G (obtained as $R_{g*}a$ from some $a \in \mathfrak{g}$). Since the change of Φ_p to $\Phi_{p'}$ is by left translation on G, and θ is left invariant, this identification of $i_x^* w$ with θ does not depend on the choice of Φ_p.
The equations (6)-(7) are affine and convex. Hence connections on P can be defined by using a partition of unity to patch local connections (which obviously exist). Also, the set of connections of P is an affine subspace $\mathcal{A}_P$ of $A_P^1(\mathfrak{g})$, the $\mathfrak{g}$-valued one-forms on P.

Curvature. The *curvature* of w is the $\mathfrak{g}$-valued 2-form on P given by

$$\Omega(X_1, X_2) = d\omega(X_1^H, X_2^H) = -\omega([X_1^H, X_2^H]).$$

We can extend the usual exterior product to $\mathfrak{g}$-valued differential forms on P just by tensoring the coefficients:

$$\cdot \wedge \cdot : A_P^k(\mathfrak{g}) \otimes A_P^l(\mathfrak{g}) \longrightarrow A_P^{k+l}(\mathfrak{g} \otimes \mathfrak{g}). \tag{10}$$

Then, by taking the Lie bracket of coefficients we obtain a map

$$[\cdot \wedge \cdot] : A_P^k(\mathfrak{g}) \otimes A_P^l(\mathfrak{g}) \longrightarrow A_P^{k+l}(\mathfrak{g})$$

which is explicitly given by

$$[\zeta \wedge \eta]\,(X_1, \ldots, X_{k+l}) = \sum_{\sigma \in S_{k+l}} \varepsilon(\sigma)\left[\zeta(X_{\sigma(1)}, \ldots, X_{\sigma(k)}), \eta(X_{\sigma(k+1)}, \ldots, X_{\sigma(k+l)})\right].$$

For instance if $\zeta \in A_P^1(\mathfrak{g})$ we have

$$[\zeta \wedge \zeta]\,(X, Y) = 2\,[\zeta(X), \zeta(Y)]\,. \tag{11}$$

If G is a matrix group the Lie bracket of $\mathfrak{g}$ is the commutator of matrices, so that $(1/2)\,[\cdot \wedge \cdot]$ is sometimes denoted by $\wedge$ (hence "$\zeta \wedge \zeta(X, Y)$" is used for $[\zeta(X) \wedge \zeta(Y)]$). We will not use this notation as we observed it often leads to confusions.
Connections on P satisfy the Cartan structure equation

$$\Omega = dw + \frac{1}{2}\,[w \wedge w] \tag{12}$$

and the Bianchi identity

$$d\Omega + [w \wedge \Omega] = 0 \tag{13}$$

Exercise 1. Prove these identities :
1. For (12) decompose vector fields into horizontal and vertical components. Remember that w associates a constant value in $\mathfrak{g}$ to any vertical field, and show, using $R_{g*}H_pP = H_{p\cdot g}P$, that $[X_1, X_2]$ is horizontal if X_1 is horizontal and X_2 vertical.
2. For (13) take the exterior derivative of (12), and use the identity

$$[[w \wedge w] \wedge w] = 0. \tag{14}$$

which follows from the Jacobi identity for the Lie bracket of $\mathfrak{g}$.

The Maurer-Cartan form (which can be interpreted as a connection on the bundle $G \to \{pt\}$) satisfies also the Cartan structure equation with vanishing curvature:

$$d\theta = -\frac{1}{2}\,[\theta \wedge \theta]\,. \tag{15}$$

This is easily deduced from (11) and

$$d\theta(X_1, X_2) = X_1.\theta(X_2) - X_2.\theta(X_1) - \theta([X_1, X_2]) = -\theta([X_1, X_2]) = -\left[\theta(X_1), \theta(X_2)\right],$$

where X_1 and X_2 are left invariant, so that $X_1.\theta(X_2)$ and $X_2.\theta(X_1)$ vanish.

Gauge transformations. By definition, a map of principal G-bundles $\varphi : P' \to P$ commutes with the G-actions. If moreover the base X and X' are equal and the induced map $\bar{\varphi} : X' \to X$ is the identity, we say that φ is a morphism.
An *automorphism*, or *gauge transformation*, is a morphism $\varphi : P \to P$ from P to itself. Then there exists a map $g_\varphi : P \to G$ such that

$$\varphi(p) = p \cdot g_\varphi(p).$$

If $\varphi : P' \to P$ is a bundle map and w a connection on P, then $\varphi^* w$ is a connection on P'. In particular, the group of gauge transformations acts on the right on the space of connections $\mathcal{A}_P$.
Let us derive this action explicitly. Let γ be a path in P, $\gamma(0) = p$ and $\gamma'(0) = V$. If φ is a gauge transformation, then

$$\begin{aligned}
\varphi^* V &= \frac{d}{dt}\left(\gamma(t) \cdot g_\varphi(\gamma(t)\right)_{t=0} \\
&= \frac{d}{dt}\left(\gamma(t) \cdot g_\varphi(\gamma(0)\right)_{t=0} + \frac{d}{dt}\left(\gamma(0) \cdot g_\varphi(\gamma(t)\right)_{t=0} \\
&= R_{g_\varphi(p)*} V + \frac{d}{dt}\left(p \cdot g_\varphi(p) g_\varphi^{-1}(p) g_\varphi(\gamma(t))\right)_{t=0} \\
&= R_{g_\varphi(p)*} V + \varphi(p) \cdot g_\varphi^* \theta(V)
\end{aligned}$$

where θ is, as usual, the Maurer-Cartan form (see (9)), and the second term means the infinitesimal action of $g_\varphi^*\theta(V) \in \mathfrak{g}$ on $\varphi(p)$. By evaluating with w we deduce

$$\varphi^* w = \mathrm{Ad}_{g_\varphi^{-1}} w + g_\varphi^* \theta. \tag{16}$$

The "correction" term $g_\varphi^*\theta$ takes care of the contribution of horizontal vector fields (see (20)). The curvature has a very different behaviour, since it transforms as a tensor. Indeed,

$$\begin{aligned}
\varphi^*\Omega(Y, Z) &= dw((R_{g_\varphi *} Y)^H, (R_{g_\varphi *} Z)^H) \\
&= dw(R_{g_\varphi *}(Y^H), R_{g_\varphi *}(Z^H)) \\
&= dR_{g_\varphi}^* w(Y^H, Z^H) \\
&= d\mathrm{Ad}_{g_\varphi^{-1}} w(Y^H, Z^H) \\
&= \mathrm{Ad}_{g_\varphi^{-1}} dw(Y^H, Z^H).
\end{aligned}$$

Hence

$$\varphi^*\Omega = \mathrm{Ad}_{g_\varphi^{-1}} \Omega. \tag{17}$$

We will comment on this property of Ω when considering associated bundles.

Let us derive two consequences of (16).

First, represent w locally as $\mathcal{A}_i = s_i^* w \in A^1_{U_i}(\mathfrak{g})$, with (U_i, s_i) as in (3), and similarly for (U_j, s_j). We have $s_j = s_i g_{ij}$, where $g_{ij} : U_i \cap U_j \to P$ is the transition function. Then, by taking $\varphi = g_{ij}$ we deduce from (16) that

$$\mathcal{A}_j = \mathrm{Ad}_{g_{ij}^{-1}} \mathcal{A}_i + g_{ij}^* \theta. \tag{18}$$

This is the local compatibility relation required to define a connection by patching local $\mathfrak{g}$-valued one forms on the sets U_i. In the particular case of a line bundle, by writing the transition functions as $g_{ij}(p) = \exp(-\chi_{ij}(p) \in U(1)$, (18) reads

$$\mathcal{A}_j = \mathcal{A}_i - d\chi_{ij}(p). \tag{19}$$

Second, suppose that γ is a path in P and $\varphi(\gamma) = \gamma \cdot g_\varphi(\gamma)$ is horizontal. Then, by applying (16) to the tangent vectors to γ we get

$$\frac{d}{dt}\left(g_\varphi(\gamma(t))\right)_{t=0} = -w_{\gamma(0)}(\gamma'(0))g_\varphi(\gamma(t)). \tag{20}$$

This is an ordinary linear differential equation, with a unique solution for each initial condition $g_\varphi(\gamma(0)) \in G$. Hence, for any path $\bar{\gamma}$ in X and any point $p \in P_{\bar{\gamma}(0)}$ there is a unique horizontal path through p lifting $\bar{\gamma}$.

The geometric meaning of this solution is the following. Since the flow of (20) depends only on the projection $\bar{\gamma} = \pi(\gamma)$ it defines a map between fibers

$$PT_{\bar{\gamma}} : P_{\bar{\gamma}(0)} \longrightarrow P_{\bar{\gamma}(1)} \tag{21}$$

which associates to $p_0 \in P_{\bar{\gamma}(0)}$ the unique point $p_1 \in P_{\bar{\gamma}(1)}$ contained in the horizontal lift $\tilde{\gamma}$ of $\bar{\gamma}$ passing through p_0. To compute explicitly that point, we have to fix a gauge (ie. a section over $\bar{\gamma}$), which is just an arbitrary lift γ of $\bar{\gamma}$, as we did above. Then, with the same notations,

$$p_0 = \gamma(0) \cdot g_\varphi(\gamma(0)) \quad , \quad p_1 = \gamma(1) \cdot g_\varphi(\gamma(1)),$$

and the points $\tilde{\gamma}(t)$ of the horizontal lift through p_0 are given by

$$\gamma(t) \cdot \exp\left(-\int_0^t w_{\gamma(t)}(\gamma'(t))dt\right). \tag{22}$$

We call $PT_{\bar{\gamma}}$ the *parallel transport*, and g_φ (or φ) a *horizontal* or *parallel* gauge transformation.

Exercise 2. Show that:

1. Parallel transport commutes with the right G-action: $PT_{\bar{\gamma}} \circ R_g = R_g \circ PT_{\bar{\gamma}}$. (Hint: show that for any two horizontal lifts $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ of a same path $\bar{\gamma}$ in X we have $\tilde{\gamma}_1 = \tilde{\gamma}_2 \cdot g$ for some fixed $g \in G$.)

2. A gauge transformation $\varphi : P \to P$ is parallel if and only if it is an automorphism of the connection: $\varphi^* w = w$. In particular, if X is connected a parallel gauge transformation is the identity at some point of P if and only if it is the identity.

Covariant derivatives and associated bundles. Let V be k-dimensional vector space with a faithful linear representation ρ of G. Let us write ρ as an action on the left (hence $\rho(g)(v) = g \cdot v$).

Define the *bundle associated to* (P and) ρ as

$$P_V = P \times V / \left((p, v) \sim (p \cdot g, g^{-1} \cdot v)\right). \tag{23}$$

This is a k-dimensional vector bundle, denoted $P \times_\rho V$, with projection π_V given by $\pi_V[(p, v)] = \pi(p)$, and transition functions given by the action of those of P: at any point $p \in \pi^{-1}(U_i \cap U_j) \neq \emptyset$ we have

$$\tau_j^{-1} \circ \tau_i : [(p, v)] \longmapsto [(p \cdot g_{ij}, v)] = [(p, g_{ij} \cdot v].$$

Hence the principal bundle π can be recovered uniquely from its associated vector bundles.

The counterpart of connections in this context are the *covariant derivatives.* Take a small path γ in X, and set $x = \gamma(0)$ and $\zeta = \gamma'(0)$. For any section s of P_V we can write

$$s(\gamma(t)) = [(\tilde{\gamma}(t), \eta(t))]$$

for some horizontal lift $\tilde{\gamma}$ of γ, by properly choosing the representatives in the class. Then

$$(\nabla_\zeta s)_x := [(\tilde{\gamma}(0), \frac{d}{dt}\,(\eta(t))_{t=0})]. \tag{24}$$

By Exercise 2 this is clearly independent of the choice of a horizontal lift. The relation with the connection w on P comes from the characterization of parallel sections. Namely, let $\sigma : U \to P$ be a local section such that

$$s(\gamma(t)) = [(\sigma(\gamma(t)), \eta)] \tag{25}$$

with η a constant vector. Hence $\sigma(\gamma) = \tilde{\gamma} \cdot g_\varphi^{-1}(\gamma)$ for some $g_\varphi : U \to G$. We have

$$\begin{array}{rcl} (\nabla_\zeta s)_x & = & [(\tilde{\gamma}(0), \dfrac{d}{dt}\left(g_\varphi^{-1}(\gamma(t))\right)_{t=0} \cdot \eta)] \\ & = & [(\tilde{\gamma}(0), -g_\varphi^{-1}(\gamma(0))\dfrac{d}{dt}\,(g_\varphi(\gamma(t)))_{t=0}\, g_\varphi^{-1}(\gamma(0)) \cdot \eta)] \\ & = & [\sigma(x), w_x(\sigma_*\zeta) \cdot \eta)]. \end{array}$$

Here we use (20). If we choose a local frame over U, ie. a basis of sections s_i with linearly independent vectors η_i in (25), for any local section $s(x) = [(\sigma(x), f^i(x)\eta_i)]$ we get

$$(\nabla_\zeta s)_x = [\sigma(x), df^i_x(\zeta)\eta_i + f^i(x)w_x(\sigma_*\zeta) \cdot \eta_i)]. \tag{26}$$

As a consequence, the covariant derivative ∇ on P_V and the connection w on P are equivalent objects.

In general we will identify the image of a section with its component in the fiber. Then we say that a section is *horizontal*, or *parallel*, when the above differential expression vanishes, and we write $(\nabla_\zeta s)_x = 0$. Parallel sections map paths in X to horizontal lifts.

Denote by $A^k_X(P_V)$ the space of k-forms on P_V, also called *V-valued differential forms on X*. Locally they are given by tensor products $\mu \otimes s$, where μ is a k-form on X and s is a section of P_V. In particular $A^0_X(P_V)$ is the space of sections of P_V, and we can rewrite equation (26) as

$$\nabla_\zeta s = (df^i \otimes \eta_i + f^i \otimes w \cdot \eta_i)(\zeta) = (ds + w \cdot s)(\zeta)$$

where $s = f^i \otimes \eta_i \in A^0_X(P_V)$ in the local trivialization $\sigma : U \to P$. Hence $\nabla = d + w\cdot$ defines a map $\nabla^0 : A^0_X(P_V) \to A^1_X(P_V)$. It extends naturally to a map

$$\nabla^k : A^k_X(P_V) \to A^{k+1}_X(P_V)$$

via the Leibniz rule

$$\nabla^k(\mu \otimes s) = d\mu \otimes s + (-1)^k \mu \wedge \nabla^0 s, \quad \mu \in A^k_X(P_V).$$

We can recover the curvature as follows. Assume without loss of generality that the section s is constant. Then

$$\begin{aligned}\nabla^{k+1}\nabla^k(\mu\otimes s) &= \nabla^{k+1}(d\mu\otimes s+(-1)^k\mu\wedge\nabla^0 s)\\ &= (-1)^{k+1}d\mu\wedge\nabla^0 s+(-1)^k d\mu\wedge\nabla^0 s+\mu\wedge\nabla^1\nabla^0 s\\ &= \mu\wedge\nabla^1(w\cdot s)\\ &= \mu\wedge(dw\cdot s-w\wedge(w\cdot s)).\end{aligned}$$

By using (11) and the fact that G acts on the right on P and on the left on V (see eg. the computation of $(\nabla_\zeta s)_x$ above), it is readily check that

$$-w\wedge(w\cdot s)(U,V)=[w(U),w(V)]\cdot s=(1/2)\,[w\wedge w]\,(U,V)\cdot s.$$

We deduce the following counterpart of Cartan's equation

$$\nabla^{k+1}\nabla^k(\mu\otimes s)=\mu\wedge(\Omega\cdot s). \tag{27}$$

On another hand,

$$\begin{aligned}(dw\cdot s-w\wedge(w\cdot s))(U,V) &= (U.w(V))\cdot s-(V.w(U))\cdot s-w([U,V])\cdot s-\\ &\qquad -w(U)\cdot(w(V)\cdot s)+w(V)\cdot(w(U)\cdot s)\\ &= \nabla^0_U(w(V)\cdot s)-\nabla^0_V(w(U)\cdot s)-\nabla^0_{[U,V]}s\\ &= \nabla^0_U\nabla^0_V s-\nabla^0_V\nabla^0_U s-\nabla^0_{[U,V]}s\end{aligned}$$

thus recovering the usual formula for the curvature.

A main example of associated bundle is the *Adjoint bundle* $P_{\mathfrak{g}}=P\times_{Ad}\mathfrak{g}$, where G acts on $\mathfrak{g}$ via the adjoint action. By (26) the covariant derivative is

$$\nabla=d+ad(w). \tag{28}$$

Also, recall that by the definition of the curvature Ω and by (17) we have

$$\begin{aligned} i_x^*\Omega &= 0\\ \varphi^*\Omega &= \mathrm{Ad}_{g_\varphi^{-1}}w.\end{aligned} \tag{29}$$

These properties show that Ω descends to a two-form on $P_{\mathfrak{g}}$.

3. The Chern-Simons action

In the course of the construction of the Chern-Simons action we will make three assumptions, two on the group G and one the manifold X.

Hypothesis 1. *There exists a bilinear symmetric Ad-invariant form*

$$\langle\cdot,\cdot\rangle:\mathfrak{g}\times\mathfrak{g}\to\mathbb{C}. \tag{30}$$

Recall that the Ad-invariance means $\langle \mathrm{Ad}_g a,\mathrm{Ad}_g b\rangle=\langle a,b\rangle$ for all $g\in G$, and that it implies the ad-invariance $\langle [c,a],b\rangle=-\langle a,[c,b]\rangle$ for all $c\in\mathfrak{g}$. We will identify $\langle\cdot,\cdot\rangle$ with an element $\langle\cdot\rangle\in S^2(\mathfrak{g}^*)$, the second symmetric power of $\mathfrak{g}^*$, thus writing $\langle a\otimes b\rangle$ for $\langle a,b\rangle$. The main example is the multiples of the Killing form for semi-simple Lie groups. The case of $SL(2,\mathbb{C})$ is detailed below.

Definition 3.1. *The Chern-Simons* 3*-form of the connection* w *is*

$$\alpha(w)=\langle w\wedge\Omega\rangle-\frac{1}{6}\langle w\wedge[w\wedge w]\rangle=\langle w\wedge dw+\frac{1}{3}w\wedge[w\wedge w]\rangle\quad\in A^3_P(\mathbb{C}).$$

Several comments are in order. First, recall the wedge product of $\mathfrak{g}$-valued forms (see (10)). We define $\langle w \wedge \Omega \rangle$ by applying $\langle \cdot \rangle \in S^2(\mathfrak{g}^*)$ to the coefficients of $w \wedge \Omega$, and so on. Hence $\alpha(w)$ is a genuine complex valued 3-form on P.

The origin of $\alpha(w)$ will be explained in Section 8. As shows the next proposition, it is a primitive of a tensor in the curvature, which we will interpret as the Chern-Weil 4-form associated to $\langle \cdot \rangle$ and representing a characteristic class of P in De Rham cohomology. We also give the behaviour under gauge transformations.

Proposition 3.2. *The following holds:*

1. $d\alpha(w) = \langle \Omega \wedge \Omega \rangle$*;*
2. *Denote as usual by θ the Maurer-Cartan form. If $\varphi : P \to P$ is a gauge transformation acting on fibers via $g_\varphi : P \to G$, then*

$$\varphi^*\alpha(w) = \alpha(w) + d\langle \mathrm{Ad}_{g_\varphi^{-1}} w \wedge g_\varphi^*\theta \rangle - \frac{1}{6} g_\varphi^* \langle \theta \wedge [\theta \wedge \theta] \rangle.$$

Proof. The proof is by direct computation. We have

$$d\alpha(w) = \langle dw \wedge \Omega \rangle - \langle w \wedge d\Omega \rangle - \frac{1}{6}\left(\langle dw \wedge [w \wedge w] \rangle - \langle w \wedge d\,[w \wedge w] \rangle \right).$$

By the Leibniz rule and the commutation rule $[\zeta \wedge \eta] = (-1)^{kk'+1}\,[\eta \wedge \zeta]$, where ζ and η are forms of degree k and k', and the ad-invariance and the symmetry of $\langle \cdot \rangle$, we get

$$\langle w \wedge d\,[w \wedge w] \rangle = -2\langle w \wedge [w \wedge dw] = 2\langle [w \wedge w] \wedge dw \rangle = 2\langle dw \wedge [w \wedge w] \rangle.$$

Then, using Cartan (6) and Bianchi (13),

$$d\alpha(w) = \langle (\Omega - \frac{1}{2}\,[w \wedge w]) \wedge \Omega \rangle + \langle w \wedge [w \wedge \Omega] \rangle - \frac{1}{2}\langle (\Omega - \frac{1}{2}\,[w \wedge w]) \wedge [w \wedge w] \rangle.$$

The ad-invariance and the Jacobi identity (14) imply $\langle [w \wedge w] \wedge [w \wedge w] \rangle = 0$. The first result then follows from $\langle w \wedge [w \wedge \Omega] \rangle = \langle [w \wedge w] \wedge \Omega \rangle$.

Let us prove (2). Put $\phi = g_\varphi^*\theta$, the pull-back of the Maurer-Cartan form. Ad-invariance and (16) yield

$$\varphi^*\langle w \wedge \Omega \rangle = \langle (\mathrm{Ad}_{g_\varphi^{-1}} w + \phi) \wedge \mathrm{Ad}_{g_\varphi^{-1}} \Omega \rangle = \langle w \wedge \Omega \rangle + \langle \phi \wedge \mathrm{Ad}_{g_\varphi^{-1}} \Omega \rangle.$$

Also, using Ad- and ad-invariance, and the commutation rule of $[\cdot \wedge \cdot]$ we get

$$\begin{aligned} \varphi^*\langle w \wedge [w \wedge w] \rangle &= \langle (\mathrm{Ad}_{g_\varphi^{-1}} w + \phi) \wedge \Big[(\mathrm{Ad}_{g_\varphi^{-1}} w + \phi) \wedge (\mathrm{Ad}_{g_\varphi^{-1}} w + \phi) \Big] \rangle \\ &= \langle w \wedge [w \wedge w] \rangle + \langle \phi \wedge [\phi \wedge \phi] \rangle + 3\langle \mathrm{Ad}_{g_\varphi^{-1}} w \wedge [\phi \wedge \phi] \rangle + \\ &\qquad + 3\langle \phi \wedge \Big[\mathrm{Ad}_{g_\varphi^{-1}} w \wedge \mathrm{Ad}_{g_\varphi^{-1}} w \Big] \rangle. \end{aligned}$$

Gathering terms we deduce

$$\begin{aligned} \varphi^*\alpha(w) = \langle w \wedge \Omega \rangle &- \frac{1}{6}\langle w \wedge [w \wedge w] \rangle \\ &- \frac{1}{2}\langle \mathrm{Ad}_{g_\varphi^{-1}} w \wedge [\phi \wedge \phi] \rangle - \frac{1}{2}\langle \phi \wedge \Big[\mathrm{Ad}_{g_\varphi^{-1}} w \wedge \mathrm{Ad}_{g_\varphi^{-1}} w \Big] \rangle \\ &- \frac{1}{6}\langle \phi \wedge [\phi \wedge \phi] \rangle + \langle \phi \wedge \mathrm{Ad}_{g_\varphi^{-1}} \Omega \rangle. \end{aligned}$$

Now, Cartan's equation (12) gives

$$\begin{aligned}\langle \phi \wedge \mathrm{Ad}_{g_\varphi^{-1}}\Omega\rangle &= \langle \phi \wedge \mathrm{Ad}_{g_\varphi^{-1}}dw\rangle + \frac{1}{2}\langle \phi \wedge \mathrm{Ad}_{g_\varphi^{-1}}[w\wedge w]\rangle \\ &= \langle \mathrm{Ad}_{g_\varphi^{-1}}dw \wedge \phi\rangle + \frac{1}{2}\langle \phi \wedge \mathrm{Ad}_{g_\varphi^{-1}}[w\wedge w]\rangle \\ &= d\langle \mathrm{Ad}_{g_\varphi^{-1}}w \wedge \phi\rangle - \langle \mathrm{Ad}_{g_\varphi^{-1}}w \wedge d\phi\rangle + \frac{1}{2}\langle \phi \wedge \mathrm{Ad}_{g_\varphi^{-1}}[w\wedge w]\rangle.\end{aligned}$$

Recall that ϕ is a pull-back of the Maurer-Cartan form, and thus satisfies (15). Then we see that the last two terms cancel those in the second line of the computation of $\varphi^*\alpha(w)$ above. This achieves the proof. □

Remark 3.3. Since $i_x^*\Omega = 0$, where $i_x : P_x \hookrightarrow P$ is the inclusion, the pull-back of the Chern-Simons 3-form to the fibers is

$$i_x^*\alpha(w) = g_\varphi^*\langle\theta\wedge[\theta\wedge\theta]\rangle.$$

By arguments similar to those used in the proof above we can check very easily that it is closed:

$$d\langle\theta\wedge[\theta\wedge\theta]\rangle = 3\langle d\theta\wedge[\theta\wedge\theta]\rangle = -\frac{3}{2}\langle[\theta\wedge\theta]\wedge[\theta\wedge\theta]\rangle = \frac{3}{2}\langle\theta\wedge[\theta\wedge[\theta\wedge\theta]] = 0.$$

where we use, in particular, the ad-invariance and the Jacobi identity (14). Alternatively, this results from $i_x^*d\alpha(w) = i_x^*\langle\Omega\wedge\Omega\rangle = 0$ (Ω is horizontal).

Recall that a principal G-bundle $\pi : Q \to Y$ is trivial when it admits a global section. Then a trivialization is given by

$$\begin{array}{rccc}\tilde{s}: & Y\times G & \longrightarrow & Q \\ & (x,g) & \longmapsto & s(x)\cdot g\end{array}$$

Obstructions to finding a section are homotopic, since once a continuous sections exists it can be smoothed within a same homotopy class. The following result is standard, and comes by taking a cellulation of Y and trying to define a section of Q skeleton by skeleton (we use also that $\pi_2(G) = 0$ always holds true).

