NORM-INFLATION WITH INFINITE LOSS OF REGULARITY FOR PERIODIC NLS EQUATIONS IN NEGATIVE SOBOLEV SPACES

by

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Abstract. — In this paper we consider Schrödinger equations with nonlinearities of odd order $2\sigma+1$ on $\mathbb{T}^d$. We prove that for $\sigma d \geqslant 2$, they are strongly illposed in the Sobolev space H^s for any $s<0$, exhibiting norm-inflation with infinite loss of regularity. In the case of the one-dimensional cubic nonlinear Schrödinger equation and its renormalized version we prove such a result for H^s with $s<-2/3$.

1. Introduction

We consider nonlinear Schrödinger (NLS) equations of the form

$$i\partial_t\psi+\frac{1}{2}\Delta\psi=\mu|\psi|^{2\sigma}\psi,\quad \psi=\psi(t,x)\in\mathbb{C},\ t\in\mathbb{R},\ x\in\mathbb{T}^d \tag{1.1}$$

and the renormalized versions

$$i\partial_t\psi+\frac{1}{2}\Delta\psi=\mu|\psi|^{2}\psi-\frac{2\mu}{(2\pi)^d}\left(\int_{\mathbb{T}^d}|\psi(t,x)|^2dx\right)\psi\,, \tag{1.2}$$

where $\sigma\geqslant 1$ is an integer, $\mathbb{T}=\mathbb{R}/2\pi\mathbb{Z}$, $\Delta=\sum_{k=1}^d\partial_{x_k}^2$, and $\mu\in\{1,-1\}$.

For any $s\in\mathbb{R}$ and $1\leqslant p\leqslant\infty$, denote by $\mathcal{F}L^{s,p}(\mathbb{T}^d)\equiv\mathcal{F}L^{s,p}(\mathbb{T}^d,\mathbb{C})$ the Fourier-Lebesgue space,

$$\mathcal{F}L^{s,p}(\mathbb{T}^d)=\{f\in\mathcal{D}'(\mathbb{T}^d,\mathbb{C});\quad \langle\cdot\rangle^s\,\hat{f}(\cdot)\in\ell^p(\mathbb{Z}^d)\}$$

with $\ell^p(\mathbb{Z}^d)\equiv\ell^p(\mathbb{Z}^d,\mathbb{C})$ denoting the standard ℓ^p sequence space. Note that for any $s\in\mathbb{R}$, $\mathcal{F}L^{s,2}(\mathbb{T}^d)$ is the Sobolev space $H^s(\mathbb{T}^d)\equiv H^s(\mathbb{T}^d,\mathbb{C})$ and for any $1\leqslant p\leqslant\infty$, $\cap_{s\in\mathbb{R}}\mathcal{F}L^{s,p}(\mathbb{T}^d)$ coincides with $C^\infty(\mathbb{T}^d)\equiv C^\infty(\mathbb{T}^d,\mathbb{C})$. The aim of this paper is to establish the following strong ill-posedness property of equations (1.1) and (1.2).

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Theorem 1.1. — *Let $\sigma, d \geqslant 1$ be integers.*
(i) Assume that $d\sigma \geqslant 2$ in the case of (1.1) and $d \geqslant 2$ in the case of (1.2). Then for any $s < 0$, there exists a sequence of initial data $(\psi_n(0))_{n\geqslant 1}$ in $C^\infty(\mathbb{T}^d)$ such that

$$\|\psi_n(0)\|_{\mathcal{F}L^{s,p}(\mathbb{T}^d)} \underset{n\to\infty}{\longrightarrow} 0, \quad \forall p \in [1,\infty],$$

and a sequence of times $t_n \to 0$ such that the corresponding solutions ψ_n to (1.1) respectively (1.2) satisfy

$$\|\psi_n(t_n)\|_{\mathcal{F}L^{r,p}(\mathbb{T}^d)} \underset{n\to\infty}{\longrightarrow} \infty, \quad \forall r \in \mathbb{R},\ \forall p \in [1,\infty].$$

(ii) If $d = \sigma = 1$, then for any $s < -2/3$, there exists a sequence of initial data $\psi_n(0) \in C^\infty(\mathbb{T})$ with

$$\|\psi_n(0)\|_{\mathcal{F}L^{s,p}(\mathbb{T})} \underset{n\to\infty}{\longrightarrow} 0, \quad \forall p \in [1,\infty],$$

and a sequence of times $t_n \to 0$ such that the corresponding solutions ψ_n to (1.1) respectively (1.2) satisfy

$$\|\psi_n(t_n)\|_{\mathcal{F}L^{r,p}(\mathbb{T})} \underset{n\to\infty}{\longrightarrow} \infty, \quad \forall r \in \mathbb{R},\ \forall p \in [1,\infty].$$

Theorem 1.1 implies the following

Corollary 1.2. — *Let $d, \sigma \geqslant 1$ be integers and let s be as in Theorem 1.1. Furthermore assume that $p_1, p_2 \in [1,\infty]$ and $T > 0$. Then for no $r \in \mathbb{R}$, there exists a neighborhood U of 0 in $\mathcal{F}L^{s,p_1}(\mathbb{T}^d)$ and a continuous function $M_r : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ such that any smooth solution ψ to (1.1) (or (1.2)) satisfy the a priori estimate*

$$\|\psi\|_{L^\infty(0,T;\mathcal{F}L^{r,p_2}(\mathbb{T}^d))} \leqslant M_r\left(\|\psi(0)\|_{\mathcal{F}L^{s,p_1}(\mathbb{T}^d)}\right).$$

In particular, for $p_1 = p_2 = 2$, there is no continuous function M_r such that smooth solutions to (1.1) respectively (1.2) satisfy the a priori estimate

$$\|\psi\|_{L^\infty(0,T;H^r(\mathbb{T}^d))} \leqslant M_r\left(\|\psi(0)\|_{H^s(\mathbb{T}^d)}\right).$$

Comments: In connection with the study of ill-posedness of nonlinear Schrödinger and nonlinear wave equations on the whole space $\mathbb{R}^d$, Christ, Colliander, and Tao introduced in [**10**] (cf. also [**11**]), the notion of norm inflation with respect to a given (Sobolev) norm, saying that there exist a sequence of smooth initial data $(\psi_n(0))_{n\geqslant 1}$ and a sequence of times $(t_n)_{n\geqslant 1}$, both converging to 0, so that the corresponding smooth solutions ψ_n, evaluated at t_n, is unbounded. Further results in this direction were obtained in [**2, 5, 6, 20**], where in particular norm inflation together with finite or infinite loss of regularity was established for various equations on $\mathbb{R}^d$. Theorem 1.1 states that such type of results (in the strongest sense, since the loss of regularity is infinite) hold true for nonlinear Schrödinger equations on the torus $\mathbb{T}^d$.
Recently, the renormalized cubic Schrödinger equation (1.2) has caught quite some attention. In particular, on $\mathbb{T}$, some well-posedness / ill-posedness results below L^2 have been established – see [**9**], [**15**] as well as [**8**], [**19**]. Although there are indications that (1.2) has better stability properties than (1.1), our results show no difference between the two equations as far as norm inflation concerns.

Finally let us remark that the scaling symmetry of (1.1), considered on the Sobolev spaces $H^s(\mathbb{R}^d)$, $\psi(t,x) \mapsto \lambda^{-2/\sigma}\psi(\frac{t}{\lambda^2}, \frac{x}{\lambda})$ with $\lambda > 0$, has as critical exponent $s_{2,\sigma} = \frac{d}{2} - \frac{1}{\sigma}$ since for this value of s, the homogeneous H^s-norm is invariant under this scaling. More generally, for any given $1 \leqslant p \leqslant \infty$, the homogeneous $W^{s,p}(\mathbb{R}^d)-$norm is invariant for $s_{p,\sigma} = \frac{d}{p} - \frac{1}{\sigma}$. It suggests that the $\mathcal{F}L^{s,p}(\mathbb{R}^d)-$norm is invariant for $s^{FL}_{p,\sigma} = \frac{d}{p'} - \frac{1}{\sigma}$ with $\frac{1}{p'} = 1 - \frac{1}{p}$. Furthermore, the Galilean invariance of (1.1), $\psi(t,x) \mapsto e^{-iv\cdot x/2}e^{i|v|^2t/4}\psi(t, x-vt)$ for arbitrary velocities v, leaves the $\mathcal{F}L^{0,p}(\mathbb{R}^d)-$norm invariant. Note that the statements of Theorem 1.1 for (1.1), considered on $H^s(\mathbb{T}^d)$, are valid in a range of s, contained in the half line $-\infty < s \leqslant \min(s_{2,\sigma}, 0)$.

