MONOKINETIC SOLUTIONS TO A SINGULAR VLASOV EQUATION FROM A SEMICLASSICAL PERSPECTIVE

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ABSTRACT. Solutions to a singular one-dimensional Vlasov equation are obtained as the semiclassical limit of the Wigner transform associated to a logarithmic Schrödinger equation. Two frameworks are considered, regarding in particular the initial position density: Gaussian initial density, or smooth initial density away from vacuum. For Gaussian initial densities, the analysis also yields global solutions to the isothermal Euler system that do not enter the frame of regular solutions to hyperbolic systems by P. D. Lax.

1. INTRODUCTION AND MAIN RESULTS

This paper is concerned by the Cauchy problem for the Vlasov equation

$$\partial_t f + \xi\partial_x f - \lambda\big(\partial_x \ln(\rho)\big)\partial_\xi f = 0, \quad f(0,x,\xi) = f_0(x,\xi), \quad t>0, (x,\xi)\in\mathbb{R}^2, \tag{1.1}$$

where $\lambda \neq 0$ and $\rho(t,x) = \int_{\mathbb{R}} f(t,x,\xi)d\xi$. For $\lambda > 0$ it arises in plasma physics, e.g. for quasineutral plasmas in the core or tokamaks when one focuses on the direction of the magnetic lines. There, f denotes the ionic distribution function and the electrons of the plasma are assumed adiabatic.

Due to the derivative of the density ρ with respect to space in the force term $\partial_x \ln(\rho)$, this equation is highly singular. The Cauchy problem can in particular be proven to be well-posed for very specific initial data like mono-kinetic distribution functions of the form

$$f_0(x,\xi) = \rho_0(x)dx \otimes \delta_{\xi=v_0(x)},$$

with time-dependent mono-kinetic solutions of the form

$$f(t,x,\xi) = \rho(t,x)dx \otimes \delta_{\xi=v(t,x)}.$$

From a fluid dynamics perspective for $\lambda > 0$, it is well-known that $f(t,x,\xi) = \rho(t,x)dx\otimes \delta_{\xi=v(t,x)}$ is a distributional solution of (1.1) if and only if its moments

$$\rho(t,x) = \int f(x,\xi)d\xi \quad \text{and} \quad \rho(t,x)v(t,x) = \int \xi f(t,x,\xi)d\xi$$

are solutions of the isothermal Euler system

$$\begin{cases} \partial_t\rho + \partial_x(\rho v) = 0, \\ \partial_t(\rho v) + \partial_x\left(\rho v^2 + \lambda\rho\right) = 0. \end{cases} \tag{1.2}$$

This paper addresses the existence of mono-kinetic solutions to (1.1), as limits of the Wigner transform of solutions to a logarithmic Schrödinger equation introduced in [4] in the context of wave mechancis. As noticed there, the evolution of initial Gaussian data can be computed rather explicitly, a remark which yields our first main result:

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Theorem 1.1. *Let* $\rho_*, \sigma_0 > 0$ *and* $\omega_0, p_0 \in \mathbb{R}$. *Set*

$$\rho_0(x) = \rho_* e^{-\sigma_0 x^2}, \quad v_0(x) = \omega_0 x + p_0,$$

and consider the ordinary differential equation

$$\ddot{\gamma} = \frac{2\lambda\sigma_0}{\gamma}, \quad \gamma(0) = 1, \quad \dot{\gamma}(0) = \omega_0. \tag{1.3}$$

Suppose that (1.3) *has a solution* $\gamma \in C^2([0, T[)$. *Set*

$$\rho(t,x) = \frac{\rho_*}{\gamma(t)} e^{-\sigma_0 (x - p_0 t)^2/\gamma(t)^2}, \quad v(t,x) = \frac{\dot{\gamma}(t)}{\gamma(t)} x + p_0.$$

Then

$$\mu = \rho(t,x) dx \otimes \delta_{\xi = v(t,x)}$$

is a measure solution to (1.1) *on* $[0, T[$, *with* $\mu_{|t=0} = \rho_0 dx \otimes \delta_{\xi = v_0}$.
In the case $\lambda > 0$, *the solution* γ *is globally defined and smooth, so* T *can be taken arbitrarily large. In addition, it safisfies*

$$\gamma(t) \underset{t\to\infty}{\sim} 2t\sqrt{\lambda\sigma_0 \ln t}, \quad \dot{\gamma}(t) \underset{t\to\infty}{\sim} 2\sqrt{\lambda\sigma_0 \ln t}.$$

On the other hand, for $\lambda < 0$, *the solution* γ *becomes singular in finite time.*

Remark 1.2. In the case $\lambda > 0$ the global smooth solutions to the isothermal Euler system that are obtained as part of the result, do not enter the frame of regular solutions developed by Lax [24]. Indeed the Lax solutions require bounded initial data, whereas the initial velocity in Theorem 1.1 is unbounded.

Remark 1.3. To our knowledge, the case $\lambda < 0$ does not correspond to a physical model related to (1.1). However, the logarithmic nonlinear Schrödinger presented in Section 3 was introduced initially exactly in the case $\lambda < 0$ ([4]). We will see that the analysis of this case requires very little extra effort.

Our second result deals with more general solutions to (1.1) such that ρ remains bounded away from zero. To do so, we do not consider $\rho \in H^s(\mathbb{R})$ the standard Sobolev space, or $\rho \in \rho_* + H^s(\mathbb{R})$ for some $\rho_* > 0$, but rather Zhidkov spaces, as introduced in [31, 30], and further analyzed in [15] in the case of Schrödinger equations. For $s \geqslant 1$, we set

$$X^s(\mathbb{R}) = \left\{ f \in L^\infty(\mathbb{R}), \quad f' \in H^{s-1}(\mathbb{R}) \right\}.$$

Note that being in dimension one, $H^s(\mathbb{R}) \subset X^s(\mathbb{R})$ holds for all $s \geqslant 1$, and $X^s(\mathbb{R})$ is an algebra.
The second result of the paper is the following theorem.

Theorem 1.4. *Let* $\lambda > 0$ *and* $s \geqslant 2$. *Suppose that* $(\rho_0, \Phi_0) \in X^s(\mathbb{R}) \times C(\mathbb{R})$ *with* $\Phi_0' \in X^s(\mathbb{R})$ *and* $\rho_0(x) \geqslant \rho_{0*}$ *for some positive constant* ρ_{0*}.
There are $T > 0$, $\rho \in C([0, T[; X^s(\mathbb{R}))$, $\Phi \in C([0, T[\times\mathbb{R})$, *with* $\partial_x \Phi \in C([0, T[; X^s(\mathbb{R}))$, *such that*

$$\mu = \rho(t,x) dx \otimes \delta_{\xi = \partial_x \Phi(t,x)} \quad \textit{with} \quad (\rho, \Phi)\Big|_{t=0} = (\rho_0, \Phi_0),$$

is a measure solution to (1.1).
Moreover, $\rho(t, \cdot)$ *is bounded from below by a positive decreasing function of time.*

Remark 1.5. In (1.1), the term $\partial_x \Big(\ln \int \mu d\xi \Big) \partial_\xi \mu$ can be considered in a weak sense as $\partial_\xi \Big(\partial_x \big(\ln \int \mu d\xi \big) \mu \Big)$, since $\int \mu d\xi = \rho \in C([0, T]; C^1(\mathbb{R}))$ and $\rho > 0$.

