Cocompactly cubulated Artin-Tits groups

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Abstract. We give a complete classification of cocompactly cubulated Artin-Tits groups, i.e. acting geometrically on a CAT(0) cube complex. A particular case is that for $n \geqslant 4$, the n-strand braid group is not cocompactly cubulated. The methods extend to give a classification of cocompactly cubulated groups among mapping class group of surfaces and (outer) automorphism groups of free groups. A key result is that any group containing a subgroup isomorphic to non-cocompactly cubulated Artin group is not cocompactly cubulated.

Introduction

Groups acting geometrically on CAT(0) spaces (called CAT(0) groups), or even better on CAT(0) cube complexes (called cocompactly cubulated groups), enjoy a liste of nice properties: they have a quadratic Dehn function, a solvable word and conjugacy problem, they have the Haagerup property, their amenable subgroups are virtually abelian and undistorted, they satisfy the Tits alternative... R. Charney conjectures that all Artin-Tits groups are CAT(0), but very few cases are known. With D. Kielak and P. Schwer (see [HKS13]), we pursued the construction of T. Brady and J. McCammond (see [Bra01] and [BM10]) to prove that for $n \leqslant 6$, the n-strand braid group is CAT(0).

In this article, we give a complete answer to the question of knowing which Artin groups act geometrically on a CAT(0) cube complex. Right-angled Artin groups are well-known to act cocompactly on their Salvetti CAT(0) cube complex, but there are a few more examples. This question was asked by D. Wise for the particular case of braid groups (see [Wis, Problem 13.4]).

Theorem A (Classification of cocompactly cubulated Artin-Tits groups)**.** Let Γ be a finite Coxeter graph. The Artin-Tits group $A(\Gamma)$ is cocompactly cubulated if and only if the connected components $\Gamma = \sqcup_{i=1}^{p} \Gamma_i$ of the graph Γ without the edges labeled 2 satisfy the following :

- For each $1 \leqslant i \leqslant p$, Γ_i is a vertex, an odd edge, or it is a star with edges labeled by even numbers different from 2, with central vertex denoted s_i.
- For every odd edge Γ_i with vertices $\{s,t\}$, for every $u \in \Gamma^{(0)} \backslash \{s,t\}$, the edge us is in Γ if and only if the edge ut is in Γ.

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- For every star Γ_i with even edges and central vertex s_i, for every $s \in \Gamma_i^{(0)} \backslash \{s_i\}$ and for every $t \in \Gamma^{(0)} \backslash \Gamma_i^{(0)}$, if the edge ts is in Γ, then the edge ts_i is in Γ.

Roughly speaking, cocompactly cubulated Artin groups are obtained from dihedral Artin groups and Artin groups of even stars, and combining them in a right-angled-like fashion.

Another way to state Theorem A is by describing local obstructions in the graph Γ, see also Figure 1. In particular, special subgroups of rank 3 and 4 determine if an Artin group is cocompactly cubulated or not.

Corollary B. Let Γ be a finite Coxeter graph. The Artin-Tits group $A(\Gamma)$ is not cocompactly cubulated if and only if at least one of the following happens

- there exist 3 pairwise distinct $a, b, c \in \Gamma^{(0)}$ such that the edge ab is in Γ and labeled by an odd number, the edge bc is in Γ and there is no edge ac labeled 2,
- there exist 3 pairwise distinct $a, b, c \in \Gamma^{(0)}$ such that the edges ab and ac are in Γ and labeled by even numbers different from 2, and the edge bc is in Γ, or
- there exist 4 pairwise distinct $a, b, c, d \in \Gamma^{(0)}$ such that the edge bc is in Γ and not labeled by 2, the edges ab and cd are in Γ, and there are no edges ac nor bd labeled 2.

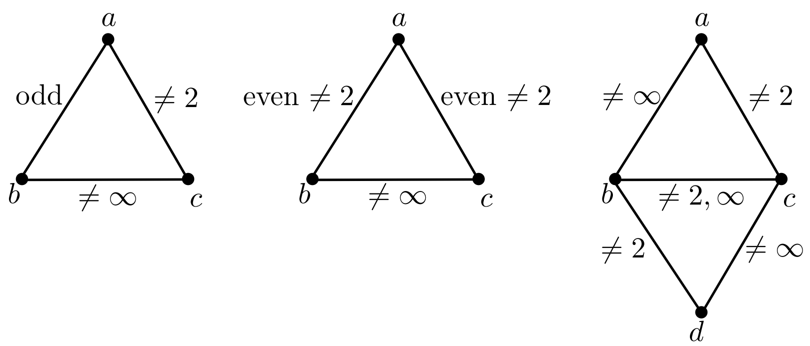

Figure 1: Local obstructions in Corollary B

The main direction of this result is a consequence of the following obstruction to cubicality theorem.

Theorem C (Obstruction to cubicality)**.** Let G be a group and $a, b \in G$ such that:

- $\langle a, b \rangle$ is a subgroup isomorphic to the dihedral Artin group $A(p)$, for some finite $p \geqslant 3$,
- there exists $g \in G$ commuting with a such that non non-zero powers of g and z_{ab} commute and
- if p is even, assume furthermore that there exists $h \in G$ commuting with b such that non non-zero powers of h and z_{ab} commute.

Then G is not cocompactly cubulated.

The most useful consequences of this result are listed below.

Corollary D. Let G be a group containing one of the following subgroups:

- the 4-strand braid group B_4,
- the central quotient $B_4/Z(B_4)$ of B_4 or
- an Artin group A, which is not cocompactly cubulated.

Then G is not cocompactly cubulated.

We will use Corollary D mostly for rank 3 Artin subgroups, but we will also need to consider rank 4 Artin subgroups.

During the proof, we also prove the following result, of independent interest.

Theorem E (Cubulation of centralizers)**.** Let G be a group acting geometrically on a CAT(0) cube complex X of dimension at most D. Let A be an abelian subgroup of G such that every element of A is the D^{th} power of a combinatorially hyperbolic isometry in G. Then the centralizer $Z_G(A)$ is cocompactly cubulated.

Another ingredient in the proof is a very recent result from D. Wise and D. Woodhouse (see [WW15] and Theorem 3.3), a flat torus theorem for maximal abelian subgroups of cocompactly cubulated groups. They used it to show that some specific groups are not cocompactly cubulated.

Using these results, we can derive a classification of cocompactly cubulated groups among mapping class groups of surfaces, automorphisms and outer automorphisms groups of free groups. The results for mapping class groups (see [Bri10]), $Aut(\mathbb{F}_n)$ (see [Ger94]) and $Out(\mathbb{F}_n)$ (see [BV06]) are not new, nor optimal if we consider more general actions on CAT(0) spaces, but the proof given here is a simple corollary of the main theorems.

Corollary F. For each group G of the list below, G is cocompactly cubulated if and only if $G/Z(G)$ does not contain $\mathbb{Z}^2$. More precisely:

- The n-strand braid group B_n (or its central quotient) is cocompactly cubulated if and only if $n \leqslant 3$.
- The mapping class group $MCG(S_{g,p})$ (of the closed surface of genus g with p punctures) is cocompactly cubulated if and only if $3g - 3 + p \leqslant 1$.
- The automorphism group $Aut(\mathbb{F}_n)$ (of the free group $\mathbb{F}_n$) is cocompactly cubulated if and only if $n = 1$.
- The outer automorphism group $Out(\mathbb{F}_n)$ is cocompactly cubulated if and only if $n \leqslant 2$.

However, according to B. Bowditch (see [Bow13]), all mapping class groups, including braid groups, are coarse median, which implies that their asymptotic cones “look like” asymptotic cones of CAT(0) cube complexes: they are not cocompactly cubulated, but “look cubical” on a large scale.

Concerning proper actions of Artin groups on CAT(0) cube complexes, even the following question is still open.

Question (Charney [Cha], Wise [Wis]). *Does the* 4*-strand braid group* B_4 *has a metrically proper action on a CAT(0) cube complex ?*

J. Huang, K. Jankiewicz and P. Przytycki proved (see [HJP15]), simultaneously and independently, a stronger version of Theorem A for 2-dimensional Artin groups: in particular, they show that a 2-dimensional Artin group is cocompactly cubulated if and only if it is virtually cocompactly cubulated. Our methods do not apply yet to cover finite index subgroups, but we intend to work together and characterize all virtually cocompactly cubulated Artin groups.

Concerning Coxeter groups, Niblo and Reeves proved (see [NR03]) that every Coxeter group acts properly on a locally finite CAT(0) cube complex. Caprace and Mühlherr proved (see [CM05]) that this action is cocompact if and only if the Coxeter diagram does not contain an affine subdiagram of rank at least 3.

O. Varghese recently described (see [Var15]) a group-theoretic condition ensuring that any (strongly simplicial) isometric action on a CAT(0) cube complex has a global fixed point. This condition is notably satisfied by $Aut(\mathbb{F}_n)$ for $n \geqslant 1$.

Outline of the proof The rough idea is to study the CAT(0) visual angle between maximal abelian subgroups in Artin groups. Using a result from J. Crisp and L. Paoluzzi (see [CP05]), we show that if a, b are the standard generators of the 3-strand braid group acting on some CAT(0) space, then the translation axis for a and $ababab$ form an acute visual angle at infinity.

On the other hand, we show that the translation axis of elements in maximal abelian subgroups of a group acting geometrically on a CAT(0) cube complex, with finite intersection, form an obtuse visual angle at infinity. This is the source of the non-cubicality results.

