DERIVED CATEGORIES AND DELIGNE-LUSZTIG VARIETIES II

by

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Abstract. **—** This paper is a continuation and a completion of [**BoRo1**]. We extend the Jordan decomposition of blocks: we show that blocks of finite groups of Lie type in non-describing characteristic are Morita equivalent to blocks of subgroups associated to isolated elements of the dual group — this is the modular version of a fundamental result of Lusztig, and the best approximation of the character-theoretic Jordan decomposition that can be obtained via Deligne-Lusztig varieties. The key new result is the invariance of the part of the cohomology in a given modular series of Deligne-Lusztig varieties associated to a given Levi subgroup, under certain variations of parabolic subgroups.

We also bring in local block theory methods: we show that the equivalence arises from a splendid Rickard equivalence. Even in the setting of [**BoRo1**], the finer homotopy equivalence was unknown. As a consequence, the equivalences preserve defect groups and categories of subpairs. We finally determine when Deligne-Lusztig induced representations of tori generate the derived category of representations. An additional new feature is an extension of the results to disconnected reductive algebraic groups, which is required to handle local subgroups.

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1. Introduction

Let $\mathbf{G}$ be a connected reductive algebraic group endowed with an endomorphism F, a power of which is a Frobenius endomorphism. Let ℓ be a prime number distinct from the defining characteristic of $\mathbf{G}$ and K a finite extension of $\mathbb{Q}_\ell$, large enough for the finite groups considered. Let $\mathcal{O}$ be the ring of integers of K over $\mathbb{Z}_\ell$ and k the residue field. We will denote by Λ a ring that is either K, $\mathcal{O}$ or k.

The main tool for the study of representations of $\mathbf{G}^F$ over Λ is the Deligne-Lusztig induction. Let $\mathbf{L}$ be an F-stable Levi subgroup of $\mathbf{G}$ contained in a parabolic subgroup $\mathbf{P}$ with unipotent radical $\mathbf{V}$ so that $\mathbf{P} = \mathbf{V} \rtimes \mathbf{L}$. Consider the Deligne-Lusztig variety

$$\mathbf{Y_P} = \{g\mathbf{V} \in \mathbf{G}/\mathbf{V} \mid g^{-1}F(g) \in \mathbf{V} \cdot F(\mathbf{V})\}.$$

It has a left action of $\mathbf{G}^F$ and a right action of $\mathbf{L}^F$ by multiplication. The corresponding complex of ℓ-adic cohomology induces a triangulated functor

$$\mathscr{R}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} : D^b(\Lambda\mathbf{L}^F) \to D^b(\Lambda\mathbf{G}^F),\ M \mapsto R\Gamma_c(\mathbf{Y_P}, \Lambda) \otimes^{\mathbb{L}}_{\Lambda\mathbf{L}^F} M$$

and a morphism

$$R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} = [\mathscr{R}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}] : G_0(\Lambda\mathbf{L}^F) \to G_0(\Lambda\mathbf{G}^F).$$

This is the usual Harish-Chandra construction when $\mathbf{P}$ is F-stable.

1.A. Jordan decomposition. — Let $\mathbf{G}^*$ be a group Langlands dual to $\mathbf{G}$, with Frobenius F^*. Consider the set $\mathrm{Irr}(\mathbf{G}^F)$ of characters of irreducible representations of $\mathbf{G}^F$ over K. Deligne and Lusztig gave a decomposition of $\mathrm{Irr}(\mathbf{G}^F)$ into rational series

$$\mathrm{Irr}(\mathbf{G}^F) = \coprod_{(s)} \mathrm{Irr}(\mathbf{G}^F, (s))$$

where (s) runs over the set of conjugacy classes of semi-simple elements of $(\mathbf{G}^*)^{F^*}$. The *unipotent characters* of $\mathbf{G}^F$ are those in $\mathrm{Irr}(\mathbf{G}^F, 1)$.

Let $\mathbf{L}$ be an F-stable Levi subgroup of $\mathbf{G}$ with dual $\mathbf{L}^* \subset \mathbf{G}^*$ containing $C_{\mathbf{G}^*}(s)$. Lusztig constructed a bijection

$$\mathrm{Irr}(\mathbf{L}^F, (s)) \xrightarrow{\sim} \mathrm{Irr}(\mathbf{G}^F, (s)),\ \psi \mapsto \pm R^{\mathbf{G}}_{\mathbf{L}}(\psi).$$

If $s \in Z(\mathbf{L}^*)$, then there is a bijection

$$\mathrm{Irr}(\mathbf{L}^F, (1)) \xrightarrow{\sim} \mathrm{Irr}(\mathbf{L}^F, (s)),\ \psi \mapsto \eta\psi$$

where η is the one-dimensional character of $\mathbf{L}^F$ corresponding to s, and we obtain a bijection

$$\mathrm{Irr}(\mathbf{L}^F, (1)) \xrightarrow{\sim} \mathrm{Irr}(\mathbf{G}^F, (s)).$$

This provides a description of irreducible characters of $\mathbf{G}^F$ in the rational series (s) in terms of unipotent characters of an other group, when $C_{\mathbf{G}^*}(s)$ is a Levi subgroup of $\mathbf{G}^*$.