Lemma 3.4. *If Y is a compact orientable manifold of dimension ≤ 3, and the homotopy groups $\pi_0(G)$ and $\pi_1(G)$ are trivial, then Q admits a global section.*

We shall meet situations where we allow $G = SO(3)$ or $PSL(2,\mathbb{C})$, which have π_1 equal to $\mathbb{Z}/2$; by the way we will only make the following assumption:

Hypothesis 2. *The manifold X is a compact oriented manifold of dimension three, and the principal G-bundle $\pi : P \to X$ is trivial.*

Hence for $G = SO(3)$ or $PSL(2,\mathbb{C})$ we will assume that the second Stiefel-Whitney characteristic class of π is zero. What is important is that the lemma shows that the following definition makes sense in a great variety of situations, for instance for all semi-simple Lie groups.

Definition 3.5. *Let $s : X \to P$ be a global section. The Chern-Simons invariant of w and s is*

$$S_X(w,s) = \int_X s^*\alpha(w). \tag{31}$$

We call

$$\begin{array}{rccc} S_X: & \mathcal{A}_P \times \Gamma(P) & \longrightarrow & \mathbb{C} \\ & (w,s) & \longmapsto & S_X(w,s) \end{array}$$

the Chern-Simons action.

An immediate consequence of Proposition 3.2 is the behaviour under a change of section:

Proposition 3.6. *Let $\varphi : P \to P$ is a gauge transformation acting on fibers via $g_\varphi : P \to G$. Denote $g = g_\varphi \circ s : X \to G$ and $\phi_g = g^*\theta$. We have*

$$\begin{aligned} S_X(w, \varphi \circ s) &= S_X(\varphi^* w, s) \\ &= S_X(w,s) + \int_{\partial X} \langle \mathrm{Ad}_{g^{-1}} s^* w \wedge \phi_g \rangle + \int_X -\frac{1}{6} \langle \phi_g \wedge [\phi_g \wedge \phi_g] \rangle. \end{aligned}$$

Recall from Remark 3.3 that the form $\langle \theta \wedge [\theta \wedge \theta] \rangle \in A^3_G(\mathbb{C})$ is closed. Then it represents a class in the cohomology group $H^3(G;\mathbb{R})$. We now make our last assumption, which is about the *normalization* of the form $\langle \cdot \rangle$ (a multiple of the Killing form for semi-simple Lie groups):

Hypothesis 3. *The class $-(1/6)\,[\langle \theta \wedge [\theta \wedge \theta] \rangle] \in H^3_G(\mathbb{C})$ is integral, that is, it is the image of a class in $H^3(G;\mathbb{Z})$ under the natural inclusion.*

Corollary 3.7. *The Chern-Simons action is a smooth function of connections and satisfies (we use the notations of Proposition 3.6):*

(1) *If X has no boundary, then*

$$S_X(w) = S_X(w,s) \mod(1)$$

is a well-defined invariant of (X,w) independent of s. Hence

$$S_X : \mathcal{A}_P/\mathcal{G} \to \mathbb{C}/\mathbb{Z}$$

well defines a smooth map[1] *on the set $\mathcal{A}_P$ of connections on P up to the action of the group $\mathcal{G}$ of gauge transformations.*

(2) *If $\partial X \neq \emptyset$, then the* Wess-Zumino-Witten functional

$$W_{\partial X}(\partial g) = \int_X -\frac{1}{6} \langle \phi_g \wedge [\phi_g \wedge \phi_g] \rangle \mod(1)$$

depends only on the restriction ∂g of $g : X \to G$ to ∂X.

Proof. The smoothness of $S_X(w,s)$ follows from the smoothness of $\alpha(w)$ with respect to w. Consider the first claim. Recall that $d\alpha(w) = \langle \Omega \wedge \Omega \rangle$ (Proposition 3.2(1)). Because of (29) and Ad-invariance, this forms descends on X. Since X has dimension three, it is zero. Hence $d\alpha(w)$ is closed, so that $S_X(w,s)$ does not change under homotopies of s. To conclude, we note that using the trivialization defined by s to identify P with $X \times G$, any other section s' appears as a map $\bar{s}' : X \to G$. Thus, the image of the fundamental class $(s_* - s'_*)\,[X] \in H_3(P;\mathbb{Z})$ reduces to a class in $H_3(G;\mathbb{Z})$ (this class may also be identified in the Künneth formula of $H_3(P;\mathbb{Z}) \cong H_3(X \times G;\mathbb{Z})$).

[1]Some (non generic) connections have non trivial isotropy groups in $\mathcal{G}$, so that the space $\mathcal{A}_P/\mathcal{G}$ has orbifold points. In fact it has even non-Haussdorff points corresponding to open orbits when G is non-compact (see Section 5 and 6). We will check smoothness away from such singularities.

From Hypothesis 3 and the fact that $i_x^*\alpha(w) = \langle\theta\wedge[\theta\wedge\theta]\rangle$ (Remark 3.3), we deduce $S_X(w,s) - S_X(w,s') \in \mathbb{Z}$.

Alternatively, and perhaps more simply, by Hypothesis 3 the class $-\left[\langle\theta\wedge[\theta\wedge\theta]\rangle\right]/6$ pulls back via g to an integral class on X, so that its integral over X is an integer. We conclude by using the fact that any two sections of P are related by a gauge transformation.

For the second claim, assume $g' : X' \to G$ coincides with g under an orientation reversing diffeomorphism $\partial X \cong \partial X'$. Glue X to X' along their boundaries by using this diffeomorphism, and consider the induced map $\tilde{g} : g\cup_\partial g' : X\cup_\partial X' \to G$. We have

$$W_{\partial X}(g) - W_{\partial X'}(g') = \int_{X-X'} -\frac{1}{6}\langle\phi_{\tilde{g}}\wedge[\phi_{\tilde{g}}\wedge\phi_{\tilde{g}}]\rangle.$$

By the first claim, this is an integer. □

Hence the behaviour of the Chern-Simons action of manifolds with boundary under gauge transformations can be controlled *along the boundary.* This will be the main ingredient of the construction of the Chern-Simons line bundle.

From Proposition 3.2 and Stockes' theorem, in the case of closed manifolds we also derive the following alternative definition of the Chern-Simons action in terms of Chern-Weil forms:

Proposition 3.8. *Let $P \to W$ be a principal G-bundle with connection w over a compact oriented 4-manifold W. Denote the curvature of w by Ω, and ∂w the restriction to the boundary. Then*

$$S_{\partial W}(\partial w) = \int_W \langle\Omega\wedge\Omega\rangle \mod(1).$$

Example: the metric Chern-Simons invariant. Let (X^3, g) be a riemannian manifold, and $\nabla_g : TX \to T^*X\otimes TX$ the Levi-Civita connection. The bundle with covariant derivative (TX, ∇_g) is associated to a principal $SO(3)$-bundle, the *frame bundle* $F(X)$, with a connection $\tilde{\nabla}_g$ (see Section 2). It is a classical result that any closed orientable manifold is parallelizable, that is, has a trivial tangent bundle. Hence $F(X)$ is trivial; this justifies his name: it can be identified with the bundle of frames of X by letting elements of $SO(3)$ act on a fixed frame at some point of X. Below we describe the Ad-invariant form on $so(3)$. Hence we can define a Chern-Simons invariant (the factor 1/2 is put in agreement with [CS])

$$S_X(g) := \frac{1}{2}S_X(\tilde{\nabla}_g). \tag{32}$$

Theorem 3.9. (Chern-Simons [CS]) *$S_X(g)$ is an invariant of the conformal class of g. If (X, g) admits a global conformal immersion in $\mathbb{R}^4$ then $S_X(g) = 0$* mod(1).

In particular, $\mathbb{R}P^3 \cong SO(3)$ with its standard riemannian metric g_s cannot be conformally immersed in $\mathbb{R}^4$, although it is locally conformally flat. This follows from a computation similar to that given at the end of Proposition 3.13 below, which identifies the Chern-Simons form of $\tilde{\nabla}_g$ with $-1/\pi^2$ times the volume form of $SO(3) \subset PSL(2,\mathbb{C})$ (see Remark 3.14). Hence

$$S_{\mathbb{R}P^3}(g_s) = \frac{1}{2} \mod(1).$$

We refer to the paper of Chern and Simons for more on this.

The case of $SL(2,\mathbb{C})$ and $SU(2)$: volumes and the Killing form. Recall that the Lie algebra $sl(2,\mathbb{C})$ of trace-free 2×2-matrices is generated over $\mathbb{C}$ by

$$h=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\quad,\quad e=\begin{pmatrix}0&1\\0&0\end{pmatrix}\quad,\quad f=\begin{pmatrix}0&0\\1&0\end{pmatrix}$$

satisfying the relations

$$[h,e]=2e\quad,\quad [h,f]=-2f\quad,\quad [e,f]=h.$$

Up to scalar multiples the Killing form K is the unique Ad-invariant symmetric bilinear form on $sl(2,\mathbb{C})$. By definition it is given by

$$K(a,b)=\mathrm{Trace}(ad_a\circ ad_b)=-4\mathrm{Trace}(ab)\quad \forall a,b\in sl(2,\mathbb{C})$$

where $ad_a : sl(2,\mathbb{C})\rightarrow sl(2,\mathbb{C})$ is the endomorphism induced by the adjoint action of a, and similarly for b. On the right-hand side, however, a and b stand for their matrix representatives in the fundamental representation given above. The equality between the traces is easily checked. Hence we can set

$$\langle a\otimes b\rangle=\frac{k}{8\pi^2}K(a,b),\quad k\in\mathbb{C}.$$

Let us compute which values of k are admissible for the Integrality Hypothesis 3. In the basis (h,e,f) the Killing form reads

$$K_{(h,e,f)}=\begin{pmatrix}-8&0&0\\0&0&-4\\0&-4&0\end{pmatrix}.$$

Denote by h^*, e^* and f^* the *complex* left-invariant 1-forms associated to h, e and f (that is, dual to them at the identity). The Maurer-Cartan form of $SL(2,\mathbb{C})$ reads

$$\theta=h\otimes h^*+e\otimes e^*+f\otimes e^*.$$

Then

$$\begin{aligned}-(1/6)\langle\theta\wedge[\theta\wedge\theta]\rangle &= -(1/6)\cdot 6\langle h\otimes[e,f]\rangle h^*\wedge e^*\wedge f^*\\ &= \frac{k}{\pi^2}h^*\wedge e^*\wedge f^*.\end{aligned}$$

We get very similar results with the *real* Lie algebras $su(2)$ and $so(3)$. Recall that $su(2)$ can be represented as the Lie algebra of trace free 2×2-skew-hermitian matrices. Under multiplication[2] by $-i$ we can take as generators the Pauli matrices

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}\quad,\quad \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix}\quad,\quad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

with the Lie bracket $[a,b]=i(ab-ba)$. We have the relations

$$[\sigma_1,\sigma_2]=-2\sigma_3\quad,\quad[\sigma_2,\sigma_3]=-2\sigma_1\quad,\quad[\sigma_3,\sigma_1]=-2\sigma_2. \tag{33}$$

[2]this choice yields real valued $SU(2)$ Chern-Simons invariants.

Since $su(2)$ is a Lie subalgebra of $sl(2,\mathbb{C})$, the above Killing form restricts to define a real-valued ad-invariant symmetric bilinear form. In the basis $(\sigma_1,\sigma_2,\sigma_3)$ it is -4 times the identity. Thus, expanding as above the Maurer-Cartan form of $SU(2)$ we find

$$\begin{aligned} -(1/6)\langle\theta\wedge[\theta\wedge\theta]\rangle &= -\langle\sigma_1\otimes[\sigma_2,\sigma_3]\rangle\sigma_1^*\wedge\sigma_2^*\wedge\sigma_3^* \\ &= -\frac{k}{\pi^2}\sigma_1^*\wedge\sigma_2^*\wedge\sigma_3^*. \end{aligned} \tag{34}$$

Consider now the Lie algebra $so(3)$. It has the standard basis (A_{12},A_{13},A_{23}) given in (42); the linear map defined by $(A_{12},A_{13},A_{23})\mapsto(\sigma_1/2,\sigma_2/2,\sigma_3/2)$ provides an isomorphism with $su(2)$. The Killing form of $so(3)$ is the trace. Computing $\mathrm{Trace}(A_{12}^2)=-2$ we get

$$-(1/6)\langle\theta\wedge[\theta\wedge\theta]\rangle = -\frac{k}{4\pi^2}A_{12}^*\wedge A_{13}^*\wedge A_{23}^*. \tag{35}$$

Equations (34)-(35) are volume forms; since $Vol(SU(2))=2\pi^2$, for $k=1$ the former is minus the pull-back of the standard volume form on $SO(3)$ by the two-fold regular covering $SU(2)\to SO(3)$. Because $SL(2,\mathbb{C})$ (resp. $PSL(2,\mathbb{C})$) is homotopy equivalent to $SU(2)$ (resp. $PSU(2)\cong SO(3)$) we deduce:

Lemma 3.10. *Let $\mathfrak{g}=sl(2,\mathbb{C})$ or $\mathfrak{g}=su(2)$, identified with their fundamental representations as matrix Lie algebras. Any Ad-invariant symmetric bilinear form $\langle\cdot,\cdot\rangle:\mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$ satisfying Hypothesis 3 is of the form*

$$\langle a\otimes b\rangle = -\frac{k}{2\pi^2}\mathrm{Trace}(ab)$$

where $k\in\mathbb{Z}[1/2]$ if $G=SL(2,\mathbb{C})$ or $SU(2)$, and $k\in\mathbb{Z}$ if $G=PSL(2,\mathbb{C})$. For $\mathfrak{g}=so(3)$ we have

$$\langle a\otimes b\rangle = \frac{k}{8\pi^2}\mathrm{Trace}(ab),\quad k\in 4\mathbb{Z}.$$

Remark 3.11. The integer k is usually called the *level.* Besides providing a unified treatment of the normalization, it is used to rescale the symplectic structure on moduli spaces of flat connections on surfaces, that we will derive from the Chern-Simons action in Section 5. This will be crucial for quantization.

Remark 3.12. The normalization $k=1$ for $G=SL(2,\mathbb{C})$ or $\mathfrak{g}=SU(2)$ is -4 times the one derived from Chern-Weil theory (see Section 8). For $G=SO(3)$ and the Riemannian Chern-Simons invariant (32), the standard normalization is $k=1/2$.

Exercise 3. Check that for a *closed* manifold and $G=SL(2,\mathbb{C})$ or $SU(2)$, the integral $\int_X -\frac{1}{6}\langle\phi_g\wedge[\phi_g\wedge\phi_g]\rangle$ is the degree of the map $g:X\to G$ (see Proposition 3.6).

In the complex case the decomposition into real and imaginary parts also make the connection with volumes:

Proposition 3.13. *Denote by $vol_{\mathbb{H}}$ the volume form of the hyperbolic 3-space, and $\tilde{\nabla}_{\mathbb{H}}$ the Levi-Civita connection on its frame bundle $F(\mathbb{H})=PSL(2,\mathbb{C})$. We have:*

$$(i/\pi^2)h^*\wedge e^*\wedge f^* = (1/\pi^2)vol_{\mathbb{H}} + d\gamma + i\alpha(\tilde{\nabla}_{\mathbb{H}})$$

where γ is an (explicit) real 2-form.

Proof. First we will identify explicitly $PSL(2,\mathbb{C})$ with the frame bundle $F(\mathbb{H})$ by fixing a point in $\mathbb{H}$ and considering the action of $PSL(2,\mathbb{C}) \cong \mathrm{Isom}^+(\mathbb{H})$. Then we will compute the standard left invariant basis of vector fields on $PSL(2,\mathbb{C})$ (considered as a real manifold) in terms of the pull-backs of a moving frame and the Levi-Civita connection on $\mathbb{H}$ under the identification $PSL(2,\mathbb{C}) \cong F(\mathbb{H})$. Finally we will put the result in the Chern-Simons 3-form to obtain the result.
Let us take the Poincaré upper half space model of $\mathbb{H}$. It is realized as the subspace of the quaternions

$$\mathbb{R}\langle 1,i,j,k\rangle = \left\{ \begin{pmatrix} z & t \\ -\bar{t} & \bar{z} \end{pmatrix},\ z,t \in \mathbb{C} \right\}$$

where the complex coordinate t is real, and $t > 0$. Here we use the standard basis

$$1 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad i := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad , \quad j := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad , \quad k := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Hence

$$\mathbb{H} = \{ z + tj \mid z = x + yi,\ x,y \in \mathbb{R},\ t > 0 \} \in M_2(\mathbb{C})$$

with the usual metric $ds^2 = (dx^2 + dy^2 + dt^2)/t^2$. The action of $PSL(2,\mathbb{C}) = \mathrm{Aut}(\mathbb{CP}^1)$ by Moëbius transformations on the boundary at infinity $\mathbb{CP}^1 = \partial_\infty \mathbb{H}$ extends to the isometric action on $\mathbb{H}$ via the Poincaré extension. It is given by

$$\begin{aligned} \gamma \cdot \zeta &= (a\zeta + b)(c\zeta + d)^{-1} \\ &= \frac{1}{det(c\zeta + d)} \left((az + b)(\overline{cz + d}) + a\bar{c}t + tj \right) \end{aligned} \tag{36}$$

where $\gamma = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{C})$ and $\zeta = z + tj \in \mathbb{H}$. Here we denote by $\cdot$ the action by Poincaré extension, and on the right hand side the operations of multiplication and taking the inverse are the matrix ones, inherited from the quaternions. For instance, if $\gamma \cdot z = \lambda z + \mu$, then $\gamma \cdot \zeta = \lambda z + \mu + |\lambda| tj$, and if $\gamma \cdot z = 1/\bar{z}$, then $\gamma \cdot \zeta = \zeta det(\zeta)^{-1}$. Then the map

$$\begin{array}{rccc} \pi : & PSL(2,\mathbb{C}) & \longrightarrow & \mathbb{H} \\ & \gamma & \longmapsto & \gamma \cdot j \end{array} \tag{37}$$

is a principal fibration; the structure group is the isotropy subgroup at j, which from (36) is computed as $PSU(2) \cong SO(3)$. By considering the action on the standard orthonormal frame at j we get the bundle automorphism

$$\begin{array}{ccc} PSL(2,\mathbb{C}) & \longrightarrow & F(\mathbb{H}) \\ \gamma & \longmapsto & (\gamma \cdot j, \gamma_*(\partial/\partial x, \partial/\partial y, \partial/\partial t)_j). \end{array}$$

Now recall the standard basis vectors h, e, f of the *complex* Lie algebra $sl(2,\mathbb{C})$. Let us denote them by $h_\mathbb{C}$, $e_\mathbb{C}$, $f_\mathbb{C}$, and write h, e, f, ih, ie, if the corresponding basis of $sl(2,\mathbb{C})$ over $\mathbb{R}$. By using (36) we easily compute that for any $u \in \mathbb{R}$ we have

$$\begin{aligned} e^{uh/2} \cdot j &= e^u j \\ e^{u(e+f)} \cdot j &= \begin{pmatrix} \cosh(u) & \sinh(u) \\ \sinh(u) & \cosh(u) \end{pmatrix} \cdot j = \frac{\sinh(2u) + j}{\cosh(2u)} \\ e^{u(ie-if)} \cdot j &= \begin{pmatrix} \cosh(u) & i\sinh(u) \\ -i\sinh(u) & \cosh(u) \end{pmatrix} \cdot j = \frac{2i\sinh(2u) + j}{\cosh(2u)}. \end{aligned}$$

Hence

$$\frac{d}{du}\left(e^{uh/2}\cdot j\right)_{u=0}=j \quad , \quad \frac{d}{du}\left(e^{u(e+f)}\cdot j\right)_{u=0}=2 \quad , \quad \frac{d}{du}\left(e^{u(ie-if)}\cdot j\right)_{u=0}=2i$$

which implies, by using the fibration π in (37),

$$\text{(38)}\quad d\pi_{e*}(h/2)=(\partial/\partial t)_j \quad , \quad d\pi_{e*}(e+f)=2(\partial/\partial x)_j \quad , \quad d\pi_{e*}(ie-if)=2(\partial/\partial y)_j.$$

Denote by $\theta=\{\theta_i\}$ the fundamental form (or *moving frame*) of the riemannian connection on $\mathbb{H}$ dual to $(\partial/\partial t, \partial/\partial x, \partial/\partial y)_j$ at j, and h^*, e^*, f^*, $(ih)^*$, $(ie)^*$, $(if)^*$ the basis of left-invariant (real) vector fields on $PSL(2,\mathbb{C})$ associated to the basis h, e, f, ih, ie, if of $sl(2,\mathbb{C})_{\mathbb{R}}$. Since both basis are left-invariant under the left action of $PSL(2,\mathbb{C})$, from (38) we deduce

$$\text{(39)}\qquad \pi^*\theta_1=2h^*=2\sigma_3^* \quad , \quad \pi^*\theta_2=e^*+f^*=\sigma_1^* \quad , \quad \pi^*\theta_3=(ie)^*-(if)^*=-\sigma_2^*$$

where we denote by σ_i the Pauli matrices. In fact, remember the polar decomposition diffeomorphism $PSL(2,\mathbb{C})\cong\mathcal{H}\times SO(3)$, where $\mathcal{H}$ denotes the set of stricly positive 2×2-hermitian matrices with unit determinant. The fibration π identifies $\mathcal{H}$ with $\mathbb{H}$. Its tangent space at the identity is $T_{id}\mathcal{H}=\mathbb{R}\langle\sigma_1,\sigma_2,\sigma_3\rangle$, the set of trace free 2×2-hermitian matrices. Since $SU(2)$ is the space of quaternions of unit norm (ie. determinant 1), we have also $su(2)=T_{id}SU(2)=\mathbb{R}\langle i,j,k\rangle$, the orthogonal complement to the identity, which is the same as $\mathbb{R}\langle i\sigma_3, i\sigma_2, i\sigma_1\rangle=iT_{id}\mathcal{H}$. Hence (39) just identifies the (dual of the) first summand of the isomorphism of tangent spaces $sl(2,\mathbb{C})_{(h,e,f)}=_{\mathbb{R}} T_{id}\mathcal{H}_{(\sigma_3,\sigma_2,\sigma_1)}\oplus_{\mathbb{R}} su(2)_{(i,j,k)}$ (the factor 2 in $f^*\theta_1=2h^*=2\sigma_3^*$ is because the map $SU(2)\to SO(3)$ is degree two). This discussion gives an alternative argument to the above computation.