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1 Definitions and notations

1.1 Artin groups

For $p \in \mathbb{N}$, let w_p denote the word $w_p(a, b) = aba...ba$ of length p. Let S be a finite set, and let Γ be a graph with vertex set S and edges labeled in $\mathbb{N}_{\geqslant 2}$, the *Artin-Tits group* $A(\Gamma)$ is defined by the following presentation:

$$A(\Gamma) = \langle s \in S \,|\, w_p(s, t) = w_p(t, s) \text{ for each edge } \{s, t\} \text{ labeled } p\rangle \,.$$

If $S = \{a, b\}$, then $A(\Gamma)$ is called a dihedral Artin group, and we will denote it by $A(p)$, where p is the label of the edge $\{a, b\}$ (or $p = \infty$ if there is no edge). For instance, $A(2) \simeq \mathbb{Z}^2$ and $A(\infty) \simeq \mathbb{F}_2$.

If a and b are different elements of S, then the subgroup of $A(\Gamma)$ spanned by a and b is isomorphic to the dihedral group $A(p)$, where p is the label of the edge $\{a, b\}$. If $p \notin \{2, \infty\}$, the center of $A(p)$ is the infinite cyclic group spanned by $z_{ab} = w_q(a, b)$, where $q = 2p$ if p is odd, and $q = p$ if p is even.

1.2 CAT(0) cube complexes

A finite dimensional cube complex X is naturally endowed with two natural distances, defined piecewise on cubes: the L^1 distance d_1 and the L^2 distance d_2 (each edge has length 1). Throughout the paper, unless we want to use both distances, we will mainly use the L^1 distance d_1 and will simply denote it d.

A cube complex X is called CAT(0) if the d_2 distance is CAT(0), or equivalently if the d_1 distance is median (see section 1.3). A discrete group G is called cocompactly cubulated if it acts geometrically, i.e. properly and cocompactly by cubical isometries, on a CAT(0) cube complex.

Let us recall the fundamental local-to-global property for CAT(0) spaces.

Theorem 1.1 (Cartan-Hadamard)**.** *A metric space is CAT(0) if and only if it is simply connected and locally CAT(0).*

Let us recall Gromov's combinatorial criterion to show that a cube complex is locally CAT(0).

Theorem 1.2 (Gromov, see [Gro87])**.** *A cube complex X is locally CAT(0) if and only if, for any 3 cubes Q,Q',Q'' of X, which pairwise intersect in codimension 1 and intersect globally in codimension 2, they are codimension 1 faces of some cube of X.*

In a CAT(0) cube complex X, a *hyperplane* H denotes the orthogonal (with respect to the CAT(0) metric d_2) of some edge $[x, y]$ at its midpoint, we denote it $H = [x, y]^{\perp}$. Each hyperplane divides X into two connected components, the closures of which are called *half-spaces* and denoted by H^+ and H^-. An automorphism g of X is said to *skewer* the half-space H^+ if $g \cdot H^+ \subset H^+$. By skewering H, we mean skewering H^+ or H^-.

If x, y are vertices of a CAT(0) cube complex X, then $d_1(x, y)$, also called the *combinatorial distance* between x and y, coincides with the number of hyperplanes separating x and y. An autorphism g of X is called *combinatorially hyperbolic* if g preserves a combinatorial (d_1) geodesic, on which it acts by a nontrivial translation.

If X is a cube complex, we can divide naturally each d-cube into 2^d smaller cubes, getting a new cube complex (up to rescaling the metric by 2) called the *cubical subdivision* of X.

Theorem 1.3 (Haglund, see [Hag07])**.** *Let G be a group acting properly on a locally finite CAT(0) cube complex X, and let $g \in G$ be of infinite order. Then g acts as a combinatorial hyperbolic isometry of the cubical subdivision of X.*

Proof. Since $g \in G$ is of infinite order and G acts properly on the locally finite CAT(0) cube complex X, then g is not combinatorially elliptic. The statement is now a consequence of Lemma 4.2 and Theorem 6.3 of [Hag07]. □

If a group G acts by cubical isometry on a CAT(0) cube complex X, the action is said to be *cube minimal* if X is the smallest non-empty convex cube subcomplex invariant by G.

If g is a cubical isometry of a CAT(0) cube complex, its *translation length* is

$$\delta_g = \min_{x \in X^{(0)}} d_1(x, g \cdot x).$$

If A is a subgroup of G, its d_1 *minimal set* is

$$\mathrm{Min}_1(A) = \{x \in X \,|\, \forall a \in A, d_1(x, a \cdot x) = \delta_a\}$$

and its d_2 *minimal set* is

$$\mathrm{Min}_2(A) = \{x \in X \,|\, \forall a \in A, d_2(x, a \cdot x) = \delta_a\}.$$

If $g \in G$, we will simply denote $\mathrm{Min}_1(g)$ or $\mathrm{Min}_2(g)$ instead of $\mathrm{Min}_1(\langle g \rangle)$ and $\mathrm{Min}_2(\langle g \rangle)$

Remark. Note that $\mathrm{Min}_1(g)$ need not be convex for the d_1 distance, nor need it be a cube subcomplex: consider for instance $X = \mathbb{R}^2$, with the standard Cayley square complex structure of $\mathbb{Z}^2$, and let $g : (x, y) \mapsto (y + 1, x + 1)$. Then $\delta_g = 2$ and $\mathrm{Min}_1(g) = \{(x, y) \in \mathbb{R}^2 \,|\, |x - y| \leqslant 1\}$ is not a cube subcomplex.

By a slight abuse of notation, we will write $\mathrm{Min}_1(A)^{(0)}$ in place of $\mathrm{Min}_1(A) \cap X^{(0)}$, even though $\mathrm{Min}_1(A)$ has not necessarily a cell structure.

If X is a CAT(0) cube complex, we will denote by $\partial_\infty X$ its visual (CAT(0)) boundary at infinity: it is endowed with the visual distance $\sphericalangle$. Each hyperbolic isometry g of X has a unique attracting fixed point $g(+\infty) \in \partial_\infty X$.

1.3 Median algebras

A *median algebra* is a set M endowed with a symmetric map $\mu : M^3 \to M$, called the *median*, satisfying the following

$$\begin{aligned} &\forall a, b \in M, \mu(a, a, b) = a \\ &\forall a, b, c, d, e \in M, \mu(a, b, \mu(c, d, e)) = \mu(\mu(a, b, c), \mu(a, b, d), e). \end{aligned} \tag{1}$$

In a metric space M, the interval between $a \in M$ and $b \in M$ denotes $I(a, b) = \{c \in M \,|\, d(a, c) + d(c, b) = d(a, b)\}$. A metric space M is called *metric median* if

$$\forall a, b, c \in M, I(a, b) \cap I(b, c) \cap I(a, c) = \{\mu(a, b, c)\},$$

which implies that μ is a median.

Median algebras and CAT(0) cube complexes are highly related, as proved by Chepoi.

Theorem 1.4 ([Che00]). *A connected graph, endowed with its combinatorial distance, is metric median if and only if it is the* 1*-skeleton of a CAT(0) cube complex.*

Starting with a more general median algebra, one has the following.

Theorem 1.5 ([CN05]). *Let M be a median algebra with intervals being finite and of rank at most D, and let G be a group of automorphisms of M. There exists is a locally finite CAT(0) cube complex $X(M)$, with vertex set $X(M)^{(0)} = M$, of dimension at most D, on which G acts as a group of cubical automorphisms.*

If X is a CAT(0) cube complex and $x, y \in X$, let $I(x,y) = \{z \in X \,|\, d_1(x,z)+d_1(z,y) = d_1(x,y)\}$ denote the d_1 interval between x and y. A subset $Y \subset X$ is said to be *convex* if for every $x, y \in Y$, we have $I(x,y) \subset Y$. If $Y \subset X$, the d_1 *convex hull* of Y is the smallest convex subset of X containing Y, denoted $\mathrm{Hull}_1(Y)$.

The median $\mu : X^3 \to X$ is defined by

$$\forall x, y, z \in X, I(x,y) \cap I(y,z) \cap I(z,x) = \{\mu(x,y,z)\}.$$

Note that μ is 1-Lipschitz with respect to the three variables, and for both distances d_1 and d_2.

If $I \subset \mathbb{R}$ is an interval, a map $c : I \to X$ is called monotone if

$$\forall s \leqslant t \leqslant u \in I, \mu(c(s), c(t), c(u)) = c(t).$$

If $C \subset X$ is a non-empty convex subset, there exists a unique map $\pi_C : X \to C$, called the gate projection onto C, such that

$$\forall x \in X, \forall c \in C, \mu(x, \pi_C(x), c) = \pi_C(x).$$

Note that π_C coincides with the nearest point projection with respect to the CAT(0) distance d_2.

2 Cubulation of centralizers

In this section, we prove the following result on cubulation of centralizers, which is more precise than Theorem E.

Theorem 2.1. *Let G be a group acting geometrically by semisimple isometries on a locally finite CAT(0) cube complex X of dimension at most D. Let A be an abelian subgroup of G such that every element of A is the $D!^{th}$ power of a combinatorially hyperbolic isometry in G. Then the centralizer $Z_G(A)$ of A acts geometrically on the locally finite CAT(0) cube complex $X(\mathrm{Min}_1(A)^{(0)})$ of dimension at most D associated to the median subalgebra $\mathrm{Min}_1(A)^{(0)}$.*

Remark. It is not always true that $Z_G(g)$ acts cocompactly on $\mathrm{Min}_1(g^{D!})^{(0)}$: consider for instance $X = \mathbb{R}^2$, with the standard Cayley square complex structure of $\mathbb{Z}^2$, and let $g : (x,y) \mapsto (y+1, x+2)$ and $h : (x,y) \mapsto (x+1, y+1)$. Consider the group G spanned by g and h. We have $Z_G(g) = \langle g \rangle \simeq \mathbb{Z}$, but $\mathrm{Min}_1(g^2) = \mathbb{R}^2$.