Let us now consider the modular version of the theory described above. Let s be a semi-simple element of $\mathbf{G}^{*F}$ of order prime to ℓ. Consider $\coprod_t \mathrm{Irr}(\mathbf{G}^F, (t))$, where (t) runs over conjugacy classes of semi-simple elements of $(\mathbf{G}^*)^{F^*}$ whose ℓ'-part is (s). Broué and Michel **[BrMi]** have show this is a union of blocks of $\mathscr{O}\mathbf{G}^F$. The sum of the corresponding block idempotents is an idempotent $e_s^{\mathbf{G}^F} \in Z(\mathcal{O}\mathbf{G}^F)$, and we obtain a decomposition

$$\mathcal{O}\mathbf{G}^F\text{-mod} = \bigoplus_{(s)} \mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}\text{-mod}$$

where (s) runs over conjugacy classes of semi-simple ℓ'-elements of $\mathbf{G}^{*F}$.

Let $\mathbf{L}$ be an F-stable Levi subgroup of $\mathbf{G}$ with dual $\mathbf{L}^*$ containing $C_{\mathbf{G}^*}(s)$. Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$ with unipotent radical $\mathbf{V}$ and Levi complement $\mathbf{L}$. Broué [**Br2**] conjectured that the $(\mathcal{O}\mathbf{G}^F, \mathcal{O}\mathbf{L}^F)$-bimodule $\mathrm{H}^{\dim \mathbf{Y_P}}(\mathbf{Y_P}, \mathcal{O})e_s^{\mathbf{L}^F}$ gives a Morita equivalence between $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$ and $\mathcal{O}\mathbf{L}^F e_s^{\mathbf{L}^F}$. This was proven by Broué [**Br2**] when $\mathbf{L}$ is a torus and in [**BoRo1**] in general.

Broué also conjectured that the truncated complex of cohomology $\mathrm{G}\Gamma_c(\mathbf{Y_V}, \mathcal{O})e_s^{\mathbf{L}^F}$ (well defined by Rickard [**Ri**] in the homotopy category) induces a splendid Rickard equivalence between $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$ and $\mathcal{O}\mathbf{L}^F e_s^{\mathbf{L}^F}$: it induces not only an equivalence of derived categories, but even an equivalence of homotopy categories, and it induces a similar equivalence for centralizers of ℓ-subgroups. One of our main results here is a proof of that conjecture. In order to show that there is a homotopy equivalence, for connected groups, we show that the global functor induces local derived equivalences for centralizers of ℓ-subgroups. Since such centralizers need not be connected, we need to extend the results of [**BoRo1**] to disconnected groups. So, part of this work involves working with disconnected groups.

We also extend the "Jordan decomposition equivalences" (Morita and splendid Rickard) to the "quasi-isolated case": assume now only $C^\circ_{\mathbf{G}^*}(s) \subset \mathbf{L}^*$. We show that the right action of $\mathbf{L}^F$ on $\mathrm{H}^{\dim \mathbf{Y_P}}(\mathbf{Y_P}, \mathcal{O})e_s^{\mathbf{L}^F}$ extends to an action of $N = N_{\mathbf{G}^F}(\mathbf{L}, e_s^{\mathbf{L}^F})$ commuting with the action of $\mathbf{G}^F$, and the resulting bimodule induces a Morita equivalence between $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$ and $\mathcal{O}N e_s^{\mathbf{L}^F}$. Similarly, we obtain a splendid Rickard equivalence between $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$ and $\mathcal{O}N e_s^{\mathbf{L}^F}$.

As a consequence, we deduce that the bijection between blocks of $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$ and $\mathcal{O}N e_s^{\mathbf{L}^F}$ preserves the local structure, and in particular, preserves defect groups. Kessar and Malle have proven this in the setting of [**BoRo1**], when one of the blocks under consideration has abelian defect groups (modulo a central ℓ-subgroup) [**KeMa1**, Theorem 1.3], an important step in their proof of half of Brauer's height zero conjecture for all finite groups [**KeMa1**, Theorem 1.1] and the second half for quasi-simple groups [**KeMa2**, Main Theorem].

Let us summarize this.

Theorem 1.1. — *Assume* $C^\circ_{\mathbf{G}^*}(s) \subset \mathbf{L}^*$.

The right action of $\mathbf{L}^F$ *on* $\mathrm{G}\Gamma_c(\mathbf{Y_V}, \mathcal{O})e_s^{\mathbf{L}^F}$ *extends to an action of* N *and the resulting complex* C *induces a splendid Rickard equivalence between* $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$ *and* $\mathcal{O}N e_s^{\mathbf{L}^F}$. *The bimodule* $\mathrm{H}^{\dim \mathbf{Y_P}}(C)$ *induces a Morita equivalence between* $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$ *and* $\mathcal{O}N e_s^{\mathbf{L}^F}$.

The bijections between blocks of $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$ and $\mathcal{O}N e_s^{\mathbf{L}^F}$ induced by those equivalences preserve the local structure.

Significant progress has been made recently on counting conjectures for finite groups, using the classification of finite simple groups, and [**BoRo1**] has proved useful. We hope this theorem will lead to simplifications and new results.

The character-theoretic consequence of this theorem is that, for groups with disconnected center, the Jordan decomposition shares many of the properties of that for the connected case (commutation with Deligne-Lusztig induction for example). In type A, the Jordan decomposition of characters links all series to unipotent series of smaller groups: even in that case, the good behaviour of those correspondences was known only when q is large (Bonnafé [**Bo3**] for SL and Cabanes [**Ca**] for SU).