Next consider the Levi-Civita connection $(\theta_{ij})\in A^1_{\mathbb{H}}(so(3))$. It is characterized by the torsion-free condition

$$\text{(40)}\qquad d\theta_i+\sum_{j=1}^{3}\theta_{ij}\wedge\theta_j=0$$

which consequently completely determines $(\pi^*\theta_{ij})$. For instance, by evaluating on left-invariant vector fields X and Y we have an identity

$$\begin{aligned}-(e^*+f^*)([X,Y])=-\pi^*\theta_2([X,Y])&=\pi^*d\theta_2(X,Y)\\ &=(-\pi^*\theta_{23}\wedge\pi^*\theta_3-\pi^*\theta_{21}\wedge\pi^*\theta_1)(X,Y).\end{aligned}$$

Proceeding in the same way for the two other $d\theta_i$ we get a linear system which, by using (39) and letting X and Y span a basis dual to h^*, …, $(if)^*$, is solved by

$$\text{(41)}\qquad \pi^*\theta_{12}=e^*-f^* \quad , \quad \pi^*\theta_{13}=(ie)^*+(if)^* \quad , \quad \pi^*\theta_{23}=-2(ih)^*.$$

By comparing with (39) and decomposing into real and imaginary parts as $h^*_{\mathbb{C}}=h^*+i(ih)^*$ etc, we eventually get

$$h^*_{\mathbb{C}}=\frac{1}{2}\pi^*(\theta_1-i\theta_{23}) \quad , \quad e^*_{\mathbb{C}}=\frac{1}{2}\pi^*((\theta_2+\theta_{12})+i(\theta_3+\theta_{13}))$$
$$f^*_{\mathbb{C}}=\frac{1}{2}\pi^*((\theta_2-\theta_{12})-i(\theta_3-\theta_{13})).$$

Let us now compute the Chern-Simons 3-form. By inserting these expressions into $h^*_{\mathbb{C}} \wedge e^*_{\mathbb{C}} \wedge f^*_{\mathbb{C}}$ we find that this complex valued 3-form has real part

$$\mathrm{Re}(h^*_{\mathbb{C}} \wedge e^*_{\mathbb{C}} \wedge f^*_{\mathbb{C}}) = \frac{1}{4}(\theta_{12} \wedge \theta_{13} \wedge \theta_{23} - \theta_1 \wedge \theta_2 \wedge \theta_{12} - \theta_2 \wedge \theta_3 \wedge \theta_{23} - \theta_1 \wedge \theta_3 \wedge \theta_{13})$$

and imaginary part

$$\mathrm{Im}(h^*_{\mathbb{C}} \wedge e^*_{\mathbb{C}} \wedge f^*_{\mathbb{C}}) = \frac{1}{4}(-\theta_1 \wedge \theta_2 \wedge \theta_3 + \theta_2 \wedge \theta_{12} \wedge \theta_{23} + \theta_1 \wedge \theta_{12} \wedge \theta_{13} + \theta_3 \wedge \theta_{13} \wedge \theta_{23}).$$

Since $\mathbb{H}$ a constant curvature -1, the riemannian curvature two-form is $\Omega_{ij} = -\theta_i \wedge \theta_j$. Hence Cartan's structure equation read as $d\theta_{ij} = -\Sigma_{k=1,2,3}\theta_{ik} \wedge \theta_{kj} - \theta_i \wedge \theta_j$. A straightforward computation then gives

$$\begin{aligned} d(\theta_1 \wedge \theta_{23}) + d(\theta_2 \wedge \theta_{31}) + d(\theta_3 \wedge \theta_{12}) = \\ 3\theta_1 \wedge \theta_2 \wedge \theta_3 + \theta_2 \wedge \theta_{12} \wedge \theta_{23} + \theta_3 \wedge \theta_{13} \wedge \theta_{23} + \theta_1 \wedge \theta_{12} \wedge \theta_{13}. \end{aligned}$$

Note that $vol_{\mathbb{H}} = \theta_1 \wedge \theta_2 \wedge \theta_3$. Hence

$$\begin{aligned} \frac{i}{\pi^2} h^*_{\mathbb{C}} \wedge e^*_{\mathbb{C}} \wedge f^*_{\mathbb{C}} = \frac{1}{\pi^2} vol_{\mathbb{H}} - \frac{1}{4\pi^2} d(\theta_1 \wedge \theta_{23} + \theta_2 \wedge \theta_{31} + \theta_3 \wedge \theta_{12}) + \\ + \frac{i}{4\pi^2}(\theta_{12} \wedge \theta_{13} \wedge \theta_{23} - \theta_1 \wedge \theta_2 \wedge \theta_{12} - \theta_2 \wedge \theta_3 \wedge \theta_{23} - \theta_1 \wedge \theta_3 \wedge \theta_{13}). \end{aligned}$$

We are left to identify the imaginary part. Consider the standard basis of $so(3)$, given by

$$A_{12} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad A_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad , \quad A_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}. \tag{42}$$

It satisfies the relations

$$[A_{12}, A_{13}] = -A_{23} \quad , \quad [A_{12}, A_{23}] = A_{13} \quad , \quad [A_{13}, A_{23}] = -A_{12}.$$

By Lemma 3.10 and Remark 3.12 the normalized bracket is $\langle A \otimes B \rangle = \mathrm{Trace}(AB)/8\pi^2$. Moreover, $\Omega_{ij} = -\theta_i \wedge \theta_j$ implies that the $so(3)$-components of Ω_{12}, Ω_{13} and Ω_{23} are $-A_{13}$, A_{23} and $-A_{12}$ (remember that the θ_i are dual to $(\partial/\partial t, \partial/\partial x, \partial/\partial y)_j$ in this order, and that these have coefficients A_{12}, A_{23} and A_{13}). From equation (40) we see that the $so(3)$-components of θ_{12}, θ_{13} and θ_{23} are A_{13}, $-A_{23}$ and A_{12}. Hence denoting by θ the connection matrix (θ_{ij}) of $F(\mathbb{H})$ and similarly for its curvature Ω we find

$$\langle \theta \wedge \Omega \rangle = (1/4\pi^2)(\theta_{12} \wedge \Omega_{12} + \theta_{13} \wedge \Omega_{13} + \theta_{23} \wedge \Omega_{23}).$$

Also, using (11) we get

$$[\theta \wedge \theta] = 4A_{12} \otimes (\theta_{12} \wedge \theta_{13}) + 4A_{23} \otimes (\theta_{12} \wedge \theta_{23}) + 4A_{13} \otimes (\theta_{13} \wedge \theta_{23})$$

and similarly

$$\theta \wedge [\theta \wedge \theta] = 4(A_{12}^{\otimes 2} + A_{13}^{\otimes 2} + A_{23}^{\otimes 2}) \otimes (\theta_{12} \wedge \theta_{13} \wedge \theta_{23}).$$

Hence

$$\langle \theta \wedge [\theta \wedge \theta] \rangle = \frac{4}{16\pi^2}\mathrm{Trace}(A_{12}^2 + A_{13}^2 + A_{23}^2)\theta_{12} \wedge \theta_{13} \wedge \theta_{23} = -\frac{3}{2\pi^2}\theta_{12} \wedge \theta_{13} \wedge \theta_{23}$$

and finally

$$\begin{aligned}\alpha(\theta) &= \langle\theta\wedge\Omega\rangle - \frac{1}{6}\langle\theta\wedge[\theta\wedge\theta]\rangle \\ &= (1/4\pi^2)(\theta_{12}\wedge\Omega_{12}+\theta_{13}\wedge\Omega_{13}+\theta_{23}\wedge\Omega_{23}+\theta_{12}\wedge\theta_{13}\wedge\theta_{23}) \\ &= \operatorname{Im}\left(\frac{i}{\pi^2}h_{\mathbb{C}}^*\wedge e_{\mathbb{C}}^*\wedge f_{\mathbb{C}}^*\right). \qquad (43)\end{aligned}$$

This concludes the proof. □

Remark 3.14. By similar computations it is readily checked that the formula

$$\alpha(\theta) = (1/4\pi^2)(\theta_{12}\wedge\Omega_{12}+\theta_{13}\wedge\Omega_{13}+\theta_{23}\wedge\Omega_{23}+\theta_{12}\wedge\theta_{13}\wedge\theta_{23})$$

holds true for the Levi-Civita connection θ of any Riemannian 3-manifold.

4. Rigidity and volumes of representations

Recall that the space $\mathcal{A}_P$ of connections on P is an affine subspace of $A^1_P(\mathfrak{g})$, and that the Chern-Simons action is a smooth function on $\mathcal{A}_P$. Next we are going to compute the derivative of S_X.
Note that by (6)-(7), a tangent vector $\dot{w}$ to $\mathcal{A}_P$ satisfies

$$\begin{aligned} i_x^*\dot{w} &= 0 \\ R_g^*\dot{w} &= \mathrm{Ad}_{g^{-1}}\dot{w}. \end{aligned} \qquad (44)$$

Hence, as for the curvature (see (29)) we see that $\dot{w}$ is a one-form on the adjoint bundle $P_{\mathfrak{g}}$.

Proposition 4.1. *Let w_t be a smooth path of connections on P which restricts to a fixed connection over ∂X. Denote by $w = w_0$, $\dot{w} = \frac{d}{dt}(w_t)_{t=0}$, and Ω the curvature of w. Then, for any section p of $\pi : P \to X$ we have*

$$\frac{d}{dt}\left(S_X(w_t,p)\right)_{t=0} = 2\int_X p^*\langle\Omega\wedge\dot{w}\rangle.$$

Proof. Consider the cylinder $[0,t]\times X$. In case X has boundary, smooth the corners $\{0\}\times\partial X$ and $\{1\}\times\partial X$. The path of connections w_s on P, with $s\in[0,t]$, forms a single connection $w.$ on the pull-back of the bundle $\pi : P\to X$ by the projection $[0,t]\times X\to X$.
Denote by Ω_s the curvature of $w.$ at time s, and $\Omega_{(s)}$ the curvature of the connection w_s on P at time s. Then

$$\begin{aligned}\Omega_s &= d(w.)_s + \frac{1}{2}[w_s\wedge w_s] \\ &= dw_s + ds\wedge\frac{d}{ds}(w_s) + \frac{1}{2}[w_s\wedge w_s] \\ &= \Omega_{(s)} + ds\wedge\frac{d}{ds}(w_s).\end{aligned}$$

Let us write $\tilde{p}$ the pull-back section to the cylinder. Because $w.$ is constant over ∂X we have

$$S_{[0,t]\times\partial X}(w.,\tilde{p}) = \int_{[0,t]\times\partial X}\tilde{p}^*\alpha(w.) = 0.$$

Hence, using Proposition 3.2(1) we get

$$\begin{aligned} S_X(w_t,p)-S_X(w_0,p) &= \int_{\partial([0,t]\times X)} p^*\alpha(w.) \\ &= \int_{[0,t]\times X} \tilde{p}^*\langle \Omega_s\wedge\Omega_s\rangle \\ &= 2\int_{[0,t]\times X} \tilde{p}^*\langle \Omega_{(s)}\wedge ds\wedge \frac{d}{ds}(w_s)\rangle \\ &= 2\int_0^t\int_X p^*\langle \Omega_{(s)}\wedge \frac{d}{ds}(w_s)\rangle. \end{aligned}$$

The result follows by differentiating at $t=0$. □

Corollary 4.2. *The Chern-Simons action is constant along any path rel(∂) of* flat connections*, that is, connections with curvature* $\Omega=0$*.*

Similarly as in the proof of Proposition 4.1, by using $d\alpha(w)=\langle\Omega\wedge\Omega\rangle$ we get:

Proposition 4.3. *Let W be a compact oriented 4-manifold with boundary $X_0\cup(\partial X_0\times[0,1])\cup(-X_1)$, and $P\to W$ a trivial principal G-bundle with a flat connection w and a section s. Denote by w_i, s_i the restrictions of w and s to X_i.*
If the restriction of s^*w to the cylinder $\partial X_0\times[0,1]$ has no dt component, then

$$S_{X_0}(w_0,s_0)=S_{X_1}(w_1,s_1).$$

We say that the Chern-Simons action is an invariant of flat cobordism rel(∂)*.*

This result applies in particular when $W=[0,1]\times X$ and X has no boundary. Then it says that the Chern-Simons action is *rigid* under continuous deformations of (gauge equivalence classes of) connections on X. To see this we use the next lemma (note that it holds true in any dimension).

Lemma 4.4. *Let $Q\to Y$ be a G-bundle over a manifold Y, and w a connection on the pull-back bundle $[0,\infty)\times Q\to[0,\infty)\times Y$ to the cylinder over Y.*
(i) There exists a unique gauge transformation φ of $[0,\infty)\times Q$ such that $\varphi_{|\{0\}\times Q}=id$ and φ^*w has no dt component. Hence φ^*w defines a path η_t of connections over Y.
(ii) If moreover the curvature of w vanishes then the path η_t is constant.
(iii) As a consequence, for any any pseudo-isotopy $\psi:[0,1]\times Y\to[0,1]\times Y$ and any flat connection η on $Q\to Y$, the connections $\psi_0^*\eta$ and $\psi_1^*\eta$ are gauge equivalent.

Since $[0,\infty)$ is contractible any bundle $P\to[0,\infty)\times Y$ is isomorphic to $[0,\infty)\times Q$, where Q is the restriction of P to $\{0\}\times Y$. Hence the lemma says that, up to gauge equivalence, connections over cylinders are paths of connections on a boundary component, and that flat connections correspond to constant paths. That is, denoting by $\mathcal{G}$ the group of gauge transformations, we have an identification

$$\mathcal{A}_P/\mathcal{G}=\mathrm{Map}([0,\infty)\,,\mathcal{A}_Q)/\mathcal{G}.$$

Moreover, denoting spaces of gauge equivalence classes of flat connections by $\mathcal{M}$, we have

$$\mathcal{M}_{[0,\infty)\times Y}\cong\mathcal{M}_Y.$$

In particular, a section s such that s^*w satisfies the hypothesis of Proposition 4.3 is the composition of a section $id \times s'$, $s' : Y \to Q$, and a gauge transformation as in the lemma. Such sections s are called *temporal gauges*.

Proof. Denote by $g_t : Q \to G$ the map associated to $\varphi_{|\{t\}\times Q}$, and $\phi_t = g_t^* w$. We can write w as

$$w = \eta_t + \zeta_t dt, \quad t \in [0, \infty)\,, \eta_t \in A^1_Q(\mathfrak{g}), \zeta_t \in A^0_Q(\mathfrak{g}).$$

From equation (16) we see that $\varphi^* w(\partial/\partial t) = 0$ if and only if

$$\mathrm{Ad}_{g_t^{-1}} \zeta_t + \phi_t(\frac{\partial}{\partial t}) = 0.$$

This is a first order ordinary differential equation, with a unique solution such that $g_0 = id$.

The curvature Ω of $\varphi^* w = \eta_t$ at time s is $\Omega_s = \Omega_{(s)} + ds \wedge \dfrac{d}{ds}(\eta_s)$, where $\Omega_{(s)}$ is the curvature of η_s (see Proposition 4.1). If the curvature of w vanishes, then so does Ω. Hence $ds \wedge \dfrac{d}{ds}(\eta_s) = 0$, which implies that η_t is constant.

Take w as the pull-back of η. Since w does not depend on t and η is flat, w is flat. Consider the connection $\psi^* w$ on $[0, 1] \times Q$. By (i)-(ii) there is a gauge transformation φ such that $\varphi^*\psi^* w$ is a constant path of connections on Q, and $\varphi_0 = id$. Hence $\psi_0^*\eta = \varphi_0^*\psi_0^* w = \varphi_1^*\psi_1^* w = \varphi_1^*\psi_1^*\eta$. $\square$

Proposition 4.3 shows the fundamental topological nature of flat connections. Note that the identity $\varphi^*\Omega = R_{g*}\Omega$ implies that flat connections make an invariant subset of $\mathcal{A}_P$ under gauge transformations.

The next proposition singles out some characterizations of this space. First we need a definition.

Definition 4.5. The trivial connection *on a trivial product bundle $U \times G \to U$ is the pull-back of the Maurer-Cartan form by the projection $proj_2 : U \times G \to G$:*

$$\theta_{(x,g)} = (L_{g^{-1}} \circ proj_2)_{(x,g)*}.$$

For any trivial principal G-bundle $\pi : P \to U$ and any trivialization $\tau : P \to U \times G$, we call $\tau^*\theta$ the trivial connection of π (induced by the trivialization).

For an arbitrary *principal G-bundle we say that a connection is flat if its curvature vanishes.*

Exercise 4. Show that the trivial connection on a trivial principal G-bundle $P \to U$ with trivialization $\tau : P \to U \times G$ is the pull-back of the Maurer-Cartan connection on the G-bundle over a point:

$$\begin{array}{ccc} E & \xrightarrow{proj_2 \circ \tau} & G \\ \big\downarrow & & \big\downarrow \\ U & \longrightarrow & \{pt\}. \end{array}$$

Proposition 4.6. *Let $\pi : P \to X$ be a principal G-bundle over a connected manifold X, and w a connection on P. The followings are equivalent:*

(1) *w is a flat connection (ie. $\Omega = 0$);*

(2) *the horizontal distribution HP is integrable;*
(3) *for each point of x there is a neighborhood U of x and a trivialization of $\pi_{|\pi^{-1}(U)}$ such that the restriction of w to $\pi^{-1}(U)$ is the trivial connection;*
(4) *there is a covering of X by open sets U_i such that $\pi_{|\pi^{-1}(U_i)}$ is trivial, the transition functions $g_{ij} : U_i \cap U_j \to G$ are constant, and w agrees with the trivial connection on $\pi_{|\pi^{-1}(U_i)}$.*
(5) *The parallel transport defines a conjugacy class of* holonomy representations

$$\rho : \pi_1(X, x) \longrightarrow G$$

which determine the gauge equivalence class of w uniquely.

Hence, denoting by $\mathcal{G}$ the group of gauge transformations and

$$\mathcal{M}_X = \{w \in \mathcal{A}_P | \Omega = 0\}/\mathcal{G},$$

we have a natural identification of sets

$$\mathcal{M}_X = \mathrm{Hom}(\pi_1(X, x), G)/G.$$

Since holonomy representations are well-defined up to conjugacy we often omit the base point x when writing $\pi_1(X, x)$.

Proof. By its very definition Ω vanishes if and only the Lie bracket of horizontal vectors fields is horizontal. Hence equivalence of (1) and (2) is Frobenius' integrability theorem, which asserts that HP defines a foliation F with tangeant space H_pP at each point $p \in P$. That (3) implies (1) is a consequence of Cartan's structure equation (15) for the Maurer-Cartan form, which holds true also for the trivial connections of product bundles and hence implies they have vanishing curvature.
Next we prove (1) $\Rightarrow$ (3). Fix $p \in P$ and put $x = \pi(p)$. Since $d\pi_p : H_pP \to T_xX$ is an isomorphism we can find a neighborhood V of p in the leaf F through p such that $\pi_{|V} : V \to U$ is a diffeomorphism. Denote by $s = \pi_{|V}^{-1} : U \to V$ the inverse section of $\pi_{|\pi^{-1}(U)}$. Hence $\pi_{|\pi^{-1}(U)}$ is trivial, with trivialization

$$\begin{array}{rccc} \tau : & U \times G & \longrightarrow & \pi^{-1}(U) \\ & (y, g) & \longmapsto & s(y) \cdot g. \end{array}$$

For any $g \in G$ and $q \in V$ we know that (see (5))

$$T_{y \cdot g}V = H_{y \cdot g}P = R_{g*}H_pP = R_{g*}T_yV.$$

Hence w and the trivial connection on $\pi_{|\pi^{-1}(U)}$ define the same horizontal subspaces in $\pi^{-1}(U)$, and therefore coincide.
Assume now that (3) holds true, and let U_i and U_j be two open sets as above, with $U_i \cap U_j \neq \emptyset$ and trivialisations τ_i and τ_j. The map $\tau_{ij} = \tau_j^{-1} \circ \tau_i : (U_i \cap U_j) \times G \to (U_i \cap U_j) \times G$ preserves the trivial connection. Hence it permutes the leaves of the horizontal foliation. This implies for all $x \in U_i \cap U_j$ and $g \in G$ we have $\tau_{ij}(x, g) = (xg_{ij}g)$ for some constant $g_{ij} \in G$ (see Exercise 2 (1)). In other words, the transition functions are constant over $U_i \cap U_j$.

Conversely, assume (4). With the same notations as above, we have a commutative diagram of bundle maps

$$\begin{tikzcd} (U_i \cap U_j) \times G \arrow[r, "\tau_{ij}"] \arrow[d, "proj_2"] & (U_i \cap U_j) \times G \arrow[d, "proj_2"] \\ G \arrow[r, "L_{g_{ij}}"] & G. \end{tikzcd}$$

Since the Maurer-Cartan form θ on G is left-invariant we have $\tau_{ij}^* proj_2^* \theta = proj_2^* \theta$. Hence the trivial connections agree on $\pi^{-1}(U_i \cap U_j)$, and we can glue them to define the connection w on P, which necessarily has vanishing curvature.
We conclude with (1) $\Leftrightarrow$ (5). For each point $p \in P_x$ and each loop $\gamma : [0,1] \to X$ based at x the parallel transport defines a map

$$\rho_{x,p} : \gamma \longmapsto g^{-1}, \quad \tilde{\gamma}_p(1) = p \cdot g$$

where $\tilde{\gamma}_p$ is the horizontal lift of γ through p. When $\Omega = 0$ we can consider the restriction $\pi_{|F_p} : F_p \to X$ of the bundle projection to the horizontal leaf F_p through p. Since $d\pi_p : H_pP \to T_xX$ is an isomorphism, $\pi_{|F_p}$ is a covering. In particular, any two homotopic loops on X based at x lift to paths in F_p having the same endpoints. Hence $\rho_{x,p}$ gives the same value to homotopic loops.
Let α, β be two loops based at x and denote by $\tilde{\alpha}$, $\tilde{\beta}$ their horizontal lifts starting at p. Since the lift of β starting at $\tilde{\alpha}(1) = p \cdot \rho_{x,p}(\alpha)^{-1}$ is $\tilde{\beta} \cdot \rho_{x,p}(\alpha)^{-1}$, the lift $\widetilde{\alpha\beta}$ of $\alpha\beta$ starting at p is $\tilde{\alpha}\tilde{\beta} \cdot \rho_{x,p}(\alpha)^{-1}$. Hence

$$p \cdot \rho_{x,p}(\alpha\beta)^{-1} = \widetilde{\alpha\beta}(1) = p \cdot \rho_{x,p}(\beta)^{-1} \rho_{x,p}(\alpha)^{-1} = p \cdot (\rho_{x,p}(\alpha)\rho_{x,p}(\beta))^{-1}.$$

This shows that $\rho_{x,p}$ is a homomorphism. Exercise 2(1) implies that the conjugacy class of $\rho_{x,p}$ does not depend on the choice of p in P_x, and $\rho_{x,p}$ is deformed isomorphically when we move p in a horizontal leaf (hence moving the base point x of loops in X). Since any gauge transformation φ conjugates the values of $\rho_{x,p}$ by $g_\varphi(p)^{-1}$, we deduce that the gauge equivalence class $[w]$ defines a *holonomy*

$$[\rho] \in \mathrm{Hom}(\pi_1(X,x), G)/G$$

represented by the maps $\rho_{x,p}$, where the action of G is by conjugation.
Finally we prove that $[\rho]$ determines the gauge equivalence class of w completely. When X is simply connected the covering $\pi_{|F_p}$ is a diffeomorphism. Hence the section $\pi_{|F_p}^{-1}$ defines a trivialization $P \cong X \times G$, which is an isomorphism of bundles with connections (ie. an *isomorphism of flat bundles*, see the proof of (1) $\Rightarrow$ (3) above) when $X \times G$ has the trivial connection.
For an arbitrary X we consider the pull-back bundle $\tilde{P} \to \tilde{X}$ defined by the commutative diagram

$$\begin{tikzcd} \tilde{P} \arrow[r, "\tilde{cov}"] \arrow[d, "\tilde{\pi}"'] & P \arrow[d, "\pi"] \\ \tilde{X} \arrow[r, "cov"] & X \end{tikzcd}$$

where $\tilde{X} \to X$ the universal covering map. We have just seen that there is an isomorphism of flat bundles $\tilde{\varphi} : \tilde{P} \to \tilde{X} \times G$. To state this isomorphism explicitely let us use the following model of $\tilde{X}$:

$$\tilde{X} = \{\gamma : [0,1] \to X \mid \gamma(1) = x_0\}/\text{homotopy rel}(\partial)$$

where $x_0 \in X$ is our base point. The covering map is $cov([\gamma]) = \gamma(0)$, and the (right) action of $\pi_1(X)$ on $\tilde{X}$ by deck transformations is $([\gamma], \alpha) \mapsto [\gamma\alpha]$. By definition we have

$$\tilde{P} = \{([\gamma], p) \in \tilde{X} \times P \mid \pi(p) = cov([\gamma]) = \gamma(0)\}.$$

Fix a base point $p_0 \in P_{x_0}$. Then

$$\begin{array}{rccc} \tilde{\varphi} : & \tilde{P} & \longrightarrow & \tilde{X} \times G \\ & ([\gamma], p) & \longmapsto & ([\gamma], g_{[\gamma]}(p)) \end{array}$$

where for each point $p \in P$ we set $p = \tilde{\gamma}(0) \cdot g_{[\gamma]}(p)$, and $\tilde{\gamma}$ is the horizontal lift of γ such that $\tilde{\gamma}(1) = p_0$. Under this isomorphism the action of $\pi_1(X)$ on $\tilde{P}$ reads

$$([\gamma]\,\alpha, p) \longmapsto ([\gamma]\,\alpha, g_{[\gamma]\alpha}(p)).$$

Since the right action of α on $[\gamma]$ induces the parallel transport along the lifts of α^{-1} we have

$$p = \tilde{\gamma}(0) \cdot g_{[\gamma]}(p) = \tilde{\gamma}(0)\rho(\alpha) \cdot g_{[\gamma]\alpha}(p).$$

Hence $g_{[\gamma]\alpha}(p) = \rho(\alpha)^{-1} g_{[\gamma]}(p)$. This proves that the action of $\pi_1(X)$ on $\tilde{X}$ lifts to $\tilde{P} \cong \tilde{X} \times G$ as follows:

$$\begin{array}{ccc} (\tilde{X} \times G) \times \pi_1(X) & \longrightarrow & \tilde{X} \times G \\ (([\gamma], g), \alpha) & \longmapsto & ([\gamma]\,\alpha, \rho(\alpha)^{-1} g). \end{array}$$

The isomorphism $\tilde{\varphi}$ thus descends to an isomorphism of flat bundles

$$\varphi : P \to \tilde{X} \times_\rho G$$

where $\tilde{X} \times_\rho G$ is the quotient of $\tilde{X} \times G$ by the above (free) action of $\pi_1(X)$. Note that the bundle projection $\pi_\rho : \tilde{X} \times_\rho G \to X$ is given by $\pi_\rho([([\gamma], g)]) = \gamma(0)$, and the flat connection on $\tilde{X} \times_\rho G$ is induced from the trivial one on $\tilde{X} \times G$, which is left-invariant and so passes to the quotient.

Since any two isomorphisms φ as above (ie. corresponding to different choices of base points x_0 and p_0) are related by a gauge transformation, the conjugacy class $[\rho]$ determines the gauge equivalence class $[w]$ uniquely. □

This proposition has the following striking consequence. Recall the form

$$-(1/6)\langle \theta \wedge [\theta \wedge \theta] \rangle \in A^3_G(\mathbb{C}).$$

Since it is left-invariant, the pull-back via the projection $proj_2 : \tilde{X} \times G \to G$ descends to a 3-form on $\tilde{X} \times_\rho G$, which is just the Chern-Simons 3-form for the canonical flat connection induced by the trivial one on $\tilde{X} \times G$. We denote it by $-(1/6)\overline{\langle \theta \wedge [\theta \wedge \theta] \rangle}$.

Corollary 4.7. *For any closed manifold X and gauge equivalence class $[w]$ of flat connection w on $\pi : P \to X$ we have*

$$S_X([w]) = \int_X -(1/6) s^* \overline{\langle \theta \wedge [\theta \wedge \theta] \rangle}$$

where $s : X \to P$ is an arbitrary section.

Proof. This is a direct consequence of the existence of an isomorphism of bundles $\varphi : P \to \tilde{X} \times_\rho G$ described in Proposition 4.6(5). Such isomorphisms are non canonical, but since S_X does not depend on the choice of section for closed manifolds it is invariant under gauge transformations.
Alternatively, by proposition 4.6(4) the Chern-Simons 3-form $\alpha(w)$ agrees locally with $\alpha(\theta_i)$ over each trivializing set U_i, where θ_i is the trivial connection on $\pi_{|\pi^{-1}(U_i)}$ induced by the trivialization $\tau_i : \pi^{-1}(U_i) \to U_i \times G$. By Remark 3.3 we have

$$\alpha(\theta_i) = \tau_i^* proj_{i,2}^*(-(1/6)\langle\theta \wedge [\theta \wedge \theta])$$

where $proj_{i,2} : \tilde{U}_i \times G \to G$. Hence $\alpha(w)$ coincides with $-(1/6)\overline{\langle\theta \wedge [\theta \wedge \theta]\rangle}$. □

We have seen in Section 3 that for $G = (P)SL(2,\mathbb{C})$ or $G = (P)SU(2)$ the form $-(1/6)\langle\theta \wedge [\theta \wedge \theta]\rangle$ can be interpreted as a normalized volume form, or a complex extension of it. So, for flat connections w it is natural to look for an interpretation of $S_X(w)$ as a kind of volume twisted by the choice of w.