Lemma 2.2. *Let X be a CAT(0) cube complex of dimension at most D, and let g be a combinatorially hyperbolic isometry of X. Then for any $x \in \mathrm{Min}_1(g)^{(0)}$, and for any hyperplane H separating x and $g^{D!} \cdot x$, $g^{D!}$ skewers H.*

Proof. Fix $x \in \mathrm{Min}_1(g)^{(0)}$, and let H be a hyperplane such that $x \in H^-$ and $g^{D!} \cdot x \in H^+$. Assume that for every $0 \leqslant i < j \leqslant D$ we have $g^i \cdot H \cap g^j \cdot H \neq \emptyset$. Since $g^{D!} \cdot H \neq H$, for every $0 \leqslant i < j \leqslant D$, we have $g^i \cdot H \neq g^j \cdot H$ and $g^i \cdot H \cap g^j \cdot H \neq \emptyset$ so $g^i \cdot H$ and $g^j \cdot H$ cross. Hence the $D+1$ hyperplanes $H, \dots, g^D \cdot H$ pairwise cross, which is impossible in the cube complex X with dimension at most D.

As a consequence, there exist $0 \leqslant i < j \leqslant D$ such that $g^i \cdot H \cap g^j \cdot H = \emptyset$. Let $k = j - i \leqslant D$, we have $H \cap g^k \cdot H = \emptyset$. Since $x \in \mathrm{Min}_1(g)$, we know that $\forall n \geqslant D!, g^n \cdot x \in H^+$ and $\forall n \leqslant 0, g^n \cdot x \in H^-$. Hence we deduce that $g^k \cdot H^+ \subset H^+$, and since k divides $D!$ we have $g^{D!} \cdot H^+ \subset H^+$. $\square$

We now prove a very similar statement, but with the weaker assumption that $x \in \mathrm{Min}_1(g^{D!})^{(0)}$ instead of $x \in \mathrm{Min}_1(g)^{(0)}$.

Lemma 2.3. *Let X be a CAT(0) cube complex of dimension at most D, and let g be a combinatorially hyperbolic isometry of X. Then for any $x \in \mathrm{Min}_1(g^{D!})^{(0)}$, and for any hyperplane H separating x and $g^{D!} \cdot x$, $g^{D!}$ skewers H.*

Proof. Fix $x \in \mathrm{Min}_1(g^{D!})^{(0)}$ and $y \in \mathrm{Min}_1(g)^{(0)}$. By contradiction, assume that there exists a hyperplane H such that $x \in H^-$, $g^{D!} \cdot x \in H^+$ and H is not skewered by g. According to Lemma 2.2, for every $n, m \in \mathbb{Z}$, H does not separate $g^{nD!} \cdot y$ and $g^{mD!} \cdot y$. By symmetry, assume that $\forall n \in \mathbb{Z}$, $g^{nD!} \cdot y \in H^-$.

As a consequence, for every $n \geqslant 0$, $g^{nD!} \cdot H$ separates x and y. Since only $d_1(x, y)$ hyperplanes separate x and y, we deduce that there exists $n > 0$ such that $g^{nD!} \cdot H = H$. This contradicts the fact that a combinatorial geodesic from x to $g^{nD!} \cdot x$ goes via $g^{D!} \cdot x$ as $x \in \mathrm{Min}_1(g^{D!})$ and crosses H, whereas $g^{nD!} \cdot H = H$ does not separate x and $g^{nD!} \cdot x$. □

Lemma 2.4. *Let X be a CAT(0) cube complex of dimension at most D, and let g be a combinatorially hyperbolic isometry of X of translation length δ. Then the set*

$$\{\textit{hyperplanes of } X \textit{ skewered by } g^{D!}\} / < g^{D!} >$$

has cardinality $D!\delta$.

Proof. Let $h = g^{D!}$, and fix $x \in \mathrm{Min}_1(h)^{(0)}$.

Let H be a hyperplane skewered by h. Since the sequence $\{h^n \cdot H\}_{n\in\mathbb{Z}}$ has no accumulation point as n goes to $\pm\infty$, we deduce that there exists $n \in \mathbb{Z}$ such that $h^n \cdot H$ separates x and $h \cdot x$. The number of hyperplanes separating x and $h \cdot x$ is equal to $D!\delta$, so the cardinality of $\{$hyperplanes of X skewered by $h\} / < h >$ is at most $D!\delta$.

Fix a hyperplane H separating x and $h \cdot x$: according to Lemma 2.3, h skewers H. For instance, $x \in H^-$, $h \cdot x \in H^+$ and $h \cdot H^+ \subset H^+$. As a consequence, if $n > 0$ then $x, h \cdot x \in h^n \cdot H^-$ so $h^n \cdot H$ does not separate x and $h \cdot x$. Similarly, if $n < 0$ then $x, h \cdot x \in h^n \cdot H^+$ so $h^n \cdot H$ does not separate x and $h \cdot x$.

We conclude that the $D!\delta$ hyperplanes separating x and $h \cdot x$ are disjoint $< h >$-orbits, hence the cardinality of $\{$hyperplanes of X skewered by $h\} / < h >$ is exactly $D!\delta$. □

Proposition 2.5. *Let X be a CAT(0) cube complex of dimension at most D, and let g be a combinatorially hyperbolic isometry of X. Then $\mathrm{Min}_1(g^{D!})^{(0)}$ is a median subalgebra of $X^{(0)}$, i.e. it is stable under the median of $X^{(0)}$.*

Remark. There exists a combinatorially hyperbolic isometry g of a locally finite CAT(0) cube complex such that for any $n \geqslant 1$, $\mathrm{Min}_1(g^n)$ is not convex.

Proof. Let $h = g^{D!}$, and let δ denote the translation length of g. Let μ denote the median of X.

Let $x, y, z \in \mathrm{Min}_1(h)^{(0)}$, and let $m = \mu(x, y, z) \in X^{(0)}$. Let H be a hyperplane separating m and $h \cdot m$, for instance $m \in H^-$ and $h \cdot m \in H^+$. Since $m = \mu(x, y, z) \in H^-$ which is convex, at least two vertices among x,y and z belong to H^-: we can assume that $x, y \in H^-$. Similarly, $h \cdot m = \mu(h \cdot x, h \cdot y, h \cdot z) \in H^+$ which is convex, at least two vertices among $h \cdot x$,$h \cdot y$ and $h \cdot z$ belong to H^+: we can assume that $h \cdot x, h \cdot z \in H^+$. As a consequence, H separates x and $h \cdot x$, so by Lemma 2.3 H is skewered by h.

Since $m \in H^-$, $h \cdot m \in H^+$ and H is skewered by h, we conclude as in the proof of Lemma 2.4 that for any $n \neq 0$, $h^n \cdot H$ does not separate m and $h \cdot m$. According to Lemma 2.4, we conclude that at most $D!\delta$ hyperplanes separate m and $h \cdot m$. Since the translation length of h is $D!\delta$, we conclude that $d(m, h \cdot m) = D!\delta$, so $m \in \mathrm{Min}_1(h)^{(0)}$. □

Proposition 2.6. *Let G be a group acting geometrically on a locally finite CAT(0) cube complex X of dimension at most D, and let A be an abelian subgroup of G consisting of hyperbolic isometries. Then the centralizer $Z_G(A)$ of A in G acts geometrically on $\mathrm{Min}_1(A)^{(0)}$.*

Proof. The action of $Z_G(A)$ on X is proper and stabilizes $\mathrm{Min}_1(A)^{(0)}$, so it induces a proper action on $\mathrm{Min}_1(A)^{(0)}$. Assume that the action is not cocompact: since G acts properly and cocompactly on X, there exist $C \geqslant 0$, $x \in \mathrm{Min}_1(A)^{(0)}$ and $(h_n)_{n\in\mathbb{N}} \in G^{\mathbb{N}}$ such that $\forall n \in \mathbb{N}, d(h_n \cdot x, \mathrm{Min}_1(A)^{(0)}) \leqslant C$ and the cosets $(h_n Z_G(A))_{n\in\mathbb{N}} \in (G/Z_G(A))^{\mathbb{N}}$ are pairwise distinct.

Fix $a_1, \dots, a_r$ some generators of A. Fix some $1 \leqslant i \leqslant r$.

Let δ_i denote the combinatorial translation length of a_i. For all $n \in \mathbb{N}$, since $d(h_n \cdot x, \mathrm{Min}_1(a_i)^{(0)}) \leqslant C$ we have $d(h_n \cdot x, a_i h_n \cdot x) \leqslant \delta_i + 2C$. So $d(x, h_n^{-1} a_i h_n \cdot x) \leqslant \delta_i + 2C$. Since X is locally finite, up to passing to a subsequence, we may assume that $\forall n, m \in \mathbb{N}, h_n^{-1} a_i h_n \cdot x = h_m^{-1} a_i h_m \cdot x$. Since the action of G on X is proper, the stabilizer of x is finite, so up to passing to a new subsequence, we may assume that $\forall n, m \in \mathbb{N}, h_n^{-1} a_i h_n = h_m^{-1} a_i h_m$. So $\forall n, m \in \mathbb{N}, h_n h_m^{-1}$ centralizes a_i.

If we apply this for every $1 \leqslant i \leqslant r$, we obtain up to passing to a new subsequence that $\forall n, m \in \mathbb{N}, h_n h_m^{-1}$ centralizes $a_1, \dots, a_r$. Since $a_1, \dots, a_r$ span A, we deduce that $\forall n, m \in \mathbb{N}, h_n h_m^{-1} \in Z_G(A)$. This contradicts the assumption that the cosets $(h_n Z_G(A))_{n\in\mathbb{N}} \in (G/Z_G(A))^{\mathbb{N}}$ are pairwise distinct.

As a consequence, the induced action of $Z_G(A)$ on $\mathrm{Min}_1(A)^{(0)}$ is proper and cocompact. □

We obtain now the proof of Theorem 2.1.