Remark 1.6. Although the Cauchy problem for the isothermal Euler (1.2) has global in time entropy weak solutions $(\rho, u) \in L^\infty(\mathbb{R}_+ \times \mathbb{R})^2$ (see [11, 12]), we cannot use them directly for our purpose. Indeed, the momentum equation in (1.2) is obtained from the kinetic equation (1.1) by multiplying (1.1) by ξ and integrating the resulting equation with respect to ξ. This leads to the product

$$\rho\, \partial_x \ln(\rho).$$

Since it is not under a conservative form, it is well known that there is no rigorous way to give a sense to this product for a general $\rho \in L^\infty$. It is why we have recourse to regular solutions to the isothermal Euler system, thus restricting for general initial bounded data to local in time solutions far from vacuum.

Remark 1.7. For (ρ, v) with values in $]0, +\infty[\times\mathbb{R}$, (1.2) is a strictly hyperbolic system. Consequently, for $(\rho_0, v_0) \in (X^2(\mathbb{R}))^2$ with $\rho_0 \geqslant \alpha$ for some $\alpha > 0$, there are $T > 0$ and $(\rho, v) \in \big(C^1([0, T] \times \mathbb{R})\big)^2$ solution to the Cauchy problem associated to (1.2) and the initial datum (ρ_0, v_0). This theorem makes an extra connection with a logarithmic Schrödinger equation, for which the above system corresponds to a limit system in the semiclassical regime.

Remark 1.8. The assumption on the initial density ρ_0 is more general than merely $\rho_0 \in \rho_{0*} + H^s(\mathbb{R})$ for some $s \geqslant 2$, since for instance, ρ_0 may have different limits as $x \to -\infty$ and $x \to +\infty$, or, even, no limit at all.

The plan of the paper is the following. Section 2 recalls the main steps of the derivation of the model for $\lambda > 0$ and related mathematical results. In Section 3, we show how (1.1) can be obtained formally from a nonlinear Schrödinger equation through the semiclassical limit. Section 4 establishes Theorem 1.1. In Section 5 , we prove Theorem 1.4. Section 6 adapts the proof of the boundedness from below of the density in the isotropic case by [10] to the isothermal case.

2. Derivation of the model and related results

In this section, we recall the main lines of the derivation of the model with $\lambda > 0$, used for studying fusion plasmas ([17]). The evolution of the ions in the core of such plasmas is well described by the Vlasov equation

$$\partial_t f + v \cdot \nabla_x f + \frac{Ze}{m_i}(-\nabla_x \Phi + v \wedge B) \cdot \nabla_v f = 0,$$

where f is the ionic distribution function depending on time, position (in the domain Ω of the plasma) and velocity (in $\mathbb{R}^3$), Ze and m_i are the ion charge and mass respectively. The electric potential Φ and the magnetic field B should be governed by the Maxwell equations. But a finite Larmor radius approximation is derived in the limit of a large and uniform external magnetic field. This leads to the following equation for the ionic distribution function $\overline{f}$ in gyro coordinates,

$$\partial_t \overline{f} + v_\parallel \partial_{x_\parallel} \overline{f} - J^0_{\rho_L}(\partial_{x_\parallel}\Phi)\partial_{v_\parallel}\overline{f} - \big(J^0_{\rho_L}\nabla_{x_\perp}\Phi\big)^\perp \cdot \nabla_{x_\perp}\overline{f} = 0. \tag{2.1}$$

Here, the index $\|$ (resp. $\perp$) refers to the direction parallel (resp. orthogonal) to the external magnetic field. For any vector $u = (u_i) \in \mathbb{R}^3$, $u^\perp$ denotes the vector $(u_2, -u_1, 0)$. The operator $J^0_{\rho_L}$ is a Bessel operator performing averages on circles of Larmor radius ρ_L in planes orthogonal to the magnetic field. Since it is not used in this paper, we do not enter into more details about it. The electrons move quite more quickly than the ions, so that their density n_e is given in terms of the electric potential Φ by the Maxwell-Boltzmann equation

$$n_e = n_0 e^{\frac{e}{T_e}(\Phi - <\Phi>)}. \tag{2.2}$$

Here, e (resp. T_e) is the electronic charge (resp. temperature), and $<\Phi>$ is the average of the potential on a magnetic field line. Due to the electroneutrality of the plasma, the Poisson equation is replaced by the electroneutrality equation

$$n_e = \rho,$$

where ρ is the ionic density. The operator $J^0_{\rho_L}$ induces some regularity in the orthogonal direction, but none in the parallel direction. The two-dimensional dynamics in the direction perpendicular to the magnetic field is studied in [21]. In order to analyze the difficulty coming from the highly singular term $J^0_{\rho_L}(\partial_{x_\parallel}\Phi)\partial_{v_\parallel}\overline{f}$, we restrict to a one-dimensional spatial setting, e.g. by considering ionic distribution functions written in the form

$$f(t,x,v) = f_\parallel(t,x_\parallel,v_\parallel)f_\perp(|v_\perp|),$$

with

$$\int_0^{+\infty} f_\perp(|v_\perp|)2\pi|v_\perp|d|v_\perp| = 1.$$

Then the term $f_\perp$ has no incidence in equation (2.1) and can be factorized. The equation that $f_\parallel$ should solve is

$$\partial_t f_\parallel + v_\parallel \partial_{x_\parallel} f_\parallel - \lambda\big(\partial_{x_\parallel}\ln(\rho_\parallel)\big)\partial_{v_\parallel} f_\parallel = 0, \quad t>0, (x_\parallel, v_\parallel)\in\mathbb{R}^2,$$

i.e. the partial differential equation in (1.1) for $f_\parallel$ (resp. $\rho_\parallel$, $x_\parallel$, $v_\parallel$) denoted by f (resp. ρ, x,ξ), and $\lambda = \frac{T_e}{e}$.

Mathematical results related to (1.1) have been obtained for a system close to equilibrium, i.e. in the case where the departure of the electric potential Φ from its average along the magnetic lines $<\Phi>$ is small. Equation (2.2) simplifies into

$$n_e = n_0\left(1 + \frac{e}{T_e}(\Phi - <\Phi>)\right),$$

so that (1.1) is replaced by

$$\partial_t f + \xi\partial_x f - \lambda\big(\partial_x\rho\big)\partial_\xi f = 0, \quad t>0, (x,\xi)\in\mathbb{R}^2. \tag{2.3}$$

The Cauchy problem for (2.3) is locally well-posed either for initial analytic data [22] or in Sobolev spaces and satisfying a Penrose stability condition [20], but is ill-posed in the sense of Hadamard for regular initial data in Sobolev spaces and arbitrarily small time [3].