1.B. Generation of the derived category. — One of the two key steps in [**BoRo1**] was the proof that the category of perfect complexes for $\mathcal{O}\mathbf{G}^F$ is generated by the complexes $\mathrm{R}\Gamma_c(\mathbf{Y}_\mathbf{B})$, where $\mathbf{B}$ runs over Borel subgroups of $\mathbf{G}$ with an F-stable maximal torus. We show here a more precise result of generation of the derived category of $\mathcal{O}\mathbf{G}^F$. Let $\mathcal{E}$ be the set $\{\mathrm{R}\Gamma_c(\mathbf{Y}_\mathbf{B})\otimes^{\mathbf{L}}_{\mathcal{O}\mathbf{T}^F} M\}$, where $\mathbf{T}$ runs over F-stable maximal tori of $\mathbf{G}$, $\mathbf{B}$ over Borel subgroups of $\mathbf{G}$ containing $\mathbf{T}$, and M over isomorphism classes of $\mathcal{O}\mathbf{T}^F$-modules.

Theorem 1.2. — *The set $\mathcal{E}$ generates $D^b(\mathcal{O}\mathbf{G}^F)$ (as a thick subcategory) if and only if all elementary abelian ℓ-subgroups of $\mathbf{G}^F$ are contained in tori.*

This, in turn, requires an extension of the results of Broué-Michel [**BrMi**] on the compatibility between Deligne-Lusztig series of characters and the Brauer morphism, to disconnected groups. We are able to achieve this by refining our result on the generation of the category of perfect complexes to a generation of the category of ℓ-permutation modules whose vertices are contained in tori (the crucial case is that of connected groups). Such a result allows us to obtain a generating result for the full derived category, under the assumption that all elementary abelian ℓ-subgroups are contained in tori.

Note that the condition on elementary abelian ℓ-subgroups is automatically satisfied if $\mathbf{G}=\mathbf{GL}_n(\mathbb{F})$ or if ℓ is *very good* for $\mathbf{G}$.

1.C. Independence of the Deligne-Lusztig induction of the parabolic in a given series. — It is known in most cases, and conjectured in general, that the map $R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}$ is actually independent of $\mathbf{P}$ ([**DeLu, Lu2**] when $\mathbf{L}$ is a torus and [**BoMi**] when $q>2$ and F is a Frobenius endomorphism over $\mathbf{F}_q$). On the other hand, the functor $\mathcal{R}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}$ does depend on $\mathbf{P}$. Our main new geometrical result proves the independence after truncating by a suitable series.

Let $\mathbf{P}_1$ and $\mathbf{P}_2$ be two parabolic subgroups admitting a common Levi complement $\mathbf{L}$. Denote by $\mathbf{V}_i^*$ the unipotent radical of the parabolic subgroup of $\mathbf{G}^*$ corresponding to $\mathbf{P}_i$.

Theorem 1.3. — *Let s be a semi-simple element of $\mathbf{L}^{*F}$ of order prime to ℓ. If*

$$C_{\mathbf{V}_1^*\cap {}^{F^*}\mathbf{V}_1^*}(s) \subset C_{\mathbf{V}_2^*}(s) \quad \textit{and} \quad C_{\mathbf{V}_2^*\cap {}^{F^*}\mathbf{V}_2^*}(s) \subset C_{{}^{F^*}\mathbf{V}_1^*}(s)$$

then there is an isomorphism of functors between

$$\mathscr{R}_{\mathbf{L}\subset\mathbf{P}_1}^{\mathbf{G}} : \mathrm{D}^b(\Lambda\mathbf{L}^F e_s^{\mathbf{L}^F}) \longrightarrow \mathrm{D}^b(\Lambda\mathbf{G}^F e_s^{\mathbf{G}^F})$$

and $$\mathscr{R}_{\mathbf{L}\subset\mathbf{P}_2}^{\mathbf{G}}[m] : \mathrm{D}^b(\Lambda\mathbf{L}^F e_s^{\mathbf{L}^F}) \longrightarrow \mathrm{D}^b(\Lambda\mathbf{G}^F e_s^{\mathbf{G}^F}),$$

where $m = \dim(\mathbf{Y}_{\mathbf{P}_2}^{\mathbf{G}}) - \dim(\mathbf{Y}_{\mathbf{P}_1}^{\mathbf{G}})$.

This is the key result to prove Theorem 1.1. This result shows that when $C_{\mathbf{G}^*}^\circ(s) \subset \mathbf{L}^*$, the $(\mathscr{O}\mathbf{G}^F, \mathscr{O}\mathbf{L}^F)$-bimodule $\mathrm{H}^{\dim \mathbf{Y}_\mathbf{P}}(\mathbf{Y}_\mathbf{P}, \mathscr{O})e_s^{\mathbf{L}^F}$ is independent of $\mathbf{P}$, a question left open in **[BoRo1]**. We deduce that the bimodule is stable under the action of $N = N_{\mathbf{G}^F}(\mathbf{L}^F, e_s^{\mathbf{L}^F})$. Using an embedding in a group with connected center, we show that the obstruction for extending the action of $\mathbf{L}^F$ to N does vanish.

1.D. Structure of the article. — We begin in §3 with the study of generation of the category of perfect complexes, then we move to complexes of ℓ-permutation modules and finally we derive our result on the derived category. A key tool, due to Rickard, is that the Brauer functor applied to the complex of cohomology of a variety is the complex of cohomology of the fixed point variety.