Proof. Let $a_1, \dots, a_r$ be some generators of A. For each $1 \leqslant i \leqslant r$, by Proposition 2.5 $\mathrm{Min}_1(a_i)^{(0)}$ is a median subalgebra of $X^{(0)}$. As a consequence, $\mathrm{Min}_1(A)^{(0)} = \bigcap_{i=1}^r \mathrm{Min}_1(a_i)^{(0)}$ is a also a median subalgebra of $X^{(0)}$, and by Proposition 2.6 $Z_G(A)$ acts cocompactly on $\mathrm{Min}_1(A)^{(0)}$. Theorem 1.5 concludes the proof.

Note that the CAT(0) cube complex $X(\mathrm{Min}_1(A)^{(0)})$ has dimension at most D, since $\mathrm{Min}_1(A)^{(0)}$ is a median subalgebra of $X^{(0)}$ which has rank at most D. □

Remark. Note that the distances induced on $\mathrm{Min}_1(A)^{(0)}$ by X and by $X(\mathrm{Min}_1(A)^{(0)})$ may be different.

3 Convex-cocompact subgroups

Definition 3.1. A subgroup A of a group G acting geometrically on a CAT(0) cube complex X is said to be *convex-cocompact in X* if for every $x \in \mathrm{Min}_2(A)$ (equivalently, for some $x \in \mathrm{Min}_2(A)$), A acts geometrically on $\mathrm{Hull}_1(A \cdot x)$.

Remark. Note that being convex-cocompact depends on the CAT(0) cube complex X: see for instance Subsection 5.2.

Definition 3.2. A virtually abelian subgroup A of a group G is called *highest* if for any virtually abelian subgroup B of G such that $A \cap B$ has finite index in A, then $A \cap B$ has finite index in B.

We now recall the following recent result from D. Wise and D. Woodhouse.

Theorem 3.3 (Cubical flat torus theorem [WW15])**.** *Let G be a group acting geometrically on a CAT(0) cube complex X. Let A be a highest virtually abelian subgroup of G. Then A is convex-cocompact in X.*

Lemma 3.4. *Let G be a group acting geometrically on a CAT(0) cube complex X, and let A, B be subgroups of G which are convex-cocompact in X. Then $A \cap B$ is convex-cocompact in X.*

Proof. Fix $x \in \mathrm{Min}_2(A \cap B)$, and consider a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathrm{Hull}_1(A \cap B \cdot x)$. Since A and B act cocompactly on $\mathrm{Hull}_1(A \cdot x)$ and $\mathrm{Hull}_1(B \cdot x)$ respectively, there exist $C > 0$ and sequences $(a_n)_{n\in\mathbb{N}}$ in A and $(b_n)_{n\in\mathbb{N}}$ in B such that $\forall n \in \mathbb{N}, d(a_n \cdot x, x_n) \leqslant C$ and $d(b_n \cdot x, x_n) \leqslant C$. As a consequence, $\forall n \in \mathbb{N}, d(b_n^{-1} a_n \cdot x, x) \leqslant 2C$. Since G acts properly on the CAT(0) cube complex, we deduce that, up to passing to subsequences, we have $\forall n, m \in \mathbb{N}, b_m^{-1} a_m = b_n^{-1} a_n$, so $a_n a_m^{-1} = b_n b_m^{-1} \in A \cap B$. So for all $n \in \mathbb{N}$, we have $d(a_n a_0^{-1} \cdot x, x_n) \leqslant d(a_n \cdot x, x_n) + d(a_n a_0^{-1} \cdot x, a_n \cdot x) \leqslant C + d(a_0^{-1} \cdot x, x)$ is bounded. Since $\forall n \in \mathbb{N}, a_n a_0^{-1} \in A \cap B$, this proves that $A \cap B$ acts cocompactly on $\mathrm{Hull}_1(A \cap B \cdot x)$. □

Proposition 3.5. *Let G be a group acting geometrically on a CAT(0) cube complex X, and let A be a virtually abelian subgroup of G. Then there exists a virtually abelian subgroup B of G virtually containing A which is convex-cocompact in X such that B is virtually minimal, i.e. for any other such subgroup B', $B \cap B'$ has finite index in B.*

Proof. Consider an intersection B of virtually abelian subgroups of G virtually containing A which are convex-cocompact in X, such that B has minimal rank. According to Theorem 3.3, B exists. For any virtually abelian subgroup B' of G virtually containing A which is convex-cocompact in X, one knows that $B \cap B'$ has the same rank as B, so $B \cap B'$ has finite index in B. □

Lemma 3.6. *Let G be a group acting geometrically, cube minimally by semisimple isometries on a CAT(0) cube complex X, and let W be a central subgroup of G which is convex-cocompact in X. Then X splits as a product of two convex cube subcomplexes $X \simeq Y \times Z$, where G preserves this splitting, and W acts finite index kernel W' on Y and cube minimally, geometrically on Z. Furthermore, G/W' acts geometrically on Y.*

Proof. Let D denote the dimension of X, and let $W^{D!} = \langle w^{D!} \,|\, w \in W \rangle$. Since W is abelian and finitely generated, $W^{D!}$ has finite index in W. Fix $x \in \mathrm{Min}_2(W)^{(0)}$, and assume (up to choosing another x) that the action of $W^{D!}$ on $Z = \mathrm{Hull}_1(W \cdot x)$ is cube minimal.

Fix a hyperplane H_0 adjacent to x such that $H_0 \cap Z = \emptyset$. Since the action of G on X is cube minimal, there exists $g \in G$ such that H_0 separates x and $g \cdot x$.

We will first show that for every $w \in W$, we have $w^{D!} \cdot H_0 = H_0$. Fix H a hyperplane such that $Z \subset H^-$ and $g \cdot x \in H^+$. We will show that H separates $\langle w \rangle \cdot x$ and $g\langle w \rangle \cdot x$. Since $\langle w \rangle \cdot x \subset W \cdot x \subset Z \subset H^-$, we want to show that $g\langle w \rangle \cdot x \subset H^+$.

By contradiction, assume that there exists $n \neq 0$ such that $gw^n \cdot x \in H^-$. Notice that $g \cdot x \in \mathrm{Min}_1(gWg^{-1}) = \mathrm{Min}_1(W)$. Up to replacing w with w^{-1}, we may assume that there exists $n_0 > 0$ such that $\forall n < n_0, gw^n \cdot x \in H^+$ and $\forall n \geqslant n_0, gw^n \cdot x \in H^-$. As a consequence, for every $n \geqslant 0$, $w^n \cdot H$ separates x and $g \cdot x$. This implies that there exists $n > 0$ such that $w^n \cdot H = H$. This contradicts $gw^{n_0} \cdot x \in H^-$, as $g \cdot x \in \mathrm{Min}_1(W)$.

As a consequence, every hyperplane separating Z and $g \cdot x$ separates Z and $g\langle w \rangle \cdot x$. So w acts as a bijection σ on the finite set of hyperplanes separating x and $g \cdot x$. Each k-cycle in σ corresponds to k pairwise crossing hyperplanes, so $k \leqslant D$. As a consequence, $\sigma^{D!} = 1$, so $w^{D!} \cdot H_0 = H_0$.

Let Z' denote the complex spanned by the vertices of Z adjacent to H_0. Z' is convex and $W^{D!}$-invariant. Since the action of $W^{D!}$ on Z is cube minimal, we deduce that $Z' = Z$.

Let Y denote the cube subcomplex spanned by the vertices of the intersection of all half-spaces in X containing x and not containing Z. By definition, Y is a convex cube subcomplex of X. Also by definition, we have $Y \cap Z = \{x\}$. Then the gate projections $(\pi_Y, \pi_Z) : X \to Y \times Z$ define an isomorphism between X and $Y \times Z$.

By definition of Z, W acts cube minimally and geometrically on Z. We have shown that $W^{D!}$ acts trivially on Y, so W acts on Y with finite index kernel $W' \supset W^{D!}$.

Since G acts geometrically on X, G acts cocompactly on Y. As W' acts geometrically on Z, we deduce that G/W' acts properly on Y. So G/W' acts geometrically on Y. □

4 Non-cubicality criterion

We now give a slightly more general version of a result from Crisp and Paoluzzi (see [CP05]), which studies proper semisimple actions of B_3 and B_4 on CAT(0) spaces. Note that there is no cocompactness assumption in this result, nor a CAT(0) cube complex.

Proposition 4.1 (Crisp-Paoluzzi). *Let $p \in \mathbb{N}_{\geqslant 3}$ and $A = A(p) = \langle a, b \mid w_p(a,b) = w_p(b,a) \rangle$. Assume A acts properly, by semisimple isometries on a CAT(0) space X. Then a, z_{ab} and b act by hyperbolic isometries, whose attracting endpoints in $\partial_\infty X$ are denoted $a(+\infty)$, $z_{ab}(+\infty)$ and $b(+\infty)$. Furthermore:*

- *If p is odd, then we have $\sphericalangle(a(+\infty), z_{ab}(+\infty)) < \frac{\pi}{2}$ and $\sphericalangle(b(+\infty), z_{ab}(+\infty)) < \frac{\pi}{2}$.*
- *If p is even, then we have $\sphericalangle(a(+\infty), z_{ab}(+\infty)) < \frac{\pi}{2}$ or $\sphericalangle(b(+\infty), z_{ab}(+\infty)) < \frac{\pi}{2}$.*

Proof. We adapt here the proof of [CP05, Theorem 4]. Without loss of generality, we may assume that the action of A on X is minimal. By properness, every infinite order element of A acts by a hyperbolic isometry, in particular a, b and z_{ab}. Then by [BH99, Theorem 6.8], X is isometric to the product $\mathbb{R} \times Y$, where Y is a CAT(0) space, and $Z(A) = \langle z_{ab} \rangle$ acts by translation on $\mathbb{R}$ and trivially on Y. Let $\delta > 0$ denote the translation length of z_{ab}. Since a and b commute with z_{ab}, they preserve the decomposition $X \simeq \mathbb{R} \times Y$. In particular, let $\alpha, \beta \in \mathbb{R}$ denote the translation lengths of a, b on the $\mathbb{R}$ factor.