3. From Schrödinger to Vlasov via Euler

In this section, we show how a logarithmic Schrödinger equation can be formally related to the isothermal Euler system (1.2), in the semiclassical limit. For $\varepsilon > 0$, consider the Cauchy problem for the Schrödinger equation

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\partial^2_{xx}u^\varepsilon = \lambda\ln(|u^\varepsilon|^2)u^\varepsilon, \quad u^\varepsilon(0,x) = \sqrt{\rho_0(x)}e^{i\Phi_0(x)/\varepsilon}. \tag{3.1}$$

Following the idea from [18], any function $u^\varepsilon = a^\varepsilon e^{i\Phi^\varepsilon/\varepsilon}$, with $(t,x)\mapsto a^\varepsilon(t,x)\in\mathbb{C}$ and $(t,x)\mapsto \Phi^\varepsilon(t,x)\in\mathbb{R}$ solutions to the quasilinear problem

$$\partial_t\Phi^\varepsilon + \frac{(\partial_x\Phi^\varepsilon)^2}{2} + \lambda\,\ln\left(|a^\varepsilon|^2\right) = 0, \quad \Phi^\varepsilon(0,x) = \Phi_0(x), \tag{3.2}$$

$$\partial_t a^\varepsilon + \partial_x\Phi^\varepsilon\partial_x a^\varepsilon + \frac{a^\varepsilon}{2}\partial^2_{xx}\Phi^\varepsilon = i\frac{\varepsilon}{2}\partial^2_{xx}a^\varepsilon, \quad a^\varepsilon(0,x) = \sqrt{\rho_0(x)} =: a_0(x), \tag{3.3}$$

is a solution to (3.1). An important remark is that by allowing a^ε to be complex-valued (even though its initial datum is real-valued), one gains a degree of freedom to dispatch terms from (3.1) into (3.2)-(3.3), and the choice introduced by Grenier is much more robust than the Madelung transform when semiclassical limit is considered (see [6]).

Determining Φ^ε solution to (3.2) turns out to be equivalent to determining $v^\varepsilon = \partial_x\Phi^\varepsilon$ and a^ε solution to

$$\begin{cases} \partial_t v^\varepsilon + v^\varepsilon\partial_x v^\varepsilon + \lambda\,\partial_x\,\ln\left(|a^\varepsilon|^2\right) = 0, \quad v^\varepsilon(0,x) = \Phi_0'(x),\\ \partial_t a^\varepsilon + v^\varepsilon\partial_x a^\varepsilon + \dfrac{a^\varepsilon}{2}\partial_x v^\varepsilon = i\dfrac{\varepsilon}{2}\partial^2_{xx}a^\varepsilon, \quad a^\varepsilon(0,x) = a_0(x). \end{cases} \tag{3.4}$$

Indeed, given $(v^\varepsilon, a^\varepsilon)$ solution to (3.4), we can define Φ^ε by

$$\Phi^\varepsilon(t,x) = \Phi_0(x) - \int_0^t\left(\frac{1}{2}|v^\varepsilon(\tau,x)|^2 + \lambda\ln\left(|a^\varepsilon(\tau,x)|^2\right)\right)d\tau. \tag{3.5}$$

We check that

$$\partial_t\left(\partial_x\Phi^\varepsilon - v^\varepsilon\right) = \partial_x\partial_t\Phi^\varepsilon - \partial_t v^\varepsilon = 0,$$

so that $v^\varepsilon = \partial_x\Phi^\varepsilon$ and Φ^ε solves (3.2), and a^ε solves (3.3).

Passing formally to the limit $\varepsilon\to 0$ in (3.4), we get the system

$$\begin{cases} \partial_t v + v\partial_x v + \lambda\partial_x\ln(|a|^2) = 0, \quad v_{|t=0} = \Phi_0',\\ \partial_t a + v\partial_x a + \dfrac{a}{2}\partial_x v = 0, \quad a_{|t=0} = \sqrt{\rho_0}, \end{cases} \tag{3.6}$$

which turns out to be the symmetrized version of (1.2), with $\rho = |a|^2$ (see [9, 28]). As noticed in the introduction, we then formally obtain a solution to (1.1) by setting

$$f(t,x,\xi) = |a(t,x)|^2 dx\otimes\delta_{\xi=v(t,x)}.$$

A more direct link from (3.1) to (1.1) is provided by the notion of Wigner measure. The Wigner transform of u^ε, solution to (4.2)–(4.3), is defined by (see e.g. [5, 16, 26])

$$W^\varepsilon(t,x,\xi) = \int_{\mathbb{R}} e^{iy\xi}u^\varepsilon\left(t,x-\frac{\varepsilon}{2}y\right)\overline{u^\varepsilon}\left(t,x+\frac{\varepsilon}{2}y\right)dy, \quad (t,x,\xi)\in[0,T]\times\mathbb{R}^2. \tag{3.7}$$

Up to the extraction of a subsequence, W^ε converges to a non-negative measure on the phase space. In general, several limits may exist (see the above references). We will see that in the framework of this paper, the limit is unique, and solves (1.1).

4. Gaussian initial data

Let us first study the ordinary differential equation (1.3). Local existence and uniqueness of a C^1 solution stem from Cauchy-Lipschitz Theorem. Indeed, the nonlinearity in the above equation is locally Lipschitzean away from $\{\gamma = 0\}$. We now address the global existence issue.

In the case $\lambda > 0$, multiplying (1.3) by $\dot\gamma$ and integrating yields

$$(\dot\gamma)^2 = \omega_0^2 + 4\lambda\sigma_0 \ln\gamma. \tag{4.1}$$

This readily shows that γ is bounded from below away from zero, so the flow is global. In the case $\lambda < 0$, suppose that γ is bounded away from zero, i.e. there is $\delta > 0$ such that $\gamma(t) \geqslant \delta$. Then (1.3) yields

$$\ddot\gamma \leqslant \frac{2\lambda\sigma_0}{\delta},$$

hence $\gamma(t) \leqslant \frac{\lambda\sigma_0}{\delta}t^2 + \omega_0 t + 1$, and a contradiction for t sufficiently large. Now suppose that $\gamma \in C^2(0,\infty)$ with $\gamma > 0$: from the above argument, there exists a sequence t_n along which $\gamma(t_n) \to 0^+$. From (1.3), $\ddot\gamma(t_n) \to -\infty$, hence a contradiction.

To prove Theorem 1.1, we start from a semi-classically scaled logarithmic Schrödinger equation,

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\partial_{xx}^2 u^\varepsilon = \lambda\,\ln(|u^\varepsilon|^2)u^\varepsilon, \tag{4.2}$$

together with the initial value

$$u^\varepsilon(0,x) = \sqrt{\rho_*}e^{-\sigma_0 x^2/2}e^{i\omega_0 x^2/(2\varepsilon)}e^{ip_0 x/\varepsilon}. \tag{4.3}$$