Section §4 is devoted to the study of rational series and their compatibility with local block theory. Broué and Michel proved a commutation formula between generalized decomposition maps and Deligne-Lusztig induction. We need to extend the compatibility between Brauer and Deligne-Lusztig theory to disconnected groups, and check that the local blocks obtained from a series satisfying $C_{\mathbf{G}^*}^\circ(s) \subset \mathbf{L}^*$ also satisfy a similar assumption $C^\circ_{(C^\circ_{\mathbf{G}}(Q))^*}(s) \subset (\mathbf{L} \cap C_{\mathbf{G}}^\circ(Q))^*$.

From §5 onwards, the group $\mathbf{G}$ is assumed to be connected. Sections §5 and §6 are devoted to the study of the dependence of the Deligne-Lusztig induction of the parabolic subgroup. The first section is devoted to the particular case of varieties associated with Borel subgroups (and generalizations involving sequences of elements). It is convenient there to work with a reference torus. This is the crucial case, from which the general one is deduced in the second section, where we go back to non-standard Levi subgroups.

The final section §7 is devoted to the Jordan decomposition. We start by providing an extension of the action of N on the cohomology bimodule by proving that the cocycle obstruction would survive in a similar setting for a group with connected center, where the action does exist. The Rickard equivalence is obtained inductively, and that induction requires working with disconnected groups.

In an appendix, we provide some results on the homotopy category of complexes of ℓ-permutation modules for a general finite group.

2. Notations

2.A. Modules. — Let ℓ be a prime number, K a finite extension of $\mathbb{Q}_\ell$ large enough for the finite groups considered, $\mathcal{O}$ its ring of integers over $\mathbb{Z}_\ell$ and k its residue field. We will denote by Λ a ring that is either K, $\mathcal{O}$ or k.

Given $\mathscr{C}$ an additive category, we denote by $\mathrm{Comp}^b(\mathscr{C})$ the category of bounded complexes of objects of $\mathscr{C}$ and by $\mathrm{Ho}^b(\mathscr{C})$ its homotopy category.

Let A be a Λ-algebra, finitely generated and projective as a Λ-module. We denote by A^{opp} the algebra opposite to A. We denote by A-mod the category of finitely generated A-modules and by A-proj its full subcategory of projective modules. We denote by $G_0(A)$ the Grothendieck group of A-mod.

We put $\mathrm{Comp}^b(A) = \mathrm{Comp}^b(A\text{-mod})$, $D^b(A) = D^b(A\text{-mod})$ and $\mathrm{Ho}^b(A) = \mathrm{Ho}^b(A\text{-mod})$. We denote by $A\text{-perf} \subset D^b(A)$ the thick full subcategory of perfect complexes (complexes quasi-isomorphic to objects of $\mathrm{Comp}^b(A\text{-proj})$).

Let $C \in \mathrm{Comp}^b(A)$. There is a unique (up to a non-unique isomorphism) complex C^{red} which is isomorphic to C in the homotopy category $\mathrm{Ho}^b(A)$ and which has no non-zero direct summand that is homotopy equivalent to 0. Note that $C \simeq C^{\mathrm{red}} \oplus C'$ for some C' homotopy equivalent to zero.

We denote by $\mathrm{End}^\bullet_A(C)$ the total Hom-complex, with degree n term $\bigoplus_{j-i=n} \mathrm{Hom}_A(C^i, C^j)$.

Let B be Λ-algebra, finitely generated and projective as a Λ-module. Let C be a bounded complex of $(A \otimes_\Lambda B^{\mathrm{opp}})$-modules, finitely generated and projective as left A-modules and as right B-modules. We say that C induces a *Rickard equivalence* between A and B if the canonical map $B \to \mathrm{End}^\bullet_A(C)$ is an isomorphism in $\mathrm{Ho}(B \otimes_\Lambda B^{\mathrm{opp}})$ and the canonical map $A \to \mathrm{End}^\bullet_{B^{\mathrm{opp}}}(C)^{\mathrm{opp}}$ is an isomorphism in $\mathrm{Ho}(A \otimes_\Lambda A^{\mathrm{opp}})$.

2.B. Finite groups. — Let G be a finite group. We denote by G^{opp} the opposite group to G. We put $\Delta G = \{(g, g^{-1}) | g \in G\} \subset G \times G^{\mathrm{opp}}$. Given $g \in G$, we denote by $|g|$ the order of g.

Let H be a subgroup of G and $x \in G$. We denote by x_* the equivalence of categories

$$x_* : \Lambda(x^{-1} H x)\text{-mod} \xrightarrow{\sim} \Lambda H\text{-mod}$$

where $x_*(M) = M$ as a Λ-module and the action of $h \in H$ on $x_*(M)$ is given by the action of $x^{-1}hx$ on M. We also denote by x_* the corresponding isomorphism of Grothendieck groups

$$x_* : G_0(\Lambda(x^{-1}Hx)) \xrightarrow{\sim} G_0(\Lambda H).$$

We assume $\Lambda = \mathcal{O}$ or $\Lambda = k$ in the remainder of §2.B.

An *ℓ-permutation ΛG-module* is defined to be a direct summand of a finitely generated permutation module. We denote by ΛG-perm the full subcategory of ΛG-mod with objects the ℓ-permutation ΛG-modules.

Let Q be an ℓ-subgroup Q of G. We consider the *Brauer functor* $\mathrm{Br}_Q : \Lambda G\text{-perm} \to k[N_G(Q)/Q]\text{-perm}$. Given $M \in \Lambda G$-perm, we define $\mathrm{Br}_Q(M)$ as the image of M^Q in $(kM)_Q$, where $(kM)_Q$ is the largest quotient of $kM = k \otimes_\Lambda M$ on which Q acts trivially.