- If p is odd, then a and b are conjugated by $w_p(a,b)$ in A, we deduce that $\alpha = \beta$. But $z_{ab} = w_{2p}(a,b)$, so we have $\delta = 2p\alpha$. As a consequence, we have $\alpha =$

$\beta > 0$. This implies that the attracting endpoints of a and z_{ab} in $\partial_\infty X$ satisfy $\sphericalangle(a(+\infty), z_{ab}(+\infty)) < \frac{\pi}{2}$.

- If p is even, then since $z_{ab} = w_p(a,b)$, we deduce that $p\alpha + p\beta = \delta > 0$. As a consequence, $\alpha > 0$ or $\beta > 0$. This implies that $\sphericalangle(a(+\infty), z_{ab}(+\infty)) < \frac{\pi}{2}$ or $\sphericalangle(b(+\infty), z_{ab}(+\infty)) < \frac{\pi}{2}$.

□

Proposition 4.2. *Let G be a group acting geometrically on a CAT(0) cube complex X, and let A, B be subgroups of G which are convex-cocompact in X, such that $A \cap B$ is finite. Then for each $a \in A$, $b \in B$ of infinite order, their attractive endpoints in $\partial_\infty X$ satisfy $\sphericalangle(a(+\infty), b(+\infty)) \geqslant \frac{\pi}{2}$.*

Proof. Let M_A, M_B denote convex cube subcomplexes of X on which A, B respectively act geometrically.

Fix $x \in X^{(0)}$, and let $R \geqslant 0$ such that $d_1(x, M_A) \leqslant R$ and $d_1(x, M_B) \leqslant R$. Let $x_A \in M_A$ and $x_B \in M_B$ such that $d_1(x, x_A) \leqslant R$ and $d_1(x, x_B) \leqslant R$. Define

$$S = \{y \in X^{(0)} \mid d_1(y, M_A) \leqslant R \text{ and } d_1(y, M_B) \leqslant R\}.$$

We have $x \in S$. We claim that S is finite: if not, since X is locally compact, we can consider a sequence $(s_n)_{n\in\mathbb{N}}$ in S going to infinity. Since A and B act geometrically on M_A and M_B respectively, we deduce that there exist vertices $y_A \in M_A$, $y_B \in M_B$ and sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ in A and B respectively, going to infinity, such that the sequence $d_1(a_n \cdot y_A, b_n \cdot y_B)$ is bounded above. Since the action of G on X is proper and X is locally compact, we can assume up to passing to a subsequence that the sequence $(b_n^{-1}a_n)_{n\in\mathbb{N}}$ is constant, hence for all $m, n \in \mathbb{N}$ we have $a_n a_m^{-1} = b_n b_m^{-1} \in A \cap B$. As $A \cap B$ is finite, this is a contradiction. So S is finite.

From now on, fix $a \in A$ and $b \in B$ of infinite order. We will show that their attractive endpoints in $\partial_\infty X$ satisfy $\sphericalangle(a(+\infty), b(+\infty)) \geqslant \frac{\pi}{2}$.

Let $\mu : X^3 \to X$ denote the median on X. Fix $(\alpha_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}}$ sequences in M_A (resp. M_B) converging to $a(+\infty)$ (resp. $b(+\infty)$). For each $n \in \mathbb{N}$, define $m_n = \mu(\alpha_n, \beta_n, x)$. Since μ is 1-Lipschitz with respect to d_1, we deduce that $d_1(m_n, M_A) \leqslant d_1(x, x_A) + d_1(\mu(\alpha_n, \beta_n, x_A), M_A)$. Since α_n and x_A belong to the convex subcomplex M_A, we deduce that $\mu(\alpha_n, \beta_n, x_A) \in M_A$, so $d_1(m_n, M_A) \leqslant R$. For the same reason, we have $d_1(m_n, M_B) \leqslant R$. As a consequence, we have $\forall n \in N, m_n \in S$.

Since S is finite, up to passing to a subsequence we may assume that $\forall n \in \mathbb{N}, m_n = x_0$ is constant.

Fix $\varepsilon > 0$, and for each $n \in \mathbb{N}$, let α'_n (resp β'_n) be the point on the CAT(0) geodesic segment between x_0 and α_n (resp. β_n) at d_2 distance ε from x_0 (see Figure 2). Since $\mu(x_0, \alpha'_n, \alpha_n) = \alpha'_n$, $\mu(x_0, \beta'_n, \beta_n) = \beta'_n$ and $\mu(x_0, \alpha_n, \beta_n) = x_0$, we deduce that $\mu(x_0, \alpha'_n, \beta'_n) = x_0$, by using several times Equation (1) from Section 1.3. But the sequence $(\alpha'_n)_{n\in\mathbb{N}}$ (resp. $(\beta'_n)_{n\in\mathbb{N}}$) actually converges to the point α' (resp. β') on the CAT(0) geodesic ray from x_0 to $a(+\infty)$ (resp. $b(+\infty)$) at d_2 distance ε from x_0. Hence we conclude that $\mu(x_0, \alpha', \beta') = x_0$. In other words, the path $[\alpha', x_0] \cup [x_0, \beta']$ is monotone.

On the other hand, we have $\sphericalangle_{x_0}(\alpha', \beta') = \sphericalangle_{x_0}(a(+\infty), b(+\infty)) \leqslant \sphericalangle(a(+\infty), b(+\infty))$. By contradiction, assume that we have $\sphericalangle(a(+\infty), b(+\infty)) < \frac{\pi}{2}$, then $\sphericalangle_{x_0}(\alpha', \beta') < \frac{\pi}{2}$.

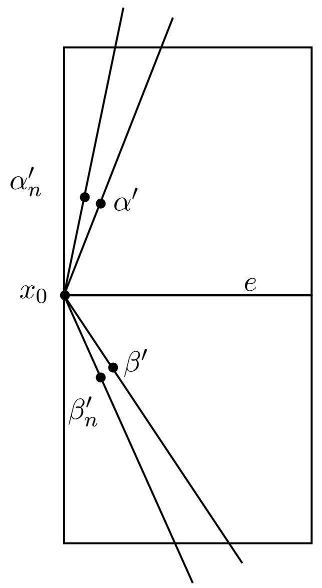

Figure 2: The proof of Proposition 4.2

There exists an edge e in X containing x_0 such that $\measuredangle_{x_0}(\alpha', e) < \frac{\pi}{2}$ and $\measuredangle_{x_0}(\beta', e) < \frac{\pi}{2}$. If we consider a shifted hyperplane H dual to e close to x_0 (the CAT(0) orthogonal of e at a point near x_0), we see that H separates x_0 and $\{\alpha', \beta'\}$: this contradicts the monotonicity of the path $[\alpha', x_0] \cup [x_0, \beta']$.

As a consequence, we have $\measuredangle(a(+\infty), b(+\infty)) \geqslant \frac{\pi}{2}$. □

Proposition 4.3. *Let G be a group acting geometrically on a CAT(0) cube complex X. Assume there exist $a, b \in G$ such that*

- *the subgroup $\langle a, b\rangle$ is isomorphic to $A(p)$, for $p \geqslant 3$,*
- *there exists a virtually abelian subgroup A virtually containing a, which is convex-cocompact in X and*
- *there exists a virtually abelian subgroup B virtually containing b, which is convex-cocompact in X.*

Then

- *if p is odd, then $A \cap \langle z_{ab}\rangle \neq \{1\}$ and $B \cap \langle z_{ab}\rangle \neq \{1\}$,*
- *if p is even, then $A \cap \langle z_{ab}\rangle \neq \{1\}$ or $B \cap \langle z_{ab}\rangle \neq \{1\}$.*

Proof. By contradiction, assume that G is a counterexample to the proposition. Assume furthermore that the dimension D of X is minimal. Without loss of generality, we may assume by Proposition 3.5 that A and B are virtually minimal virtually abelian subgroups virtually containing a and b respectively, which are convex-cocompact in X. We can furthermore assume that A and B are abelian.

Up to replacing X by its cubical subdivision (see Theorem 1.3), we can assume that every element of G is a semisimple isometry of X. According to Proposition 4.1, we know that $\measuredangle(a(+\infty), z_{ab}(+\infty)) < \frac{\pi}{2}$ (up to swapping a and b if p is even). Since G does not satisfy the conclusion of the proposition, we know that $A \cap \langle z_{ab} \rangle = \{1\}$ (up to swapping a and b if p is odd). If p is even, we also know that $B \cap \langle z_{ab} \rangle = \{1\}$.

Let C denote a highest virtually abelian subgroup of G virtually containing $\langle z_{ab}, b \rangle$: up to passing to a finite index subgroup, we may assume that C is abelian. By Theorem 3.3, C is convex-cocompact in X.

Since A virtually contains a, there exists $n > 0$ such that $a^n \in A$; similarly, since C virtually contains z_{ab}, there exists $m > 0$ such that $z_{ab}^m \in C$. If $A \cap C$ is finite, since A and C are convex-cocompact subgroups in X, by Proposition 4.2, we know that $\sphericalangle(a^n(+\infty), z_{ab}^m(+\infty)) \geqslant \frac{\pi}{2}$, which is a contradiction since $a^n(+\infty) = a(+\infty)$ and $z_{ab}^m(+\infty) = z_{ab}(+\infty)$. As a consequence, $A \cap C$ is infinite.

Let $W = \langle w^{D!} \mid w \in A \cap C \rangle$: it is an abelian, central subgroup of G which is convex-cocompact in X.

Since B is minimal and C is a convex-cocompact subgroup of G virtually containing b, up to passing to a finite index subgroup of B we may assume that $B \subset C$.

Let $H = Z_G(W)$: according to Theorem 2.1, H acts geometrically on the CAT(0) cube complex $X' = X(\mathrm{Min}_1(W)^{(0)})$ associated to the median subalgebra $\mathrm{Min}_1(W)^{(0)}$, and X' has dimension at most D. Furthermore, A is abelian and contains W, so $A \subset H$. Similarly, since W and B are subgroups of the abelian group C, we know that $B \subset H$.