Method of proof: Let us give a brief outline of the proof of item (i) of Theorem 1.1 in the case of equation (1.1). Following the approach, developed in [**5**] and [**6**] for equations such as nonlinear Schrödinger equations on the whole space $\mathbb{R}^d$, we introduce the following version of (1.1),

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = \varepsilon|u^\varepsilon|^{2\sigma}u^\varepsilon, \quad x \in \mathbb{T}^d$$

with ε being a small parameter. The equation is in a form, referred to as weakly nonlinear geometric optics. A solution u^ε of it, which is $2\pi-$periodic in its $x-$variables, is related to a solution ψ of (1.1) by

$$u^\varepsilon(t,x) = \varepsilon^{\beta/(2\sigma)}\psi\left(\varepsilon^\beta t, \varepsilon^{\frac{\beta-1}{2}}x\right)$$

where $\beta > 0$ is a free parameter, but chosen so that ψ is also $2\pi-$periodic in the $x-$variables. We then construct a first order approximate solution $u^\varepsilon_{\text{app}}(t,x)$ of $u^\varepsilon(t,x)$ of the form $u^\varepsilon_{\text{app}}(t,x) = \sum_{j\in\mathbb{Z}^d} a_j(t)e^{i\phi_j(t,x)/\varepsilon}$ where the phase function $\phi_j(t,x)$ and the amplitude $a_j(t)$ are determined in such a way that $u^\varepsilon_{\text{app}}(t,x)$ solves the above equation for u^ε up to $\mathcal{O}(\varepsilon)$. It turns out that $\phi_j(t,x) = j\cdot x - \frac{|j|^2}{2}t$ and that the a_j's satisfy a system of ODEs, defined in terms of the resonance sets

$$\text{Res}_j = \left\{(k_\ell)_{1\leqslant\ell\leqslant 2\sigma+1} \in \mathbb{Z}^{(2\sigma+1)d}\,; \quad \sum_{\ell=1}^{2\sigma+1}(-1)^{\ell+1}k_\ell = j\,; \quad \sum_{\ell=1}^{2\sigma+1}(-1)^{\ell+1}|k_\ell|^2 = |j|^2\right\}.$$

The strategy to prove Theorem 1.1 in the case considered is then to choose initial data for u^ε of the form $u^\varepsilon(0,x) = \sum_{j\in S}\alpha_j e^{i\phi_j(0,x)/\varepsilon}$ with $S \subset \mathbb{Z}^d$ finite and $0 \notin S$ so that the zero mode $a_0(t)e^{i\phi_0(t,x)/\varepsilon}$ is created by resonant interaction of nonzero modes at leading order, $\dot{a}_0(0) \neq 0$. With an appropriate choice of the scaling parameter β, the zero mode of ψ comes with a factor which is increasing in ε. Since the absolute value of the zero mode bounds the norm $\|\cdot\|_{\mathcal{F}L^{s,p}(\mathbb{T}^d)}$ of any Fourier Lebesgue space from below, it follows that for any $s < 0, 1 \leqslant p \leqslant \infty$, the sequence $(\|u^{\varepsilon_n}(t_n)\|_{\mathcal{F}L^{s,p}(\mathbb{T}^d)})_{n\geqslant 1}$ is unbounded for appropriate sequences $(\varepsilon_n)_{n\geqslant 1}$, $(t_n)_{n\geqslant 1}$, converging both to 0. The proofs of the remaining statements of Theorem 1.1 are similar, although a little bit more involved.

Related work: There are numerous works on ill-posedness for equations such as (1.1). Besides the papers already cited, we refer to the dispersive wiki page [**1**]. In [**19**] one finds a quite detailed account of existing results on the one-dimensional cubic NLS equation below L^2.

Organization: In Section 2 we recall the geometrical optics approximation of first order and a refined version of it, the latter being needed for the proof of item (ii) of Theorem 1.1. In the subsequent section, we provide estimates for the approximations of first and second order in the functional setup of the Wiener algebra. In Section 4, the resonant sets of integer vectors, coming up in the construction of the approximate solutions, are studied in more detail. Finally, in Section 5, we study the geometrical optics approximation for the renormalized NLS equation (1.2). With these preparations, we then prove Theorem 1.1 in Section 6.
The case of focusing ($\mu = -1$) NLS equations can be treated in exactly the same fashion as the case of defocusing ($\mu = 1$) ones. Hence to simplify notation, in what follows we will only consider equations (1.1) and (1.2) with $\mu = 1$. As already pointed out in [**11**], results of the type stated in Theorem 1.1 for defocusing NLS equations maybe considered as more surprising as the corresponding results for focusing ones.

Added in proof: After this work has been completed, Nobu Kishimoto informed us that in unpublished work, he obtained results similar to ours, using techniques introduced by Bejenaru and Tao ([**3**], further developed in [**16**]). In fact, his method of proof, being different from ours (it is based on a multiscale analysis), allows him to prove norm inflation in the Sobolev spaces $H^s(\mathbb{T})$ with $s \leqslant -1/2$ for the cubic NLS equation in one dimension and its renormalized version. In the preprint [**18**], posted after our work was made public, Tadahiro Oh shows norm inflation in homogeneous Sobolev spaces for the one-dimensional cubic NLS equation, $d = \sigma = 1$, for any $s \leqslant -1/2$, using the method of proof, introduced in [**10**]. However, the latter method seems not to be suited to establish norm inflation with an (infinite) loss of regularity.

2. Geometrical optics approximation: generalities

2.1. Setup. — For $0 < \varepsilon \leqslant 1$, we consider

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = \varepsilon |u^\varepsilon|^{2\sigma} u^\varepsilon, \quad x \in \mathbb{T}^d, \tag{2.1}$$

along with initial data which are superpositions of plane waves,

$$u^\varepsilon(0,x) = \sum_{j\in\mathbb{Z}^d} \alpha_j e^{ij\cdot x/\varepsilon}, \quad \alpha_j \in \mathbb{C}. \tag{2.2}$$

To insure that $u^\varepsilon(0,x)$ is $2\pi-$periodic in x we will assume throughout the paper that the parameter ε is of the from $\varepsilon = 1/N$ for some $N \in \mathbb{N}$. The goal of this and the next two sections is to describe the solution u^ε in the limit $\varepsilon \to 0$. Let us begin by briefly recalling the results detailed in [**5**]. We construct first order approximations of solutions of (2.1)–(2.2) as a superposition of modes,

$$u^{\varepsilon}_{\text{app}}(t,x) = \sum_{j\in\mathbb{Z}^d} a_j(t) e^{i\phi_j(t,x)/\varepsilon}. \tag{2.3}$$