We denote by $\mathrm{br}_Q : (\Lambda G)^Q \to kC_G(Q)$ the algebra morphism given by $\mathrm{br}_Q(\sum_{g\in G} \lambda_g g) = \sum_{g\in C_G(Q)} \lambda_g g$ where $\lambda_g \in \Lambda$ for $g \in G$. Given $M \in \Lambda G$-perm and $e \in Z(\Lambda G)$ an idempotent, we have $\mathrm{Br}_Q(Me) = \mathrm{Br}_Q(M)\mathrm{br}_Q(e)$.

Let H be a subgroup of G, let b be an idempotent of $Z(\Lambda G)$ and c an idempotent of $Z(\Lambda H)$. Let $C \in \mathrm{Comp}^b(\Lambda G b \otimes (\Lambda H c)^{\mathrm{opp}})$. We say that C is *splendid* if the C^i's are ℓ-permutation modules whose indecomposable direct summands have a vertex contained in ΔH.

2.C. Varieties. — Let p be a prime number different from ℓ and $\mathbb{F}$ an algebraic closure of $\mathbb{F}_p$. By variety, we mean a quasi-projective algebraic variety over $\mathbb{F}$.

Let $\mathbf{X}$ be a variety acted on by a finite group G. We denote by $\mathrm{G\Gamma}_c(\mathbf{X}, \Lambda)$ the complex of étale Λ-cohomology with compact support of $\mathbf{X}$ constructed as $\tau_{\leqslant 2\dim \mathbf{X}}$ of the Godement resolution (cf [**Rou1**, §2], [**DuRou**, §1.2], and [**Ri**]). This is an object of $\mathrm{Ho}^b(\Lambda G\text{-perm})$, well defined up to a unique isomorphism. We will denote by $\mathrm{R\Gamma}_c(\mathbf{X}, \Lambda)$ its image in $D^b(\Lambda G)$.

Assume $\Lambda = \mathcal{O}$ or k and let Q be an ℓ-subgroup of G. The inclusion $\mathbf{X}^Q \hookrightarrow \mathbf{X}$ induces an isomorphism [**Ri**, Theorem 4.2]

$$\mathrm{G\Gamma}_c(\mathbf{X}^Q, k) \xrightarrow{\sim} \mathrm{Br}_Q(\mathrm{G\Gamma}_c(\mathbf{X}, \Lambda)) \text{ in } \mathrm{Ho}^b(kN_G(Q)\text{-perm}).$$

2.D. Reductive groups. — Let $\mathbf{G}$ be a (possibly disconnected) reductive algebraic group endowed with an endomorphism F, a power F^δ of which is a Frobenius endomorphism defining a rational structure over a finite field $\mathbb{F}_q$ of characteristic p. We refer to [**DigMi2, DigMi3**] for basic results on disconnected groups.

A *parabolic subgroup* $\mathbf{P}$ of $\mathbf{G}$ is a subgroup containing a parabolic subgroup $\mathbf{P}_\circ$ of $\mathbf{G}^\circ$ and normalizing $\mathbf{P}_\circ$ (then $\mathbf{P}^\circ = \mathbf{P}_\circ$). Let $\mathbf{V}$ be the unipotent radical of $\mathbf{P}_\circ$. A *Levi complement* to $\mathbf{V}$ in $\mathbf{P}$ is a subgroup of $\mathbf{P}$ of the form $N_{\mathbf{P}}(\mathbf{L}_\circ)$, where $\mathbf{L}_\circ$ is a Levi complement of $\mathbf{V}$ in $\mathbf{P}_\circ$ (then $\mathbf{L}^\circ = \mathbf{L}_\circ$ and $\mathbf{P} = \mathbf{V} \rtimes \mathbf{L}$).

We denote by $\nabla(\mathbf{G}, F)$ the set of pairs $(\mathbf{T}, \theta)$ where $\mathbf{T}$ is an F-stable maximal torus of $\mathbf{G}$ and θ is an irreducible character of $\mathbf{T}^F$.

Given an integer d, we denote by $\nabla_{d'}(\mathbf{G}, F)$ the set of pairs $(\mathbf{T}, \theta) \in \nabla(\mathbf{G}, F)$ such that the order of θ is prime to d. We put $\nabla_\Lambda(\mathbf{G}, F) = \nabla(\mathbf{G}, F)$ if $\Lambda = K$ and $\nabla_\Lambda(\mathbf{G}, F) = \nabla_{\ell'}(\mathbf{G}, F)$ if $\Lambda = \mathscr{O}$ or k.