First a general definition. Let X be a closed manifold with holonomy representation $\rho : \pi_1(X) \to G$. Assume that G acts transitively and faithfully on the left by isometries on a 3-dimensional riemannian manifold F with volume form v_F. Consider the F-bundle $P \times_\rho F \to X$ associated to the flat G-bundle $\pi_\rho : P = \tilde{X} \times_\rho G \to X$. It is isomorphic to $\pi_F : \tilde{X} \times_\rho F \to X$ since both have the same holonomy. Since π_ρ is trivial, so is π_F. In fact, for any section $s : X \to \tilde{X} \times_\rho G$ and any point $f_0 \in F$ we have a trivialization

$$\begin{array}{ccc} P \times_\rho F & \longrightarrow & X \times F \\ [(s(x)g, f_0)] & \longmapsto & (x, g \cdot f_0). \end{array}$$

Denote by $\overline{v}_F$ the 3-form on $\tilde{X} \times_\rho F$ obtained by pulling-back v_F via $proj_2 : \tilde{X} \times F \to F$, and then projecting down to $\tilde{X} \times_\rho F$ by using invariance under the action of G by isometries.

Definition 4.8. *The v_F-volume of ρ associated to a section $s : X \to \tilde{X} \times_\rho F$ is the integral*

$$Vol(\rho, s) = \int_X s^* \overline{v}_F.$$

In general $Vol(\rho, s)$ depends on s. If F is contractible, for instance when $F = \mathbb{H}$ is the hyperbolic 3-space and $G = PSL(2,\mathbb{C})$ is identified with the group of orientation preserving isometries of $\mathbb{H}$, then any two sections of π_F are homotopic, so that $Vol(\rho) = Vol(\rho, s)$ is independent of s. When $G = SU(2)$ acts on $F = S^3$ with the spherical metric, then $Vol(\rho) = Vol(\rho, s)$ is well-defined mod(1). In fact Corollary 4.7 just says that the Chern-Simons action of flat $su(2)$-connections is this spherical volume of representations.
Here is the analog for $G = PSL(2,\mathbb{C}) \cong \mathrm{Isom}^+(\mathbb{H})$. Recall that fixing a frame over a point of $\mathbb{H}$ identifies its frame bundle $F(\mathbb{H})$ with $PSL(2,\mathbb{C})$. Given any representation $\rho : \pi_1(X) \to PSL(2,\mathbb{C})$ we can construct a *pseudo-developing map* $\bar{D} : \tilde{X} \to \mathbb{H}$ such that $\bar{D}(\gamma \cdot \tilde{x}) = \rho(\gamma)\bar{D}(\tilde{x})$ for all $\gamma \in \pi_1(X)$, $\tilde{x} \in \tilde{X}$. This map is well-defined up to post-composition by an hyperbolic isometry, and can be obtained by fixing a chart

$U_i \to \mathbb{H}$ arbitrarily, and then proceeding by analytic continuation to develop $\tilde{X}$ in $\mathbb{H}$ using the transition functions of the bundle $\tilde{X} \times_\rho \mathbb{H}$. This yields a map of frame bundles

$$D : F(\tilde{X}) \to F(\mathbb{H}) = PSL(2, \mathbb{C}).$$

Let $\tilde{\nabla}_{\mathbb{H}}$ be the Levi-Civita connection of $F(\mathbb{H})$. Since ρ acts by isometries on $\mathbb{H}$, the pull-back $D^*\tilde{\nabla}_{\mathbb{H}}$ descends to a connection $\nabla^\rho_{\mathbb{H}}$ on the frame bundle $F(X)$. On another hand, on bases $\bar{D}^* vol_{\mathbb{H}}$ descends to a volume form on X.
When X is hyperbolic it is identified with a quotient $\mathbb{H}/\Gamma$, where Γ is a discrete subgroup of $PSL(2, \mathbb{C})$. Then ρ maps $\pi_1(X)$ to a conjugate of Γ in $PSL(2, \mathbb{C})$, and D is an isometric diffeomorphism. We call such a ρ a *hyperbolic holonomy*. Under the identification of $\tilde{X}$ with $\mathbb{H}$ the above connection $\nabla^\rho_{\mathbb{H}}$ is just the usual Levi-Civita connection on the frame bundle, and $D^* vol_{\mathbb{H}}$ the volume form of X.

Proposition 4.9. *For any closed manifold X and any gauge equivalence class $[w]$ of flat $sl(2, \mathbb{C})$-connection w on $\pi : P \to X$ with holonomy representation $\rho : \pi_1(X) \to PSL(2, \mathbb{C})$, we have*

$$S_X([w]) = S_X(\nabla^\rho_{\mathbb{H}}) - \frac{i}{\pi^2} Vol(\rho).$$

In particular, when (X, g) is hyperbolic and ρ is the hyperbolic holonomy we have

$$S_X([w]) = S_X(g) - \frac{i}{\pi^2} Vol(X)$$

where $Vol(X)$ is the volume of X and $S_X(g)$ its metric Chern-Simons invariant.

Proof. Consider the projection $r : \tilde{X} \times PSL(2, \mathbb{C}) \to PSL(2, \mathbb{C})$. By Proposition 3.13, the Chern-Simons 3-form for the trivial connection on the product bundle $\tilde{X} \times PSL(2, \mathbb{C}) \to \tilde{X}$ is just $-ir^*C$, where

$$C = (i/\pi^2) h^* \wedge e^* \wedge f^* = (1/\pi^2) vol_{\mathbb{H}} + d\gamma + i\alpha(\tilde{\nabla}_{\mathbb{H}}).$$

Denote by $p : F(\tilde{X}) \to \tilde{X}$ the projection of the frame bundle. We have a map

$$\tilde{q} : p \times D : F(\tilde{X}) \to \tilde{X} \times PSL(2, \mathbb{C})$$

which is equivariant with respect to the action of $\pi_1(X)$ and hence descends to give a map of principal bundles covering the identity map of X:

$$q : F(X) \to \tilde{X} \times_\rho PSL(2, \mathbb{C}).$$

In particular, given a section s of $F(X)$, we have a section

$$\hat{s} = q \circ s : \hat{s} : X \to \tilde{X} \times_\rho PSL(2, \mathbb{C}).$$

From $r \circ \tilde{q} = D$ we deduce

$$S_X([w]) = \int_X \hat{s}^* \alpha(w) = -i \int_X s^* q^* \overline{r^* C} = -i \int_X s^* \overline{D^* C} = S_X(\nabla^\rho_{\mathbb{H}}) - \frac{i}{\pi^2} Vol(\rho).$$

which proves the result. $\square$

5. The Chern-Simons line bundle, I

In this section we consider the Chern-Simons action of manifolds X *with boundary.* Our aim is to prove the following theorem.

Recall that we denote by

$$\mathcal{M}_X = \{w \in \mathcal{A}_P | \Omega = 0\}/\mathcal{G}$$

the moduli space of flat connections on X. Hence, points in $\mathcal{M}_X$ are *represented* by flat connections on bundles $Q \to X$ related by gauge transformations. We use the same notation $\mathcal{M}_Y$ for oriented surfaces Y. Recall that for any G-bundle $Q \to Y$ and connection $\eta \in \mathcal{A}_Q$ the pairing $\langle\ \wedge\ \rangle : T_\eta\mathcal{A}_Q \wedge T_\eta\mathcal{A}_Q \to \mathbb{C}$ descends to a two-form on Y.

Theorem 5.1. *The Chern-Simons action determines :*

(a) *for each* closed *oriented surface Y, a smooth hermitian line bundle $\mathcal{L}_Y \to \mathcal{M}_Y$ with connection ϑ, whose curvature times $i/2\pi$ is given by*

$$\bar{\Xi}_{[\eta]}([\dot{\eta}_1]\,,[\dot{\eta}_2]) = -2\int_Y \langle \dot{\eta}_1 \wedge \dot{\eta}_2 \rangle, \quad \dot{\eta}_i \in T_\eta\mathcal{A}_Q$$

where η is a flat connection on an arbitrary G-bundle $Q \to Y$ with section $q : Y \to Q$, and $[\ ]$ denotes the gauge equivalence class.

When the bracket $\langle\quad\rangle$ is non degenerate the (closed) two-form $\bar{\Xi}$ defines a symplectic form on $\mathcal{M}_X$ (complex if G is complex).

(b) *for each compact oriented 3-manifold X a parallel section*

$$e^{2i\pi S_X(\cdot)} : \mathcal{M}_X \longrightarrow r_X^*\mathcal{L}_{\partial X}$$

of the pull-back bundle by the restriction map $r_X : \mathcal{M}_X \to \mathcal{M}_{\partial X}$ (we put $L_\emptyset := \mathbb{C}$ and $\mathcal{M}_\emptyset := \{pt\}$).

These assignments are functorial, additive, oriented, and satisfy gluing laws. If the Lie group G is compact, the connection ϑ is unitary and $e^{2i\pi S_X(\cdot)}$ has unit norm.

Definition 5.2. *We call $(\mathcal{L}_Y \to \mathcal{M}_Y, \vartheta)$ the Chern-Simons line bundle.*

Let us draw some important consequences of Theorem 5.1. The assertion that $e^{2i\pi S_X(\cdot)}$ is a parallel section of the bundle $r_X^*\mathcal{L}_{\partial X} \to \mathcal{M}_X$ implies that the variation of the Chern-Simons action along a path of connections on X can be computed as a parallel transport, by integrating the connection ϑ along the restriction of the path to the boundary. This yields a generalization of Corollary 4.2.

The symplectic form $\bar{\Xi}$ is well-known for $G = SU(2)$ or $G = (P)SL(2,\mathbb{C})$ since the works of Atiyah-Bott [AB] and Goldman [Go1] (see Section 6); on the Teichmüller component of the moduli space of flat $sl(2,\mathbb{R})$-connections it coincides with the Weil-Petersson form [Go1, §2]. For dimensional reasons, the map r_X is then a Lagrangian map (in fact, since $r_X^*\mathcal{L}_{\partial X}$ has no holonomy we have $r_X^*\bar{\Xi} = 0$).

Finally, by Chern-Weil theory the cohomology class $\left[\bar{\Xi}\right] \in H^2(\mathcal{M}_Y;\mathbb{C})$ is integral, and is the first Chern class of the bundle $\mathcal{L}_Y \to \mathcal{M}_Y$ (see section 8).

The proof of Theorem 5.1 will be done in several steps. It consists of a process of symplectic reduction, by defining a line bundle with connection over the whole space of connections on a fixed G-bundle over Y, and then passing to the quotient by a suitable action of the group $\mathcal{G}$ of gauge transformations. A first step should certainly be to give a smooth structure to the space $\mathcal{M}_X$, which is endowed with the quotient topology. As it is not essential in order to understand Theorem 5.1 we prefer to postpone it to the end of the section.

Fix a G-bundle $Q \to Y$, and denote by $\Gamma(Q)$ the set of its sections. For each $\eta \in \mathcal{A}_Q$ set

$$L_\eta = \{f : \Gamma(Q) \to \mathbb{C} \mid \forall \varphi \in \mathcal{G}, f(\varphi s) = c(s^*\eta, g_\varphi s)^{-1} f(s)\}$$

where, as usual, for any gauge transformation $\varphi : Q \to Q$ we denote by $g_\varphi : Q \to G$ the map given by $\varphi(q) = q \cdot g_\varphi(q)$, we write φs for $\varphi \circ s$ etc., and for all $a \in A^1_Y(\mathfrak{g})$ and maps $g : Y \to G$ we put

$$c(a, g) = \exp\left(2\pi i \int_Y \langle Ad_{g^{-1}} a \wedge \phi_g\rangle + 2\pi i W_Y(g)\right). \tag{45}$$

Here we use the notations $\phi_g = g^*\theta$, where θ is the Maurer-Cartan form, and (see Proposition 3.7(2))

$$W_Y(g) = \int_X -\frac{1}{6}\langle \phi_{\tilde{g}} \wedge [\phi_{\tilde{g}} \wedge \phi_{\tilde{g}}]\rangle$$

where X is an arbitrary compact oriented 3-manifold with boundary $\partial X = Y$, and $\tilde{g} : X \to G$ extends g to X.

Lemma 5.3. *The set L_η is a well-defined complex hermitian line, with a trivialization given for each section $q : Y \to Q$ by*

$$\begin{array}{rccc} s_{q,\eta} : & L_\eta & \longrightarrow & \mathbb{C} \\ & f & \longmapsto & f(s). \end{array}$$

Moreover, for any connection w on a principal G-bundle $P \to X$ over a compact oriented 3-manifold X we have (∂w denotes the restriction to ∂X)

$$e^{2\pi i S_X(w)} \in L_{\partial w}.$$

When the Lie group G is compact the trivializations s_q are unitary and $e^{2\pi i S_X(w)}$ has unit norm.

Proof. First note that $\varphi = id$ implies g_φ is the constant map to the identity element of G, so that $c(s^*\eta, g_\varphi s) = c(s^*\eta, s) = 1$ and thus $f(id \circ s) = f(s)$. We have to check the coherence of the definition of L_η with respect to the action of the group of gauge transformations, that is

$$\forall f \in L_\eta, \forall \varphi, \psi \in \mathcal{G} \quad f((\psi\varphi)s) = f(\psi(\varphi s)).$$

In other words, L_η is the set of invariant sections of the functor mapping any section $s : Y \to Q$ to $\mathbb{C}$, and any gauge transformation φ of $Q \to Y$, which is a morphism $\varphi : s \mapsto s' = \varphi s$, to the morphism of $\mathbb{C}$ given by multiplication with $c(s^*\eta, g_\varphi s)$.
Abusing of notations, let us denote by g_1 and g_2 the maps $g_\varphi s$ and $g_\psi s$, respectively, and at the same time arbitrary extensions of these maps to a compact oriented 3-manifold with boundary $\partial X = Y$. Let us write $g_1 g_2 : Y \to G$ the map obtained by pointwise multiplication. Setting $\omega(g) = -(1/6)\langle \phi_g \wedge [\phi_g \wedge \phi_g]\rangle$ we have

$$c(a, g_1 g_2) = \exp\left(2\pi i \int_Y \langle \mathrm{Ad}_{(g_1 g_2)^{-1}} a \wedge \phi_{g_1 g_2}\rangle + 2\pi i \int_X \omega(g_1 g_2)\right).$$

Since

$$\phi_{g_1 g_2} = \mathrm{Ad}_{g_2^{-1}}\phi_{g_1} + \phi_{g_2},$$

by using Ad-invariance and Cartan's equation (15) we find

$$\begin{aligned}\omega(g_1g_2) &= \omega(g_1) + \omega(g_2) - \frac{1}{2}\langle \mathrm{Ad}_{g_2^{-1}}\phi_{g_1} \wedge [\phi_{g_2}\wedge\phi_{g_2}]\rangle - \frac{1}{2}\langle \phi_{g_2} \wedge \left[\mathrm{Ad}_{g_2^{-1}}\phi_{g_1}\wedge \mathrm{Ad}_{g_2^{-1}}\phi_{g_1}\right]\rangle \\ &= \omega(g_1) + \omega(g_2) + \langle \mathrm{Ad}_{g_2^{-1}}\phi_{g_1}\wedge d\phi_{g_2}\rangle + \langle \phi_{g_2}\wedge \mathrm{Ad}_{g_2^{-1}} d\phi_{g_1}\rangle \\ &= \omega(g_1) + \omega(g_2) + d\langle \phi_{g_1}\wedge \mathrm{Ad}_{g_2}\phi_{g_2}\rangle.\end{aligned}$$

Hence, using again Ad-invariance we get

$$\begin{aligned}c(a, g_1g_2) = \exp\Bigg(2\pi i \int_Y \langle \mathrm{Ad}_{g_1^{-1}} a \wedge \phi_{g_1}\rangle + 2\pi i W_Y(g_1) + \\ + \int_Y \mathrm{Ad}_{g_2^{-1}}(\mathrm{Ad}_{g_1^{-1}} a + \phi_{g_1})\wedge\phi_{g_2}\rangle + 2\pi i W_Y(g_2)\Bigg) = \\ = c(a, g_1) c(a^{g_1}, g_2)\end{aligned}$$

where we put $a^{g_1} = \mathrm{Ad}_{g_1^{-1}} a + \phi_{g_1}$. Therefore

$$\begin{aligned}f((\psi\varphi)s) = c(s^*\eta, (g_\varphi s)(g_\psi \varphi s))^{-1} f(s) &= c(s^*\eta, g_\varphi s)^{-1} c((s^*\eta)^{g_\varphi s}, g_\psi\varphi s)^{-1} f(s) \\ &= c((s^*\eta)^{g_\varphi}, g_\psi\varphi s)^{-1} f(\varphi s) = f(\psi(\varphi s))\end{aligned}$$

which proves our claim.
That $e^{2\pi i S_X(w)} \in L_{\partial w}$ follows from Proposition 3.7(2); it is well-defined because of our integrality Hypothesis 3. Since any pair $(Q \to Y, \eta)$ is the boundary of a pair $(P \to X, w)$ we have $L_\eta \neq \emptyset$ for all $\eta \in \mathcal{A}_Q$. Also, an element $f \in L_\eta$ is completely determined by its value at some section $s : Y \to Q$. Hence L_η has complex dimension 1, with hermitian metric given in each trivialization by the standard one in $\mathbb{C}$. The last claim follows from the fact that for compact Lie groups the connections and the bracket $\langle\ \ \rangle$ are real-valued, so that $c(a, g) \in S^1$ and $e^{2\pi i S_X(w)}$ has unit norm. $\square$

Proposition 5.4. *For any G-bundles $Q \to Y$ and $P \to X$ the assignments*

$$\begin{array}{llll}\text{(i)} & \eta & \longmapsto & L_\eta, & \eta \in \mathcal{A}_Q \\ \text{(ii)} & w & \longmapsto & e^{2\pi i S_X(w)}, & w \in \mathcal{A}_P\end{array}$$

define:

(i) *a smooth hermitian line bundle $L_Q \to \mathcal{A}_Q$ endowed with an action of the gauge group $\mathcal{G}$ lifting the one on $\mathcal{A}_Q$,*
(ii) *a smooth $\mathcal{G}$-invariant section of the pull-back bundle $r_X^* L_{\partial P}$, where ∂P denotes the restriction of the bundle $P \to \partial X$ to ∂X, and $r_X : \mathcal{A}_P \to \mathcal{A}_{\partial P}$ is the restriction map.*

Moreover these assignments satisfy the following properties:

(i) (Functoriality) *Any bundle map $\psi : Q' \to Q$ covering an orientation preserving diffeomorphism $\bar\psi : Y' \to Y$ induces isomorphisms (isometries if G is compact)*

$$\psi^* : L_{Q,\eta} \longrightarrow L_{Q',\psi^*\eta}, \quad \eta \in \mathcal{A}_Q$$

such that $\psi_1^* \circ \psi_2^* = (\psi_1 \circ \psi_2)^*$, *and for any bundle map* $\varphi : P' \to P$ *covering an orientation preserving diffeomorphism* $\varphi : X' \to X$ *and with induced map* $\partial\varphi : \partial P' \to \partial P$ *over the boundary, we have*

$$(\partial\varphi)^* e^{2\pi i S_X(w)} = e^{2\pi i S_{X'}(\varphi^* w)}.$$

(ii) (Orientation) *Changing the orientation of* Y *induces natural isomorphisms* $L_{-Q,\eta} \cong \bar{L}_{Q,\bar{\eta}}$, *where* $^-$ *denotes the complex conjugation, and these isomorphisms identify the sections* $e^{2\pi i S_{-X}(w)}$ *and* $\overline{e^{2\pi i S_X(\bar{w})}}$.

(iii) (Additivity) *The disjoint union of bundles* $Q_i \to Y_i$ *with connections* w_i *induces natural isomorphisms* $L_{Q_1 \bigsqcup Q_2} \cong L_{Q_1} \otimes L_{Q_2}$, *and these isomorphisms identify the sections* $e^{2\pi i S_{X_1 \sqcup X_2}(w_1 \bigsqcup w_2)}$ *and* $e^{2\pi i S_{X_1}(w_1)} \otimes e^{2\pi i S_{X_2}(w_2)}$.

(iv) (Cutting/Gluing) *Assume that* Y *is embedded in* X, *and* $P \to X$ *restricts to* $Q \to Y$. *Denote by* $X^{cut} = X \setminus Y$, *and* $P^{cut} = f^{cut*}P \to X^{cut}$ *the pull-back bundle of* P *via the natural gluing map* $f^{cut} : X^{cut} \to X$. *Let* $w^{cut} \in \mathcal{A}_{P^{cut}}$ *be such that there exists* $\eta \in \mathcal{A}_Q$ *whose pull-back* $f^{cut*}\eta$ *coincides with* w^{cut} *over* $(f^{cut})^{-1}(Y) \cong Y \bigsqcup (-Y)$.

Then we can extend η *to connections* w *on* P, *smooth on* $X \setminus Y$ *and continuous on* X, *which are well-defined up to gauge equivalence, such that* $f^{cut*}w$ *is gauge equivalent to* w^{cut} *and equal to* w^{cut} *over* $(f^{cut})^{-1}(\partial X \bigsqcup Y)$, *and satisfying*

$$e^{2\pi i S_X(w)} = \mathrm{Trace}_\eta \left(e^{2\pi i S_{X^{cut}}(w^{cut})} \right)$$

where

$$\mathrm{Trace}_\eta : L_{\partial P^{cut}, \partial w^{cut}} \cong L_{\partial P, \partial w} \otimes L_{Q,\eta} \otimes \bar{L}_{Q,\bar{\eta}} \to L_{\partial P, \partial w}$$

is induced by additivity and the hermitian metric on L_η.

Remark 5.5. The functoriality applied to gauge transformations means that the Chern-Simons action is invariant under orientation preserving diffeomorphism. The cutting/gluing properties are very powerful; they allow one to compute the Chern-Simons action by decomposing manifolds in more elementary blocks.

Proof. Lemma 5.3 shows that the assignments (i)-(ii) well define the bundle $L_Q \to \mathcal{A}_Q$ and a section of $r_X^* L_{\partial P}$, and that the former is trivialized by the maps $s_q : L_Q \to \mathcal{A}_Q \times \mathbb{C}$ associated to sections $q : Y \to Q$. Transition functions are associated to gauge transformations; in a given trivialization s_q they are the maps $\eta \mapsto c(q^*\eta, g_\psi q)$, $\psi \in \mathcal{G}$, which are clearly smooth over smooth families of connections. Hence $\eta \mapsto L_\eta$ and the trivializations s_q are smooth.
In particular, the restriction to ∂X of a section $p : X \to P$ determines a smooth trivialization $s_{\partial p} : L_{\partial P} \to \mathcal{A}_{\partial P} \times \mathbb{C}$. In this trivialization $e^{2\pi i S_X(\cdot)}$ is the function $w \mapsto e^{2\pi i S_X(w,s)}$, which varies smoothly with w (see Corollary 3.7). Hence $e^{2\pi i S_X(w)}$ is smooth.
The $\mathcal{G}$-action on $L_Q \to \mathcal{A}_Q$ follows from the functoriality property applied to gauge transformations. Let us prove the latter. Any section $q' : Y' \to Q'$ defines a section

$q = \psi q' \bar{\psi}^{-1} : Y \to Q$. We define ψ^* by the commutative diagram

(46)
$$\begin{tikzcd} L_{Q,\eta} \arrow[r, "\psi^*"] \arrow[d, "s_q"', "\cong"] & L_{Q',\psi^*\eta} \arrow[d, "s_{q'}", "\cong"'] \\ \mathbb{C} \arrow[r, "id"] & \mathbb{C} \end{tikzcd}$$

We have to show that this does not depend on the choice of q'. In fact, if $q_2' : Y' \to Q'$ is another section and $\varphi : Q' \to Q'$ the gauge transformation such that $q_2' = \varphi q'$, we have commutative diagrams

$$\begin{tikzcd} L_{Q',\psi^*\eta} \arrow[r, "s_{q'}"] \arrow[dr, "s_{q_2'}"'] & \mathbb{C} \arrow[d, "\times c((q')^*\psi^*\eta, g_\varphi q')"] \\ & \mathbb{C} \end{tikzcd} \qquad \begin{tikzcd} L_{Q,\eta} \arrow[r, "s_q"] \arrow[dr, "s_{q_2}"'] & \mathbb{C} \arrow[d, "\times c(q^*\eta, g_{\varphi^\psi} q)"] \\ & \mathbb{C} \end{tikzcd}$$

where $q_2 = \psi q_2' \bar{\psi}^{-1} = \psi \varphi q' \bar{\psi}^{-1}$; since ψ is a bundle map we have $q_2 = \varphi^\psi q$ for a gauge transformation $\varphi^\psi : Q \to Q$ such that the associated map $g_{\varphi^\psi} : Q \to G$ satisfies $g_{\varphi^\psi}\psi = g_\varphi$. Using that $\bar{\psi}$ has degree one we get $c(q^*\eta, g_{\varphi^\psi} q) = c((q')^*\psi^*\eta, g_\varphi q')$. This shows that changing the trivialization q' to q_2' compensates in (46), which therefore well defines the isomorphism ψ^*. The last claim of Lemma 5.3 implies that ψ^* is an isometry when G is compact.

We prove the identity $\psi_1^* \circ \psi_2^* = (\psi_1 \circ \psi_2)^*$ by working in a fixed trivialization $s_{q,\eta} : L_\eta \xrightarrow{\cong} \mathbb{C}$ for each $\eta \in \mathcal{A}_Q$. Then it reduces to the cocycle property $c(a, g_1 g_2) = c(a, g_1) c(a^{g_1}, g_2)$ as in Lemma 5.3 (the extension to bundle maps of the argument given there for gauge transformations is immediate).