Since $A \subset H$, A stabilizes $\mathrm{Min}_1(W)^{(0)}$. As a consequence, there exists $x \in \mathrm{Min}_2(A) \cap \mathrm{Min}_1(W)^{(0)}$. Since A is convex-cocompact, we know that $\mathrm{Hull}_1(A \cdot x)$ is at bounded distance from $A \cdot x \subset \mathrm{Min}_1(W)^{(0)}$. Since $\mathrm{Hull}_1(A \cdot x) \cap \mathrm{Min}_1(W)^{(0)}$ is a convex subalgebra of the median algebra $\mathrm{Min}_1(W)^{(0)}$, we deduce that in the CAT(0) cube complex $X' = X(\mathrm{Min}_1(A)^{(0)})$, the convex hull of $A \cdot x$ is at bounded distance from $A \cdot x$. In other words, A is also convex-cocompact in X'. Similarly, B is convex-cocompact in X'.

According to Lemma 3.6 applied to W, X' splits as a product of two convex cube subcomplexes $X' = Y \times Z$, where W acts with finite index kernel W' on Y and cube minimally, geometrically on Z.

Since A is convex-cocompact in X', it acts cocompactly on a convex cube subcomplex M_A of X'. Since A contains W, we deduce that A acts cocompactly on the convex cube subcomplex $\pi_Y(M_A)$ of Y. As a consequence, $A' = A/W'$ acts geometrically on $\pi_Y(M_A)$. Similarly, since B is convex-cocompact in X', it acts cocompactly on a convex cube subcomplex M_B of X'. So B acts on the convex cube subcomplex $\pi_Y(M_B)$ of Y. We conclude that BW' acts geometrically on the convex cube subcomplex $\pi_Y(M_B) \times Z$ of $Y \times Z \simeq X'$. As a consequence, $B' = BW'/W'$ acts geometrically on $\pi_Y(M_B)$.

Let $G' = H/W'$. According to Lemma 3.6, G' acts geometrically on the CAT(0) cube complex Y. We will show that G' is also a counterexample to the proposition. Since $\langle a, b \rangle \cap W' \subset \langle z_{ab} \rangle \cap A = \{1\}$, the images a', b' of a, b in G' span a subgroup isomorphic to $A(p)$. Finally we have $A \cap \langle z_{ab} \rangle W' \subset W'$, so $A' \cap \langle z_{a'b'} \rangle = \{1\}$. If p is even, we know furthermore that $B \cap \langle z_{ab} \rangle W' \subset W'$, so $B' \cap \langle z_{a'b'} \rangle = \{1\}$.

As a consequence, G' is a counterexample to the proposition. But G' acts geometrically on the CAT(0) cube complex Y, which has dimension at most D minus the rank of W'. Since W' is infinite, the dimension of Y is smaller than D, which contradicts the minimality of G. This concludes the proof. $\square$

We can now give the proof of Theorem C, which we restate here.

Theorem C. Let G be a group and $a, b \in G$ such that:

- $\langle a, b\rangle$ is a subgroup isomorphic to the dihedral Artin group $A(p)$, for some finite $p \geqslant 3$,
- there exists $g \in G$ commuting with a such that non non-zero powers of g and z_{ab} commute and
- if p is even, assume furthermore that there exists $h \in G$ commuting with b such that non non-zero powers of h and z_{ab} commute.

Then G is not cocompactly cubulated.

Proof. By contradiction, assume that G acts geometrically on a CAT(0) cube complex X. Let A and B be virtually highest abelian subgroups of G such that

- A virtually contains $\langle a, g\rangle$,
- B virtually contains $\langle b\rangle$ if p is odd and
- B virtually contains $\langle b, h\rangle$ if p is even.

According to Theorem 3.3, A and B are convex-cocompact in X. Since no non-zero powers of g and z_{ab} commute, we know that $A \cap \langle z_{ab}\rangle = \{1\}$. If p is odd, this contradicts Proposition 4.3.

If p is even, then since no non-zero powers of h and z_{ab} commute, we know furthermore that $B \cap \langle z_{ab}\rangle = \{1\}$. This contradicts Proposition 4.3. □

In order to apply Theorem C to Artin groups, we need the following technical result.

Lemma 4.4. *Let $p, q, r \in \mathbb{N}_{\geqslant 2} \cup \{\infty\}$ such that $p \neq 2, \infty$ and $r \neq \infty$. Assume furthermore that*

- *if p is odd, then $q \neq 2$ and*
- *if p is even, then $(q, r) \neq (2, 2)$.*

Let $A = A(p, q, r) = \langle a, b, c \,|\, w_p(a, b) = w_p(b, a), w_q(b, c) = w_q(c, b), w_r(a, c) = w_r(c, a)\rangle$. Then z_{ac} commutes with a and no non-trivial powers of z_{ab} and z_{ac} commute.

Proof. Assume there exist $n, m > 0$ such that z_{ab}^n and z_{ac}^m commute.

- Assume first that $q \neq 2$. Hence $z_{ab}^n z_{ac}^m = z_{ac}^m z_{ab}^n$ is an equality between positive words, so by [Par02] they are equal in the positive monoid: one can pass from one to the other by applying only the standard relations of A. But the relation $w_q(b, c) = w_q(c, b)$ cannot be used since $q \neq 2$, no such subword can appear. As a consequence, starting from $z_{ab}^n z_{ac}^m$ it is not possible to obtain a word with a letter c on the left of a letter b. This is a contradiction: no non-trivial powers of z_{ab} and z_{ac} commute.
- Assume now that $q = 2$. By assumption, $p \neq 2$ is even and $r \neq 2$. If $r = 3$, then choosing the order (r, p, q) instead of (p, q, r) comes back to the first case, so assume that $r \geqslant 4$. Since $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \geqslant \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1$, A is not of spherical type. By Charney and Davis (see [CD95b, Theorem B] and [CD95a, Corollary 1.4.2]), the cohomological

dimension of A is 2. In particular, A is torsion free. Since the subgroup $\langle a, z_{ab}^n, z_{ac}^m \rangle$ is free abelian, we deduce that there exist $(x, y, z) \in (\mathbb{Z} \backslash \{0\})^3$ such that $a^x z_{ab}^{ny} z_{ac}^{mz} = 1$. Writing this as an equality between positive powers of a, z_{ab}^n and z_{ac}^m, we obtain an equality bewteen positive words, so by [Par02] they are equal in the positive monoid: one can pass from one to the other by applying only the standard relations of A. Since the letter b appears only on one side of the equality, this is a contradiction.

□

We can now prove the following, which implies one direction of Corollary B.

Theorem 4.5. *Let Γ be a finite Coxeter graph. Assume that at least one of the following holds*

- *there exist 3 pairwise distinct $a, b, c \in \Gamma^{(0)}$ such that the edge ab is in Γ and labeled by an odd number, the edge bc is in Γ and there is no edge ac labeled 2,*
- *there exist 3 pairwise distinct $a, b, c \in \Gamma^{(0)}$ such that the edges ab and ac are in Γ and labeled by even numbers different from 2, and the edge bc is in Γ, or*
- *there exist 4 pairwise distinct $a, b, c, d \in \Gamma^{(0)}$ such that the edge bc is in Γ and not labeled by 2, the edges ab and cd are in Γ, and there are no edges ac nor bd labeled 2.*

Then any group containing a subgroup isomorphic to $A(\Gamma)$ is not cocompactly cubulated.

Proof.
- Assume that there exist 3 pairwise distinct $a, b, c \in \Gamma^{(0)}$ such that the edge ab is in Γ and labeled by an odd number p, the edge bc is in Γ (labeled r) and there is no edge ac labeled 2 ($q \neq 2$). Then by Lemma 4.4 applied to b, a, c in that order, the element $g = z_{bc}$ commutes with a, and no non-trivial powers of g and z_{ab} commute. By Theorem C, G is not cocompactly cubulated.
- Assume that there exist 3 pairwise distinct $a, b, c \in \Gamma^{(0)}$ such that the edges ab and ac are in Γ and labeled by even numbers p, r different from 2, and the edge bc is in Γ and labeled $q \neq \infty$. Then by Lemma 4.4 the element $g = z_{ac}$ commutes with a, and no non-trivial powers of g and z_{ab} commute. Similarly by Lemma 4.4 the element $h = z_{bc}$ commutes with b, and no non-trivial powers of h and z_{ab} commute. By Theorem C, G is not cocompactly cubulated.
- Assume that there exist 4 pairwise distinct $a, b, c, d \in \Gamma^{(0)}$ such that the edge bc is in Γ and is labeled by $p \neq 2$, the edges ab and cd are in Γ, and there are no edges ac nor bd labeled 2. If p is odd, then by considering b, c, a in that order we are in the first case of the proof, so G is not cocompactly cubulated. Assume now that p is even.

  By Lemma 4.4 the element $g = z_{ab}$ commutes with b, and no non-trivial powers of g and z_{bc} commute. Similarly by Lemma 4.4 the element $h = z_{cd}$ commutes with c, and no non-trivial powers of h and z_{bc} commute. By Theorem C, G is not cocompactly cubulated.

□

We can now give the proof of one direction of Theorem A, namely that if an Artin group is cocompactly cubulated, then it satisfies the conditions of Theorem A.

Proof. Assume that the Artin group $A(\Gamma)$ is cocompactly cubulated. Let $\Gamma = \sqcup_{i=1}^{p}\Gamma_i$ be the connected components of the graph Γ without the edges labeled 2.

According to Theorem 4.5, for each $1 \leqslant i \leqslant p$, Γ_i is a vertex, an odd edge, or has all its edges labeled by an even number. Furthermore, for every odd edge Γ_i with vertices $\{s,t\}$, for every $u \notin \{s,t\}$, the edge us is in Γ if and only if the edge ut is in Γ.