2.E. Deligne-Lusztig varieties. — Given $\mathbf{P}$ a parabolic subgroup of $\mathbf{G}$ with unipotent radical $\mathbf{V}$ and F-stable Levi complement $\mathbf{L}$, we define the Deligne-Lusztig variety

$$\mathbf{Y}_\mathbf{V} = \mathbf{Y}_\mathbf{V}^\mathbf{G} = \mathbf{Y}_\mathbf{P} = \mathbf{Y}_\mathbf{P}^\mathbf{G} = \{g\mathbf{V} \in \mathbf{G}/\mathbf{V} \mid g^{-1}F(g) \in \mathbf{V} \cdot F(\mathbf{V})\}.$$

This is a smooth variety, with a left action by multiplication of $\mathbf{G}^F$ and a right action by multiplication of $\mathbf{L}^F$ (note that the left and right actions of $Z(\mathbf{G})^F$ coincide). This provides a triangulated functor

$$\begin{array}{rccc} \mathscr{R}_{\mathbf{L}\subset\mathbf{P}}^\mathbf{G}: & D^b(\Lambda\mathbf{L}^F) & \longrightarrow & D^b(\Lambda\mathbf{G}^F) \\ & M & \longmapsto & R\Gamma_c(\mathbf{Y}_\mathbf{V}, \Lambda) \otimes^{\mathbb{L}}_{\Lambda\mathbf{L}^F} M \end{array} \tag{2.1}$$

and a morphism

$$R_{\mathbf{L}\subset\mathbf{P}}^\mathbf{G} = [\mathscr{R}_{\mathbf{L}\subset\mathbf{P}}^\mathbf{G}] : G_0(\Lambda\mathbf{L}^F) \to G_0(\Lambda\mathbf{G}^F).$$

We put $\mathbf{X}_\mathbf{P}^\mathbf{G} = \{g\mathbf{P} \in \mathbf{G}/\mathbf{P} \mid g^{-1}F(g) \in \mathbf{P} \cdot F(\mathbf{P})\} = \mathbf{Y}_\mathbf{P}^\mathbf{G}/\mathbf{L}^F$.

3. Generation

The aim of this section is to extend [**BoRo1**, Theorem A] to the case of disconnected groups, and to deduce a generation theorem for the derived category.

In this section §3, $\mathbf{G}$ is a (possibly disconnected) reductive algebraic group.

3.A. Centralizers of ℓ-subgroups. — Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$ admitting an F-stable Levi complement $\mathbf{L}$, and let $\mathbf{V}$ denote the unipotent radical of $\mathbf{P}$. It is easily checked [**DigMi2**, Proof of Proposition 2.3] that

$$\mathbf{Y}_\mathbf{V}^\mathbf{G} = \coprod_{g \in \mathbf{G}^F/\mathbf{G}^{\circ F}} g\mathbf{Y}_\mathbf{V}^{\mathbf{G}^\circ}. \tag{3.1}$$

It follows immediately from (3.1) that

$$\mathscr{R}_{\mathbf{L}\subset\mathbf{P}}^\mathbf{G} \circ \operatorname{Ind}_{\mathbf{L}^{\circ F}}^{\mathbf{L}^F} \simeq \mathscr{R}_{\mathbf{L}^\circ\subset\mathbf{P}^\circ}^\mathbf{G} \simeq \operatorname{Ind}_{\mathbf{G}^{\circ F}}^{\mathbf{G}^F} \circ \mathscr{R}_{\mathbf{L}^\circ\subset\mathbf{P}^\circ}^{\mathbf{G}^\circ}. \tag{3.2}$$

If $\mathbf{G} = \mathbf{P} \cdot \mathbf{G}^\circ$, then we have

$$\mathscr{R}_{\mathbf{L}^\circ\subset\mathbf{P}^\circ}^{\mathbf{G}^\circ} \circ \operatorname{Res}_{\mathbf{L}^{\circ F}}^{\mathbf{L}^F} \simeq \operatorname{Res}_{\mathbf{G}^{\circ F}}^{\mathbf{G}^F} \circ \mathscr{R}_{\mathbf{L}\subset\mathbf{P}}^\mathbf{G}. \tag{3.3}$$

Proposition 3.4. — *Let Q be an ℓ-subgroup of $\mathbf{L}^F$. Then:*

(a) *The group $C_{\mathbf{G}}(Q)$ is reductive.*

(b) *$C_{\mathbf{P}}(Q)$ is a parabolic subgroup of $C_{\mathbf{G}}(Q)$ whose unipotent radical is $C_{\mathbf{V}}(Q)$ and admitting $C_{\mathbf{L}}(Q)$ as an F-stable Levi complement. In particular, $C_{\mathbf{V}}(Q)$ is connected.*

(c) *The natural map $C_{\mathbf{G}}(Q)/C_{\mathbf{V}}(Q) \to (\mathbf{G}/\mathbf{V})^{\Delta Q}$ is an isomorphism of $(C_{\mathbf{G}}(Q), C_{\mathbf{L}}(Q))$-varieties.*

(d) $(\mathbf{V} \cdot {}^F\mathbf{V})^{\Delta Q} = C_{\mathbf{V}}(Q) \cdot {}^F C_{\mathbf{V}}(Q)$.

(e) *The natural map $\mathbf{Y}_{C_{\mathbf{V}}(Q)}^{C_{\mathbf{G}}(Q)} \to (\mathbf{Y}_{\mathbf{V}}^{\mathbf{G}})^{\Delta Q}$ is an isomorphism of $(C_{\mathbf{G}}(Q)^F, C_{\mathbf{L}}(Q)^F)$-varieties.*

Proof. — Every non-trivial finite ℓ-group contains a non-trivial normal central subgroup. So an easy induction argument shows that it is enough to prove all the statements of this proposition whenever Q is cyclic. So let $l \in \mathbf{L}^F$ be an ℓ-element and let $Q = \langle l \rangle$.

(a) and (b) follow from [**DigMi2**, Proposition 1.3, Theorem 1.8, Proposition 1.11].

(c) Note that that both varieties are smooth (for $(\mathbf{G}/\mathbf{V})^{\Delta Q}$, this follows from the fact that Q is a p'-group and $\mathbf{G}/\mathbf{V}$ is smooth). The injectivity of the map is clear.