The equation $(\partial\varphi)^* e^{2\pi i S_X(w)} = e^{2\pi i S_{X'}(\varphi^* w)}$ follows from the invariance of integrals under degree one maps, using that $\bar{\varphi}$ is degree one and fixing a trivialization p' of $P' \to X'$ to identify $e^{2\pi i S_{X'}(\varphi^* w)}$ with a function $\varphi^* w \mapsto e^{2\pi i S_X(\varphi^* w, p')}$. The section $e^{2\pi i S_X(w)}$ is then identified with $w \mapsto e^{2\pi i S_X(w,p)}$, where $p = \varphi p' \bar{\varphi}^{-1} : P \to X$ (recall that $(\partial\phi)^* : L_{\partial P} \to L_{\partial P'}$ relates the trivializations given by ∂p and $\partial p'$).

Orientation and additivity are straightforward consequences of the facts that after fixing some trivialization, $c(a, g)$ and the Chern-Simons action change sign when the orientation of Y and X are reversed, and that the integral over a disjoint union is the sum of the integrals. In particular, we get a hermitian pairing

$$\begin{array}{ccc} L_{Q,\eta} \otimes \bar{L}_{Q,\bar{\eta}} & \longrightarrow & \mathbb{C} \\ f \otimes g & \longmapsto & f(q)\overline{g(q)}, \quad q \in \Gamma(Q) \end{array}$$

which does not depend on the choice of section q and coincides in each trivialization with the standard hermitian metric on $\mathbb{C}$.

For the cutting property we have to show at first that w^{cut} can be glued continuously along Y up to gauge transformations. Denote by Y_1, Y_2 the two submanifolds of ∂X^{cut} mapping diffeomorphically onto Y via the gluing map f^{cut}. Since $f^{cut*}\eta = w^{cut}_{|Y_1 \bigsqcup Y_2}$, the connections $w^{cut}_{|Y_1}$ and $w^{cut}_{|Y_2}$ agree under the identification of the bundles $P^{cut}_{|Y_1}$ and $P^{cut}_{|Y_2}$ induced by f^{cut}. Hence it is enough to find a gauge transformation $\tilde{\varphi} : P^{cut} \to P^{cut}$ such that $\tilde{\varphi}^* w^{cut}_{|Y_i} = w^{cut}_{|Y_i}$ and $\tilde{\varphi}^* w^{cut}$ has no transverse component near Y_i. Then we will define w by gluing $\tilde{\varphi}^* w^{cut}$ along the identification $P^{cut}_{|Y_1} \cong P^{cut}_{|Y_2}$.

To do it we fix tubular neighborhoods $N_i \cong [0,\infty) \times Y_i$ of Y_i in X^{cut} and bundle isomorphisms $P^{cut}_{|N_i} \cong [0,\infty) \times P^{cut}_{|Y_i}$, and we use the gauge transformations φ_i over N_i given by Lemma 4.4. Namely, we define $\tilde{\varphi}$ as the identity outside of N_i and we put

$$\forall p \in P^{cut}_{|Y_i}, \quad \tilde{\varphi}_{|N_i}(t,p) = \begin{cases} \varphi_i(\rho(t),p), & 0 \leq t \leq 1 \\ \varphi_i(1-\rho(t-1),p), & 1 \leq t \leq 2 \\ id, & 2 \leq t \end{cases} \tag{47}$$

where $\rho : [0,1] \to [0,1]$ is a monotone increasing smooth function equal to 0 near 0 and equal to 1 near 1. Then clearly $\tilde{\varphi}^* w^{cut}$ satisfies the above requirements.
The last claim follows from $\displaystyle\int_{X\setminus Y} = \int_X$, functoriality for the bundle map $f^{cut} : P^{cut} \to P$ (f^{cut} is a diffeomorphism off of Y), the equality $\mathrm{Trace}((\partial f^{cut})^*) = id$, and the fact that the Chern-Simons action is the same for connections related by gauge transformations equal to the identity near boundary components. □

Proposition 5.6. *For any G-bundle $Q \to Y$ the section $e^{2\pi i S_{[0,1]\times Y}} \in L_Q \otimes L_{-Q}$ is the parallel transport of a canonically defined $\mathcal{G}$-invariant connection on $L_Q \to \mathcal{A}_Q$, whose curvature times $i/2\pi$ is given by*

$$\Xi(\dot{\eta}_1, \dot{\eta}_2) = -2 \int_Y \langle \dot{\eta}_1 \wedge \dot{\eta}_2 \rangle, \quad \dot{\eta}_i \in T\mathcal{A}_Q.$$

When G is compact the connection is unitary and the parallel transport is an isometry.

Proof. Take a path η_t in $\mathcal{A}_Q$. It defines a connection $\eta.$ on the pull-back bundle $[0,1] \times Q \to [0,1] \times Y$ to the cylinder over Y, via the projection map $proj_2 : [0,1] \times Y \to Y$. Fix a section $q : Y \to Q$ and put $\tilde{q} = proj_2^* q$. Then at time t we have

$$\begin{aligned} \alpha(\eta.)_t &= \langle \eta. \wedge \Omega. \rangle_t - \frac{1}{6} \langle \eta. \wedge [\eta. \wedge \eta.] \rangle_t \\ &= \langle \eta_t \wedge (\Omega_t + dt \wedge \dot{\eta}_t) \rangle - \frac{1}{6} \langle \eta_t \wedge [\eta_t \wedge \eta_t] \rangle \\ &= \langle \eta_t \wedge \Omega_t \rangle - \langle \eta_t \wedge \dot{\eta}_t \rangle \wedge dt - \frac{1}{6} \langle \eta_t \wedge [\eta_t \wedge \eta_t] \rangle. \end{aligned}$$

Hence $\tilde{q}^* \alpha(\eta.) = -q^* \langle \eta_t \wedge \dot{\eta}_t \rangle \wedge dt$, and

$$S_{[0,1]\times Y}(\eta., \tilde{q}) = -\int_0^t \int_Y q^* \langle \eta_t \wedge \dot{\eta}_t \rangle. \tag{48}$$

Formula (22) gives a guess for the desired connection on $L_Q \to \mathcal{A}_Q$. Namely, set

$$(\theta_q)_\eta(\dot{\eta}) = 2\pi i \int_Y q^* \langle \eta \wedge \dot{\eta} \rangle, \quad \eta \in \mathcal{A}_Q, \dot{\eta} \in T_\eta \mathcal{A}_Q. \tag{49}$$

We are going to show that the family $\{\theta_q\}_q$, for q varying among all sections of $Q \to Y$, satisfies the local compatibility condition (18) of connections given in local charts (here the trivializations $s_q : L_Q \to \mathcal{A}_Q \times \mathbb{C}$).

Let $\psi : Q \to Q$ be a gauge transformation. We have

$$\begin{aligned}(\theta_{\psi q})_\eta(\dot\eta) &= 2\pi i \int_Y q^*\langle \psi^*\eta \wedge \psi^*\dot\eta\rangle \\ &= 2\pi i \int_Y q^*\langle \mathrm{Ad}_{g_\psi^{-1}}\eta \wedge \mathrm{Ad}_{g_\psi^{-1}}\dot\eta\rangle + 2\pi i \int_Y q^*\langle \phi_{g_\psi} \wedge \mathrm{Ad}_{g_\psi^{-1}}\dot\eta\rangle \\ &= (\theta_q)_\eta(\dot\eta) - 2\pi i \int_Y \langle \mathrm{Ad}_{g^{-1}} q^*\dot\eta \wedge \phi_g\rangle.\end{aligned}$$

where we put $g = g_\psi q$. Note that

$$\frac{dc_Y(q^*\eta, g_\psi q)(\dot\eta)}{c_Y(q^*\eta, g_\psi q)} = 2\pi i \int_Y \langle \mathrm{Ad}_{g^{-1}} q^*\dot\eta \wedge \phi_g\rangle.$$

From formula (19) and the fact that the trivializations s_q and $s_{\psi q}$ are related by $s_q(\eta)c_Y(q^*\eta, g_\psi q)^{-1} = s_{\psi q}(\eta)$, we conclude that $\{\theta_q\}_q$ defines a connection on $L_Q \to \mathcal{A}_Q$ satisfying

$$\exp(2\pi i S_{[0,1]\times Y}(\eta_\cdot, \tilde q)) = \exp(-\int_0^1 (\theta_q)_{\eta_t}(\dot\eta_t)dt).$$

Denote by $\partial\eta_\cdot$ the restriction of $\eta_\cdot$ to the disjoint union $Q \cup Q \to (\{0\}\times Y)\cup(\{1\}\times(-Y))$. By using the metric on L_{η_0} we can identify $L_{\partial\eta_\cdot} \cong L_{\eta_0} \otimes \bar L_{\bar\eta_1}$ with the line of linear maps $L_{\eta_0} \to L_{\eta_1}$. Then equation (22) shows that $e^{2\pi i S_{[0,1]\times Y}(\eta_\cdot)}$ is the parallel transport of the connection $\{\theta_q\}_q$ along the path of connections η_t:

$$e^{2\pi i S_{[0,1]\times Y}(\eta_\cdot)} = PT_{\eta_t} : L_{\eta_0} \longrightarrow L_{\eta_1}. \tag{50}$$

The curvature times $i/2\pi$ of the connection $\{\theta_q\}_q$ is

$$\begin{aligned}w_\eta(\dot\eta_1, \dot\eta_2) = \frac{i}{2\pi} d(\theta_q)_\eta(\dot\eta_1,\dot\eta_2) &= -\int_Y q^* d\langle \eta \wedge \dot\eta\rangle(\dot\eta_1, \dot\eta_2) \\ &= -2\int_Y \langle \dot\eta_1 \wedge \dot\eta_2\rangle.\end{aligned}$$

Note that the result does not depend on the choice of trivialization q.
Because of the functoriality property of Proposition 5.4, the Chern-Simons action is preserved under the action of the gauge group $\mathcal{G}$ on L_Q. Equivalently, the parallel transport (50), whence the connection $\{\theta_q\}_q$ and its curvature, is preserved. The last claim follows from Lemma 5.3. □

Proof of Theorem 5.1. Since the line bundle $L_Q \to \mathcal{A}_Q$ with its metric, connection and curvature is $\mathcal{G}$-invariant, by restricting to flat connections they pass to a hermitian line bundle $\mathcal{L}_X \to \mathcal{M}_X$ with connection on the quotient. This is the Chern-Simons line bundle. It clearly satisfies all properties stated in Proposition 5.4.
We are left to prove that the section $e^{2i\pi S_X(\cdot)} : \mathcal{M}_X \longrightarrow r_X^*\mathcal{L}_{\partial X}$ is parallel (or *flat*). We can work on a fixed bundle $P \to X$ and consider representatives of gauge equivalence classes of connections on X. Let w_t be a path of flat connections on P, and $w_\cdot$ the corresponding connection on $[0,1]\times P$. Denote the curvature of $w_\cdot$ by Ω. Since w_t is flat for all t, we have $\Omega = dt \wedge \dot w_t$. Hence

$$\exp(2\pi i S_{\partial([0,1]\times X)}(w_\cdot)) = \exp\left(2\pi i \int_{[0,1]\times X} \langle \Omega \wedge \Omega\rangle\right) = 1.$$

On another hand

$$\exp(2\pi i S_{\partial([0,1]\times X)}(w.)) = \exp(2\pi i S_X(w_0))\exp(-2\pi i S_X(w_1))\times \\ \times \exp(2\pi i \int_{[0,1]\times\partial X}\alpha(w.)).$$

Because of (50) we see that $\exp(2\pi i S_X(w_1))$ is obtained from $\exp(2\pi i S_X(w_0))$ via the parallel transport $PT_{\partial w_t} : L_{\partial w_0} \to L_{\partial w_1}$, which proves that the section $e^{2i\pi S_X(\cdot)}$ is parallel. In particular, for loops $w. : S^1 \to \mathcal{M}_X$ of flat connections on X we get a trivial holonomy. This can be seen directly from the fact that the holonomy of $w.$ is $\exp(2\pi i S_{S^1\times\partial X}(w.))$, which is equal to 1 by the same reasoning as above. □

As promised at the beginning of the section we prove now that $(\mathcal{M}_Y, \bar{\Xi})$ can be identified with a symplectic reduction of the affine space of flat connections on Q under the symmetry group $\mathcal{G}$ (the non degeneracy of $\bar{\Xi}$ is proved in Section 6). We obtain this result by combining the next proposition and Theorem 5.9 (iii).

Proposition 5.7. *The action of $\mathcal{G}$ on the bundle $L_Q \to \mathcal{A}_Q$ determines a moment map $\mu : \mathcal{A}_Q \to Lie(\mathcal{G})^*$ for the action of $\mathcal{G}$ on $(\mathcal{A}_Q, \Xi)$, which is given by*

$$\mu_\zeta(\eta) = 2\int_Y \langle \Omega(\eta)\wedge\zeta\rangle$$

where $\zeta \in A^0_Y(Q_{\mathfrak{g}}) \cong Lie(\mathcal{G})$ is an infinitesimal gauge transformation. In particular $\mathcal{M}_Y = \mu^{-1}(0)/\mathcal{G}$.

Proof. For any hermitian line bundle $L \to M$ with connection θ, curvature $-2\pi i\Xi$, and G-action $\rho : G \to \mathrm{Aut}(L)$ preserving the metric and the connection, a moment map $\mu : M \to \mathfrak{g}^*$, $m \mapsto \mu.(m)$, for the quotient G-action on (M, Ξ) is given by the obstruction to descend the connection θ to the quotient L/G. In formula:

$$\mu_\zeta(m) = -\frac{i}{2\pi}\theta_l(\dot{\rho}(\zeta)) \tag{51}$$

where $l \in L_m$ and $\dot{\rho}(\zeta)$ is the vector field on L generated by $\zeta \in \mathfrak{g}$. In fact, by the very definition of a moment map we have

$$\begin{aligned} d\mu_\zeta(m)(V) = \Xi_l(\dot{\rho}(\zeta), V) &= \frac{i}{2\pi} d\theta_l(\dot{\rho}(\zeta), V) \\ &= \frac{i}{2\pi}\left(\dot{\rho}(\zeta)\cdot\theta_l(V) - V\cdot\theta_l(\dot{\rho}(\zeta)) - \theta_l([\dot{\rho}(\zeta), V])\right) \\ &= -\frac{i}{2\pi} V\cdot\theta_l(\dot{\rho}(\zeta)). \end{aligned}$$

Here we identify $V \in T_mM$ with its horizontal lift at $l \in L_m$, so that $\theta_l(V) = 0$ and $\theta_l([\dot{\rho}(\zeta), V]) = 0$ since ρ preserve the connection and consequently the Lie derivative $[\dot{\rho}(\zeta), V]$ is horizontal. The formula (51) follows.
In our situation we will compute the analog of $\theta_l(\dot{\rho}(\zeta))$ by working in a fixed trivialization $s_q : L_Q \to \mathcal{A}_Q \times \mathbb{C}$ associated to a section $q : Y \to Q$, and then comparing the action of an infinitesimal gauge transformation with the infinitesimal parallel transport it defines.

Recall that in the trivialization q a gauge transformation $\Psi : Q \to Q$ is a map $g = g_\psi q : Y \to G$, and that in the associated trivialization s_q the isomorphism $\psi : L_\eta \to L_{\psi^*\eta}$ is identified with the multiplication by $\exp(2\pi i c_Y(a,g))$ (as usual we put $a = q^*\eta$). An infinitesimal gauge transformation read as a map $\zeta : Y \to \mathfrak{g}$, acting as multiplication by the derivative

$$\frac{d}{dt}\left(\exp(2\pi i c_Y(a, e^{t\zeta}))\right)_{t=0} = 2\pi i \int_Y \langle a \wedge d\zeta\rangle. \tag{52}$$

On another hand, denoting by $a^{e^{t\zeta}}$ the action of the gauge transformation $e^{t\zeta}$ on the connection a in the trivialization q, the formula (16) reads as $a^{e^{t\zeta}} = \mathrm{Ad}_{e^{-t\zeta}} a + e^{t\zeta *}\theta$. Hence

$$\frac{d}{dt}\left(a^{e^{t\zeta}}\right)_{t=0} = -\left[\zeta \wedge a\right] + d\zeta = d_a\zeta. \tag{53}$$

This is the covariant derivative of ζ viewed as a section of the adjoint bundle $Q_{\mathfrak{g}}$, associated to the connection a (see (28)). Hence the infinitesimal parallel transport in the direction of ζ acts as multiplication by

$$-(\theta_q)_a(d_a\zeta) = -2\pi i \int_Y \langle a \wedge d_a\zeta\rangle. \tag{54}$$

Using Stockes' theorem we see that the difference of (52) and (54) times $(-i/2\pi)$ is

$$\begin{aligned}
\int_Y \langle a \wedge (2d\zeta + [a \wedge \zeta])\rangle &= 2\int_Y \langle da \wedge \zeta\rangle + \int_Y \langle a \wedge [a \wedge \zeta]\rangle \\
&= 2\int_Y \langle da \wedge \zeta\rangle + \int_Y \langle [a \wedge a] \wedge \zeta\rangle \\
&= 2\int_Y \langle \Omega(a) \wedge \zeta\rangle.
\end{aligned}$$

This computes the vertical component of the infinitesimal action of ζ in the trivialization q, and thus coincides with $\mu_\zeta(a)$. □

Remark 5.8. Consider the commutative diagram

$$\begin{tikzcd}
\mathcal{M}_X \arrow[r] \arrow[d, "r_X"] & \mathcal{A}_P/\mathcal{G} \arrow[d, "r_X"] \\
\mathcal{M}_{\partial X} \arrow[r] & \mathcal{A}_{\partial P}/\mathcal{G}
\end{tikzcd} \tag{55}$$

where the horizontal arrows are the natural inclusions, and the vertical arrows are the restriction maps.

At the bottom right we have a hermitian line bundle[3] with connection $(L_{\partial P}/\mathcal{G} \to \mathcal{A}_{\partial P}/\mathcal{G}, \vartheta)$. At the top right we have a section of the pull-back bundle $\exp(2\pi i S_X(\cdot)) : \mathcal{A}_P/\mathcal{G} \to r_X^* L_{\partial P}/\mathcal{G}$, which in general is *not* parallel. The Chern-Simons line bundle and parallel section over $\mathcal{M}_X$ are obtained by pulling back via the inclusions, and so are the solutions of the Euler-Lagrange equation $\Omega = 0$ for the Chern-Simons action (Proposition 4.1).

[3]I don't want to care about the topology, smoothness etc. of that quotient space here.

We conclude the section with a brief "gauge-theoretic" discussion about the smoothness of the moduli spaces $\mathcal{M}_Y$. It will be completed in Section 6.

Given a bundle $R \to Z$ with a connection η over a finite dimensional manifold Z, denote by $Stab(\eta)$ the stabilizer of η in the group of gauge transformations. That is, $\psi \in Stab(\eta)$ if $\psi^*\eta = \eta$, or equivalently if ψ is parallel (see Exercise 2(2)), or equivalently if the associated map $g_\psi : Q \to G$ commutes with the holonomy representations of η. There is an inclusion $Z(G) \subset Stab(\eta)$ of the center of G, identified with the set of constant gauge transformations. We will say that η is *irreducible* if $Lie(Stab(\eta)) = \{0\}$, and denote by $\mathcal{A}_R^{irr}$, $\mathcal{M}_Z^{irr}$ etc. the corresponding subspaces.

Theorem 5.9. *Let X be a compact oriented 3-manifold, and Y a closed oriented surface Y with a fixed G-bundle $Q \to Y$.*

(i) *The moduli space $\mathcal{M}_Y^{irr} \subset \mathcal{M}_Y$ is a smooth top dimensional orbifold of dimension $-\dim(\mathrm{G})\chi(Y)$.*

(ii) *At each point $\eta \in \mathcal{A}_Q$ the action of $Stab(\eta)$ on the Chern-Simons line L_η factors through an action of the (finite) group $\pi_0(Stab(\eta)/Z(G))$. Hence the line bundle $\mathcal{L}_Y \to \mathcal{M}_Y^{irr}$ is smooth.*

(iii) *If the bracket $\langle \quad \rangle$ is non degenerate then so is the pairing*
$$\bar{\Xi} : T\mathcal{M}_Y \times T\mathcal{M}_Y \to \mathbb{C}.$$
Hence the closed two-form $\bar{\Xi}$ defines a symplectic form on $\mathcal{M}_Y^{irr}$ (complex valued when G is complex, continuous at orbifold points) and $\mathcal{M}_Y^{irr} = \mathcal{A}_Q^{flat\ irr} \cap \mu^{-1}(0)/\mathcal{G}$ is the symplectic reduction of $\mathcal{A}_Q$.

(iii) *The restriction map $r_X^* : \mathcal{M}_X \to \mathcal{M}_{\partial X}$ is Lagrangian.*

Sketch of proof. (i) By definition the action of $\mathcal{G}/Z(G)$ on $\mathcal{A}_Q^{irr}$ is locally free, and it is free at the subset $\mathcal{A}_Q^0$ of connections η where $Stab(\eta) = Z(G)$. By restricting to flat connections this action is also proper, so $\mathcal{M}_Y^{irr} = \mathcal{A}_Q^{flat-irr}/(\mathcal{G}/Z(G))$ is an orbifold which is smooth at $\mathcal{A}_Q^{flat-0}/(\mathcal{G}/Z(G))$. When G is compact this is clear. When G is non compact this follows from a result of Culler and Shalen ([CuSh], Prop. 1.5.2), using the identification of $\mathcal{M}_Y$ as a space of holonomy representations, described in Section 6. Open $\mathcal{G}$-orbits of (reducible) connections must be identified in order to make the full space $\mathcal{M}_Y = \mathcal{A}_Q^{flat}/\mathcal{G}$ Haussdorf. Dimensions are computed below.

(ii) Take $\psi \in Stab(\eta)$. Consider the bundle $Q_\psi \to S^1 \times Y$ formed by gluing the ends of $[0,1] \times Q$ using ψ. The connection η defines a connection on $[0,1] \times Q$, which glues to define a connection η_ψ on $Q_\psi \to S^1 \times Y$. By the gluing property of the Chern-Simons action we have
$$e^{2\pi i S_{S^1\times Y}(\eta_\psi)} = \mathrm{Trace}(e^{2\pi i S_{[0,1]\times Y}(\eta_\psi,.)}),$$
where (see (50))
$$e^{2\pi i S_{[0,1]\times Y}(\eta_\psi,.)} = PT_{\eta_t} : L_\eta \longrightarrow L_{\psi^*\eta}$$
is parallel transport along the path η_t in $\mathcal{A}_Q$ associated via Lemma 4.4 to the restriction of η_ψ over the cylinder $[0,1] \times Y$. Clearly it computes the action of ψ on L_η.

Now, if ψ and ψ' are connected by a path ψ_t in $Stab(\eta)$, there is a gauge transformation $\psi_. : [0,1] \times Q \to [0,1] \times Q$ such that $\psi_{.|\{0\}\times Q} = id$ and $\psi_{.|\{1\}\times Q}\psi = \psi'$, defined by sending (t,q) to $(t, \psi_t\psi^{-1}(q))$. It yields a gauge transformation $Q_\psi \to Q_{\psi'}$ such that

$\psi^*_{\cdot}\eta_\psi = \eta_{\psi'}$. Since $e^{2\pi i S_{S^1\times Y}(\cdot)}$ is invariant under gauge transformations, we deduce that $e^{2\pi i S_{S^1\times Y}(\eta_\psi)} = e^{2\pi i S_{S^1\times Y}(\eta_{\psi'})}$. Hence the actions of ψ and ψ' on L_η are the same.
Finally, in any fixed trivialization $s_q : L_\eta \to \mathbb{C}$ the action of an element g of the center $Z(G)$ on L_η is multiplication by $c_Y(q^*\eta, g) = 1$. This proves (ii).

(iii) First we have to specify the tangent spaces to moduli spaces. Fix a G-bundle $R \to Z$. Recall that we denote by $R_{\mathfrak{g}}$ the adjoint bundle, and by $A^i_Z(P_{\mathfrak{g}})$ the forms of degree i on Z with values in $P_{\mathfrak{g}}$. We have already seen that for any connection $\eta \in \mathcal{A}_R$, any tangent vector $\dot\eta \in T_\eta\mathcal{A}_R$ belongs to $A^1_Z(R_{\mathfrak{g}})$. Also, the *infinitesimal* gauge transformations, that is, the Lie algebra of $\mathcal{G}$, clearly coincide with $A^0_Z(R_{\mathfrak{g}})$.
Assuming that η is flat the covariant derivative $d^0_\eta = d_\eta = d + ad_\eta$ on $R_{\mathfrak{g}}$ satisfies $d^2_\eta = 0$ (see (27)). Then we can consider the cohomology groups $H^i(Z, R_{\mathfrak{g}}(\eta))$ of the complex

$$C^*(R_{\mathfrak{g}}(\eta)) : 0 \to A^0_Z(R_{\mathfrak{g}}) \xrightarrow{d^0_\eta} A^1_Z(R_{\mathfrak{g}}) \xrightarrow{d^1_\eta} \dots \tag{56}$$

For any $\xi \in A^0_Z(R_{\mathfrak{g}})$, we have $d_\eta\xi = d\xi + ad_{-\xi}\eta = 0$ if and only if ξ is parallel. Hence

$$H^0(Z, R_{\mathfrak{g}}(\eta)) = Lie(Stab(\eta)). \tag{57}$$

Also, by differentiating the relation $d\eta + [\eta\wedge\eta]/2 = \Omega = 0$ in the direction of a vector $\dot\eta \in T_\eta\mathcal{A}_R$ which is tangent to the space of flat connections, we find $d_\eta\dot\eta = d\dot\eta + [\eta\wedge\dot\eta] = 0$. For any path φ_t of gauge transformations we have $d(\varphi_t^*\eta)_{t=0} = -[\dot\varphi\wedge\eta] + d\dot\varphi = d_\eta\dot\varphi$. Hence

$$T^{flat}_\eta\mathcal{A}_R = Z^1(Z, R_{\mathfrak{g}}(\eta)) \quad , \quad T_\eta\mathcal{G}^*\eta = B^1(Z, R_{\mathfrak{g}}(\eta)).$$

Therefore, the tangent space at a smooth equivalence class of flat irreducible connections $\eta \in \mathcal{A}^{irr}_R$ is given by

$$T_{[\eta]}\mathcal{M}_Z = H^1(Z, R_{\mathfrak{g}}(\eta)). \tag{58}$$

We wish to compute the dimension of this vector space for $Z = Y$ a closed oriented surface. The key ingredient is the De Rham isomorphism for local systems, which allows to define the cohomology groups $H^*(Z;, R_{\mathfrak{g}}(\eta))$ in terms of simplicial or singular cohomology (see eg. [Ra], [BT]). In particular, the Euler characteristic of the complex (56) is a topological invariant; by using the trivial connection η we see that it is given by

$$\chi(C^*(R_{\mathfrak{g}}(\eta)) = \dim(G)\chi(Y). \tag{59}$$

Then, take a non-degenerate Ad-invariant symmetric bilinear pairing $\langle\ ,\ \rangle' : \mathfrak{g}\times\mathfrak{g} \to \mathbb{C}$, and an *arbitrary flat connection* $\eta \in \mathcal{A}_R$. Consider the composition of the natural cup product on $H^*(Z;, R_{\mathfrak{g}}(\eta))$ followed by this coefficient pairing and evaluation on the fundamental class:

$$\begin{aligned}\langle \cup \rangle' : H^i(Z;, R_{\mathfrak{g}}(\eta)) \times H^{n-i}(Z, \partial Z;, R_{\mathfrak{g}}(\eta)) &\xrightarrow{\cup} \\ H^n(Z, \partial Z;, R_{\mathfrak{g}}\otimes R_{\mathfrak{g}}) &\xrightarrow{\langle\ ,\ \rangle'} H^n(Z, \partial Z; \mathbb{C}) \xrightarrow{\cap[Z,\partial Z]} \mathbb{C}.\end{aligned}$$

where n is the dimension of Z and $R_{\mathfrak{g}}\otimes R_{\mathfrak{g}}$ the tensor product of the adjoint bundle by itself, with its natural covariant derivative (obtained from d_η via the Leibniz rule).