The regime (2.1) goes under the name of weakly nonlinear geometric optics (see e.g. [**4**]) since according to the considerations below, the phase functions ϕ_j turn out to be not affected by the nonlinearity in (2.1), while the amplitudes a_j are. To find ϕ_j and a_j, substitute the ansatz (2.3) into (2.1) and for each $j\in\mathbb{Z}^d$, consider the terms containing $e^{i\phi_j/\varepsilon}$ separately. We then determine ϕ_j and a_j so as to cancel the terms of lowest orders in ε. Since the initial data are assumed to be of the form (2.2), we find for any given $j\in\mathbb{Z}^d$ at order $\mathcal{O}\left(\varepsilon^0\right)$,

$$\mathcal{O}\left(\varepsilon^0\right): \quad \partial_t\phi_j + \frac{1}{2}|\nabla\phi_j|^2 = 0, \quad \phi_j(0,x) = j\cdot x,$$

hence

$$\phi_j(t,x) = j\cdot x - \frac{|j|^2}{2}t. \tag{2.4}$$

In particular, for $j=0$ one has $\phi_0 = 0$ and hence the zero mode $a_0 e^{i\phi_0/\varepsilon}$ equals a_0 and is thus independent of ε. At next order, we obtain the following evolution equation for the amplitude a_j

$$\mathcal{O}\left(\varepsilon^1\right): \quad i\dot{a}_j = \sum_{(k_1,k_2,\cdots,k_{2\sigma+1})\in\text{Res}_j} a_{k_1}\bar{a}_{k_2}\dots a_{k_{2\sigma+1}}, \quad a_j(0) = \alpha_j, \tag{2.5}$$

where $\dot{a}_j$ denotes the $t-$derivative of a_j and $\text{Res}_j \subset \mathbb{Z}^{(2\sigma+1)d}$ the resonant set, associated to $j\in\mathbb{Z}^d$ and the nonlinearity $|u^\varepsilon|^{2\sigma}u^\varepsilon$. It is given by

$$\text{Res}_j = \left\{ (k_\ell)_{1\leqslant\ell\leqslant 2\sigma+1} \in \mathbb{Z}^{(2\sigma+1)d}\,; \quad \sum_{\ell=1}^{2\sigma+1}(-1)^{\ell+1}k_\ell = j\,; \quad \sum_{\ell=1}^{2\sigma+1}(-1)^{\ell+1}|k_\ell|^2 = |j|^2 \right\}.$$

We describe these sets in more detail in Section 4. First we want to explain why the above sum is restricted to the resonant set, preparing in this way the justification of the geometrical optics approximation, presented in Section 3.

Duhamel's formulation of (2.1)–(2.2) reads

$$u^\varepsilon(t) = e^{i\frac{t}{2}\varepsilon\Delta}u^\varepsilon(0) - i\int_0^t e^{i\frac{t-\tau}{2}\varepsilon\Delta}\left(|u^\varepsilon|^{2\sigma}u^\varepsilon\right)(\tau)d\tau\,. \tag{2.6}$$

Substituting the expression of the approximate solution (2.3) into the above formula, we get

$$\sum_{j\in\mathbb{Z}^d} a_j(t)e^{i\phi_j(t,x)/\varepsilon} \approx \sum_{j\in\mathbb{Z}^d}\alpha_j e^{i\frac{t}{2}\varepsilon\Delta}\left(e^{i\phi_j(0,x)/\varepsilon}\right)$$

$$-i\int_0^t e^{i\frac{t-\tau}{2}\varepsilon\Delta}\sum_{k_1,k_2,\cdots,k_{2\sigma+1}\in\mathbb{Z}^d} a_{k_1}(\tau)e^{i\phi_{k_1}/\varepsilon}\bar{a}_{k_2}(\tau)e^{-i\phi_{k_2}/\varepsilon}\cdots a_{k_{2\sigma+1}}(\tau)e^{i\phi_{k_{2\sigma+1}}/\varepsilon}d\tau\,,$$

where the symbol "≈" means that left and right hand sides in the formula above are equal up to $\mathcal{O}(\varepsilon)$. Taking into account the identity

$$e^{i\frac{t}{2}\varepsilon\Delta}\left(e^{i\phi_j(0,x)/\varepsilon}\right) = e^{i\phi_j(t,x)/\varepsilon}, \tag{2.7}$$

we conclude that modulo ε,

$$\sum_{j\in\mathbb{Z}^d} a_j(t)e^{i\phi_j(t)/\varepsilon} = \sum_{j\in\mathbb{Z}^d} \alpha_j e^{i\phi_j(t)/\varepsilon}$$
$$- i \sum_{k_1,k_2,\cdots,k_{2\sigma+1}\in\mathbb{Z}^d} \int_0^t e^{i\frac{t-\tau}{2}\varepsilon\Delta}\left(a_{k_1}\bar{a}_{k_2}\cdots a_{k_{2\sigma+1}}e^{i\left(\sum_{\ell=1}^{2\sigma+1}(-1)^{1+\ell}\phi_{k_\ell}\right)/\varepsilon}\right)(\tau)d\tau\,.$$

The aim of the next subsection is to analyze terms of the form as in the above sum in order to infer (2.5).

2.2. An explicit formula and a first consequence. — Given $\omega\in\mathbb{Z}$, $j\in\mathbb{Z}^d$, and $A\in L^\infty([0,T])\equiv L^\infty([0,T],\mathbb{C})$ with $T>0$, introduce

$$D^\varepsilon(t,x) := \int_0^t e^{i\frac{t-\tau}{2}\varepsilon\Delta}\left(A(\tau)e^{ij\cdot x/\varepsilon - i\omega\tau/(2\varepsilon)}\right)d\tau.$$

By the identity (2.7),

$$D^\varepsilon(t,x) = e^{ij\cdot x/\varepsilon - i|j|^2t/(2\varepsilon)}\int_0^t A(\tau)e^{i(|j|^2-\omega)\tau/(2\varepsilon)}d\tau. \tag{2.8}$$

***Lemma 2.1* (From [5], Lemma 5.6).** — *Suppose that $A,\dot{A}\in L^\infty([0,T])$ for some $T>0$. Then the following holds:*
(i) The function D^ε is in $C([0,T]\times\mathbb{T}^d)$ and

$$\|D^\varepsilon\|_{L^\infty([0,T]\times\mathbb{T}^d)} \leqslant \int_0^T |A(t)|dt.$$

(ii) Assume in addition that $\omega\neq|j|^2$. Then there exists a constant C independent of j, ω, and A such that

$$\|D^\varepsilon\|_{L^\infty([0,T]\times\mathbb{T}^d)} \leqslant \frac{C\varepsilon}{||j|^2-\omega|}\left(\|A\|_{L^\infty([0,T])} + \|\dot{A}\|_{L^\infty([0,T])}\right).$$

Sketch of the proof. — Item (i) is obvious and item (ii) follows from (2.8) by integrating by parts. □

Back to the above Duhamel's formula, we have

$$\sum_{j\in\mathbb{Z}^d} a_j(t)e^{i\phi_j(t)/\varepsilon} = \sum_{j\in\mathbb{Z}^d}\alpha_j e^{i\phi_j(t)/\varepsilon} - i\sum_{j\in\mathbb{Z}^d} e^{i\phi_j(t)/\varepsilon}E_j(t)$$

where

$$E_j(t) := \sum_{\substack{k_1,k_2,\cdots,k_{2\sigma+1}\in\mathbb{Z}^d\\ k_1-k_2+\cdots+k_{2\sigma+1}=j}} \int_0^t \left(a_{k_1}\bar{a}_{k_2}\ldots a_{k_{2\sigma+1}}\right)(\tau)e^{i\left(|j|^2-\sum_{\ell=1}^{2\sigma+1}(-1)^{1+\ell}|k_\ell|^2\right)\tau/(2\varepsilon)}d\tau.$$

By item (ii) of Lemma 2.1, all non-resonant terms yield a contribution of order $\mathcal{O}(\varepsilon)$, hence are discarded in (2.5).