Let us prove the surjectivity. Let $g\mathbf{V} \in (\mathbf{G}/\mathbf{V})^{\Delta Q}$. Then, $g^{-1}lgl^{-1} \in \mathbf{V}$ or, in other words, $g^{-1}lg \in \mathbf{V}l = l\mathbf{V}$. But $g^{-1}lg$ is an ℓ-element, so it is semisimple, hence it normalises a maximal torus of $\mathbf{P}^\circ$ (see [**St**, Theorem 7.5]). We deduce that $g^{-1}lg$ belongs to the unique Levi complement $\mathbf{L}'$ of $\mathbf{P}$ containing this maximal torus (see [**St**, Theorem 7.5]). Since all Levi complements are conjugate under the action of $\mathbf{V}$, there exists $v \in \mathbf{V}$ such that $v^{-1}g^{-1}lgv \in \mathbf{L}$. But $v^{-1}g^{-1}lgv \in l\mathbf{V}$, so $gv \in C_{\mathbf{G}}(l) = C_{\mathbf{G}}(Q)$, as desired.

The tangent space at $\mathbf{V}$ of $(\mathbf{G}/\mathbf{V})^{\Delta Q}$ is the ΔQ-invariant part of the tangent space of $\mathbf{G}/\mathbf{V}$ at $\mathbf{V}$. That last tangent space is a quotient of the tangent space of $\mathbf{G}$ at the origin. It follows that the canonical map $C_{\mathbf{G}}(Q) = \mathbf{G}^{\Delta Q} \to (\mathbf{G}/\mathbf{V})^{\Delta Q}$ induces a surjective map between tangent spaces at the origin. Consequently, the canonical map $C_{\mathbf{G}}(Q)/C_{\mathbf{V}}(Q) \to (\mathbf{G}/\mathbf{V})^{\Delta Q}$ induces a surjective map between tangent spaces at the origin. We deduce that the map is an isomorphism.

(d) The number of F-stable maximal tori of $\mathbf{L}$ is a power of p (see [**St**, Corollary 14.16]). Since Q is an ℓ-group, it normalizes some F-stable maximal torus. Using now the root system with respect to this maximal torus, we deduce that there exists a Q-stable subgroup $\mathbf{V}'$ of $\mathbf{V}$ such that $\mathbf{V} = \mathbf{V}' \cdot (\mathbf{V} \cap F(\mathbf{V}))$ and $\mathbf{V}' \cap F(\mathbf{V}) = 1$. Therefore, $\mathbf{V} \cdot F(\mathbf{V}) = \mathbf{V}' \cdot F(\mathbf{V})$ and the result follows.

(e) follows immediately from (c) and (d). □

To complete the previous proposition, note the following result.

Lemma 3.5. — *Let P be an ℓ-subgroup of $\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}$ such that $(\mathbf{Y}_{\mathbf{V}}^{\mathbf{G}})^P \neq \varnothing$. Then there exists an ℓ-subgroup Q of $\mathbf{L}^F$ such that P and ΔQ are conjugate under the action of $\mathbf{G}^F \times 1$.*

Proof. — Let $Q \subset \mathbf{L}^F$ (respectively $R \subset \mathbf{G}^F$) denote the image of P through the second (respectively first) projection and let $y\mathbf{V} \in (\mathbf{Y}_{\mathbf{V}}^{\mathbf{G}})^P$.

If $g \in R$, then there exists $l \in Q$ such that $(g, l) \in P$. Therefore, $g\,y\,l\mathbf{V} = y\mathbf{V}$, hence $y^{-1}g\,y\mathbf{V} = l^{-1}\mathbf{V}$. This implies that $y^{-1}Ry \subset Q\mathbf{V}$. We denote by $\eta : R \to Q$ the composition $R \xrightarrow{\sim} y^{-1}Ry \hookrightarrow Q\mathbf{V} \twoheadrightarrow Q$. Since R (respectively Q) acts freely on $\mathbf{G}/\mathbf{V}$, the previous computation shows that η is an isomorphism, and that

$$P = \{(g, \eta(g)) \mid g \in R\}.$$

Now, there exists a positive integer m such that $F^m(\mathbf{P}) = \mathbf{P}$ and $y^{-1}Ry \subset \mathbf{P}^{F^m}$. So $y^{-1}Ry$ acts by left translation on $\mathbf{P}^{F^m}/\mathbf{L}^{F^m}$. Since $y^{-1}Ry$ is a finite ℓ-group and $|\mathbf{P}^{F^m}/\mathbf{L}^{F^m}| = |\mathbf{V}^{F^m}|$ is a power of p, it follows that $y^{-1}Ry$ has a fixed point in $\mathbf{P}^{F^m}/\mathbf{L}^{F^m}$. Consequently, there exists $v \in \mathbf{V}$ such that $y^{-1}l\,y\,v\mathbf{L} = v\mathbf{L}$ for all $l \in R$. In other words, $(y\,v)^{-1}R(y\,v) \subset \mathbf{L}$. This means that, by replacing y by $y\,v$ if necessary, we may assume that $y^{-1}Ry \subset \mathbf{L}$. Therefore, $y^{-1}Ry = Q$ and $P = \{(y\,l\,y^{-1}, l) \mid l \in Q\}$.