Fix $1 \leqslant i \leqslant p$ such that all edges of Γ_i have even labels. According to Theorem 4.5, Γ_i is a star, with central vertex denoted s_i and all its edges have labels different from 2. Note that if Γ_i is only one edge, one has two choices for s_i.

By contradiction, assume that there exist $s \in \Gamma_i^{(0)} \backslash \{s_i\}$ and $t \in \Gamma^{(0)} \backslash \Gamma_i^{(0)}$ such that the edge ts is in Γ, but the edge ts_i is not in Γ. We will consider two cases.

1. Assume first that Γ_i has at least 3 vertices $\{s_i, s, s'\}$. Then the subgraph spanned by t, s, s_i, s' in that order contradicts Theorem 4.5.

2. Assume that Γ_i is just one edge $\{s, s'\}$. If no choice of the star $s_i \in \{s, s'\}$ satisfies the conditions of Theorem A, then there exist $t, t' \in \Gamma^{(0)} \backslash \{s, s'\}$ such that the edges ts and $t's'$ are in Γ, but the edges ts' and $t's$ are not. Then the subgraph spanned by t, s, s', t' in that order contradicts Theorem 4.5.

As a conclusion, the graph Γ satisfies the conditions of Theorem A. □

We can also prove the following result.

Proposition 4.6. *Let G be a group containing either the n-strand braid group B_n, or its central quotient $B_n/Z(B_n)$, for some $n \geqslant 4$. Then G is not cocompactly cubulated.*

Proof. If B_n is a subgroup of G, by Theorem 4.5 G is not cocompactly cubulated.

If $B_4/Z(B_4)$ is a subgroup of G, let a, b, c be the images in G of the standard generators of B_4, and let $g = z_{bc} = bcbcbc$. The assumptions of Theorem C for b, a and g in that order are satisfied, so G is not cocompactly cubulated.

If $B_n/Z(B_n)$ is a subgroup of G for some $n \geqslant 5$, since B_4 is a subgroup of $B_n/Z(B_n)$, by Theorem 4.5 G is not cocompactly cubulated. □

5 Cubulation of Artin groups

5.1 Cubulation of dihedral Artin groups

Brady and McCammond showed (see [BM00]) that for all $p \in \{2, \ldots, \infty\}$, the dihedral Artin group $A(p)$ is cocompactly cubulated. Let us recall their construction, which will be useful. We will need this construction when $p \notin \{2, \infty\}$, but it works as well when $p = 2$, so let us fix $p \neq \infty$ (when $p = \infty$, the Artin group is just the rank 2 free group).

The Artin group $A(p)$ has the following presentation, due to Brady and McCammond:

$$A(p) = \langle x, a_1, \ldots, a_p \,|\, \forall 1 \leqslant i \leqslant p, a_i a_{i+1} = x \rangle,$$

where $a_{p+1} = a_1$. This can easily be seen, with a_1 and a_2 corresponding to the standard generators of $A(p)$. The presentation 2-complex K is a $K(\pi, 1)$ for $A(p)$ consisting of 1 vertex v, $p+1$ loops $a_1, \ldots, a_p, x$ and p triangles $a_1 a_2 x^{-1}, \ldots, a_p a_1 x^{-1}$ (see Figure 3).

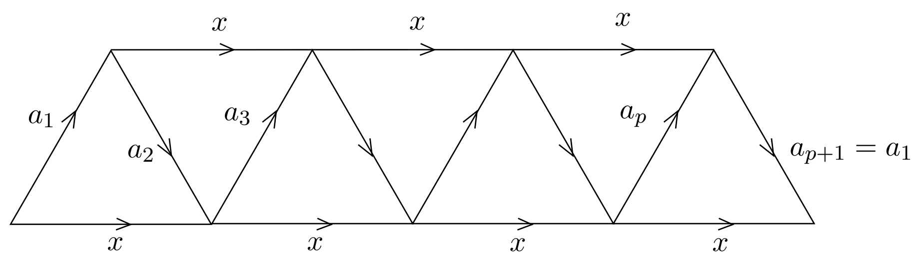

Figure 3: Brady and McCammond's presentation 2-complex K

We will define another $K(\pi, 1)$ for $A(p)$, which will be cubical and will have the same underlying topological space as K. Start with two vertices v and w, and $p+2$ oriented edges between v and w labelled $\alpha_1, \ldots, \alpha_p, \beta^{-1}, \gamma$. Finally, add the p squares with boundary labeled by $\alpha_1\beta^{-1}\alpha_2\gamma^{-1}, \ldots, \alpha_p\beta^{-1}\alpha_1\gamma^{-1}$ and let $X(A(p))$ denote the resulting cube complex. It is easy to see that the underlying topological space of $X(A(p))$ is homoeomorphic to K: w corresponds to the midpoint of the edge x, the edge x corresponds to the path $\gamma\beta^{-1}$, and each square corresponds to the union of the halves of two triangles of K (see Figure 4).

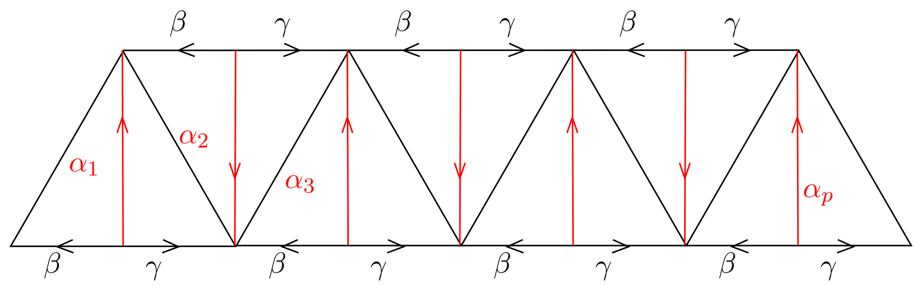

Figure 4: The square complex $X(A(p))$

Hence $X(A(p))$ is also a $K(\pi, 1)$ for $A(p)$. Since every triple of squares in $X(A(p))$ pairwise sharing an edge share the same edge, $X(A(p))$ is a locally CAT(0) square complex, hence $A(p)$ is cocompactly cubulated.

Remark. Notice that $X(A(p))$ is naturally isometric to the product of $\mathbb{R}$ and the infinite p-regular tree. In the case of the 3-strand braid group $B_3 \simeq A(3)$, one recovers in the central quotient the action of $B_3/Z(B_3) \simeq \mathrm{PSL}(2, \mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ on its Bass-Serre 3-regular tree.

5.2 Recubulation of even dihedral Artin groups

In the case where p is even, there are two other natural CAT(0) square complexes on which the dihedral Artin group $A(p)$ acts geometrically. Each will be associated with one of the two generators a,b of $A(p)$. We will describe the first one, associated with $a = a_1$.

Start with the same presentation 2-complex K as before, and remove all edges $a_2, a_4, \ldots, a_p$ with even labels, and replace each pair of triangles $(a_{2i+1}a_{2i+2}x^{-1}, a_{2i+2}a_{2i+3}x^{-1})$, for $0 \leqslant i \leqslant \frac{p}{2} - 1$, by a square with edges $a_{2i+1}xa_{2i+3}^{-1}x^{-1}$. We obtain a square complex $X_a(A(p))$ with one vertex v, $p+1$ edges $x, a_1, a_3, \ldots, a_{p-1}$ and $\frac{p}{2}$ squares $a_1xa_3{}^{-1}x^{-1}, \ldots, a_{p-1}xa_1{}^{-1}x^{-1}$ (see Figure 5).

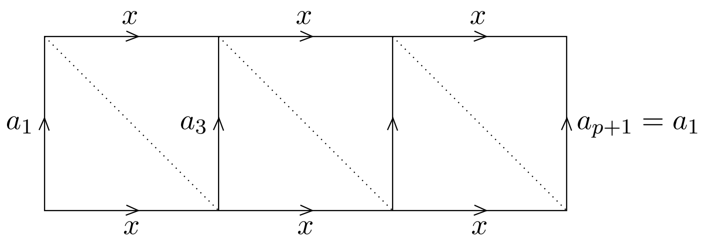

Figure 5: The square complex $X_a(A(p))$

The underlying topological space of $X_a(A(p))$ is K, so it is also a $K(\pi, 1)$ for $A(p)$. Since every triple of squares in $X_a(A(p))$ pairwise sharing an edge share the same edge, $X_a(A(p))$ is a locally CAT(0) square complex.

The other locally CAT(0) square complex, denoted $X_b(A(p)$, is obtained by keeping only the edges with even labels and removing those with odd labels.

The fundamental difference of $X_a(A(p))$ and $X(A(p))$ is that, in the universal covers, the visual angles between the attractive fixed points of a and z_{ab} differ: in $X(A(p))$ that angle is acute, while in $X_a(A(p))$ it is equal to $\frac{\pi}{2}$. This is due to the fact that, in $X_a(A(p))$, the edge $a = a_1$ belongs to the complex, so the subgroup $\langle a \rangle$ is convex-cocompact in $X_a(A(p))$ but not in $X(A(p))$. This illustrates the case where p is even in Proposition 4.1.

5.3 Cubulation of Artin groups of even stars

Let Γ be a finite Coxeter graph with vertex set S, which is a star with central vertex s and with all edges having even labels. We will now prove a particular case of the converse direction of Theorem A, namely showing that $A(\Gamma)$ is cocompactly cubulated. Note that J. Huang, K. Jankiewicz and P. Przytycki independently gave the same construction in [?].

Write $S \backslash \{s\} = \{s_1, \dots, s_m\}$. For each $1 \leqslant i \leqslant m$, the subgroup $A(\{ss_i\})$ of $A(\Gamma)$ spanned by s and s_i is a dihedral Artin group with even integer: let $X_s(A(\{ss_i\}))$ denote the previously constructed locally CAT(0) square complex with fundamental group $A(\{ss_i\})$, where some edge e_i in $X_s(A(\{ss_i\}))$ represents s.