Such an initial datum does not fit into the framework of [2], since it goes to zero at infinity. However, as noted in [4], for *fixed* $\varepsilon > 0$, the solution to (4.2) with such an initial datum can be computed rather explicitly. Indeed, bearing in mind the propagation of coherent states in the semi-classical regime ([19], see also [13]) it is consistent to look for a solution of the form

$$u^\varepsilon(t,x) = b^\varepsilon(t)e^{-\Omega^\varepsilon(t)(x-q(t))^2/2+ip(t)(x-q(t))/\varepsilon+iS(t)/\varepsilon}, \tag{4.4}$$

with $b^\varepsilon, \Omega^\varepsilon \in \mathbb{C}$, and $q,p,S \in \mathbb{R}$. Plugging this ansatz into (3.1), we find:

$$i\varepsilon\dot b^\varepsilon - i\varepsilon\dot\Omega^\varepsilon\frac{(x-q)^2}{2}b^\varepsilon + i\varepsilon\Omega^\varepsilon\dot q(x-q)b^\varepsilon - \dot p(x-q)b^\varepsilon + p\dot q b^\varepsilon - \frac{p^2}{2}b^\varepsilon - \dot S b^\varepsilon - \varepsilon^2\frac{\Omega^\varepsilon}{2}b^\varepsilon$$
$$+\,\varepsilon^2(\Omega^\varepsilon)^2\frac{(x-q)^2}{2}b^\varepsilon - ip\varepsilon\Omega^\varepsilon(x-q)b^\varepsilon = \lambda b^\varepsilon\ln|b^\varepsilon|^2 - \lambda b^\varepsilon(x-q)^2\operatorname{Re}\Omega^\varepsilon.$$

Cancelling the polynomial in $x-q$, sufficient conditions for the previous equation to hold are

$$\dot q = p, \quad \dot p = 0, \quad \dot S = p\dot q - \frac{p^2}{2} = \frac{p^2}{2}, \tag{4.5}$$

$$i\varepsilon\dot b^\varepsilon = \frac{\varepsilon^2}{2}\Omega^\varepsilon b^\varepsilon + \lambda b^\varepsilon\ln|b^\varepsilon|^2, \tag{4.6}$$

$$i\varepsilon\dot\Omega^\varepsilon = \varepsilon^2(\Omega^\varepsilon)^2 + 2\lambda\operatorname{Re}\Omega^\varepsilon. \tag{4.7}$$

Equation (4.5) corresponds to the classical Hamiltonian flow in the absence of external force, and the associated classical action. We compute exactly

$$p(t) = p_0, \quad q(t) = p_0 t, \quad S(t) = \frac{p_0^2}{2}t.$$

Since we are eventually interested only in the modulus of b^ε, we infer from (4.6):

$$\frac{d}{dt}|b^\varepsilon|^2 = 2\operatorname{Re}\bar b^\varepsilon\dot b^\varepsilon = \varepsilon|b^\varepsilon|^2\operatorname{Im}\Omega^\varepsilon,$$

hence

$$|b^\varepsilon(t)|^2 = \rho_* e^{\varepsilon\int_0^t \operatorname{Im}\Omega^\varepsilon(s)ds}. \tag{4.8}$$

For fixed $\varepsilon > 0$ and $\operatorname{Re}\Omega^\varepsilon(0) > 0$, (4.7) has a unique, global solution $\Omega^\varepsilon \in C(\mathbb{R})$, whose large time behavior depends on the sign of λ. Indeed, in a similar fashion as in [25], we seek Ω^ε of the form

$$\Omega^\varepsilon = -i\frac{\dot{\eta}^\varepsilon}{\varepsilon\eta^\varepsilon}.$$

Then (4.7) becomes

$$\ddot{\eta}^\varepsilon = \frac{2\lambda}{\varepsilon}\eta^\varepsilon \operatorname{Im}\frac{\dot{\eta}^\varepsilon}{\eta^\varepsilon}.$$

Introducing the scaled polar decomposition $\eta^\varepsilon = \gamma^\varepsilon e^{i\varepsilon\alpha^\varepsilon}$, Ω^ε is given by

$$\Omega^\varepsilon = \dot{\alpha}^\varepsilon - \frac{i}{\varepsilon}\frac{\dot{\gamma}^\varepsilon}{\gamma^\varepsilon}, \tag{4.9}$$

and the above equation reads:

$$\ddot{\gamma}^\varepsilon - \varepsilon^2\gamma^\varepsilon(\dot{\alpha}^\varepsilon)^2 = 2\lambda\gamma^\varepsilon\dot{\alpha}^\varepsilon, \qquad \ddot{\alpha}^\varepsilon\gamma^\varepsilon + 2\dot{\alpha}^\varepsilon\dot{\gamma}^\varepsilon = 0.$$

The second equation yields

$$\frac{d}{dt}\left((\gamma^\varepsilon)^2\dot{\alpha}^\varepsilon\right) = 0.$$

In view of (4.9), we have a degree of freedom to set $\gamma^\varepsilon(0)$. Setting $\gamma^\varepsilon(0) = 1$, we have

$$\dot{\alpha}^\varepsilon(0) = \operatorname{Re}\Omega^\varepsilon(0) = \sigma_0, \quad \dot{\gamma}^\varepsilon(0) = -\varepsilon\operatorname{Im}\Omega^\varepsilon(0) = \omega_0.$$

Therefore, we have

$$(\gamma^\varepsilon)^2\dot{\alpha}^\varepsilon = \sigma_0. \tag{4.10}$$

The equation for γ^ε boils down to

$$\ddot{\gamma}^\varepsilon = \frac{\varepsilon^2\sigma_0^2}{(\gamma^\varepsilon)^3} + \frac{2\lambda\sigma_0}{\gamma^\varepsilon},$$

along with the initial data that we recall:

$$\gamma^\varepsilon(0) = 1, \quad \dot{\gamma}^\varepsilon(0) = \omega_0.$$

Local existence and uniqueness of a C^1 solution stem from Cauchy-Lipschitz Theorem. Indeed, the nonlinearity in the above equation is locally Lipschitzean away from $\{\gamma^\varepsilon = 0\}$. We now address the global existence issue. Multiplying the above equation by $\dot{\gamma}^\varepsilon$ and integrating in time, we infer:

$$(\dot{\gamma}^\varepsilon)^2 = 4\lambda\sigma_0\ln\gamma^\varepsilon - \frac{\varepsilon^2\sigma_0^2}{(\gamma^\varepsilon)^2} + \varepsilon^2\sigma_0^2 + \omega_0^2.$$

This shows that for fixed $\varepsilon > 0$, γ^ε remains bounded away from zero, for if we had $\gamma^\varepsilon(t_n) \to 0$ for some sequence t_n, then the above right hand side would become negative for n large enough, hence a contradiction. Thus, for fixed $\varepsilon > 0$, (4.7) has a unique, global solution $\Omega^\varepsilon \in C^1(\mathbb{R})$.