Since $\langle\ ,\ \rangle'$ is non degenerate, $\langle\ \cup\ \rangle' \to \mathbb{C}$ is non degenerate. Hence we have a Poincaré duality isomorphism

$$\begin{array}{rccc} PD: & H^i(Z;,R_{\mathfrak{g}}(\eta)) & \longrightarrow & H^{n-i}(Z,\partial Z;,R_{\mathfrak{g}}(\eta))^* \\ & \mu & \longmapsto & \langle \mu \cup \cdot \rangle'. \end{array}$$

For instance, when $Z = Y$ is a closed oriented surface we have $H^0(Z;,R_{\mathfrak{g}}(\eta)) \cong H^2(Z;,R_{\mathfrak{g}}(\eta))$. From (59) we deduce

$$\begin{aligned} \dim(H^1(Y,R_{\mathfrak{g}}(\eta))) &= -\chi(C^*(R_{\mathfrak{g}}(\eta)) + 2\dim(H^0(Y,R_{\mathfrak{g}}(\eta))) \\ &= -\dim(G)\chi(Y) + 2\dim(Lie(Stab(\eta))). \end{aligned} \tag{60}$$

This is minimal at flat irreducible connections, and (58) gives at smooth points

$$\dim(\mathcal{M}_Y) = \dim T_{[\eta]}\mathcal{M}_Y = -\dim(G)\chi(Y). \tag{61}$$

When $\langle\ ,\ \rangle' = \langle\ ,\ \rangle$, the De Rham isomorphism (58) identifies the cup product

$$\langle\ \cup\ \rangle : H^1(Y;,R_{\mathfrak{g}}(\eta)) \times H^1(Y;,R_{\mathfrak{g}}(\eta)) \longrightarrow \mathbb{C} \tag{62}$$

with the two-form

$$\bar{\Xi} : T_{[\eta]}\mathcal{M}_Y \times T_{[\eta]}\mathcal{M}_Y \longrightarrow \mathbb{C}. \tag{63}$$

Hence the latter is non-degenerate. It remains to show that r_x is Lagrangian. Poincaré duality and the exact sequence of the pair $(X,\partial X)$ give for any connection $\eta \in \mathcal{A}_P$ a commutative diagram

$$\begin{tikzcd} H^1(X;R_{\mathfrak{g}}(\eta)) \arrow[r, "r_X"] \arrow[d, "PD"] & H^1(\partial X;R_{\mathfrak{g}}(\eta)) \arrow[r, "\delta"] \arrow[d, "PD"] & H^2(X,\partial X;R_{\mathfrak{g}}(\eta)) \arrow[d, "PD"] \\ H^2(X,\partial X;R_{\mathfrak{g}}(\eta))^* \arrow[r, "\delta^*"] & H^1(\partial X;R_{\mathfrak{g}}(\eta))^* \arrow[r, "r_X^*"] & H^1(X;R_{\mathfrak{g}}(\eta))^* \end{tikzcd}$$

where δ^* and r_X^* denote the adjoint maps of δ and r_X. We have $\mathrm{Im}(\delta) \cong \mathrm{Im}(r_X^*)$, and $\mathrm{Ker}(\delta) = \mathrm{Im}(r_X)$ is the annihilator of $\mathrm{Ker}(r_X^*)$. Hence

$$\dim(H^1(\partial X;R_{\mathfrak{g}}(\eta))) = \dim(\mathrm{Ker}(\delta)) + \dim(\mathrm{Im}(\delta)) = 2\dim\mathrm{Im}(r_X).$$

This concludes the proof. □

6. The Chern-Simons line bundle, II

Here we propose to define the Chern-Simons line bundle directly from volumes of representations (see Corollary 4.7). Though very direct, this approach will have to wait for the results of Section 9 to be usable, for instance to recover the variation formulas obtained in Section 7 via gauge theory.

First we need to recall some fundamental results on character varieties. We present them via a discussion which will be formalized in Theorem 6.7.

6.1. **A very brief review on character varieties.** We restrict to the case of $G = (P)SL(2,\mathbb{C})$, though for surfaces most of the results could extend to other Lie groups having an Ad-invariant bilinear form on their Lie algebra. References are [CuSh], [Go1, Go2], [BZ], and [S].

6.1.1. *The case of a closed oriented surface* Y. Fix a base point $y \in Y$, and put $\pi = \pi_1(Y, y)$. We assume that the genus g of Y is ≥ 2. The case $g = 1$ is detailed in Section 7.

Since $SL(2,\mathbb{C})$ and $PSL(2,\mathbb{C}) \cong SO(3,\mathbb{C})$ are affine algebraic groups, the space of representations $\mathrm{Hom}(\pi, G)$ is an irreducible affine algebraic set. The group G acts algebraically on $\mathrm{Hom}(\pi, G)$ by conjugation.

Proposition 6.1. (Goldman [Go1, Prop. 1.2-3.7] and Culler-Shalen [CuSh, Prop. 1.5.2]) *The simple points of* $\mathrm{Hom}(\pi, G)$ *are the representations* ρ *such that*

$$\dim(Z(\rho)/Z(G)) = 0 \tag{64}$$

where $Z(\rho)$ *is the centralizer of* ρ *and* $Z(G)$ *the center of* G. *The action of* $G/Z(G)$ *on simple points is locally free, but it is proper only on the subset* $\mathrm{Hom}(\pi, G)^{irr}$ *of irreducible representations.*

For $G = (P)SL(2,\mathbb{C})$ the representations satisfying equation (64) are all the non abelian representations: the *irreducible* ones, which have no fixed point on $\mathbb{CP}^1$ under the projective action by fractional transformations, and also the reducible connections fixing a unique point on $\mathbb{CP}^1$. Note that $\mathrm{Hom}(\pi, SU(2))$ is a real analytic G-invariant subspace of $\mathrm{Hom}(\pi, SL(2,\mathbb{C}))$. The smooth points of $\mathrm{Hom}(\pi, SU(2))$ are the irreducible representations.

Exercise 5. Take Y of genus two and a standard presentation of its fundamental group

$$\pi = \langle \gamma_1, \mu_1, \gamma_2, \mu_2 \mid [\gamma_1, \mu_1]\,[\gamma_2, \mu_2] = 1 \rangle.$$

Consider a sequence of representations $\rho_n : \pi \to SL(2,\mathbb{C})$ such that $\rho_n(\gamma_1) = \rho_n(\mu_2)$ is a fixed diagonal element g with eigenvalues a, a^{-1} ($a > 1$), and

$$\rho_n(\mu_1) = \rho_n(\gamma_2) = \begin{pmatrix} (1 + a^{-2n})^{1/2} & a^{-2n} \\ 1 & (1 + a^{-2n})^{1/2} \end{pmatrix}.$$

Hence ρ_n is completely determined. Check that ρ_n (resp. $g^n \rho_n g^{-n}$) converges to a lower (resp. upper) triangular representation when $n \to \infty$. Hence the G-orbits of the limit representations do not have disjoint open neighborhoods in $\mathrm{Hom}(\pi, G)$, though they are distinct.

Since the action of G is in general not proper, the quotient $\mathrm{Hom}(\pi, G)/G$ is not Haussdorff. Then one usually considers a further quotient, the *character variety* $\mathcal{X}_Y = \mathrm{Hom}(\pi, G)//G$, which is defined as the variety whose coordinate ring is the ring

$$\mathbb{C}\left[\mathcal{X}_Y\right] = \mathbb{C}\left[\mathrm{Hom}(\pi, G)\right]^G$$

of regular functions on $\mathrm{Hom}(\pi, G)$ invariant under the action of G. The ring $\mathbb{C}\left[\mathcal{X}_Y\right]$ is generated by a finite set of characters

$$\begin{array}{rccl} \chi_\gamma : & \mathrm{Hom}(\pi, G) & \longrightarrow & \mathbb{C} \\ & \rho & \longmapsto & \mathrm{Trace}(\rho(\gamma)), \quad \gamma \in \pi. \end{array} \tag{65}$$

Hence, points in $\mathrm{Hom}(\pi, G)$ are identified in $\mathcal{X}_Y$ when they have the same characters.

Proposition 6.2. *The quotient space* $\mathcal{X}_Y^{irr} = \mathrm{Hom}(\pi, G)^{irr}/G$ *is a smooth complex submanifold of the variety* $\mathcal{X}_Y$*, of dimension*

$$\dim(\mathcal{X}_Y^{irr}) = (2g-2)\dim(G).$$

When $G = SL(2,\mathbb{C})$ *or* $SU(2)$ *the manifold* $\mathcal{X}_Y^{irr}$ *is connected and simply-connected, and* $\pi_2(\mathcal{X}_Y^{irr}) \cong \mathbb{Z}$.

Since characters do not distinguish between Abelian representations and representations into a Borel subgroup, $\mathcal{X}_Y$ can be identified with the set of orbits of both Abelian and irreducible representations, and $\mathcal{X}_Y^{irr}$ is a Zariski open subset of $\mathcal{X}_Y$. The real part of $\mathrm{Hom}(\pi, SL(2,\mathbb{C}))^{irr}/SL(2,\mathbb{C})$ is $\mathrm{Hom}(\pi, SU(2))^{irr}/SU(2)$, and similarly for $PSL(2,\mathbb{C})$.
The computation we give of the first homotopy groups can be adapted for $PSL(2,\mathbb{C})$ and higher homotopy groups.

Proof. The first claim follows directly from the fact that $G/Z(G)$ acts freely and properly on $\mathrm{Hom}(\pi, G)^{irr}$ (Proposition 6.1). The dimension is computed in equation (70) below.
We compute the first homotopy groups of $\mathcal{X}_Y^{irr}$ as follows. Consider the case of $G = SL(2,\mathbb{C})$. Denote by R^{a-irr} the G-orbit of the manifolds of Abelian and non Abelian reducible representations in $\mathrm{Hom}(\pi, G)$. At each point it has complex codimension $\geq 6g - ((2g-1)2+3) = 2g-1$ in $R = G^{2g}$. Hence for $g \geq 2$ we have the homotopy groups

$$\pi_0(R \setminus R^{a-irr}) = \pi_1(R \setminus R^{a-irr}) = \pi_2(R \setminus R^{a-irr}) = 0. \tag{66}$$

Consider the map

$$\begin{array}{cccc} R: & R \setminus R^{a-irr} & \longrightarrow & G \\ & \rho & \longmapsto & [\rho(\gamma_1), \rho(\mu_1)] \ldots [\rho(\gamma_n), \rho(\mu_n)] \end{array}$$

where the γ_i and μ_i are standard generators of π, as in Exercise 5. The map R is a proper submersion (see eg. [Go1, Prop 3.7] where it is used to show that the action of G on $\mathrm{Hom}(\pi, G) \setminus R^{ab}$ is locally free). Hence it defines a fibration, with fiber $\mathrm{Hom}(\pi, G)^{irr}$ at the identity. From (66) and the exact sequence in homotopy we deduce

$$\pi_0(\mathrm{Hom}(\pi, G)^{irr}) = \pi_1(\mathrm{Hom}(\pi, G)^{irr}) = 0 \quad , \quad \pi_2(\mathrm{Hom}(\pi, G)^{irr}) \cong \mathbb{Z}.$$

Since the projection $\mathrm{Hom}(\pi, G)^{irr} \to \mathrm{Hom}(\pi, G)^{irr}/G = \mathcal{X}_Y^{irr}$ is a locally trivial fibration with fiber $PSL(2,\mathbb{C})$ we get $\pi_0(\mathcal{X}_Y^{irr}) = \pi_1(\mathcal{X}_Y^{irr}) = 0$ and $\pi_2(\mathcal{X}_Y^{irr}) \cong \mathbb{Z}$. The same computation works for $SU(2)$. $\square$

Tangent spaces and dimensions. It is classical that the Zariski tangent space $T_\rho \mathrm{Hom}(\pi, G)$ is the linear space $Z^1(\pi; \mathfrak{g}_{Ad_\rho})$ of 1-*cocycles* $u : \pi \to \mathfrak{g}_{Ad_\rho}$ (see eg. [Ra] and Exercise 6). Recall that these are the maps to the Lie algebra $\mathfrak{g}$ considered as a module for the Adjoint action of $\rho(\pi)$, and satisfying

$$u(\alpha\beta) = u(\alpha) + Ad_{\rho(\alpha)} u(\beta).$$

We have [Go1, Prop. 1.2-3.7]

$$\dim(Z^1(\pi, \mathfrak{g}_{Ad_\rho})) = (2p-1)\dim(G) + \dim(Z(\rho)). \tag{67}$$

On another hand, the Zariski tangent space $T_\rho(G\cdot\rho)$ of the G-orbit coincides with the linear subspace $B^1(\pi;\mathfrak{g}_{Ad_\rho})$ of 1-*coboundaries*, which are the cocycles $u:\pi\longrightarrow\mathfrak{g}_{Ad_\rho}$ satisfying $u(\alpha)=\mathrm{Ad}_{\rho(\alpha)}a-a$ for some $a\in\mathfrak{g}$. Hence

$$\dim(G\cdot\rho)=\dim(B^1(\pi;\mathfrak{g}_{Ad_\rho}))=\dim(G)-\dim(Z(\rho)). \tag{68}$$

These dimensions are minimal when $\dim(Z(\rho)/Z(G))=0$; it is achieved at the simple points of $\mathrm{Hom}(\pi,G)$. Also, it follows from (68) that the action of $G/Z(G)$ is locally free at the simple points.

By definition the quotient vector space is the first cohomology group of π with coefficients in the $\rho(\pi)$-module $\mathfrak{g}_{Ad_\rho}$ (see eg. [Br]):

$$Z^1(\pi;\mathfrak{g}_{Ad_\rho})/B^1(\pi;\mathfrak{g}_{Ad_\rho})=H^1(\pi;\mathfrak{g}_{Ad_\rho}).$$

Hence

$$T_{[\rho]}\mathcal{X}_Y=H^1(\pi;\mathfrak{g}_{Ad_\rho}) \tag{69}$$

and

$$\dim(T_{[\rho]}\mathcal{X}_Y)=(2g-2)\dim(G)+2\dim(Z(\rho)). \tag{70}$$

The reader should compare this computation with (60).

Exercise 6. Prove the above description of Zariski tangent spaces in terms of group cocycles. Namely, show that if ρ_t is a differentiable path in $\mathrm{Hom}(\pi,G)$ which is written as $\rho_t(\gamma)=\exp(tu(\gamma)+o(t))\rho(\gamma)$ at first order, then the homomorphism condition implies that $u\in Z^1(\pi,\mathfrak{g}_{Ad_\rho})$. Show that if $\rho_t(\gamma)=g_t^{-1}\rho(\gamma)g_t$ with $g_t=\exp(ta+o(t))$ for some $a\in\mathfrak{g}$, then the cocycle corresponding to ρ_t is $u(\gamma)=\mathrm{Ad}_{\rho(\gamma)}a-a$.

For each $\rho\in\mathrm{Hom}(\pi,G)$ there is an identification

$$H^1(\pi;\mathfrak{g}_{Ad_\rho})=H^1(Y;P_{\mathfrak{g}}(\rho)) \tag{71}$$

where the right-hand side is the singular first cohomology group of Y with coefficients in the bundle $P_{\mathfrak{g}}(\rho)$ [Br, Ch. 2]. In particular the pairing (62) defines a non degenerate skew-symmetric bilinear form

$$\langle\cup\rangle_\pi : T_{[\rho]}\mathcal{X}_Y\times T_{[\rho]}\mathcal{X}_Y\longrightarrow\mathbb{C}. \tag{72}$$

As it is, this definition is only ponctual, given at each $[\rho]\in\mathcal{X}_Y$. However, $\langle\cup\rangle_\pi$ can be defined directly via the cup product in group cohomology, and since Y is an Eilenberg-MacLane space (71) extends to an identification of rings with cup products. By using Fox calculus Goldman proved the following:

Proposition 6.3. (Goldman [Go1, §3.10]) $\langle\cup\rangle_\pi$ *defines an algebraic, whence holomorphic, two-form on* $\mathcal{X}_Y^{irr}$*, which is continous on all of* $\mathcal{X}_Y$.

6.1.2. *The case of compact oriented* 3*-manifolds* X. The character variety $\mathcal{X}_X$ is defined in the same way, but is in general a much more complicated object than for surfaces. By [CuSh, Prop. 1.5.2] the action of G on irreducible representations is still proper, but their characters may be non smooth points of $\mathcal{X}_X$, depending on their behaviour along ∂X. Moreover $\mathcal{X}_X$ can have an arbitrary number of irreducible components, of arbitrary dimensions.

In fact, the identifications (69)-(71) hold true also for X. Then the arguments at the end of the proof of Theorem 5.9 imply that

$$\dim(T_{[\rho]}\mathcal{X}_X) = \frac{1}{2}\dim(H^1(\partial X; P_{\mathfrak{g}}(\rho)) + \dim(\mathrm{Im} : H^1(X, \partial X; P_{\mathfrak{g}}(\rho)) \to H^1(X; P_{\mathfrak{g}}(\rho))).$$

In particular, when X contains a closed incompressible surface we can expect to find irreducible components of $\mathcal{X}_X$ of dimension ≥ 2 by deforming representations on one side. In general the irreducible components of $\mathcal{X}_X$ containing the character of an irreducible representation have dimension $\geq t - \chi(X)$, where t is the number of incompressible torus components of ∂X [CuSh, Prop. 3.2.1]. In the other direction, if X contains no closed incompressible surface and its boundary is a single torus, then every component of $\mathcal{X}_X$ has dimension 1 [CCGLS, Prop. 2.4].
Except for the case of graph manifolds, where $\mathcal{X}_X$ is in principle computable, there is a large variety of situations coming from hyperbolic geometry where more can be said. Here are two main examples.
Assume that the interior of X supports a finite volume complete hyperbolic metric. By Mostow rigidity, this metric is unique up to orientation preserving isometry (see eg. [BP]). It corresponds to a conjugacy class of faithful and discrete representations $\chi_{df} \in \mathcal{X}_X(PSL(2,\mathbb{C}))$. If X is not closed then ∂X is a disjoint union of, say, n tori. Assume that $n = 1$.

Theorem 6.4. (Dunfield [Dun]) *Let X_0 be a component of $\mathcal{X}_X$ containing χ_{df}. The restriction map $r_X : X_0 \to \mathcal{X}_{\partial X}$ is a birational isomorphism onto its image.*

We will prove this result in Section 7. In general, when X has boundary a single torus, the algebraic set $r_X(\mathcal{X}_X)$ is closed and defines a plane curve which has been extensively studied since the work [CCGLS]. This plane curve is known to be non trivial for all knot complements.
More in general, assume that ∂X is a disjoint union of n tori, and there is a representation $\rho \in \mathrm{Hom}(\pi_1(X), PSL(2,\mathbb{C}))$ that gives the interior $\mathrm{Int}(X)$ a structure of smooth *but not necessarily complete* hyperbolic manifold, whose metric completion is a closed manifold and the completed metric has cone-like singularities along a link with cone angles at most 2π. The following extends a well-known result of Thurston for cusped hyperbolic manifolds:

Theorem 6.5. (Hogdson and Kerckhoff [HK]) *Under the above hypothesis $\mathcal{X}_X$ is smooth of complex dimension n near $[\rho]$. Moreover, if $\mu_1, \ldots, \mu_n$ are homotopy classes of meridian curves of ∂X, then the complex length maps*

$$\begin{array}{rccc} L: & \mathcal{X}_X & \longrightarrow & \mathbb{C}^n \\ & [\rho] & \longmapsto & (L(\rho(\mu_1)), \ldots, L(\rho(\mu_n))) \end{array}$$

where $2\cosh(L(\gamma)) = \mathrm{Trace}(\gamma)$, *is a local diffeomorphism near* $[\rho]$.

6.2. **Moduli spaces and character varieties.** Consider a G-bundle $R \to Z$ over a compact oriented manifold Z (in any dimension). Fix a base point $z \in Z$. As above put $\pi = \pi_1(Z, z)$. In Proposition 4.6 we defined for any $r \in R_z$ a holonomy map

$$\begin{array}{rccc} \widetilde{hol}: & \mathcal{A}_R^{flat} & \longrightarrow & \mathrm{Hom}(\pi, G) \\ & w & \longmapsto & \rho_{w,r} : \gamma \longmapsto g^{-1}, \quad \tilde{\gamma}_r(1) = r \cdot g \end{array}$$

where $\tilde{\gamma}_q$ is the horizontal lift of γ through r for the flat connection w. We have a commutative diagram

$$\begin{tikzcd} \mathcal{A}_R^{flat} \arrow[r, "\widetilde{hol}"] \arrow[d, "/\mathcal{G}"] & \mathrm{Hom}(\pi, G) \arrow[d, "//G"] \\ \mathcal{M}_Z \arrow[r, "hol"] & \mathcal{X}_Z(G). \end{tikzcd}$$

Any smooth family of flat connections w_m on $\mathcal{A}_R$ is equivalently given by a smooth family[4] of horizontal foliations of R. The image by $\widetilde{hol}$ of w_m is then a continuous family of holonomy representations $\rho_m \in \mathrm{Hom}(\pi, G)$, which is smooth at simple points. By Proposition 4.6 (5) the quotient map *hol* is a bijection. Hence it is a homeomorphism, and a diffeomorphism on each smooth stratum of $\mathcal{M}_Z$ mapping to a smooth stratum of $\mathcal{X}_Z$.

Remark 6.6. *(A canonical smooth structure on $\mathcal{M}_X$?)* The above observation may seem really surprising, since the complex structure of $\mathcal{X}_Z(G)$ is canonical. In fact we never had to specify the smooth structure of $\mathcal{M}_X$ because topological structures on compact oriented 3-manifolds X can be smoothed in a unique way, so that the smooth structure of principal G-bundles $P \to X$, as well as that of their spaces of connections, is canonically determined. The same is true for surfaces. Hence the diffeomorphism types of the moduli spaces $\mathcal{M}_X$ and $\mathcal{M}_{\partial X}$ are canonical, and the above identification with $\mathcal{X}_X$ is canonical. This is no longer true in higher dimensions.

The Chern-Simons action of *flat* connections also depends only on the homeomorphism type of X, together with the identification of ∂X as a submanifold of X up to isotopy (see Theorem 5.1 (b)). Indeed, by Lemma 4.4 (iii) pseudo-isotopies of ∂X induce the identity map on $\mathcal{M}_{\partial X}$, and thus preserve the restriction map $r_X : \mathcal{M}_X \to \mathcal{M}_{\partial X}$. The diffeomorphism invariance of the Chern-Simons action (Proposition 5.4, functoriality property) then proves our claim.

Summing up the above discussion and the results of Section 6.1.1-6.1.2 we get (see eg. (72)):

Theorem 6.7. *The holonomy map hol provides:*

(i) *a* canonical *identification of the Chern-Simons line bundle as a smooth line bundle $(\mathcal{L}_Y \to \mathcal{X}_Y, \vartheta)$ over the (smooth strata of) the character variety.*
(ii) *a* canonical *identification of the Chern-Simons action as a parallel section $e^{2i\pi S_X(\cdot)} : \mathcal{X}_X \longrightarrow r_X^* \mathcal{L}_{\partial X}$.*

In particular, the closed *two-form $\bar{\Xi}$ on $\mathcal{M}_Y$ is equal to the algebraic two-form $\langle \cup \rangle_\pi$ on $\mathcal{X}_Y$, which consequently defines a complex holomorphic symplectic structure on $\mathcal{X}_Y^{irr}$, continuous on all of $\mathcal{X}_Y$.*

Note that it is not clear a priori that $\langle \cup \rangle$ is closed. By definition, it is invariant under the natural action of mapping classes on $\mathrm{Hom}(\pi, G)$ by precomposition, $\rho \mapsto \rho \circ f^{-1}$, but this follows also from the fonctoriality of the Chern-Simons action.

[4]We will not bother here with technicalities regarding smooth infinite dimension manifolds; for our purpose it is enough to take families of connections w_m parametrized by a finite dimensional manifold M

Remark 6.8. The moduli spaces $\mathcal{M}_Y$ have a very rich Hamiltonian geometry. The trace functions (65) provide Darboux coordinates on $\mathcal{X}_Y$ where this geometry is transparent. They define complex Hamiltonian flows extending the classical Fenchel-Nielsen flows on Teichmüller space (see eg. [JW], [Go2] and the references therein).