2.3. Refined ansatz. — In the cubic one-dimensional case, we will need to go one step further in the asymptotic description of the solution u^ε. To simplify notations we therefore restrict the presentation to the cubic defocusing NLS equation with $d=1$,

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\partial_x^2 u^\varepsilon = \varepsilon|u^\varepsilon|^2 u^\varepsilon, \quad x\in\mathbb{T}. \tag{2.9}$$

For initial data as in (2.2), we construct an approximate solution of the form

$$u^\varepsilon_{\mathrm{app}}(t,x) = \sum_{j\in\mathbb{Z}} \left(a_j(t)+\varepsilon b_j(t)\right) e^{i\phi_j(t,x)/\varepsilon}, \tag{2.10}$$

introducing terms of order ε in the amplitude. It turns out that for our applications, we may assume that $b_j(0)=0$ for all j. Following the procedure of the previous section, we get, using again formula (2.8),

$$b_j(t) = -i\sum_{(k,\ell,m)\in\mathrm{Res}_j}\int_0^t \left(a_k\bar{a}_\ell b_m + a_k\bar{b}_\ell a_m + b_k\bar{a}_\ell a_m\right)(\tau)d\tau \tag{2.11}$$

$$-\frac{i}{\varepsilon}\sum_{\substack{k-\ell+m=j\\ k^2-\ell^2+m^2\neq j^2}}\int_0^t \left(a_k\bar{a}_\ell a_m\right)(\tau)e^{i\left(j^2-k^2+\ell^2-m^2\right)\tau/(2\varepsilon)}d\tau\,. \tag{2.12}$$

Note that despite the prefactor $\frac{i}{\varepsilon}$, the latter term is in fact of order $\mathcal{O}\left(\varepsilon^0\right)$ since each of the summands is non-resonant and hence can be integrated by parts (cf. item (ii) of Lemma 2.1). To be consistent, the above expression for $b_j(t)$ should be considered modulo $\mathcal{O}(\varepsilon)$, but we may choose to keep some terms of order ε for convenience. In this case, $b_j(t)$ might depend on ε and we therefore write $b_j^\varepsilon(t)$ instead of $b_j(t)$. To give a precise definition of $b_j^\varepsilon(t)$, let us analyze the above expression for b_j in more detail. Let $A := a_k\bar{a}_\ell a_m$ and assume that $A,\dot{A},\ddot{A}\in L^\infty([0,T])$ for some $T>0$. Furthermore assume that $\delta_{j,k,\ell,m} := j^2-k^2+\ell^2-m^2\in\mathbb{Z}\setminus\{0\}$. Then integrating by parts, one obtains (cf. item (ii) of Lemma 2.1)

$$\begin{aligned}\frac{i}{\varepsilon}\int_0^t A(\tau)e^{i\left(j^2-k^2+\ell^2-m^2\right)\tau/(2\varepsilon)}d\tau &= \frac{2}{\delta_{j,k,\ell,m}}\left(A(t)e^{i\left(j^2-k^2+\ell^2-m^2\right)t/(2\varepsilon)} - A(0)\right)\\ &\quad - \frac{2}{\delta_{j,k,\ell,m}}\int_0^t \dot{A}(\tau)e^{i\left(j^2-k^2+\ell^2-m^2\right)\tau/(2\varepsilon)}d\tau\,.\end{aligned}$$

As by assumption, $\ddot{A}\in L^\infty([0,T])$, the latter term can be integrated by parts once more and is hence of order $\mathcal{O}(\varepsilon)$. Taking into account the assumption $b_j(0)=0$, we define b_j^ε as follows:

$$b_j^\varepsilon(t) = -i\sum_{(k,\ell,m)\in\mathrm{Res}_j}\int_0^t \left(a_k\bar{a}_\ell b_m^\varepsilon + a_k\bar{b}_\ell^\varepsilon a_m + b_k^\varepsilon\bar{a}_\ell a_m\right)(\tau)d\tau \tag{2.13}$$

$$- \sum_{\substack{k-\ell+m=j \\ k^2-\ell^2+m^2\neq j^2}} \frac{2}{j^2-k^2+\ell^2-m^2} \left(\left(a_k\bar{a}_\ell a_m\right)(t) e^{i\left(j^2-k^2+\ell^2-m^2\right)\frac{t}{2\varepsilon}} - \alpha_k \bar{\alpha}_\ell \alpha_m \right).$$

Note that (2.13) is a linear system for the coefficients b_j^ε. They might indeed depend on ε through the inhomogeneity given by the latter term. We also note that the expression $-i\sum_{(k,\ell,m)\in\mathrm{Res}_j}\int_0^t \left(a_k\bar{a}_\ell b_m^\varepsilon + a_k\bar{b}_\ell^\varepsilon a_m + b_k^\varepsilon \bar{a}_\ell a_m\right)(\tau)d\tau$ may have the effect of coupling the b_j^ε's. We will make explicit computations on a simple example in Subsection 6.4.

3. Geometrical optics: justification of the approximation

3.1. Functional setting. — As in [**5**] (and following successively [**17**] and [**12**], in the context of geometrical optics for hyperbolic equations), we choose to work in the Wiener algebra.

***Definition 3.1* (Wiener algebra).** — *The Wiener algebra consists of functions of the form*

$$f(y) = \sum_{j\in\mathbb{Z}^d} \alpha_j e^{ij\cdot y}, \quad \alpha_j \in \mathbb{C}$$

with $(\alpha_j)_{j\in\mathbb{Z}^d} \in \ell^1(\mathbb{Z}^d)$. *It is endowed with the norm*

$$\|f\|_W = \sum_{j\in\mathbb{Z}^d} |\alpha_j|.$$

Note that $W = \mathcal{F}L^{0,1}(\mathbb{T}^d)$. The following properties of W are discussed in [**5**]:

Lemma 3.2. — *(i) For* f *in* W *and* $\varepsilon\, (= 1/N,\ N\in\mathbb{N},)$ *one has* $f(\cdot/\varepsilon)\in W$ *and*

$$\|f(\cdot/\varepsilon)\|_W = \|f\|_W.$$

(ii) W *is a Banach space and continuously embeds into* $L^\infty(\mathbb{T}^d)$.
(iii) W *is an algebra and*

$$\|fg\|_W \leqslant \|f\|_W\|g\|_W \quad \forall f,g\in W.$$

(iv) If $F:\mathbb{C}\to\mathbb{C}$ *maps* u *to a finite sum of terms of the form* $u^p\overline{u}^q$, $p,q\in\mathbb{N}$, *then it extends to a map from* W *into itself which is Lipschitz on bounded subsets of* W.
(v) For any $t\in\mathbb{R}$, *the operator* $e^{i\frac{t}{2}\varepsilon\Delta}$ *is unitary on* W.

3.2. Existence results. — It turns out that the Wiener algebra is very well suited for constructing both exact and approximate solutions of (2.1)–(2.2) and for proving error estimates. By [**5**, Proposition 5.8], one has the following results:

Proposition 3.3. — *Let* $\sigma, d\geqslant 1$ *be integers. Then for any* $u_0^\varepsilon\in W$, *there exists* $T^\varepsilon>0$ *so that* (2.1) *admits a unique solution* $u^\varepsilon\in C([0,T^\varepsilon];W)$ *satisfying* $u^\varepsilon_{|t=0} = u_0^\varepsilon$.

An existence result for the resonant system (2.5) is given in [**5**, Proposition 5.12]. In [**7**, Lemma 2.3], extra regularity properties are established in the cubic case $\sigma=1$ which can be readily proved to extend to higher order nonlinearities, yielding the following proposition.

Proposition 3.4. — *Let $\sigma \geqslant 1$ be an integer and $(\alpha_j)_{j\in\mathbb{Z}^d} \in \ell^1(\mathbb{Z}^d)$. Then there exists $T > 0$ so that* (2.5) *admits a unique solution $(a_j)_{j\in\mathbb{Z}^d} \in C^\infty([0,T];\ell^1(\mathbb{Z}^d))$.*

Note that $(a_j)_{j\in\mathbb{Z}^d}$ needs to be in $C^2([0,T];\ell^1(\mathbb{Z}^d))$ in order to justify in the analysis of the previous subsection that $u^\varepsilon_{\rm app}$ solves Duhamel's formula associated to (2.9) up to $\mathcal{O}(\varepsilon^2)$. For the linear system (2.13), the following result holds:

Lemma 3.5. — *Let $T > 0$ and $(a_j)_{j\in\mathbb{Z}^d} \in C([0,T];\ell^1(\mathbb{Z}^d))$. Then* (2.13) *has a unique solution $(b^\varepsilon_j)_{j\in\mathbb{Z}^d} \in C([0,T];\ell^1(\mathbb{Z}^d))$. In addition, $(b^\varepsilon_j)_{j\in\mathbb{Z}^d} \in C^1([0,T];\ell^1(\mathbb{Z}^d))$ and $\|b^\varepsilon_j\|_{L^\infty([0,T];\ell^1)} + \|\dot b^\varepsilon_j\|_{L^\infty([0,T];\ell^1)}$ is bounded uniformly in $\varepsilon \in (0,1]$.*