Consider now the square complex $X(A(\Gamma))$ which is the glueing of the square complexes $X_s(A(\{ss_1\})), \dots, X_s(A(\{ss_m\}))$ where all edges $e_1, \dots, e_m$ are identified with a single edge e. By Van Kampen Theorem, the fundamental group of $X(A(\Gamma))$ is the free product of $A(\{ss_1\}), \dots, A(\{ss_m\})$ amalgamated over the cyclic subgroup $\langle s \rangle$, which is precisely isomorphic to the Artin group $A(\Gamma)$.

The only three squares in $X(A(\Gamma))$ which pairwise share an edge are squares containing the edge e. As a consequence, their triple intersection does not have codimension 2, so $X(A(\Gamma))$ is a locally CAT(0) cube complex. As a consequence, $A(\Gamma)$ is cocompactly cubulated.

5.4 General case

Let Γ be a finite Coxeter graph satisfying the assumptions of Theorem A. We will show that the Artin group $A(\Gamma)$ is freely cocompactly cubulated.

Let S denote the vertex set of Γ. Let S_0 denote the set of vertices with all incident edges labeled 2.

Let $S_1 = \{a_1, b_1\}, \dots, S_n = \{a_n, b_n\}$ denote the pairs of vertices of S for which the edge $a_i b_i$ has an odd label.

Let $S_{n+1}, \dots, S_{n+p}$ denote the set of vertices for which the induced graph of S_i is a star with central vertex s_i and with edges labeled with an even integer bigger than 2.

By assumption, we have $S = \bigsqcup_{0 \leqslant i \leqslant n+p} S_i$.

We will consider cube complexes with edges labeled in $\mathcal{P}(S)$, the power set of S.

Let X_0 be the Salvetti cube complex of the right-angled Artin group of the graph induced by S_0: we will recall here its construction (see [Sal87]). It has one vertex and its edge set is S_0: each edge is labeled by some element of S_0. For each simplex $T \subset S_0$, we add a $|T|$-cube, by identifying each of the $|T|$ parallel classes of edges of $[0,1]^{|T|}$ with the edges T. Then by Theorem 1.2, X_0 is locally CAT(0) cube complex. To be precise in the following construction, each edge in X_0 is labeled by some $\{s\}$, where $s \in S_0$.

For each $1 \leqslant i \leqslant n$, let X_i denote a copy of the previously constructed cube complex $X(A(p_i))$ for the subgroup generated by a_i and b_i, where p_i is odd. Label each edge of X_i by $\{a_i, b_i\}$.

For each $n+1 \leqslant i \leqslant n+p$, let X_i denote a copy of the previously constructed cube complex $X(A(S_i))$ for the subgroup generated S_i. Label the edge corresponding to the element s_i by $\{s_i\}$, and label each other edge coming from the square complex $X_{s_i}(A(\{s_i s\}))$ by $\{s_i, s\}$, for every $s \in S_i \backslash \{s_i\}$.

Consider the following cube complex X, which will be a cube subcomplex of the direct product $\prod_{i=0}^{n+p} X_i$. For each set of cubes $Q_0, \dots, Q_{n+p}$ of $X_0, \dots, X_{n+p}$ respectively, we will add the cube $Q_0 \times \cdots \times Q_{n+p}$ to X if and only if the set of labels of edges of $Q_0, \dots, Q_{n+p}$ if and only if the following holds:

$$\begin{aligned} &\forall 0 \leqslant i \neq j \leqslant n+p, \text{for any } t_i \text{ belonging to the label of some edge of } Q_i, \\ &\text{for any } t_j \text{ belonging to the label of some edge of } Q_j,\ t_i \text{ and } t_j \text{ commute.} \end{aligned}$$

Proposition 5.1. *X is a locally CAT(0) cube complex, so $A(\Gamma)$ is cocompactly cubulated.*

Proof. Let Q,Q', Q'' be cubes of X, which pairwise intersect in codimension 1, and intersect globally in codimension 2. Write $Q = \prod_{i=0}^{n+p} Q_i$, $Q' = \prod_{i=0}^{n+p} Q'_i$ and $Q'' = \prod_{i=0}^{n+p} Q''_i$.

Since Q,Q' and Q'' pairwise intersect in codimension 1, there exists a unique $k \in [\![0, n+p]\!]$ such that $\forall i \neq k, Q_i = Q'_i = Q''_i$. Furthermore, the three cubes Q_k,Q'_k and Q''_k of X_k pairwise intersect in codimension 1 and globally intersect in codimension 2. Since X_k is locally CAT(0), there exists a cube K_k in X_k such that Q_k,Q'_k and Q''_k are codimension 1 faces of K_k. Since for every $1 \leqslant i \leqslant n+p$, X_i is a square complex and K_k has dimension at least 3, we deduce that $k = 0$.

Let $K = K_0 \times \prod_{i=1}^{n+p} K_i$, where $\forall 1 \leqslant i \leqslant n+p, K_i = Q_i = Q'_i = Q''_i$.

We will check that the cube K belongs to X: fix $0 \leqslant i \neq j \leqslant n+p$, and choose t_i belonging to the label of some edge of K_i and t_j belonging to the label of some edge of K_j.

- If $i, j \neq 0$, then $K_i = Q_i$ and $K_j = Q_j$, and since Q is a cube of X, t_i and t_j commute.

- If $i = 0$ or $j = 0$, assume that $i = 0$. Then some edge of K_0 has label $\{t_0\}$. By definition of X_0, parallel edges in K_0 have the same labels, so $\{t_0\}$ is also the label of some edge of Q_0, Q_0' or Q_0'': assume that $\{t_0\}$ is the label of some edge of Q_0. Since Q is a cube of X, t_0 and t_j commute.

As a consequence, K is a cube of X. According to Theorem 1.2, X is a locally CAT(0) cube complex.

The fundamental group of X is given by its 2-skeleton, and it is the quotient of the free product of $A(\Gamma|_{S_0}), \dots, A(\Gamma|_{S_{n+p}})$ obtained by adding the following commutation relations:

$$\forall 0 \leqslant i \neq j \leqslant n+p, \text{if } s_i \in S_i \text{ and } s_j \in S_j \text{ commute in } A(\Gamma), s_i \text{ and } s_j \text{ commute in } \pi_1(X).$$

The group $\pi_1(X)$ is therefore isomorphic to $A(\Gamma)$.

As a consequence, $A(\Gamma)$ is cocompactly cubulated. □

Theorem C and Proposition 5.1 complete the proof of Theorem A. Also Proposition 4.6, Theorem 4.5 and Theorem A complete the proof of Corollary D.

6 Mapping class groups and automorphisms of free groups

We can now give the proof of Corollary F.

Proof.

- ⋆ For $n = 3$, the 3-strand braid group B_3 is cocompactly cubulated, because it acts geometrically on Brady and McCammond's complex $X(A(3))$. The central quotient of B_3 is isomorphic to $\mathrm{PSL}(2, \mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$, and acts geometrically on its Bass-Serre tree.

  ⋆ For $n \geqslant 4$, Proposition 4.6 shows that B_n and $B_n/Z(B_n)$ are not cocompactly cubulated.

- ⋆ If $3g - 3 + p \leqslant 1$ and $MCG(S_{g,p})$ is not trivial, then either $g = 0$ and $p = 4$, or $g = 1$ and $p \leqslant 1$.
  If $g = 0$ and $p = 4$, then $MCG(S_{0,4})$ surjects onto $\mathrm{PSL}(2, \mathbb{Z})$ with finite kernel, so it is cocompactly cubulated.
  If $g = 1$ and $p = 0$, then $MCG(S_{1,0}) = MCG(\mathbb{T}^2) \simeq \mathrm{SL}(2, \mathbb{Z})$, so it is cocompactly cubulated.
  If $g = 1$ and $p = 1$, then $MCG(S_{1,1}) \simeq B_3$, so it is cocompactly cubulated.

  ⋆ If $3g - 3 + p \geqslant 2$, assume first that $g \geqslant 1$. Then there exists 3 simple closed (oriented) curves γ_1, γ_2, and γ_3 such that γ_1 and γ_2 intersect in one point, γ_2 and γ_3 intersect in one point, and γ_1 and γ_3 are disjoint. Then the subgroup of $MCG(S_{g,p})$ spanned by the Dehn twists around γ_1, γ_2, and γ_3 is isomorphic to the 4-strand braid group B_4 (see for instance [PV96]). So by Proposition 4.6, $MCG(S_{g,p})$ is not cocompactly cubulated.

  Assume now that $g = 0$, and so $p \geqslant 5$. Then the subgroup of $MCG(S_{0,p})$ fixing one puncture is isomorphic to the $(p-1)$-strand braid group B_{p-1}. Since $p - 1 \geqslant 4$, by Proposition 4.6, $MCG(S_{0,p})$ is not cocompactly cubulated.

- If $n = 2$, then by [KR07] $Aut(\mathbb{F}_2)$ has an index 2 subgroup isomorphic to $B_4/Z(B_4)$, so by Proposition 4.6 it is not cocompactly cubulated.

  If $n \geqslant 3$, then B_{n+1} is a subgroup of $Aut(\mathbb{F}_n)$ (see [PV96]), so by Proposition 4.6 it is not cocompactly cubulated.

- If $n = 2$, $Out(\mathbb{F}_2) \simeq \mathrm{GL}(2, \mathbb{Z})$ is cocompactly cubulated.

  If $n \geqslant 3$, we see $\mathbb{F}_n$ as the fundamental group of the $(n-1)$-times punctured torus $S_{1,n-1}$. Since B_4 is a subgroup of $MCG(S_{1,n-1})$, which itself is a subgroup of $\mathrm{Out}(\mathbb{F}_n)$, we deduce by Proposition 4.6 that $Out(\mathbb{F}_n)$ is not cocompactly cubulated.

□

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