6.3. **Chern-Simons invariants of marked cobordisms.** In this section we reformulate in purely topological terms the line bundle $(\mathcal{L}_Y \to \mathcal{X}_Y, \vartheta)$ and the parallel section $e^{2i\pi S_X(\cdot)} : \mathcal{X}_X \longrightarrow r_X^*\mathcal{L}_{\partial X}$. In particular we will loose all informations coming from the right-hand side of the commutative diagram (55).

Let X be a compact oriented 3-manifold with a base point $x \in X$. Given a representation $\rho \in \mathrm{Hom}(\pi_1(X,x),G)$ consider the associated G-bundle $\tilde{X} \times_\rho G \to X$ with its canonical flat connection, as defined before Corollary 4.7. For any section $s : X \to \tilde{X} \times_\rho G$, put

$$S_X(\rho, s) = \int_X -(1/6)s^*\overline{\langle \theta \wedge [\theta \wedge \theta]\rangle}.$$

We call (X, ρ, s) a *represented cobordism*; we define *represented surfaces* (Y, ρ, s) in the same way. The following is easy.

Lemma 6.9. *The Chern-Simons action $S_X(\rho, s)$ is invariant under conjugation of ρ and homotopy of s rel(∂). Moreover we have:*

(i) (Diffeomorphism invariance) *For any orientation preserving diffeomorphism $\varphi : X' \to X$, representation $\rho' \in \mathrm{Hom}(\pi_1(X',x'),G)$ and section $s' : X' \to \tilde{X}' \times_\rho G$, we have*

$$S_X(\rho' \circ \varphi_*^{-1}, \tilde{\varphi}s'\varphi^{-1}) = S_{X'}(\rho', s').$$

(ii) (Gluing) *Any two represented cobordisms (X_1, ρ_1, s_1), (X_2, ρ_2, s_2) with a same represented boundary component (Y, ρ, s) can be glued to form a represented cobordism $(X_1 \cup_Y X_2, \rho_1 * \rho_2, s_1 \cup s_2)$ satisfying*

$$S_{X_1 \cup_Y X_2}(\rho_1 * \rho_2, s_1 \cup s_2) = S_{X_1}(\rho_1, s_1) + S_{X_2}(\rho_2, s_2).$$

(iii) (Rigidity) *$S_X(\rho_t, s)$ is constant along any path $\rho_t \in \mathrm{Hom}(\pi_1(X,x),G)$ which is constant on the peripheral subgroups.*

Denote by $\Gamma(Q)$ the set of sections of the G-bundle $\tilde{Y} \times_\rho G \to Y$. The group $\mathcal{G}_{Y_\rho}$ of bundle automorphisms $\varphi : \tilde{Y} \times_\rho G \to \tilde{Y} \times_\rho G$ acts naturally on $\Gamma(Q)$ on the left.

Lemma 6.10. *For any representation $\rho \in \mathrm{Hom}(\pi_1(Y,y),G)$, bundle automorphism $\varphi \in \mathcal{G}_{Y_\rho}$ and section $s : Y \to \tilde{Y} \times_\rho G$, the number*

$$z_Y(\rho, \varphi, s) = \exp\left(S_X(\tilde{\rho}, \tilde{s}) - S_X(\tilde{\rho}, \tilde{\varphi} \cdot \tilde{s})\right)$$

does not depend on the choice of represented cobordism $(X, \tilde{\rho}, \tilde{s})$ with boundary (Y, ρ, s), and G-bundle $\tilde{X} \times_\rho G \to X$ and automorphism $\tilde{\varphi} \in \mathcal{G}_{X_\rho}$ restricting to $\tilde{Y} \times_\rho G \to Y$ and φ on (Y, ρ, s).

Proof. □

For any $\rho \in \mathrm{Hom}(\pi_1(Y,y),G)$ define

$$L_\rho = \{f : \Gamma(Q) \to \mathbb{C} \mid \forall \varphi \in \mathcal{G}_{Y_\rho}, f(\varphi \cdot s) = z_Y(\rho, \varphi, s)f(s)\}$$

and

$$\mathcal{L}_Y = \{(\rho, f)\ |\rho \in \mathrm{Hom}(\pi_1(Y, y), G), f \in L_\rho\}/G.$$

Proposition 6.11. *The natural projection $\mathcal{L}_Y \to \mathcal{X}_Y$ defines a hermitian line bundle which is smooth at characters of irreducible representations.*

Proof. □

Definition 6.12. A *marked cobordism* is a compact oriented 3-manifold X with a base point $x \in X$, and for each basepoint on ∂X a fixed homotopy class of path to x. We consider marked cobordisms up to orientation preserving diffeomorphism isotopic to the identity on ∂X.

Lemma 6.13. *Any marked cobordism X determines a section*

$$exp(2\pi i S_X(\cdot)) : \mathcal{X}_X \longrightarrow r_X^* \mathcal{L}_{\partial X}$$

lifting the restriction map $r_X : \mathcal{X}_X \to \mathcal{X}_{\partial X}$.

Proof. □

The next result is the key to prove the existence of a canonical connection on the line bundle $\mathcal{L}_Y \to \mathcal{X}_Y$ (it may be used also as an alternative argument in Proposition 5.6).

Proposition 6.14. *Let $\pi : L \to M$ be a smooth hermitian line bundle over a manifold M (in any dimension, possibly infinite), such that for any path $l : [0,1] \to M$ we have an isomorphism*

$$PT_l : L_{l(0)} \longrightarrow L_{l(1)}$$

depending smoothly on l and satisfying the following properties:

(i) (Diffeomorphism invariance) $PT_{l_1} = PT_{l_2}$ *for any two paths l_1, l_2 equal up to reparametrization (ie. such that $l_2(t) = l_1(s(t))$ for some orientation preserving diffeomorphism $s : [0,1] \to [0,1]$);*

(ii) (Multiplicativity) $PT_{l_1 \cdot l_2} = PT_{l_2} PT_{l_1}$ *for any two paths l_1, l_2 with $l_1(1) = l_2(0)$.*

Then PT is the parallel transport of a unique connection on L. The connection is unitary if and only if PT_l is an isometry.

Note that when M is infinite dimensional, the set $\mathcal{P}(M)$ of smooth (parametrized) paths is also a manifold: a tangent vector to a path $l \in \mathcal{P}(M)$ is a smooth function

$$v : [0,1] \ni t \to v(t) \in T_{l(t)}(M).$$

Also, note that multiplicativity makes sense because of the diffeomorphism invariance (the concatenation of paths can be reparametrized on $[0,1]$), and that the two properties imply $PT_l = id$ on constant paths.

Proof. Denote by $\mathcal{P}(L)$ the set of smooth paths in L. Any path $\tilde{l} \in \mathcal{P}(L)$ projects to a smooth path $l = \pi(\tilde{l})$ in M. Since PT_l is an isomorphism the element $PT_l(\tilde{l}(0)) \in L_{l(1)}$ differs from $\tilde{l}(1)$ by a non zero complex number, which moreover depends smoothly on $\tilde{l}$. Hence there exists a smooth function $h : \mathcal{P}(L) \to \mathbb{C}^*$ such that

$$PT_l(\tilde{l}(0)) \cdot h(\tilde{l}) = \tilde{l}(1) \tag{73}$$

where $\cdot$ is the multiplication in the fiber. The image of h is a simply connected domain. Indeed, since $h \equiv 1$ on constant paths (this because of $PT_l = id$), if there would be a non contractible loop in $\mathrm{Im}(h)$ based at 1, it would be the image of some loop in $\mathcal{P}(L)$ based at a constant loop in L. This contradicts the fact that $\mathcal{P}(L)$ retracts onto the space of constant paths in L. Hence we can define the logarithm of h,

$$g = log(h) : \mathcal{P}(L) \rightarrow \mathbb{C}.$$

The function g is smooth. Like PT it is additive under concatenation of paths and invariant under reparametrization.
We claim that these properties garantee the existence of a complex-valued one-form ϑ on L such that

$$g(\tilde{l}) = \int_{\tilde{l}} \vartheta. \tag{74}$$

We define ϑ as a limit of Riemann sums, as follows. For any path $\tilde{l} \in \mathcal{P}(L)$ and any integer $n \geq 1$, setting $\tilde{l}_i = \tilde{l}_{[(i-1)/n, i/n]}$ we have additivity

$$g(\tilde{l}) = \sum_{i=1}^{n} g(\tilde{l}_i). \tag{75}$$

We would like to rewrite $g(\tilde{l}_i)$ as

$$g(\tilde{l}_i) = \frac{1}{n}\vartheta\left(\dot{\tilde{l}}_i\left(\frac{i-1}{n}\right)\right) + \mathcal{O}\left(\frac{1}{n^2}\right). \tag{76}$$

Then, by taking the limit of (75) as $n \rightarrow \infty$ we would get (74). Note that for each point $p \in L$ and each vector $v \in T_pL$ the map $t \mapsto tv$ defines a tangent vector to the constant path l_p at p. Hence it is natural to put

$$\vartheta(v) = dg(i(v))$$

where

$$\begin{array}{rccc} i: & T_pL & \longrightarrow & T_{l_p}\mathcal{P}(L) \\ & v & \longmapsto & (t \mapsto tv). \end{array}$$

Clearly ϑ is smooth and linear for each p, and thus defines a one-form on L. Let us check equation (76). First reparametrize $\tilde{l}_i$ as a path defined on $[0,1]$. Namely, set

$$\tilde{l}_i(t) = \tilde{l}\left(\frac{t+i-1}{n}\right), \quad 0 \leq t \leq 1.$$

Then consider the path γ in $\mathcal{P}(L)$ obtained by convex combination of $\tilde{l}_i$ and the constant path $\tilde{l}((i-1)/n)$. That is,

$$\begin{array}{rccc} \gamma: & [0,1] & \longrightarrow & \mathcal{P}(L) \\ & \varepsilon & \longmapsto & \left(\tilde{l}_i^{\varepsilon} : t \mapsto \tilde{l}\left(\frac{\varepsilon t + i - 1}{n}\right)\right). \end{array}$$

We have

$$\begin{array}{rccc} \left(\frac{d\gamma}{d\varepsilon}\right)_{\varepsilon=0} : & [0,1] & \longrightarrow & T_{\tilde{l}((i-1)/n)}\mathcal{P}(L) \\ & t & \longmapsto & \frac{t}{n}\dot{\tilde{l}}\left(\frac{i-1}{n}\right). \end{array}$$

Hence for n large enough the Taylor expansion of g between $\gamma(0) = \tilde{l}((i-1)/n)$ and $\gamma(1) = \tilde{l}_i$ is

$$\begin{aligned} g(\tilde{l}_i) &= dg\left(\left(\frac{d\gamma}{d\varepsilon}\right)_{\varepsilon=0}\right) + \mathcal{O}\left(\left\|\left(\frac{d\gamma}{d\varepsilon}\right)_{\varepsilon=0}\right\|^2\right) \\ &= \frac{1}{n}\vartheta\left(\dot{\tilde{l}}_i\left(\frac{i-1}{n}\right)\right) + \mathcal{O}\left(\frac{1}{n^2}\right). \end{aligned}$$

This proves (76). To conclude the proof of the proposition, we have to check that the one-form ϑ on L is invariant under the action of the gauge group $\mathbb{C}^*$. In fact, the equations (73) and (74) show that the restriction of ϑ to a fiber is the Maurer Cartan form on $\mathbb{C}^*$. Moreover, (74) implies that for any $a \in \mathbb{C}^*$ we have

$$\begin{aligned} PT_l(\tilde{l}(0)\cdot a)\cdot h(\tilde{l}\cdot a) = \tilde{l}(1)\cdot a = PT_l(\tilde{l}(0))\cdot h(\tilde{l})\cdot a &= PT_l(\tilde{l}(0))\cdot a\cdot h(\tilde{l}) \\ &= PT_l(\tilde{l}(0)\cdot a)\cdot h(\tilde{l}). \end{aligned}$$

Hence $h(\tilde{l}\cdot a) = h(\tilde{l})$. Therefore h, and thus ϑ, is invariant under the action of $\mathbb{C}^*$. □

Corollary 6.15. *The line bundle with connection* $(\mathcal{L}_Y \to \mathcal{X}_Y, \vartheta)$ *and the section* $exp(2\pi i S_X(\cdot)) : \mathcal{X}_X \longrightarrow r_X^*\mathcal{L}_{\partial X}$ *defined in Proposition 6.11-6.14 and Lemma 6.13 coincide with the Chern-Simons line bundle and parallel section of Theorem 5.1.*

Proof. □

7. Variation formulas and applications

In this section we will use again and again the following immediate consequence of Corollary 4.2, its generalization Theorem 5.1 (b) for manifolds with boundary, and Theorem 6.7 (ii), we have:

Corollary 7.1. *Let X be a compact oriented* 3*-manifold, and $\mathcal{X}_X$ the variety of characters of $\pi_1(X)$ in G. The following holds:*

- (i) *If $\partial X = \emptyset$, the Chern-Simons action of X is constant on connected components of $\mathcal{X}_X$.*
- (ii) *If $\partial X \neq \emptyset$, the variation of the Chern-Simons action of X along a path χ_t in $\mathcal{X}_X$ is independent of the choice of a trivialization of a G-bundle over ∂X, and is computed as the parallel transport of the Chern-Simons connection ϑ on $\mathcal{L}_{\partial X} \to \mathcal{X}_{\partial X}$ along the boundary trace of χ_t.*

Recall Proposition 4.9. The following result shows that the Chern-Simons invariants distinguish compact hyperbolic manifolds from Seifert-fibered manifolds:

Theorem 7.2. (A. Rezhnikov [Re]) *The Chern-Simons invariants of characters of closed oriented Seifert fibered* 3*-manifolds in $PSL(2,\mathbb{C})$ are real rational numbers.*

The proof uses, among other things, Corollary 7.1 (i). For explicit computations of the Chern-Simons invariants of Seifert fibered manifolds, see [KK] and the references therein.

In the rest of this section we give explicit formulas of the connection and curvature of the Chern-Simons line bundles in the case of $G = PSL(2,\mathbb{C})$, and apply this to hyperbolic manifolds.

7.1. **The Chern-Simons line over a torus.** Fix a meridian m and a longitude l of the torus T^2. Denote by Δ the set of diagonal representation of $\pi_1(T^2)$ in $PSL(2,\mathbb{C})$. We have an isomorphism

$$\begin{array}{rccl} \mathfrak{d}: & \Delta & \longrightarrow & \mathbb{C}^* \times \mathbb{C}^* \\ & \rho & \longmapsto & (\mu(\rho)^2, \lambda(\rho)^2) \end{array} \tag{77}$$

where $\mu(\rho)$ and $\lambda(\rho)$ are the top left eigenvalues of $\rho(m)$ and $\rho(l)$, which are well-defined up to sign:

$$(\rho(m), \rho(l)) = \left(\pm \begin{pmatrix} \mu(\rho) & 0 \\ 0 & \mu^{-1}(\rho) \end{pmatrix}, \pm \begin{pmatrix} \lambda(\rho) & 0 \\ 0 & \lambda^{-1}(\rho) \end{pmatrix} \right).$$

The conjugation action of G identifies representations of Δ with the same trace, that is, with the same pair of eigenvalues. Hence the following diagram is commutative:

$$\begin{tikzcd} \Delta \arrow[r, "\mathfrak{d}"] \arrow[d, "/G"] & \mathbb{C}^* \times \mathbb{C}^* \arrow[d, "/\tau"] \\ \mathcal{X}_{T^2} \arrow[r] & (\mathbb{C}^* \times \mathbb{C}^*)^\tau \end{tikzcd} \tag{78}$$

where τ is the algebraic, generically $2:1$, map given by

$$\begin{array}{rccl} \tau: & \mathbb{C}^* \times \mathbb{C}^* & \longrightarrow & \mathbb{C}^* \times \mathbb{C}^* \\ & (x,y) & \longmapsto & (x^{-1}, y^{-1}). \end{array}$$

The bottom map is thus an isomorphism of varieties; in particular, $\mathcal{X}_{T^2}$ is singular at the character of the trivial representation. If we would have taken $G = SL(2,\mathbb{C})$, we would have removed the squares in the right-hand side of (77), and obtain four singular points at the characters of $(\pm Id, \pm Id)$.

We will use the following covering of $\mathcal{X}_T^2$, which follows from the diagram (78) by taking the log in (77):

$$\begin{array}{ccc} \mathrm{Hom}(\pi_1(T^2), \mathbb{C}) & \longrightarrow & \mathcal{X}_T^2 \\ (\alpha(\tilde\rho), \beta(\tilde\rho)) & \longmapsto & \left[(e^{2\pi i \alpha(\tilde\rho)}, e^{2\pi i \beta(\tilde\rho)})\right]. \end{array} \tag{79}$$

Here $\tilde\rho \in \mathrm{Hom}(\pi_1(T^2), \mathbb{C})$, which we identify with $\mathbb{C}\times\mathbb{C}$ by using the meridian-longitude basis (m,l) on T^2, and we denote by $[(x,y)]$ the class of $(x,y) \in \mathbb{C}^* \times \mathbb{C}^*$ under the involution τ.

The group of deck transformations of this covering is

$$G = \langle x, y, b \mid [x,y] = bxbx = byby = b^2 = 1 \rangle. \tag{80}$$

The action of G on $\mathrm{Hom}(\pi_1(T^2), \mathbb{C})$ is given by

$$x \cdot (\alpha, \beta) = (\alpha + \frac{1}{2}, \beta) \quad , \quad y \cdot (\alpha, \beta) = (\alpha, \beta + \frac{1}{2}) \quad , \quad b \cdot (\alpha, \beta) = (-\alpha, -\beta). \tag{81}$$

Next we wish to identify the Chern-Simons line bundle over $\mathcal{X}_{T^2}$. First we compute the Chern-Simons connection ϑ.

Fix a G-bundle $Q \to T^2$, and a section $q: T^2 \to Q$. The section q induces a trivialization $\mathcal{A}_Q \cong \Omega^1_{T^2} \otimes sl(2,\mathbb{C})$. Give coordinates (x,y) to $T^2 = S^1 \times S^1$ so that the meridian-longitude are $m = \{(e^{2\pi i x}, 1)\}$ and $l = \{(1, e^{2\pi i x})\}$. In this basis, for any connection $\eta \in \mathcal{A}_Q$ there are functions $A(x,y),\ B(x,y) : T^2 \to sl(2,\mathbb{C})$ such that $q^*\eta$ reads as

$$q^*\eta = A(x,y)dx + B(x,y)dy.$$

We know that $\mathcal{M}_{T^2}$ can be identified with $\mathcal{X}_{T^2}$. Hence, by (78)-(79) it is enough to consider sections q such that $q^*\eta$ has A and B constant and diagonal. That is,

$$q^*\eta = \begin{pmatrix} 2\pi i\alpha & 0 \\ 0 & -2\pi i\alpha \end{pmatrix} dx + \begin{pmatrix} 2\pi i\beta & 0 \\ 0 & -2\pi i\beta \end{pmatrix} dy. \tag{82}$$

In fact this can be seen directly: by conjugating we can put $A(x,y)$ and $B(x,y)$ simultaneously in Jordan form with constant diagonal, say

$$A = \begin{pmatrix} 2\pi i\alpha & a(x,y) \\ 0 & -2\pi i\alpha \end{pmatrix} \quad , \quad B = \begin{pmatrix} 2\pi i\beta & b(x,y) \\ 0 & -2\pi i\beta \end{pmatrix}$$

for some functions $a(x,y)$, $b(x,y)$. Then, consider a path η_t in $\mathcal{A}_Q$ which is in this form. We have

$$\begin{aligned} q^*\langle \eta_t \wedge \dot\eta_t\rangle = &-\frac{1}{2\pi^2}\left(\mathrm{Trace}\left(\begin{pmatrix} 2\pi i\alpha_t & a_t(x,y) \\ 0 & -2\pi i\alpha_t \end{pmatrix}\begin{pmatrix} 2\pi i\dot\beta_t & \dot b_t(x,y) \\ 0 & -2\pi i\dot\beta_t \end{pmatrix}\right)\right) dx\wedge dy - \\ &- \frac{1}{2\pi^2}\left(\mathrm{Trace}\left(\begin{pmatrix} 2\pi i\beta_t & b_t(x,y) \\ 0 & -2\pi i\beta_t \end{pmatrix}\begin{pmatrix} 2\pi i\dot\alpha_t & \dot a(x,y) \\ 0 & -2\pi i\dot\alpha_t \end{pmatrix}\right)\right) dy\wedge dx \\ = &\ 4(\alpha_t\dot\beta_t - \beta_t\dot\alpha_t)dx\wedge dy \end{aligned}$$

where, as usual, $\dot\alpha_t$ is the derivative of α with respect to t, and so on. Note that this expression depends only on the diagonal of A_t and B_t. Removing the t to simplify notations, we get

$$(\theta_q)_\eta(\dot\eta) \overset{def}{=} 2\pi i \int_{T^2} q^*\langle \eta\wedge\dot\eta\rangle = 8\pi i(\alpha\dot\beta - \beta\dot\alpha) \tag{83}$$

That is,

$$\theta_q = 8\pi i(\alpha d\beta - \beta d\alpha). \tag{84}$$

This is the lift on $\mathcal{A}_Q$ of the Chern-Simons connection ϑ, given in the trivialization q. Hence, the lift of the symplectic form on $\mathcal{X}_{T^2}$ is

$$\Xi \overset{def}{=} \frac{i}{2\pi} d\theta_q = -8 d\alpha\wedge d\beta. \tag{85}$$

We can now describe explicitely the Chern-Simons line bundle

$$\mathcal{L}_{T^2} \to \mathcal{X}_{T^2}.$$

By construction, in the trivialization induced by q it is given by the diagonal action of the gauge group $\mathcal{G}$ on $\mathcal{A}_Q \times \mathbb{C}$, where the action on $\mathbb{C}$ is the inverse of the corresponding parallel transport on $\mathcal{A}_Q$, with respect to the connection θ_q (see (23)). We have just seen that $\mathcal{A}_Q$ can be reduced to the subspace of connections with coordinates $(\alpha,\beta) \in \mathbb{C}^2$ as in (82), and that the gauge group action reduces to that of the discrete group G defined in (80) on this subspace.

For instance, take the path of connections η_t with $\alpha_t = \alpha + t/2$, $t \in [0,1]$, and β constant. Then the parallel transport along η_t from (α,β) to $(\alpha+1/2;\beta)$ is

$$\exp\left(-\int_0^1 (\theta_q)_{\eta_t}(\dot{\eta}_t)\right) = \exp(-4\pi i\beta).$$

By working similarly with the other generators of G we find that that

$$\mathcal{L}_{T^2} = \mathrm{Hom}(\pi_1(T^2),\mathbb{C}) \times \mathbb{C}/G,$$

where the G action is given by

$$x\cdot(\alpha,\beta;z) = (\alpha+\frac{1}{2},\beta;ze^{4\pi i\beta}) \quad , \quad y\cdot(\alpha,\beta;z) = (\alpha,\beta+\frac{1}{2};ze^{-4\pi i\alpha}) \tag{86}$$
$$b\cdot(\alpha,\beta;z) = (-\alpha,-\beta;z).$$

Finally, consider a compact oriented 3-manifold X with boundary a single torus, and a path of representations $\rho_t : \pi_1(X) \to PSL(2,\mathbb{C})$, $t \in [0,1]$. Fix a meridian and longitude basis of ∂X, and let (α_t,β_t) be coordinates of a a lift of $(\rho_t)_{\partial X}$ to $\mathrm{Hom}(\pi_1(T^2),\mathbb{C}) \cong \mathbb{C}^2$ in this basis. Then the parallel section

$$\begin{array}{rccc} e^{2\pi i S_X(\cdot)}: & \mathcal{X}_{T^2} & \longrightarrow & r_{T^2}^*\mathcal{L}_{T^2} \\ & [\rho_t] & \longmapsto & [(\alpha_t,\beta_t;z(t))] \end{array}$$

satisfies

$$z(1)z(0)^{-1} = \exp\left(\int_{(\alpha_t,\beta_t)} \theta_q\right) = \exp\left(8\pi i\int_0^1 \left((\alpha_t\frac{d\beta_t}{dt} - \beta_t\frac{d\alpha_t}{dt}\right)dt\right). \tag{87}$$

The first Chern class of $\mathcal{L}_{T^2} \to \mathcal{X}_{T^2}$ is

$$\boxed{\Xi} = -8\,\boxed{d\alpha\wedge d\beta} \in H^{2,orb}(\mathcal{X}_{T^2};\mathbb{Z})$$

Integrating over the fundamental domain $[0,1/4]\times[0,1/2]$ for the action of G we find this is -1 times the generator.

These results generalize immediately to an arbitrary number of torus boundary components.

7.2. **Applications to cusped hyperbolic manifolds.** Let X be a compact oriented 3-manifold with n boundary tori, such that the interior admits a smooth finite volume hyperbolic metric. By Mostow rigidity, this structure is unique up to orientation preserving isometries; it corresponds to a unique conjugacy class of discrete and faithful representations of $\pi_1(X)$ in $PSL(2,\mathbb{C})$. Denote by $\chi_{df} \in \mathcal{X}_X$ the corresponding character. Recall the hermitian pairing in Proposition 5.4 (4).

Corollary 7.3. (see eg. [Ho], [Yo], [Ne0]) *Let $\chi \in \mathcal{X}_X$ be the holonomy of a smooth incomplete hyperbolic metric in a neighborhood of χ_{df}, whose completion is a closed smooth hyperbolic manifold $X(\chi)$ obtained by Thurston's hyperbolic Dehn surgery theorem. Denote by γ_k the geodesic added to the k-th cusp of* $\mathrm{Int}(X)$ *to form $X(\chi)$, and $(D^2\times S^1)_k$ a tubular neighborhood of γ_k in $X(\chi)$. We have*

$$\left(\exp(2\pi i S_X(\chi)),\exp(2\pi i S_{\cup_k(D^2\times S^1)_k}(\chi))\right) = e^{\frac{2}{\pi}(Vol(X(\chi))+2\pi i CS(X(\chi)))}\prod_{k=1}^n e^{long(\gamma_k)+itors(\gamma_k)}. \tag{88}$$

We have to recall some fundamental results of hyperbolic geometry that we need in the sequel. For details we refer to [BP] (see also [PP] for Thurston's hyperbolic Dehn surgery theorem). Compare also with Section 6.1.2.