In view of (4.10), (4.9) also reads

$$\Omega^\varepsilon = \frac{\sigma_0}{(\gamma^\varepsilon)^2} - \frac{i}{\varepsilon}\frac{\dot{\gamma}^\varepsilon}{\gamma^\varepsilon}. \tag{4.11}$$

Given (4.8) and (4.11), the Wigner transform (3.7) of u^ε is given by

$$W^\varepsilon(t,x,\xi) = \frac{\rho_*}{\gamma^\varepsilon(t)}e^{-\frac{\sigma_0(x-p_0t)^2}{(\gamma^\varepsilon(t))^2}}\int e^{-\frac{\sigma_0\varepsilon^2 y}{4(\gamma^\varepsilon(t))^2}}e^{iy\left(\xi - \frac{\dot{\gamma}^\varepsilon}{\gamma^\varepsilon}(x-p_0t)-p_0\right)}dy. \tag{4.12}$$

On every time interval such that γ is bounded away from zero, the Gronwall lemma shows that

$$\gamma^\varepsilon - \gamma = \mathcal{O}(\varepsilon), \quad \dot\gamma^\varepsilon - \dot\gamma = \mathcal{O}(\varepsilon).$$

Consequently, when $\varepsilon \to 0$, the Wigner transform W^ε of u^ε weakly converges to the bounded measure

$$\mu(t,dx,d\xi) = \frac{\rho_*}{\gamma(t)} e^{-\frac{\sigma_0(x-p_0t)^2}{(\gamma(t))^2}} dx \otimes \delta_{\xi=\frac{\dot\gamma}{\gamma}(x-p_0t)+p_0}. \tag{4.13}$$

Straightforward computations show that

$$(\rho(t,x),v(t,x)) := \Big(\frac{\rho_*}{\gamma(t)} e^{-\frac{\sigma_0(x-p_0t)^2}{(\gamma(t))^2}}, \frac{\dot\gamma(t)}{\gamma(t)}(x-p_0t)+p_0\Big)$$

is a solution to the isothermal Euler system (1.2) on the time interval of the existence of γ.

To conclude, we consider the large time behavior of γ for $\lambda > 0$. If γ was bounded from above, $\gamma \leqslant M$, then (1.3) would yield

$$\ddot\gamma \geqslant \frac{2\lambda\sigma_0}{M} > 0,$$

hence a contradiction. Therefore, for $\underline{t}$ sufficiently large, $\gamma(\underline{t}) \geqslant 1$ and $\dot\gamma(\underline{t}) > 0$. Since $\ddot\gamma \geqslant 0$, integration then shows

$$\gamma(t) \geqslant \dot\gamma(\underline{t})(t-\underline{t}) + 1 \mathop{\longrightarrow}_{t\to\infty} \infty,$$

and $\dot\gamma > 0$ for t sufficiently large. The asymptotic behavior announced in Theorem 1.4 then follows by integrating the identity

$$\frac{d\gamma}{\sqrt{\omega_0^2 + 4\lambda\sigma_0 \ln\gamma}} = dt,$$

to obtain the asymptotic behavior of γ.

$$t = \int \frac{d\gamma}{\sqrt{\omega_0^2 + 4\lambda\sigma_0 \ln\gamma}} = \frac{1}{2\lambda\sigma_0}\int e^{(y^2-\omega_0^2)/(4\lambda\sigma_0)}dy,$$

where we have changed the variable as

$$y = \sqrt{4\lambda\sigma_0\ln\gamma + \omega_0^2}.$$

Recall that the Dawson function, defined by

$$F(x) = e^{-x^2}\int_0^x e^{y^2}dy$$

satisfies

$$F(x) \mathop{\sim}_{x\to+\infty} \frac{1}{2x},$$

(see e.g. [1]), we infer that

$$\gamma(t) \mathop{\sim}_{t\to+\infty} 2t\sqrt{\lambda\sigma_0 \ln t}.$$

Since

$$\dot\gamma = \sqrt{\omega_0^2 + 4\lambda\sigma_0\ln\gamma},$$

we conclude

$$\dot\gamma(t) \mathop{\sim}_{t\to+\infty} 2\sqrt{\lambda\sigma_0 \ln t}.$$

In the case $\lambda < 0$, suppose that γ is bounded away from zero, $\gamma(t) \geqslant \delta > 0$. Then (1.3) yields

$$\ddot{\gamma} \leqslant \frac{2\lambda\sigma_0}{\delta},$$

hence $\gamma(t) \leqslant \frac{\lambda\sigma_0}{\delta}t^2 + \omega_0 t + 1$, and a contradiction for t sufficiently large. Now suppose that $\gamma \in C^2(0,\infty)$ with $\gamma > 0$: from the above argument, there exists a sequence t_n along which $\gamma(t_n) \to 0^+$. From (1.3), $\ddot{\gamma}(t_n) \to -\infty$, hence a contradiction.

This ends the proof of Theorem 1.1.

5. WKB analysis

In this section, we justify the formal approach presented in Section 3 in the framework of Theorem 1.4, thus proving this result.

5.1. Constructing the solution.

Note that for fixed $\varepsilon > 0$, the Cauchy problem for (3.1) has been considered in [8] (see also [7, Section 9.1]), for initial data in the class

$$W = \left\{ f \in H^1(\mathbb{R}), \quad \int_{\mathbb{R}} |f(x)|^2 \left| \ln |f(x)|^2 \right| dx < \infty \right\}.$$

This class is not compatible with the assumption $\rho(x) \geqslant \rho_{0*} > 0$ from Theorem 1.1, which is equivalent to $|u^\varepsilon(0,x)|^2 = |a_0(x)|^2 \geqslant \rho_{0*} > 0$ in the approach that we follow. Therefore, we choose to rather work in Zhidkov spaces $X^s(\mathbb{R})$. The system (3.4) has a unique smooth solution as stated in the following proposition, which includes the case $\varepsilon = 0$.

Proposition 5.1. *Let $s > 5/2$ and $\lambda > 0$. Suppose that $\rho_0, \Phi_0' \in X^s(\mathbb{R})$, with*

$$\rho_0(x) \geqslant \rho_{0*} > 0.$$

Then there exists T independent of $s > 5/2$ and $\varepsilon \in [0,1]$, and a unique solution $(a^\varepsilon, v^\varepsilon) \in C([0,T]; X^s \times X^s)$ to (3.4).

Proof. This result is a rather direct consequence of [2, Proposition 2.1], whose proof we recall the main idea. Separate real and imaginary parts of a^ε, $a^\varepsilon = a_1^\varepsilon + ia_2^\varepsilon$, and introduce

$$\mathbf{u}^\varepsilon = \begin{pmatrix} a_1^\varepsilon \\ a_2^\varepsilon \\ v^\varepsilon \end{pmatrix}, \quad \mathbf{u}_0 = \begin{pmatrix} \sqrt{\rho_0} \\ 0 \\ \Phi_0' \end{pmatrix}, \quad L = \begin{pmatrix} 0 & -\partial_{xx}^2 & 0 \\ \partial_{xx}^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{and} \quad A(\mathbf{u}) = \begin{pmatrix} v & 0 & \frac{a_1}{2} \\ 0 & v & \frac{a_2}{2} \\ \frac{2\lambda a_1}{a_1^2+a_2^2} & \frac{2\lambda a_2}{a_1^2+a_2^2} & v \end{pmatrix}.$$

We now have the system:

$$\partial_t \mathbf{u}^\varepsilon + A(\mathbf{u}^\varepsilon)\partial_x \mathbf{u}^\varepsilon = \frac{\varepsilon}{2} L\mathbf{u}^\varepsilon \quad ; \quad \mathbf{u}^\varepsilon_{|t=0} = \mathbf{u}_0. \tag{5.1}$$

Since ρ_0 is bounded away from zero, its square root is also in X^s, so that $\mathbf{u}_0 \in X^s(\mathbb{R})^3$. The matrix A is symmetrized by the matrix

$$S = \begin{pmatrix} I_2 & 0 \\ 0 & \frac{a_1^2+a_2^2}{4\lambda} \end{pmatrix},$$

which is symmetric positive if and only if $a_1^2 + a_2^2 > 0$, that is, so long as no vacuum appears. By assumption,

$$(a_1^\varepsilon)^2 + (a_2^\varepsilon)^2\Big|_{t=0} \geqslant \rho_{0*} > 0.$$