3.3. Error estimates. — In the case of the first order expansion presented in Subsection 2.1, the approximate solution $u^\varepsilon_{\rm app}$, defined by Proposition 3.4 on an interval $[0,T]$, satisfies

$$i\varepsilon\partial_t u^\varepsilon_{\rm app} + \frac{\varepsilon^2}{2}\Delta u^\varepsilon_{\rm app} = \varepsilon|u^\varepsilon_{\rm app}|^{2\sigma}u^\varepsilon_{\rm app} - \varepsilon r^\varepsilon, \quad u^\varepsilon_{{\rm app}|t=0} = u^\varepsilon_{|t=0}$$

where the term $r^\varepsilon \equiv r^\varepsilon(t,x)$ is given by

$$r^\varepsilon = \sum_{j\in\mathbb{Z}^d} \sum_{\substack{k_1-k_2+\dots+k_{2\sigma+1}=j \\ |k_1|^2-|k_2|^2+\dots+|k_{2\sigma+1}|^2\neq |j|^2}} a_{k_1}\bar a_{k_2}\cdots a_{k_{2\sigma+1}} e^{i(\phi_{k_1}-\phi_{k_2}+\dots+\phi_{k_{2\sigma+1}})/\varepsilon}.$$

Since the k_ℓ's are integer vectors and hence there are no issues of small nonzero divisors, the integrated source term

$$R^\varepsilon(t,x) = \int_0^t e^{i\frac{t-\tau}{2}\varepsilon\Delta} r^\varepsilon(\tau,x)d\tau$$

can be estimated in view of item (ii) of Lemma 2.1 by

$$\|R^\varepsilon\|_{L^\infty([0,T];W)} \leqslant C\varepsilon$$

where the constant C is independent of ε.

In the case of the second order expansion presented in Subsection 2.3 for the cubic NLS equation on the circle ($d = 1, \sigma = 1$), one has by Proposition 3.4 and Lemma 3.5 that the approximate solution $u^\varepsilon_{\rm app}$ is defined on the interval $[0,T]$ with T as in Proposition 3.4. Hence

$$i\varepsilon\partial_t u^\varepsilon_{\rm app} + \frac{\varepsilon^2}{2}\partial_x^2 u^\varepsilon_{\rm app} = \varepsilon|u^\varepsilon_{\rm app}|^2 u^\varepsilon_{\rm app} - \varepsilon r^\varepsilon_b, \quad u^\varepsilon_{{\rm app}|t=0} = u^\varepsilon_{|t=0}$$

where $r^\varepsilon_b \equiv r^\varepsilon_b(t,x)$ is given by an explicit formula, similar to the one for r^ε. Using again item (ii) of Lemma 2.1, one shows that the integrated source term

$$R^\varepsilon_b(t,x) = \int_0^t e^{i\frac{t-\tau}{2}\varepsilon\partial_x^2} r^\varepsilon_b(\tau,x)d\tau$$

satisfies the estimate

$$\|R^\varepsilon_b\|_{L^\infty([0,T];W)} \leqslant C\varepsilon^2$$

with a constant C independent of ε. In view of Proposition 3.3, a bootstrap argument applies, yielding the following error estimate:

Proposition 3.6. — *Let $\sigma, d \geqslant 1$ be integers, $(\alpha_j)_{j\in\mathbb{Z}^d}$ be a sequence in $\ell^1(\mathbb{Z}^d)$, and T be given as in Proposition 3.4 . Then there exists a constant $C > 0$ independent of ε so that the following holds:*
(i) The first order approximation $u^\varepsilon_{\rm app}$, constructed in Subsection 2.1, satisfies

$$\|u^\varepsilon - u^\varepsilon_{\rm app}\|_{L^\infty([0,T];W)} \leqslant C\varepsilon\, .$$

(ii) In the case $d = \sigma = 1$, the second order approximation $u^\varepsilon_{\rm app}$, constructed in Subsection 2.3, satisfies

$$\|u^\varepsilon - u^\varepsilon_{\rm app}\|_{L^\infty([0,T];W)} \leqslant C\varepsilon^2\, .$$

Remark 3.7 **(Renormalized equation).** — In the case of (1.2), one simply has to substract $\sum_\ell |a_\ell|^2$ from the right hand side of (2.5). It is fairly easy to check that taking this modification into account (as well as the modification regarding the b_j^ε's in the cubic one-dimensional case, as explained in Subsection 5.1), Proposition 3.6 is readily extended to the case of (1.2).

4. Description of the approximate solution

4.1. Resonant sets and the creation of modes in the cubic case. — Using arguments developed in [**5**] in connection with [**13**], the resonant sets Res_j, introduced in Subsection 2.1, can be characterized in the cubic case as follows:

Proposition 4.1. — *Let $\sigma = 1$ and $j \in \mathbb{Z}^d$.*
(i) If in addition $d = 1$, then

$$\mathrm{Res}_j = \{(j,\ell,\ell),\ (\ell,\ell,j)\ ;\ \ell \in \mathbb{Z}\setminus\{j\}\} \cup \{(j,j,j)\}.$$

(ii) If in addition $d \geqslant 2$, then $(k,\ell,m) \in \mathrm{Res}_j$ if and only if either the endpoints of the vectors k,ℓ,m,j are the four corners of a nondegenerate rectangle with ℓ and j opposing each other or this quadruplet corresponds to one of the following three degenerate cases: (j,ℓ,ℓ) with $j \neq \ell$, (ℓ,ℓ,j) with $j \neq \ell$, or (j,j,j).

By item (i) of Proposition 4.1, we see that in the case $d = \sigma = 1$, (2.5) becomes

$$i\dot a_j = \left(2\sum_{k\in\mathbb{Z}} |a_k|^2 - |a_j|^2\right) a_j, \quad a_j(0) = \alpha_j. \tag{4.1}$$

It then follows that for any $j \in \mathbb{Z}$, $\frac{d}{dt}(|a_j|^2) = 0$ and hence

$$a_j(t) = \alpha_j \exp\left(-i\left(2\sum_{k\in\mathbb{Z}}|\alpha_k|^2 - |\alpha_j|^2\right)t\right) \tag{4.2}$$

In particular, if initially the j-mode vanishes, $\alpha_j e^{\phi_j(0,\cdot)/\varepsilon} = 0$, then $a_j(t) = 0$ for any $t > 0$. The situation is different in higher dimensions. The example considered in [**6**] also plays an important role here: for $d \geqslant 2$, let

$$u^\varepsilon(0,x) = e^{ix_1/\varepsilon} + e^{ix_2/\varepsilon} + e^{i(x_1+x_2)/\varepsilon}. \tag{4.3}$$

Let $k := (1, 0, 0_{\mathbb{Z}^{d-2}})$, $\ell := (1, 1, 0_{\mathbb{Z}^{d-2}})$, and $m := (0, 1, 0_{\mathbb{Z}^{d-2}})$. Then (k, ℓ, m) is in Res_0 and the initial data can be written as $u^\varepsilon(0, x) = e^{ix\cdot k/\varepsilon} + e^{ix\cdot m/\varepsilon} + e^{ix\cdot \ell/\varepsilon}$. The zero mode $a_0(t)$ then becomes instantaneously nonzero for $t > 0$ since by (2.5),

$$i\dot{a}_{0|t=0} = 2\alpha_k\alpha_\ell\alpha_m = 2.$$

In such a case we say that the zero mode is created by resonant interaction of nonzero modes. Furthermore, by item (ii) of Proposition 4.1, no other modes are created.

4.2. Creation of modes for higher order nonlinearities. — The key idea to prove Theorem 1.1 is to choose initial data, causing instantaneous transfer of energy from nonzero modes to the zero mode. In the previous subsection we provided an example for such initial data in the cubic multidimensional case ($\sigma = 1, d \geqslant 2$). It turns out that for $d \geqslant 2$, a similar example also works for higher order nonlinearities, based on the following observation: if in the case $\sigma = 1$, one has $(k, \ell, m) \in \mathrm{Res}_j$, then for any $\sigma \geqslant 2$

$$(k, \ell, m, \underbrace{k, \cdots, k}_{2\sigma-2 \text{ times}}), (k, \ell, m, \underbrace{\ell, \cdots, \ell}_{2\sigma-2 \text{ times}}), (k, \ell, m, \underbrace{m, \cdots, m}_{2\sigma-2 \text{ times}}) \in \mathrm{Res}_j.$$

For proving Theorem 1.1, it therefore remains to consider the case $\sigma \geqslant 2$ in the one-dimensional case. In view of the above observation, it suffices to treat the case of the quintic nonlinearity ($\sigma = 2$).