7.3. 1-dimensional representations of mapping classes of surfaces.

8. Chern-Weil theory and secondary characteristic classes

8.1. Construction of Cheeger-Chern-Simons classes. As usual, let G be our Lie group and $\mathfrak{g}$ its Lie algebra. For each integer k consider the k-th symmetric power $S^k(\mathfrak{g}^*)$ of the dual Lie algebra, that is, the linear space of multilinear symmetric functions on $\mathfrak{g}$. For any $\tilde{P} \in S^k(\mathfrak{g}^*)$ put

$$P(a) = \tilde{P}(a, a, \dots, a), \quad a \in \mathfrak{g}.$$

Clealry, P is a homogeneous polynomial of degree k in the coefficients of a. In fact, by expanding $P(t_1a_1 + t_2a_2 + \ldots + t_ka_k)$ we see that $\tilde{P}(a_1, \dots, a_k)$ is $1/k!$ times the coefficient of $t_1t_2 \dots t_k$. Fixing basis elements $e_i \in \mathfrak{g}$ and writing $a = x_1e_1 + \dots x_ne_n$ we thus get an isomorphism

$$\begin{array}{rccc} \tilde{}: & \mathbb{C}[x_1, \dots, x_n]^k & \longrightarrow & S^k(\mathfrak{g}^*) \\ & P & \longmapsto & \tilde{P} \end{array}$$

where $n = \dim(\mathfrak{g})$ (to simplify notations here and below we assume that $\mathfrak{g}$ is complex, so that P is complex valued; everything goes the same way by replacing with $\mathbb{R}$). We call $\tilde{P}$ the *polarization* of the polynomial P. Similarly as for differential forms we define a product

$$\circ : S^k(\mathfrak{g}^*) \otimes S^k(\mathfrak{g}^*) \longrightarrow S^{k+l}(\mathfrak{g}^*)$$

by[5]

$$(P \circ Q)(a_1, \dots, a_{k+l}) = \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+k}} P(a_{\sigma(1)}, \dots, a_{\sigma(k)}) Q(a_{\sigma(k+1)}, \dots, a_{\sigma(k+l)}). \tag{89}$$

It is immediate to check that $\tilde{}$ extends to an isomorphism of graded algebras from $\mathbb{C}[x_1, \dots, x_n]$ to $S^*(\mathfrak{g}^*)$.

The Adjoint representation induces an action of G on $S^k(\mathfrak{g}^*)$, given by

$$(g \cdot P)(a_1, \dots, a_k) = P(\mathrm{Ad}_g^{-1} a_1, \dots, \mathrm{Ad}_g^{-1} a_k), \quad g \in G, \ a_i \in \mathfrak{g}. \tag{90}$$

Let $I^*(G)$ denote the G-invariant part of $S^*(\mathfrak{g}^*)$. Because of the isomorphism $\tilde{}$ we call $I^*(G)$ the algebra of *invariant polynomials* on $\mathfrak{g}$. Beware that in general $I^*(G)$ depends on G, not only on its Lie algebra.

[5] the normalization factor corresponds to giving the volume 1 to standard simplices rather than to unit cubes. This is convenient for characteristic classes, and differs from our preceeding conventions regarding exterior derivative and exterior product of differential forms. Note there are no signatures here !

Example: symmetrized traces. Let G be a matrix group. Then $\mathfrak{g}$ is a subalgebra of $M_n(\mathbb{C})$. Take $P(a) = \mathrm{Trace}(a^k)$. Then

$$\widetilde{P}(a_1, \ldots, a_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{Trace}(a_{\sigma(1)} \ldots a_{\sigma(k)}). \tag{91}$$

Let $\pi : E \to M$ be a G-bundle over an oriented manifold M (in any dimension). We extend the domain of invariant polynomials from $\mathfrak{g}$ to $\mathfrak{g}$-valued differential forms on E as follows: for any $\widetilde{P} \in I^k(G)$ and $w_1, \ldots, w_k \in A_E^{p_i}(\mathfrak{g})$ with $w_i = \eta_i \otimes a_i$ ($\eta_i \in A_E^{p_i}(\mathbb{C})$, $a_i \in \mathfrak{g}$), we put

$$\widetilde{P}(w_1 \wedge \ldots \wedge w_k) = \widetilde{P}(a_1, \ldots, a_k)\eta_1 \wedge \ldots \wedge \eta_k, \quad \in A_E^{p_1+\ldots+p_k}(\mathbb{C}) \tag{92}$$

and then extend by linearity to arbitrary $w \in A_E^{p_i}(\mathfrak{g})$. In particular, suppose w is a connection on $E \to M$ with curvature form $\Omega \in A_E^2(\mathfrak{g})$. Then $\Omega^k = \Omega \wedge \ldots \wedge \Omega \in A_E^{2k}(\mathfrak{g}^{\otimes k})$, and

$$P(\Omega) = \widetilde{P}(\Omega^k) \in A_E^{2k}(\mathbb{C}).$$

Since Ω is horizontal and equivariant (see (29)), and P is an invariant polynomial, $P(\Omega)$ is the lift of a $2k$-form on M which we also denote by $P(\Omega)$.

Theorem 8.1. (Chern-Weil) *Let $\pi : E \to M$ be a G-bundle with connection w, and $P \in I^k(G)$. The following holds:*

(i) *$P(\Omega)$ is a closed form.*
(ii) *The cohomology class $w_E(P) = [P(\Omega)] \in H^{2k}(A^*(M; \mathbb{R}))$ is real and does not depend on the choice of connection w, but only on the isomorphism class of E.*
(iii) *The map*

$$\begin{array}{rccc} w_E : & I^*(G) & \longrightarrow & H^*(A^*(M;\mathbb{R})) \\ & P & \longmapsto & [P(\Omega)] \end{array}$$

is an algebra homomorphism, called the Chern-Weil *homomorphism.*
(iv) *For any differentiable map $\bar{f} : N \to M$ we have $w_{f^*E} = \bar{f}^* w_E$.*

The class $w_E(P)$ is called the characteristic class of E corresponding to P. We stress the fact that it is given in De Rham cohomology, and thus depends *a priori* on the differentiable structure of E. As we will see later this is not the case.

Proof. (iii) is an immediate consequence of (89)-(92), and (iv) follows from $\bar{f}^*P(\Omega) = P(f^*\Omega))$ for any curvature form Ω on E.
Let us prove (i). Since $\pi^* : A^*(M) \to A^*(E)$ is injective it is enough to show that $dP(\Omega) = 0$ in $A^*(E)$. Take $\widetilde{P} \in I^k(G)$. Since Ω is a two-form, by symmetry of P and the Bianchi identity (13) we have

$$dP(\Omega) = \sum_{i=1}^{k} \widetilde{P}(\Omega^{i-1} \wedge d\Omega \wedge \Omega^{k-i}) = k\widetilde{P}([\Omega, w] \wedge \Omega^{k-1}).$$

Now for any a, $b \in \mathfrak{g}$ and $g_t = \exp(ta)$, the Ad-invariance and symmetry of P imply

$$\frac{d}{dt}\widetilde{P}(\mathrm{Ad}_{g_t}b, \ldots, \mathrm{Ad}_{g_t}b)_{t=0} = k\widetilde{P}([a, b], b, \ldots, b).$$

The two identities yield $dP(\Omega) = 0$. The fact that $[P(\Omega)]$ is real follows from Remark 8.4 below.

For the proof of (ii) we need the following lemma:

Lemma 8.2. *Let $h : A^k(M \times [0,1]) \to A^{k-1}(M)$ $(k = 1, 2, \ldots)$ be the operator sending $w = ds \wedge \alpha + \beta$ to*

$$h(w) = \int_{s=0}^{1} \alpha \; ds,$$

where $\alpha = (\partial/\partial s) \lrcorner w$ and β have no ds component. Here we integrate the coefficients of α with respect to $s \in [0,1]$. Then h is a cochain homotopy:

$$dh(w) + h(dw) = i_1^* w - i_0^* w$$

where $i_0(m) = (m, 0)$ and $i_1(m) = (m, 1)$ for all $m \in M$.

Proof. Denote by $d_M \alpha = d\alpha - ds \wedge (\partial\alpha/\partial s)$. Then

$$dw = -ds \wedge d_M \alpha + ds \wedge \frac{\partial \beta}{\partial s} + \ldots$$

where we have only written the terms involving ds. Hence

$$h(dw)_m = \int_{s=0}^{1} \left(\frac{\partial \beta}{\partial s}(m, s) - d_M \alpha_{(m,s)} \right) \; ds = \beta_{(m,1)} - \beta_{(m,0)} - dh(w)_m.$$

The result then follows from $i_0^* w_m = \beta_{(m,0)}$ and $i_1^* w_m = \beta_{(m,1)}$. □

End of the proof of Theorem 8.1 (ii). Consider the pull-back G-bundle $E \times [0,1] \to M \times [0,1]$ to the cylinder. For any two connections w_0 and w_1 on E with curvatures Ω_0 and Ω_1, define $w \in A^1_{E \times [0,1]}$ by

$$w_{(x,s)} = (1-s)(w_0)_x + s(w_1)_x, \quad (x, s) \in E \times [0,1] \, .$$

Since convex combinations of connections are connections, w is a connection on $E \times [0,1]$. Denote its curvature by Ω. Since $i_0^* w = w_0$ and $i_1^* w = w_1$ we have $i_0^* \Omega = \Omega_0$ and $i_1^* \Omega = \Omega_1$. By (i) the form $P(\Omega)$ on $E \times [0,1]$ is closed. Therefore by Lemma 8.2

$$dh(P(\Omega)) = i_1^* P(\Omega) - i_0^* P(\Omega) = P(\Omega_1) - P(\Omega_0). \tag{93}$$

Hence $P(\Omega_1)$ and $P(\Omega_0)$ represent the same cohomology class in $H^{2k}(A^*(M))$. This shows that $w_E(P)$ does not depend on the choice of connection. □

Exercise 7. (Poincaré's Lemma) Replace in Lemma 8.2 the manifolds $M \times [0,1]$ and M by a subset $U \subset \mathbb{R}^n$ which is star-shaped with respect to $u_0 \in U$. By considering the retraction $g : U \times [0,1] \to U$ given by $g(u, s) = (1-s)u + su_0$ and the forms $g^* w = ds \wedge \alpha + \beta$ for $w \in A^k(U)$, show that

$$h(d(g^* w))_u = \begin{cases} -dh(g^* w)_u - w_u, & \text{if } k > 0 \\ w_{u_0} - w_u, & \text{if } k = 0. \end{cases}$$

Deduce that $H^k(A^*(U)) = 0$ if $k > 0$, and $\mathbb{R}$ if $k = 0$.

Definition 8.3. Let $G = GL(n, \mathbb{R})$. Then $\mathfrak{g} = gl(n, \mathbb{R}) = \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ is the Lie algebra of real matrices with Lie bracket $[A, B] = AB - BA$. For each positive integer k the *Pontrjagin polynomial* $p_{k/2}$ of $A \in gl(n, \mathbb{R})$ is given by

$$\det(\lambda Id - \frac{A}{2\pi}) = \sum_{k=0}^{n} p_{k/2}(A) \lambda^{n-k}, \quad A \in gl(n, \mathbb{R}).$$

Similarly, for $G = GL(n,\mathbb{C})$ the *Chern polynomials* c_k are given by

$$\det(\lambda Id - \frac{A}{2\pi i}) = \sum_{k=0}^{n} c_k(A)\lambda^{n-k}, \quad A \in gl(n,\mathbb{C}).$$

Since $p_{k/2}$ and c_k are homogeneous polynomials of degree k in the roots of the (rescaled) characteristic polynomial of A, they are invariant under conjugation by G and define invariant polynomials

$$p_{k/2} \in I^k(GL(n,\mathbb{R})) \quad , \quad c_k \in I^k(GL(n,\mathbb{C})).$$

Note that the restriction of c_k to $gl(n,\mathbb{R})$ satisfies

$$i^k c_k(A) = p_{k/2}(A), \quad A \in gl(n,\mathbb{R}). \tag{94}$$

By restricting to $G = O(n) \subset GL(n,\mathbb{R})$, the Pontrjagin polynomials $p_{k/2}$ with k odd vanish. Indeed, $o(n)$ is the Lie subalgebra of skew-symmetric matrices in $gl(n,\mathbb{R})$, so that

$$\det(\lambda Id - \frac{A}{2\pi}) = \det(\lambda Id - \frac{A}{2\pi})^t = \det(\lambda Id + \frac{A}{2\pi}), \quad A \in o(n). \tag{95}$$

Also, the Chern polynomials are real-valued when restricted to $G = U(n) \subset GL(n,\mathbb{C})$, since $u(n)$ is the Lie subalgebra of $gl(n,\mathbb{C})$ of skew-hermitian matrices and so

$$\det(\lambda Id - \frac{A}{2\pi i}) = \det(\lambda Id - \frac{A}{2\pi i})^t = \overline{\det(\lambda Id - \frac{A}{2\pi i})}, \quad A \in u(n). \tag{96}$$

Assume that $A \in gl(n,\mathbb{C})$ is diagonalizable, and let $(2\pi/i)x_j$, $j = 1, \ldots n$, denote its eigenvalues. For instance, take $A \in u(n)$, so that $A/2\pi i$ is a Hermitian matrix which can be diagonalized by some matric $g \in U(n)$. Then

$$\det(\lambda Id - \frac{A}{2\pi i}) = \prod_{j=1}^{n}(\lambda + x_j) = 1 + (x_1 + \ldots + x_n)\lambda + (x_1x_2 + \ldots + x_{n-1}x_n)\lambda^2 + \\ + (x_1x_2\ldots x_n)\lambda^n.$$

Hence

$$\begin{aligned} c_0(A) &= 1 \\ c_1(A) &= \frac{i}{2\pi}\mathrm{Trace}(A) \\ c_2(A) &= \frac{1}{2}\left(\frac{i}{2\pi}\right)^2 \left(\mathrm{Trace}^2(A) - \mathrm{Trace}(A^2)\right) \\ &\vdots \\ c_n(A) &= \left(\frac{i}{2\pi}\right)^n \det(A). \end{aligned} \tag{97}$$

Remark 8.4. *(Reduction).* Any $GL(n,\mathbb{R})$-bundle is isomorphic to an $O(n)$-bundle, and similarly any $GL(n,\mathbb{C})$-bundle is isomorphic to an $U(n)$-bundle (we say that the bundles are *reduced*). In fact, for any given trivializing atlas (U_i, τ_i) the Gram-Schmidt process allows ont to define a gauge transformation $\varphi : E \to E$ such that φ^*E has transition functions g_{ij} with values in $O(n)$ (resp. $U(n)$). From (95)-(96) and Theorem

8.1 (iv) we deduce that the Chern-Weil images $w_E(p_{k/2})$ vanish for k odd, and that $w_E(c_k)$ is real for all k.

Definition 8.5. For any $GL(n,\mathbb{R})$-bundle $E \to M$ we call $p_k(E) = w_E(p_k) \in H^{4k}(M;\mathbb{R})$ the *k-th Pontrjagin class* of E. For any $GL(n,\mathbb{C})$-bundle $E \to M$ we call $c_k(E) = w_E(c_k) \in H^{2k}(M;\mathbb{R})$ the *k-th Chern class* of E.

There were some fundamental informations hidden at the end of the proof of Theorem 8.1. Let us extract them.
Consider the connection $w = (1-s)w_0 + sw_1$ on $E \times [0,1]$. As usual, let us denote by Ω_s the curvature of the connection w_s on E. The curvature of w is

$$\begin{aligned}\Omega &= ds \wedge (\partial w/\partial s) + \Omega_s \\ &= ds \wedge (w_1 - w_0) + (1-s)dw_0 + sdw_1 + \\ &\qquad + \frac{1}{2}\left((1-s)^2 [w_0 \wedge w_0] + s^2 [w_1 \wedge w_1] + 2s(1-s)[w_0 \wedge w_1]\right) \\ &= ds \wedge (w_1 - w_0) + (1-s)\Omega_0 + s\Omega_1 + \\ &\qquad \frac{1}{2}\left(-s(1-s)[w_0 \wedge w_0] + (s^2-s)[w_1 \wedge w_1] + 2s(1-s)[w_0 \wedge w_1]\right).\end{aligned} \tag{98}$$

Then we have

$$h(P(\Omega)) = \int_0^1 k\tilde{P}((w_1 - w_0) \wedge \Omega^{k-1}).$$

In particular, let us apply this formula in the following situation. Fix a connection θ on $\pi : E \to M$. Denote its curvature by Ω_θ. Consider the pull-back bundle $\bar{\pi}^*E \to E$ via the projection π:

$$\begin{array}{ccc} \bar{\pi}^*E & \xrightarrow{\bar{\pi}} & E \\ \downarrow{\scriptstyle \bar{\pi}^*\pi} & & \downarrow{\scriptstyle \pi} \\ E & \xrightarrow{\pi} & M. \end{array}$$

The bundle $\bar{\pi}^*E \to E$ is trivial, with a canonical section $i : E \to \bar{\pi}^*E$. Hence there are two natural connections on $\bar{\pi}^*E$: the trivial flat connection w_0 corresponding to the canonical section i, and $w_1 = \bar{\pi}^*\theta$. We consider the connection $w = (1-s)w_0 + sw_1$ on $\bar{\pi}^*E \times [0,1]$, and its curvature Ω.
Since w_0 is the trivial flat connection we have $i^*w_0 = 0$. Also, $i^*w_1 = \theta$ gives $i^*\Omega_1 = \Omega_\theta$. Then from (98) we find

$$i^*\Omega = ds \wedge \theta + s\Omega_\theta + \frac{1}{2}(s^2 - s)[\theta \wedge \theta] \quad \in A^{2k}_{E\times[0,1]}(\mathfrak{g})$$

and also

$$h(P(i^*\Omega)) = \int_0^1 k\tilde{P}(\theta \wedge \varphi_s^{k-1}) \in A_E^{2k-1}(\mathbb{C}) \tag{99}$$

where we set

$$\varphi_s = s\Omega_\theta + \frac{1}{2}(s^2 - s)[\theta \wedge \theta].$$

Finally, by (93) and the fact that w_0 is flat we get

$$dh(P(i^*\Omega)) = i^*dh(P(\Omega)) = i^*P(\Omega_1) = P(\Omega_\theta).$$

We have proved:

Proposition 8.6. *For any G-bundle $E \to M$ with a connection θ and any invariant polynomial $P \in I^k(G)$, the form*

$$TP(\theta) = \int_0^1 k\tilde{P}(\theta \wedge \varphi_s^{k-1})ds \quad \in A_E^{2k-1}(\mathbb{C})$$

is a canonical *antiderivative of $P(\Omega_\theta)$. That is,*

$$dTP(\theta) = P(\Omega_\theta). \tag{100}$$

Definition 8.7. We call $TP(\theta) \in A_E^{2k-1}(\mathbb{C})$ the Chern-Simons form of $(E \to M, \theta)$ associated to the polynomial P.

Remark 8.8. By naturality (Theorem 8.1 (iv)), $[P(\Omega)]$ is a characteristic class of the bundle $\bar{\pi}^*E \to E$. Since the latter is trivial, by obstruction theory it vanishes (see eg. [MiSt]). This ultimately justifies (100). Proposition 8.6 "just" provides a canonical primitive of this 0 class.

Exercise 8. The form $TP(\theta)$ can be written without the integral. Show that when $P \in I^k(G)$ we have

$$TP(\theta) = \sum_{i=0}^{k-1} c_i \widetilde{P}(\theta \wedge [\theta \wedge \theta]^i \wedge \Omega^{k-i-1})$$

where $c_i = \dfrac{(-1)^i k!(k-1)!}{2^i(k+i)!(k-1-i)!}$. In particular, when θ is flat $c_i = 0$ for $i \neq k-1$, and $c_{k-1} = 1/(1)^{k-1}2^k \begin{pmatrix} 2k-1 \\ k \end{pmatrix}$.

Example: Chern-Simons 3-forms. Take $G = SL(n, \mathbb{C})$. From (97) we get

$$\begin{aligned} Tc_2(\theta) &= \frac{1}{4\pi^2}\int_0^1 \mathrm{Trace}\left(\theta \wedge (s\Omega_\theta + \frac{s^2-s}{2}[\theta \wedge \theta])\right) ds \\ &= \frac{1}{8\pi^2}\mathrm{Trace}\left(\theta \wedge \Omega_\theta - \frac{1}{6}\theta \wedge [\theta \wedge \theta]\right). \end{aligned}$$

For $G = SO(n)$ we can use $p_1(\theta)$ instead, which is given by -1 times the same formula. Compare with Remark 3.12.

Since connections are not horizontal, the Chern-Simons forms are in general not horizontal and do not lift forms on M.

Proposition 8.9. *Let $\pi : E \to M$ be a G-bundle with connection θ, and $P \in I^k(G)$. Set $n = \dim(M)$. We have:*

(i) *If $2k-1 \geq n$, then $P(\Omega) = 0$. Hence the class $[TP(\theta)] \in H^{2k-1}(E; \mathbb{C})$ is defined.*
(ii) *If $2k-1 > n$, the class $[TP(\theta)]$ is independent of θ.*
(iii) *Assume that $P(\Omega) = 0$ and P is* integral. *That is, for the Maurer-Cartan form θ_{MC} on $G \to \{pt\}$ we have $[TP(\theta_{MC})] \in H^{2k-1}(G; \mathbb{Z})$. Then there exists $\hat{P}(\theta) \in H^{2k-1}(M; \mathbb{C}/\mathbb{Z})$ such that*

$$[TP(\theta)] = \pi^*(\hat{P}(\theta)) \mod(\mathbb{Z}).$$

Note that since the class $[TP(\theta)]$ depends on θ we cannot apply Remark 8.4. Hence, when P is complex valued, the class $\hat{P}(\theta) \in H^{2k-1}(M;\mathbb{C}/\mathbb{Z})$ is not not necessarily real. The proposition shows also that for closed 3-manifolds and any Lie group G the most interesting classes are obtained for $k=2$. For matrix groups this corresponds to $P = p_1$ and $P = c_2$.

Proof. (i) If $2k-1 \geq n$ then $2k > n$. Since the horizontal distribution has dimension n and $P(\Omega)$ is a horizontal $2k$-form, it vanishes.

In order to prove (ii) we need the following lemma (compare with Proposition 4.1):

Lemma 8.10. *For any smooth path of connections θ_t on $E \to M$ we have*

$$\frac{d}{dt}(TP(\theta_t))_{t=0} = kP(\dot{\theta} \wedge \Omega^{k-1}) + \ exact\ form \tag{101}$$

where $\theta = \theta_0$ and $\dot{\theta} = (d/dt)(\theta_t)_{t=0}$.

Proof. The symmetry of P and (100) give

$$d\left(\frac{d}{dt}(TP(\theta_t))_{t=0}\right) = \frac{d}{dt}(dTP(\theta_t))_{t=0} = \frac{d}{dt}(P(\Omega_t))_{t=0} = kP\left(\frac{d}{dt}(\Omega_t)_{t=0} \wedge \Omega^{k-1}\right).$$

On another hand, by using the Bianchi identity and the Ad-invariance of P we find

$$\begin{aligned} d(kP(\dot{\theta} \wedge \Omega^{k-1})) &= kP(d\dot{\theta} \wedge \Omega^{k-1}) - k(k-1)P(\dot{\theta} \wedge d\Omega \wedge \Omega^{k-2}) \\ &= kP(d\dot{\theta} \wedge \Omega^{k-1}) - k(k-1)P(\dot{\theta} \wedge [\Omega \wedge \theta] \wedge \Omega^{k-2}) \\ &= kP(d\dot{\theta} \wedge \Omega^{k-1}) + kP(\left[\dot{\theta} \wedge \theta\right] \wedge \Omega^{k-2}). \end{aligned}$$

The identity (101) then follows from

$$\begin{aligned} d\dot{\theta} = d((d/dt)(\theta_t)_{t=0}) = (d/dt)(d\theta_t)_{t=0} &= (d/dt)(\Omega_t - \frac{1}{2}[\theta_t \wedge \theta_t])_{t=0} \\ &= \frac{d}{dt}(\Omega_t)_{t=0} - \left[\dot{\theta} \wedge \theta\right]. \end{aligned}$$

□

End of the proof of Proposition **??**. The claim (ii) is a consequence of $P(\dot{\theta} \wedge \Omega^{k-1}) = 0$, which follows, similalry as in (i) from the fact that this $(2k-1)$-form is horizontal ($\dot{\theta}$ vanishes on vertical vector fields).

(iii)

□

Definition 8.11. We call $\hat{P}(\theta) \in H^{2k-1}(M;\mathbb{C}/\mathbb{Z})$ the Cheeger-Chern-Simons class of $(E \to M, \theta)$ associated to the polynomial P.

8.2. **The universal canonical cochain on $BPSL(2,\mathbb{C})^\delta$.** Let X be a compact oriented 3-manifold, and $\pi : P \to X$ a $PSL(2,\mathbb{C})$-bundle with a flat connection w. By Proposition 4.6 (4) we can find a trivializing atlas $\mathcal{U} = \{(U_i, \tau_i)\}$ such that $w_{|\pi^{-1}(U_i)}$ is the trivial connection. Up to refining $\mathcal{U}$ we can assume that the dual complex is a triangulation T of X; such a triangulation may be singular, ie. the 3-simplices $\Delta_{\mathcal{U}}$ may have multiple as well as self adjacencies; only the interiors of required to embed.

Our aim is to show that there exists decompositions

$$S_X(w,s) = \int_X s^*\alpha(w) = \sum_{\Delta_{\mathcal{U}}} \int_{\Delta_{\mathcal{U}}} s^*\alpha(w)$$

where all summands have a *uniform* expression, given in terms of some special, *universal*, function.

9. Simplicial formulas

10. Quantum Chern-Simons theories

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