Then the main idea is that the operator L is skew-symmetric, and so does not appear in L^2-based energy estimates. Standard tame estimates (see e.g. [27, 29]) do not involve the L^2 norm of $\mathbf{u}^\varepsilon$, and so the only aspect remaining is that L^∞-estimates can be obtained rather directly. So long as, say,

$$(a_1^\varepsilon(t,x))^2 + (a_2^\varepsilon(t,x))^2 \geqslant \frac{\rho_{0*}}{2}, \quad \forall x \in \mathbb{R}, \tag{5.2}$$

we have:

$$\begin{aligned}
\|\mathbf{u}^\varepsilon(t)\|_{L^\infty} &\leqslant \|\mathbf{u}_0\|_{L^\infty} + \int_0^t \|A(\mathbf{u}^\varepsilon(\tau))\partial_x \mathbf{u}^\varepsilon(\tau)\|_{L^\infty} d\tau + \int_0^t \|\partial_{xx}^2 a^\varepsilon(\tau)\|_{L^\infty} d\tau \\
&\leqslant \|\mathbf{u}_0\|_{L^\infty} + C\int_0^t \|\mathbf{u}^\varepsilon(\tau)\|_{L^\infty}\|\partial_x \mathbf{u}^\varepsilon(\tau)\|_{L^\infty} d\tau + \int_0^t \|\partial_{xx}^2 a^\varepsilon(\tau)\|_{L^\infty} d\tau \\
&\leqslant \|\mathbf{u}_0\|_{L^\infty} + C\int_0^t \|\mathbf{u}^\varepsilon(\tau)\|_{X^s}^2 d\tau + \int_0^t \|\mathbf{u}^\varepsilon(\tau)\|_{X^s} d\tau,
\end{aligned}$$

where we have used Sobolev embedding under the assumption $s > 5/2$. Indeed, by definition, $X^s \subset L^\infty$, and for $\mathbf{u}^\varepsilon \in X^s$, $\partial_x \mathbf{u}^\varepsilon \in H^{s-1} \subset L^\infty$, provided that $s > 3/2$, and similarly, $\partial_{xx}^2 \mathbf{u}^\varepsilon \in L^\infty$ for $s > 5/2$.
Now we set $P = (I - \partial_{xx}^2)^{(s-1)/2}\partial_x$, so that $\|f\|_{X^s} \approx \|f\|_{L^\infty} + \|Pf\|_{L^2}$, and denote by

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx,$$

the scalar product in L^2. Since L is skew-symmetric and S is real-valued,

$$\begin{aligned}
&\frac{d}{dt}\left\langle SP\mathbf{u}^\varepsilon(t), P\mathbf{u}^\varepsilon(t)\right\rangle \\
&= \left\langle (\partial_t S)P\mathbf{u}^\varepsilon(t), P\mathbf{u}^\varepsilon(t)\right\rangle + 2\operatorname{Re}\left\langle S\partial_t P\mathbf{u}^\varepsilon(t), P\mathbf{u}^\varepsilon(t)\right\rangle \\
&= \left\langle (\partial_t S)P\mathbf{u}^\varepsilon(t), P\mathbf{u}^\varepsilon(t)\right\rangle + \varepsilon\operatorname{Re}\left\langle SLP\mathbf{u}^\varepsilon(t), P\mathbf{u}^\varepsilon(t)\right\rangle \\
&\quad - 2\operatorname{Re}\left\langle SP\Big(A(u^\varepsilon(t))\partial_x u^\varepsilon(t)\Big), P\mathbf{u}^\varepsilon(t)\right\rangle.
\end{aligned}$$

So long as (5.2) holds, we have the following set of estimates. First,

$$\begin{aligned}
\left\langle (\partial_t S)P\mathbf{u}^\varepsilon(t), P\mathbf{u}^\varepsilon(t)\right\rangle &\leqslant \|\partial_t S\|_{L^\infty}\|P\mathbf{u}^\varepsilon(t)\|_{L^2}^2 \\
&\leqslant C\left(\|\mathbf{u}^\varepsilon(t)\|_{L^\infty}\right)\|\partial_t \mathbf{u}^\varepsilon(t)\|_{L^\infty}\|\mathbf{u}^\varepsilon(t)\|_{X^s}^2.
\end{aligned}$$

Directly from (5.1), we have:

$$\begin{aligned}
\|\partial_t \mathbf{u}^\varepsilon(t)\|_{L^\infty} &\leqslant C\left(\|\mathbf{u}^\varepsilon(t)\|_{L^\infty}\right)\|\partial_x \mathbf{u}^\varepsilon(t)\|_{L^\infty} + \|\partial_{xx}^2 a^\varepsilon(t)\|_{L^\infty} \\
&\leqslant C\left(\|\mathbf{u}^\varepsilon(t)\|_{X^s}\right)\|\mathbf{u}^\varepsilon(t)\|_{X^s}.
\end{aligned}$$

Since SL is skew-symmetric, we have

$$\operatorname{Re}\left\langle SLP\mathbf{u}^\varepsilon(t), P\mathbf{u}^\varepsilon(t)\right\rangle = 0,$$

which prevents any loss of regularity in the estimates. For the quasi-linear term involving the matrix A, we note that since SA is symmetric, commutator estimates (see [23]) yield:

$$\begin{aligned}
\left\langle SP\left(A(\mathbf{u}^\varepsilon)\partial_x \mathbf{u}^\varepsilon\right), P\mathbf{u}^\varepsilon(t)\right\rangle &\leqslant C\left(\|\mathbf{u}^\varepsilon(t)\|_{L^\infty}\right)\|P\mathbf{u}^\varepsilon(t)\|_{L^2}^2\|\partial_x \mathbf{u}^\varepsilon(t)\|_{L^\infty} \\
&\leqslant C\left(\|\mathbf{u}^\varepsilon(t)\|_{X^s}\right)\|P\mathbf{u}^\varepsilon(t)\|_{L^2}^2.
\end{aligned}$$

Finally, we have:

$$\frac{d}{dt}\left\langle SP\mathbf{u}^\varepsilon(t), P\mathbf{u}^\varepsilon(t)\right\rangle \leqslant C\left(\|\mathbf{u}^\varepsilon(t)\|_{X^s}\right)\|\mathbf{u}^\varepsilon(t)\|^2_{X^s}.$$

This estimate, along with the L^∞-estimate, shows that on a sufficiently small time interval $[0,T]$, with $T>0$ independent of $\varepsilon\in[0,1]$, (5.2) holds, hence the existence of a unique solution. The fact that the local existence time does not depend on $s>5/2$ follows from the continuation principle based on Moser's calculus and tame estimates (see e.g. [27, Section 2.2] or [29, Section 16.1]). □

Corollary 5.2. *Under the assumptions of Proposition 5.1, if we suppose in addition that* $\Phi_0'\in L^2(\mathbb{R})$*, then* (3.1) *has a unique solution* $u^\varepsilon\in C([0,T];X^s(\mathbb{R}))$*, where* T *is given by Proposition 5.1.*