For $d = 1$ and $\sigma = 2$, the zero mode is created by resonant interaction of nonzero modes if we can find $k_1, k_2, k_3, k_4, k_5 \in \mathbb{Z} \setminus \{0\}$ such that

$$\begin{cases} k_1 - k_2 + k_3 - k_4 + k_5 = 0, \\ k_1^2 - k_2^2 + k_3^2 - k_4^2 + k_5^2 = 0. \end{cases}$$

Squaring the first identity, written as $k_1 + k_3 + k_5 = k_2 + k_4$, and using the second identity, this system is equivalent to

$$\begin{cases} k_1 + k_3 + k_5 = k_2 + k_4, \\ k_1k_3 + k_1k_5 + k_3k_5 = k_2k_4. \end{cases}$$

Assume that k_1, k_3, k_5 are given. Then k_2 and k_4 are the zeroes of the quadratic polynomial

$$X^2 - (k_1 + k_3 + k_5)X + k_1k_3 + k_1k_5 + k_3k_5 = 0\,,$$

whose discriminant is

$$\begin{aligned} \Delta &= (k_1 + k_3 + k_5)^2 - 4\,(k_1k_3 + k_1k_5 + k_3k_5) \\ &= k_1^2 + k_3^2 + k_5^2 - 2k_1k_3 - 2k_1k_5 - 2k_3k_5\,. \end{aligned}$$

Assuming that k_2 and k_4 are listed in increasing order, they are then given by

$$k_2 = \frac{k_1 + k_3 + k_5 - \sqrt{\Delta}}{2}, \quad k_4 = \frac{k_1 + k_3 + k_5 + \sqrt{\Delta}}{2}.$$

In particular, Δ must be of the form $\Delta = N^2$ with N an integer, having the same parity as $k_1 + k_3 + k_5$. One readily sees that k_1, k_3, and k_5 cannot be all equal. Furthermore, one can construct infinitely many solutions of the form

$$(k_1, k_3, k_5) = (a, -a, b), \quad a, b \neq 0, \quad b \notin \{a, -a\}.$$

Indeed, for (k_1, k_3, k_5) of this form, $\Delta = b^2 + 4a^2$. Hence we look for integer solutions of

$$b^2 + (2a)^2 = N^2,$$

meaning that $(b, 2a, N)$ must be a Pythagorean triplet. We infer:

Lemma 4.2. — *For any $p, q \in \mathbb{Z}$ with $p, q \neq 0$ and $p \neq q$, the 5-tuple*

$$(k_1, k_2, k_3, k_4, k_5) = (pq, -q^2, -pq, p^2, p^2 - q^2)$$

creates the zero mode by resonant interaction of nonzero modes.

Example 4.3. — With $p = 2$ and $q = 1$, we find

$$(k_1, k_2, k_3, k_4, k_5) = (2, -1, -2, 4, 3).$$

Remark 4.4. — In [**14**], the creation of a mode k_6 by resonant interaction of the modes k_1, k_2, k_3, k_4, k_5 is studied. Under the specific assumptions that $k_j = k_\ell$ for two distinct odd and $k_n = k_m$ for two distinct even indices in $\{1, 2, 3, 4, 5\}$, a complete characterization of the corresponding resonant set is provided. It implies that for the special case $k_6 = 0$, any 5-tuple of the form $(k, 3k, k, 3k, 4k)$, $(k, 3k, 4k, 3k, k)$, or $(4k, 3k, k, 3k, k)$ with $k \in \mathbb{Z} \setminus 0$ creates the zero mode by resonant interaction. Note that the 5-tuples proposed in Lemma 4.2, are not of the above form and hence are complementary to the ones found in [**14**].

5. Geometrical optics for the modified NLS equation

In this section, we consider the equation

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = \varepsilon|u^\varepsilon|^2 u^\varepsilon - \frac{2\varepsilon}{(2\pi)^d}\left(\int_{\mathbb{T}^d} |u^\varepsilon(t,x)|^2 dx\right) u^\varepsilon, \quad x \in \mathbb{T}^d, \tag{5.1}$$

along with the initial data (2.2).

5.1. One-dimensional case. — In view of the analysis of Subsection 2.3, one has

$$\begin{aligned}
\int_{\mathbb{T}} |u^\varepsilon_{\rm app}(t,x)|^2 dx &= \int_{\mathbb{T}} \left|\sum_{j\in\mathbb{Z}} \left(a_j(t) + \varepsilon b_j^\varepsilon(t)\right) e^{i\phi_j(t,x)/\varepsilon}\right|^2 dx \\
&= \int_{\mathbb{T}} \sum_{j,k\in\mathbb{Z}} \left(a_j(t) + \varepsilon b_j^\varepsilon(t)\right)\left(\bar a_k(t) + \varepsilon \bar b_k^\varepsilon(t)\right) e^{i(\phi_j(t,x)-\phi_k(t,x))/\varepsilon} dx \\
&= 2\pi \sum_{j\in\mathbb{Z}} \left(|a_j(t)|^2 + \varepsilon\left(\bar a_j(t) b_j^\varepsilon(t) + a_j(t)\bar b_j^\varepsilon(t)\right) + \varepsilon^2 |b_j^\varepsilon(t)|^2\right),
\end{aligned}$$

since the family $(e^{i\phi_j(t,\cdot)/\varepsilon})_{j\in\mathbb{Z}}$ is orthogonal in $L^2(\mathbb{T})$ and $|\mathbb{T}|=2\pi$. It then follows that for any $j\in\mathbb{Z}$, the formula corresponding to (4.2) in the case of (5.1), becomes

$$i\dot{a}_j = -|a_j|^2 a_j, \quad a_j(0)=\alpha_j, \tag{5.2}$$

and thus $a_j(t)=\alpha_j e^{i|\alpha_j|^2 t}$, showing that the a_j's are no longer coupled. (This is an indication that equation (5.1) might be more stable than (1.1).) Furthermore (2.13) becomes

$$\begin{aligned} b_j^\varepsilon(t) = &- i \sum_{(k,\ell,m)\in \mathrm{Res}_j} \int_0^t \left(a_k\bar{a}_\ell b_m^\varepsilon + a_k \bar{b}_\ell^\varepsilon a_m + b_k^\varepsilon \bar{a}_\ell a_m\right)(\tau)d\tau \\ &- \sum_{\substack{k-\ell+m=j\\ k^2-\ell^2+m^2\neq j^2}} \frac{2}{j^2-k^2+\ell^2-m^2}\Big(\left(a_k\bar{a}_\ell a_m\right)(t)e^{i\left(j^2-k^2+\ell^2-m^2\right)\frac{t}{2\varepsilon}} - \alpha_k\bar{\alpha}_\ell\alpha_m\Big) \\ &+ 2i\int_0^t \left(b_j^\varepsilon \sum_{k\in\mathbb{Z}}|a_k|^2 + a_j\sum_{k\in\mathbb{Z}}\left(\bar{a}_k b_k^\varepsilon + a_k\bar{b}_k^\varepsilon\right)\right)(\tau)d\tau. \end{aligned}$$

5.2. Multi-dimensional case. — When $d\geqslant 2$, we argue as in Subsection 4.1, choosing as initial data

$$u^\varepsilon(0,x) = e^{ix_1/\varepsilon} + e^{ix_2/\varepsilon} + e^{i(x_1+x_2)/\varepsilon}\,.$$

The characterization of the resonant sets Res_j, described in item (ii) of Proposition 4.1, shows that the only possible new mode created by cubic interaction is the zero mode. Setting

$$k := (1,0,0_{\mathbb{Z}^{d-2}}), \quad \ell := (1,1,0_{\mathbb{Z}^{d-2}}), \quad m := (0,1,0_{\mathbb{Z}^{d-2}}),$$

the resonant set Res_0 is given by

$$\{(k,\ell,m),\ (m,\ell,k),\ (k,k,0),\ (0,k,k),\ (\ell,\ell,0),\ (0,\ell,\ell),\ (m,m,0),\ (0,m,m),\ (0,0,0)\}$$

and the zero mode a_0 satisfies

$$i\dot{a}_0 = 2a_k\bar{a}_\ell a_m - |a_0|^2 a_0, \quad a_{0|t=0}=0.$$

In particular, $i\dot{a}_0(0)=2$, meaning that the zero mode is created through cubic interaction of nonzero modes.