Proof. From Proposition 5.1, (3.4) has a solution $(v^\varepsilon,a^\varepsilon)\in C([0,T];X^s\times X^s)$. Plugging this information into (3.4), we infer

$$\begin{aligned}\|v^\varepsilon(t)\|_{L^2}\leqslant \|\Phi_0'\|_{L^2}&+\int_0^t\|v^\varepsilon(\tau)\|_{L^\infty}\|\partial_x v^\varepsilon(\tau)\|_{L^2}d\tau\\ &+C\int_0^t\left\|\frac{1}{a^\varepsilon(\tau)}\right\|_{L^\infty}\|\partial_x a^\varepsilon(\tau)\|_{L^2}d\tau.\end{aligned}$$

Therefore, $v^\varepsilon\in C([0,T];L^2)$, and Φ^ε, stemming from v^ε *via* the formula (3.5), satisfies $\Phi^\varepsilon\in C([0,T];X^{s+1})$. The existence part follows readily, since $X^s(\mathbb{R})$ is an algebra, by setting $u^\varepsilon=a^\varepsilon e^{i\Phi^\varepsilon/\varepsilon}$.

For the uniqueness property, consider two such solutions $u^\varepsilon,\tilde u^\varepsilon\in C([0,T];X^s)$, and set $w^\varepsilon=u^\varepsilon-\tilde u^\varepsilon$. It satisfies

$$i\varepsilon\partial_t w^\varepsilon+\frac{\varepsilon^2}{2}\partial^2_{xx}w^\varepsilon=\lambda\left(\ln(|u^\varepsilon|^2)u^\varepsilon-\ln(|\tilde u^\varepsilon|^2)\tilde u^\varepsilon\right),\quad w^\varepsilon_{|t=0}=0. \tag{5.3}$$

Recall the pointwise estimate from [8] (see also [7, Lemma 9.3.5]),

$$\left|\operatorname{Im}\left(\ln(|u^\varepsilon|^2)u^\varepsilon-\ln(|\tilde u^\varepsilon|^2)\tilde u^\varepsilon\right)\left(u^\varepsilon-\tilde u^\varepsilon\right)\right|\leqslant 4|u^\varepsilon-\tilde u^\varepsilon|^2.$$

Multiply (5.3) by $\overline{w^\varepsilon}$, integrate on an interval $I=[M_-,M_+]$, and take the imaginary part. This yields, along with the above estimate,

$$\frac{\varepsilon}{2}\frac{d}{dt}\int_I|w^\varepsilon(t,x)|^2dx+\frac{\varepsilon^2}{2}\operatorname{Im}\int_I\overline{w^\varepsilon}\partial^2_{xx}w^\varepsilon\leqslant 4\lambda\int_I|w^\varepsilon(t,x)|^2dx.$$

We have, by integration by parts,

$$\operatorname{Im}\int_I\overline{w^\varepsilon}\partial^2_{xx}w^\varepsilon=\operatorname{Im}\overline{w^\varepsilon}(t,M_+)\partial_x w^\varepsilon(t,M_+)-\operatorname{Im}\overline{w^\varepsilon}(t,M_-)\partial_x w^\varepsilon(t,M_-).$$

Since $w^\varepsilon\in C([0,T];X^1)$, we can choose sequences $M_\pm^n\to\pm\infty$ along which the above term goes to zero, and the Gronwall lemma implies $\|w^\varepsilon(t)\|_{L^2}\equiv 0$. □

5.2. **Asymptotic expansion.** Proposition 5.1 with $\varepsilon=0$ yields the existence of a unique solution $(v,a)\in C([0,T];(X^s(\mathbb{R}))^2)$ to (3.6) As a direct consequence of Proposition 5.1 and [2, Proposition 3.1], we have:

Proposition 5.3. *Under the assumptions of Proposition 5.1, there exists* C *independent of* $\varepsilon\in[0,1]$ *such that*

$$\|\partial_x\left(\Phi^\varepsilon-\Phi\right)\|_{L^\infty([0,T];X^{s-2})}+\|a^\varepsilon-a\|_{L^\infty([0,T];X^{s-2})}\leqslant C\varepsilon.$$

5.3. **Convergence of the Wigner transform.** For $u^\varepsilon = a^\varepsilon e^{i\Phi^\varepsilon/\varepsilon}$, with $(\Phi^\varepsilon, a^\varepsilon)$ solution to (3.2)–(3.3), or equivalently $(\partial_x\Phi^\varepsilon, a^\varepsilon)$ solution to (3.4), the Wigner transform defined in (3.7) is equal to

$$W^\varepsilon(t,x,\xi) = \int e^{i\xi v} a^\varepsilon\left(t, x-\frac{\varepsilon}{2}y\right)\overline{a^\varepsilon}\left(t,x+\frac{\varepsilon}{2}y\right)e^{i\varphi^\varepsilon(t,x,y)/\varepsilon}dy, \tag{5.4}$$

where

$$\varphi^\varepsilon(t,x,y) = \Phi^\varepsilon\left(t,x-\frac{\varepsilon}{2}y\right) - \Phi^\varepsilon\left(t,x+\frac{\varepsilon}{2}y\right).$$

Theorem 5.4. *When $\varepsilon \to 0$, the Wigner transform W^ε of u^ε weakly converges to the bounded measure*

$$\mu(t,dx,d\xi) = |a(t,x)|^2 dx \otimes \delta_{\xi=\partial_x\Phi(t,x)},$$

where $(\partial_x\Phi, a)$ is a solution of (3.6)*. Moreover, μ is a solution to* (1.1)*.*

Proof. In view of Proposition 5.3, $a^\varepsilon = a + r_a^\varepsilon$ and $\partial_x\Phi^\varepsilon = \partial_x\Phi + r_v^\varepsilon$, with

$$\|r_a^\varepsilon\|_{L^\infty([0,T];X^{s-2})} + \|r_v^\varepsilon\|_{L^\infty([0,T];X^{s-2})} \leqslant C\varepsilon.$$

Therefore,

$$W^\varepsilon(t,x,\xi) = \int e^{iy(\xi-\partial_x\Phi(t,x))} a\left(t,x-\frac{\varepsilon}{2}y\right)\bar{a}\left(t,x+\frac{\varepsilon}{2}y\right)dy + R_1^\varepsilon + R_2^\varepsilon + R_3^\varepsilon,$$

where

$$\begin{aligned}
&R_j^\varepsilon(t,x,\xi) = \int e^{iy(\xi-\partial_x\Phi(t,x))} r_j^\varepsilon(t,x,y)dy, \quad 1\leqslant j\leqslant 3,\\
&r_1^\varepsilon(t,x,y) = a^\varepsilon\left(t,x-\frac{\varepsilon}{2}y\right)\overline{a^\varepsilon}\left(t,x+\frac{\varepsilon}{2}y\right)\left(e^{i(\varphi^\varepsilon(t,x,y)+\varepsilon\partial_x\Phi(t,x))/\varepsilon}-1\right),\\
&r_2^\varepsilon(t,x,y) = \bar{a}\left(t,x+\frac{\varepsilon}{2}y\right)r_a^\varepsilon\left(t,x-\frac{\varepsilon}{2}y\right) + a\left(t,x-\frac{\varepsilon}{2}y\right)\overline{r_a^\varepsilon}\left(t,x+\frac{\varepsilon}{2}y\right),\\
&r_3^\varepsilon(t,x,y) = r_a^\varepsilon\left(t,x-\frac{\varepsilon}{2}y\right)\overline{r_a^\varepsilon}\left(t,x+\frac{\varepsilon}{2}y\right).
\end{aligned}$$

Propositions 5.1 and 5.3 yield, along with Taylor's formula for the term r_1^ε,

$$\|r_j^\varepsilon\|_{L^\infty([0,T]\times\mathbb{R}^2)} \leqslant C\varepsilon, \quad 1\leqslant j\leqslant 3.$$

Consequently, W^ε tends to $|a|^2dx\otimes\delta_{\xi=\partial_x\Phi}$ in $\mathcal{M}_b([0,T]\times\mathbb{R}^2)$ when ε tends to zero. Moreover, denote by $(\cdot,\cdot)$ the duality between bounded measures on $[0,T]\times\mathbb{R}^2$ and continuous functions with compact support in $[0,T]\times\mathbb{R}^2$. For any test function $\alpha(t,x,\xi)\in C^1$

with compact support in $[0,T]\times\mathbb{R}^2$, it holds

$$\begin{aligned}
&\left(\mu, \partial_t\alpha + \xi\partial_x\alpha - \lambda\left(\partial_x \ln|a|^2\right)\partial_\xi\alpha\right)\\
&= \int |a(t,x)|^2 \left(\partial_t\alpha(t,x,\partial_x\Phi(t,x)) + \partial_x\Phi(t,x)\partial_x\alpha(t,x,\partial_x\Phi(t,x))\right) dxdt\\
&\quad - \lambda \int |a(t,x)|^2 \left(\left(\partial_x \ln|a|^2\right)\partial_\xi\alpha(t,x,\partial_x\Phi(t,x))\right) dxdt\\
&= \int |a|^2 \left(\partial_t\left(\alpha(t,x,v(t,x))\right) - \partial_T v\partial_\xi\alpha(t,x,v(t,x))\right) dxdt\\
&+ \int |a|^2 \left(\partial_x\left(v\,\alpha(t,x,v(t,x))\right) - \partial_x v\,\alpha(t,x,v) - v\partial_x v\,\partial_\xi\alpha(t,x,v)\right) dxdt\\
&\quad - \lambda \int |a|^2 \left(\left(\partial_x \ln|a|^2\right)\partial_\xi\alpha(t,x,v(t,x))\right) dxdt\\
&= - \int \alpha(t,x,v)\left(\partial_t|a|^2 + v\partial_x|a|^2 + |a|^2\partial_x v\right) dxdt\\
&\quad - \int |a|^2\partial_\xi\alpha(t,x,v)\left(\partial_t v + v\partial_x v + \lambda\partial_x \ln|a|^2\right) dxdt.
\end{aligned}$$

This is zero, in view of (3.6), since

$$\partial_t|a|^2 + \partial_x\Phi\partial_x|a|^2 + |a|^2\partial^2_{xx}\Phi = 2\,\mathrm{Re}\,\overline{a}\left(\partial_t a + \partial_x\Phi\partial_x a + \frac{a}{2}\partial^2_{xx}\Phi\right) = 0.$$

Therefore, any solution to (3.6) yields a solution to (1.1). □

6. The bound from below of the density

It remains to prove that the density is bounded from below to complete the proof of Theorem 1.4. The result from [10] yields:

Proposition 6.1. *Under the assumptions of Theorem 1.4, the density ρ solution to* (1.2) *satisfies*

$$\rho(t) \geqslant \frac{\rho_{0*}}{1+Ct}, \quad t\in[0,+\infty[,$$

for some constant $C>0$.

Proof. In [10], the proof is presented for the pressure law $p(\rho)=\rho$ in (1.2) replaced by the isentropic one, $p(\rho)=\rho^\gamma$, with $\gamma>1$. We simply perform the standard modification to adapt the proof to the isothermal case (see e.g. [11, 14]). Note that Proposition 6.1 is also a consequence of [10, Remark 3.3]: we sketch the proof in the isothermal case for $\lambda=1$ for the convenience of the reader.

First, the isothermal Euler equation in Lagrangian coordinates reads (see e.g. [14])

$$\tau_t - u_x = 0, \quad u_t - \frac{\tau_x}{\tau^2} = 0, \tag{6.1}$$

with $\tau = 1/\rho$. Setting

$$s = u - \ln\tau, \quad r = u + \ln\tau,$$

we find

$$\partial_+ s = \partial_- r = 0,$$

where

$$\partial_\pm = \partial_t \pm \frac{1}{\tau}\partial_x.$$

Following [10], introduce

$$\alpha = s_x, \quad \beta = r_x.$$

By assumption, we know that u, τ, α and β are uniformly bounded at time $t = 0$: there exists $M > 0$ such that

$$\sup_{x\in\mathbb{R}}(\alpha(0,x), \beta(0,x)) < M.$$

We check

$$\partial_+\alpha = \frac{1}{2\tau}\alpha(\beta-\alpha), \tag{6.2}$$

$$\partial_-\beta = \frac{1}{2\tau}\beta(\alpha-\beta). \tag{6.3}$$

As in [10],

$$\sup_{(t,x)\in[0,T)\times\mathbb{R}}(\alpha(t,x), \beta(t,x)) < M, \tag{6.4}$$

where $T > 0$ is such that (6.1) has a C^1 solution on $[0,T)$. (6.4) is established by contradiction: assume for instance that $\alpha(t_0, x_0) = M$. Because the wave speed $\frac{1}{\tau}$ is bounded from above, we can consider the characteristic triangle with vertex (t_0, x_0) and lower boundary at time $t = 0$, denoted by Ω. Let $t_1(\leqslant t_0)$ be the first time such that $\alpha(t_1, x_1) = M$ or $\beta(t_1, x_1) = M$ in Ω. Assume $\alpha(t_1, x_1) = M$ for instance (the other case is similar), and let Ω_1 denote the characteristic triangle with vertex (t_1, x_1):

$$\sup_{(t,x)\in\Omega_1,\ t<t_1}(\alpha(t,x), \beta(t,x)) < M.$$

Since $\alpha(t_1, x_1) = M$, there exists $t_2 \in [0, t_1)$ such that

$$\inf_{(t,x)\in\Omega_1,\ t\geqslant t_2}\alpha(t,x) > 0.$$

Let $t_2 < t < t_1$. On $[t_2, t]$, (6.2) yields

$$\partial_+\alpha \leqslant K(M\alpha - \alpha^2).$$

Integrating along some charateristic between t_2 and t, we find

$$\frac{1}{M}\ln\frac{\alpha(t)}{M-\alpha(t)} \leqslant \frac{1}{M}\ln\frac{\alpha(t_2)}{M-\alpha(t_2)} + K(t-t_2).$$

Letting $t \to t_1$ then leads to a contradiction.

We infer that u_x is bounded from above, hence τ_t is bounded from above, in view of (6.1): τ grows at most linearly in time, and since $\rho = 1/\tau$, the proposition follows. □

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