6. Proof of Theorem 1.1

6.1. Scaling. — We follow the same strategy as in [**6**]: as a first step, we relate equations (1.1) and (2.1) respectively (1.2) and (5.1) by an appropriate scaling of all the quantities involved: let $\psi(t,x)$ be a solution of (1.1) and u^ε be of the form

$$u^\varepsilon(t,x) = \varepsilon^\alpha \psi(\varepsilon^\beta t, \varepsilon^\gamma x)\,.$$

Such a function solves (2.1) iff

$$1+\beta = 2+2\gamma = 1+2\sigma\alpha.$$

Keeping β as the only parameter, we have

$$u^\varepsilon(t,x) = \varepsilon^{\beta/(2\sigma)}\psi\left(\varepsilon^\beta t, \varepsilon^{\frac{\beta-1}{2}}x\right). \tag{6.1}$$

In order that the initial data for u^ε is of the form (2.2), the one for ψ is chosen so that $\varepsilon^{\beta/(2\sigma)}\psi\left(0,\varepsilon^{\frac{\beta-1}{2}}x\right) = \sum_{j\in\mathbb{Z}^d}\alpha_j e^{ij\cdot x/\varepsilon}$. It means that

$$\psi(0,x) = \varepsilon^{-\beta/(2\sigma)}\sum_{j\in\mathbb{Z}^d}\alpha_j e^{ij\cdot x/\varepsilon^{\frac{1+\beta}{2}}}. \tag{6.2}$$

Furthermore, to assure that both ψ and u^ε are periodic functions and hence welldefined on $\mathbb{T}^d$, we require that $1/\varepsilon = N^\kappa \in \mathbb{N}$, for some integers N,κ, where κ is chosen so that for a given *rational number* $\beta > 0$,

$$\frac{1}{\varepsilon^{\frac{1+\beta}{2}}} = N^{\kappa\frac{1+\beta}{2}} \text{ is an integer.}$$

In the sequel, for any given rational number $\beta > 0$, we will consider sequences $\varepsilon_n \to 0$ so that the above requirements are fulfilled.

The strategy for proving the statements of Theorem 1.1 is the following one: the initial data for u^ε (or equivalently for ψ), is chosen to be a finite sum of *nonzero* modes, which create the zero mode by resonant interaction at leading order, $\dot{a}_0(0)\neq 0$, except in the cubic one-dimensional case, where the zero mode is created at the level of the corrector b_0. Due to the choice of the scaling, the zero mode of ψ comes with a factor which is increasing in ε. Since the absolute value of the zero mode bounds the norm $\|\cdot\|_{\mathcal{F}L^{s,p}(\mathbb{T}^d)}$ of any Fourier Lebesgue space from below, it follows that for any $s<0, 1\leqslant p\leqslant\infty$, the sequence $(\|u^{\varepsilon_n}(t_n)\|_{\mathcal{F}L^{s,p}(\mathbb{T}^d)})_{n\geqslant 1}$ is unbounded for appropriate sequences $(\varepsilon_n)_{n\geqslant 1}$, $(t_n)_{n\geqslant 1}$, converging both to 0.

6.2. Norm inflation in the multidimensional case. — Suppose $d\geqslant 2$, $\sigma\geqslant 1$. For any fixed $s<0$, there exists a rational number $\beta>0$ so that

$$|s|\frac{\beta+1}{2} > \frac{\beta}{2\sigma}.$$

Note that $\beta\to 0$ as $s\to 0$. We then choose a sequence $(\varepsilon_n)_{n\geqslant 1}$ with $\varepsilon_n\to 0$ as above. Taking into account the discussion at the beginning of Subsection 4.2, it suffices to consider example (4.3). With the above scaling, $\psi_n(0,x)$ is then given by

$$\psi_n(0,x) = \varepsilon_n^{-\beta/(2\sigma)}\left(e^{ix_1/\varepsilon_n^{\frac{1+\beta}{2}}} + e^{ix_2/\varepsilon_n^{\frac{1+\beta}{2}}} + e^{i(x_1+x_2)/\varepsilon_n^{\frac{1+\beta}{2}}}\right).$$

For any $p\in[1,\infty]$, we have

$$\|\psi_n(0)\|_{\mathcal{F}L^{s,p}(\mathbb{T}^d)} \approx \varepsilon_n^{-\beta/(2\sigma)-s(\beta+1)/2} = \varepsilon_n^{-\beta/(2\sigma)+|s|(\beta+1)/2},$$

implying that

$$\|\psi_n(0)\|_{\mathcal{F}L^{s,p}(\mathbb{T}^d)} \underset{n\to\infty}{\longrightarrow} 0.$$

In Section 4 we have seen that there exists $\tau > 0$ with $a_0(\tau) \neq 0$. Setting $t_n = \tau\varepsilon_n^\beta$, one has $t_n \underset{n\to\infty}{\longrightarrow} 0$. With $\psi_{n,\mathrm{app}}(t,x)$ obtained from $u^{\varepsilon_n}_{\mathrm{app}}(t,x)$ by the above scaling, it follows that for any $r \in \mathbb{R}$ and $p \in [1,\infty]$,

$$\|\psi_{n,\mathrm{app}}(t_n)\|_{\mathcal{F}L^{r,p}(\mathbb{T}^d)} \geqslant \varepsilon_n^{-\beta/(2\sigma)}|a_0(\tau)| \underset{n\to\infty}{\longrightarrow} +\infty\,.$$

Note that $W \hookrightarrow \mathcal{F}L^{r,p}(\mathbb{T}^d)$ for any $r \leqslant 0$ and $p \in [1,\infty]$ and hence

$$\|\psi_n(t) - \psi_{n,\mathrm{app}}(t)\|_{\mathcal{F}L^{r,p}(\mathbb{T}^d)} \lesssim \|\psi_n(t) - \psi_{n,\mathrm{app}}(t)\|_W\,.$$

In view of (6.1) and the scaling invariance of the norm $\|\cdot\|_W$ (see item (i) of Lemma 3.2), Proposition 3.6 then implies

$$\|\psi_n(t_n) - \psi_{n,\mathrm{app}}(t_n)\|_{\mathcal{F}L^{r,p}(\mathbb{T}^d)} \lesssim \varepsilon_n^{1-\beta/(2\sigma)} \lesssim \varepsilon_n\|\psi_{n,\mathrm{app}}(t_n)\|_{\mathcal{F}L^{r,p}(\mathbb{T}^d)}.$$

Altogether we have shown that $\|\psi_n(t_n)\|_{\mathcal{F}L^{r,p}(\mathbb{T}^d)} \sim \|\psi_{n,\mathrm{app}}(t_n)\|_{\mathcal{F}L^{r,p}(\mathbb{T}^d)} \to \infty$ and item (i) of Theorem 1.1 is proved in the case $d \geqslant 2$, $\sigma \geqslant 1$.

6.3. Norm inflation in the quintic one-dimensional case. — The case $d = 1$, $\sigma \geqslant 2$, is dealt with along the same lines as the case $d \geqslant 2$, $\sigma \geqslant 1$, treated in the previous subsection. Since by Lemma 4.2, it is possible to create the zero mode by quintic interaction of nonzero modes, the above argument is readily adapted by choosing, for instance, initial data as in Example 4.3,

$$\psi_n(0,x) = \varepsilon_n^{-\beta/(2\sigma)}\left(e^{2ix/\varepsilon_n^{\frac{1+\beta}{2}}} + e^{-ix/\varepsilon_n^{\frac{1+\beta}{2}}} + e^{-2ix/\varepsilon_n^{\frac{1+\beta}{2}}} + e^{4ix/\varepsilon_n^{\frac{1+\beta}{2}}} + e^{3ix/\varepsilon_n^{\frac{1+\beta}{2}}}\right).$$

6.4. Norm inflation in the cubic one-dimensional case. — In the cubic one-dimensional case, we have seen in Subsection 4.1 that $\alpha_j = 0$ implies $a_j(t) = 0$ for any t. The same phenomena is true in the case of (5.1). Therefore, the previous analysis has to be modified. We consider the higher order approximation, discussed in Subsection 2.3. We want to show that for appropriate initial data $\psi_n(0,x)$, $b_0^\varepsilon(\tau^\varepsilon) \approx 1$ for some $\tau^\varepsilon > 0$ with $\tau^\varepsilon \approx \varepsilon$. Note that in view of (2.13), initial data with only one nonzero mode is not sufficient to ensure that b_j^ε has this property. We therefore choose

$$\psi_n(0,x) = \varepsilon_n^{-\beta/2}\left(e^{ix/\varepsilon_n^{\frac{1+\beta}{2}}} + e^{2ix/\varepsilon_n^{\frac{1+\beta}{2}}}\right)$$

as initial data. By (6.2), the corresponding initial data for u^ε is given by

$$u^\varepsilon(0,x) = e^{ix/\varepsilon} + e^{2ix/\varepsilon}\,.$$

It means that $\alpha_1 = 1$, $\alpha_2 = 1$, and $\alpha_j = 0$ for all $j \in \mathbb{Z}\setminus\{1,2\}$. By the analysis of Subsection 4.1,

$$a_1(t) = a_2(t) = e^{-3it}, \quad a_j(t) \equiv 0 \text{ for } j \in \mathbb{Z}\setminus\{1,2\}.$$

The creation of b_j^ε's can have two causes:

- the source term (2.12) is not zero, or
- the coupling between the b_j^ε's, due to (2.11), causes the creation of b_j^ε's after others have been created by a nonzero source term.

We examine the two possibilities separately. Let us begin with the analysis of (2.12). The only non-resonant configurations $k,\ell,m\in\{1,2\}$ in the sum in (2.12) are

$$(k,\ell,m)=(1,2,1)\quad\text{and}\quad(k,\ell,m)=(2,1,2).$$

Since $1-2+1=0$ and $2-1+2=3$, b_0^ε respectively b_3^ε are created through these configurations. Furthermore, for $j\in\mathbb{Z}\setminus\{0,3\}$, (2.12) is zero. To address the possibility of creation of b_j^ε's through coupling, consider the first term in the integral of (2.11):

$$a_k\bar a_\ell b_m^\varepsilon,\quad(k,\ell,m)\in\mathrm{Res}_j.$$

For this term to be non-zero, we have necessarily $k,\ell\in\{1,2\}$. Then, in view of item (i) of Proposition 4.1, $m\in\{1,2\}$, and we infer $j\in\{1,2\}$. The same argument can be repeated for the other two terms, $a_k\bar b_\ell^\varepsilon a_m$ and $b_k^\varepsilon\bar a_\ell a_m$. Therefore, the terms b_1^ε and b_2^ε are coupled. But since they solve a homogeneous system with zero initial data, they remain identically zero.

Since by (2.11)–(2.12), $\dot b_0^\varepsilon(0)=-i/\varepsilon$ and $\dot b_3^\varepsilon(0)=-i/\varepsilon$, altogether we have proved that precisely b_0^ε and b_3^ε are created. In particular, we compute

$$b_0^\varepsilon(t)=-4i\int_0^t b_0^\varepsilon(\tau)d\tau-\left(e^{-3it+it/\varepsilon}-1\right),$$

yielding the following explicit solution

$$b_0^\varepsilon(t)=-\frac{1-3\varepsilon}{1+\varepsilon}e^{-4it}\left(e^{it+it/\varepsilon}-1\right)$$

and hence the following formula

$$|b_0^\varepsilon(t)|=2\frac{1-3\varepsilon}{1+\varepsilon}\left|\sin\left((1+\varepsilon)\frac{t}{2\varepsilon}\right)\right|.$$

Thus, for $0<\varepsilon\ll 1$, there exists $\tau_\varepsilon\approx\varepsilon$ such that $|b_0^\varepsilon(\tau_\varepsilon)|=1$. From this point on we can argue as in the previous subsections. For any $p\in[1,\infty]$,

$$\|\psi_n(0)\|_{\mathcal{F}L^{s,p}(\mathbb{T})}\approx\varepsilon^{-\beta/2+|s|(\beta+1)/2}.$$

Hence to ensure that $\|\psi_n(0)\|_{\mathcal{F}L^{s,p}(\mathbb{T})}\to 0$ as $n\to\infty$, we need to impose that

$$|s|>\frac{\beta}{\beta+1}. \tag{6.3}$$

By taking into account only the term $\varepsilon b_0^\varepsilon(t)e^{i\phi(t,x)/\varepsilon}$ in $u^\varepsilon_{\mathrm{app}}(t,x)$, it follows that for $t_n=\varepsilon_n^\beta\tau_{\varepsilon_n}$,

$$\|\psi_{n,\mathrm{app}}(t_n)\|_{\mathcal{F}L^{r,p}(\mathbb{T})}\geqslant\varepsilon^{-\beta/2+1}, \tag{6.4}$$

where the extra power of ε stems from the factor in front of b_0^ε. Finally, for $r\leqslant 0$,

$$\begin{aligned}\|\psi_n(t_n)-\psi_{n,\mathrm{app}}(t_n)\|_{\mathcal{F}L^{r,p}(\mathbb{T})}&\lesssim\|\psi_n(t_n)-\psi_{n,\mathrm{app}}(t_n)\|_W\\&\lesssim\varepsilon_n^{-\beta/2}\|u^{\varepsilon_n}(\tau_{\varepsilon_n})-u^{\varepsilon_n}_{\mathrm{app}}(\tau_{\varepsilon_n})\|_W\,.\end{aligned}$$

By item (ii) of Proposition 3.6, it then follows that

$$\|\psi_n(t_n)-\psi_{n,\mathrm{app}}(t_n)\|_{\mathcal{F}L^{r,p}(\mathbb{T})}\lesssim\varepsilon_n^{2-\beta/2},$$

implying that

$$\|\psi_n(t_n) - \psi_{n,\mathrm{app}}(t_n)\|_{\mathcal{F}L^{r,p}(\mathbb{T})} \lesssim \varepsilon_n \|\psi_{n,\mathrm{app}}(t_n)\|_{\mathcal{F}L^{r,p}(\mathbb{T})},$$

and hence $\|\psi_n(t_n)\|_{\mathcal{F}L^{r,p}(\mathbb{T})} \approx \|\psi_{n,\mathrm{app}}(t_n)\|_{\mathcal{F}L^{r,p}(\mathbb{T})}$ as $n \to \infty$. By (6.4), the sequence $(\|\psi_n(t_n)\|_{\mathcal{F}L^{r,p}(\mathbb{T})})_{n\geqslant 1}$ is thus unbounded provided that $\beta > 2$. Taking into account that s is assumed to be negative, the condition $\beta > 2$ is compatible with (6.3) provided that $s < -2/3$.

6.5. Norm inflation for equation (1.2)**.** — To complete the proof of Theorem 1.1, it remains to consider equation (1.2). We already noted in Subsection 6.1 that the scaling introduced there establishes a one-to-one correspondence between solutions of (1.2) and those of (5.1). In the one-dimensional case, as initial data for u^ε we again choose

$$u^\varepsilon(0,x) = e^{ix/\varepsilon} + e^{2ix/\varepsilon}.$$

By (5.2),

$$a_1(t) = a_2(t) = e^{it}, \quad a_j(t) \equiv 0 \quad \forall j \in \mathbb{Z} \setminus \{1,2\}.$$

A similar combinatorial analysis as above shows that only b_0^ε and b_3^ε are created. In the case considered, b_0 is given by

$$b_0^\varepsilon(t) = -\left(e^{it+it/\varepsilon} - 1\right),$$

implying that

$$|b_0^\varepsilon(t)| = 2\left|\sin\left((1+\varepsilon)\frac{t}{2\varepsilon}\right)\right|.$$

To finish the proof, we then can argue in the same way as in the previous subsection. In particular, the multi-dimensional case can be handled in exactly the same fashion as the one considered in Subsection 6.2.

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