Now, $y^{-1}F(y) \in \mathbf{V} \cdot F(\mathbf{V})$ but, since $F(y\,l\,y^{-1}) = y\,l\,y^{-1}$ for all $l \in Q$, we deduce that $y^{-1}F(y) \in C_{\mathbf{G}}(Q)$. So

$$y^{-1}F(y) \in (\mathbf{V} \cdot F(\mathbf{V})) \cap C_{\mathbf{G}}(Q) = C_{\mathbf{V}}(Q) \cdot F(C_{\mathbf{V}}(Q)) \subset C_{\mathbf{G}}^{\circ}(Q)$$

(see Proposition 3.4(b) and (d)). So, by Lang's Theorem, there exists $x \in C_{\mathbf{G}}^{\circ}(Q)$ such that $y^{-1}F(y) = x^{-1}F(x)$. This implies that $h = y\,x^{-1} \in \mathbf{G}^F$, and

$$P = \{(h\,l\,h^{-1}, l) \mid l \in Q\},$$

as expected. □

3.B. Perfect complexes and disconnected groups. — Given M a simple $\Lambda\mathbf{G}^F$-module, we denote by $\mathscr{Y}(M)$ the set of pairs $(\mathbf{T}, \mathbf{B})$ such that $\mathbf{T}$ is an F-stable maximal torus of $\mathbf{G}$ and $\mathbf{B}$ is a Borel subgroup of $\mathbf{G}$ containing $\mathbf{T}$ such that $\mathrm{RHom}^{\bullet}_{\Lambda\mathbf{G}^F}(R\Gamma_c(\mathbf{Y}_{\mathbf{B}}, \Lambda), M) \neq 0$. We then set $d(M) = \min_{(\mathbf{T},\mathbf{B}) \in \mathscr{Y}(M)} \dim(\mathbf{Y}_{\mathbf{B}})$. The following two theorems are proved in [**BoRo1**, Theorem A] whenever $\mathbf{G}$ is connected.

Theorem 3.6. — *Let M be a simple $\Lambda\mathbf{G}^F$-module. Then $\mathscr{Y}(M) \neq \varnothing$. Moreover, given $(\mathbf{T}, \mathbf{B}) \in \mathscr{Y}(M)$ such that $d(M) = \dim(\mathbf{Y}_{\mathbf{B}})$, we have*

$$\mathrm{Hom}_{\mathrm{D}^b(\Lambda\mathbf{G}^F)}(R\Gamma_c(\mathbf{Y}_{\mathbf{B}}, \Lambda), M[-i]) = 0$$

for all $i \neq d(M)$.

Proof. — By (3.2), we have

$$\mathrm{Hom}_{\mathrm{D}^b(\Lambda\mathbf{G}^F)}(R\Gamma_c(\mathbf{Y}_{\mathbf{B}}^{\mathbf{G}},\Lambda),M[-i]) = \mathrm{Hom}_{\mathrm{D}^b(\Lambda\mathbf{G}^{\circ F})}(R\Gamma_c(\mathbf{Y}_{\mathbf{B}}^{\mathbf{G}^\circ},\Lambda),\mathrm{Res}_{\mathbf{G}^{\circ F}}^{\mathbf{G}^F} M[-i]).$$

Since M is simple and $\mathbf{G}^{\circ F} \lhd \mathbf{G}^F$, it follows that $\mathrm{Res}_{\mathbf{G}^{\circ F}}^{\mathbf{G}^F} M$ is semisimple. Since the theorem holds in $\mathbf{G}^{\circ F}$ (see [**BoRo1**, Proof of Theorem A]), we know that $\mathscr{Y}(M)$ is not empty. The second statement follows from the fact that, if two simple $\Lambda\mathbf{G}^{\circ F}$-modules M_1 and M_2 occur in the semisimple module $\mathrm{Res}_{\mathbf{G}^{\circ F}}^{\mathbf{G}^F} M$, then they are conjugate under $\mathbf{G}^F$, and so $d(M_1) = d(M_2) = d(M)$. □

Theorem 3.7. — *The triangulated category* $\Lambda\mathbf{G}^F$-perf *is generated by the complexes* $R\Gamma_c(\mathbf{Y}_{\mathbf{B}},\Lambda)$, *where* $\mathbf{T}$ *runs over the set of* F*-stable maximal tori of* $\mathbf{G}$ *and* $\mathbf{B}$ *runs over the set of Borel subgroups of* $\mathbf{G}$ *containing* $\mathbf{T}$.

3.C. Generation of the derived category. — In this section §3.C, we assume $\Lambda = \mathscr{O}$ or k.

Let Q be an ℓ-subgroup of $\mathbf{G}^F$ and let M be an indecomposable ℓ-permutation $\Lambda[\mathbf{G}^F \times Q^{\mathrm{opp}}]$-module with vertex ΔQ. We denote by $\mathscr{Y}[M]$ the set of pairs $(\mathbf{T},\mathbf{B})$ such that $\mathbf{T}$ is an F-stable maximal torus of $\mathbf{G}$ contained in a Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ such that Q normalizes $(\mathbf{T},\mathbf{B})$ and such that M is a direct summand of a term of the complex $\big(\mathrm{Res}_{\mathbf{G}^F\times Q^{\mathrm{opp}}}^{\mathbf{G}^F\times\mathbf{T}^{F\mathrm{opp}}} \mathrm{G}\Gamma_c(\mathbf{Y}_{\mathbf{B}},\Lambda)\big)^{\mathrm{red}}$. We set $d[M] = \min_{(\mathbf{T},\mathbf{B})\in\mathscr{Y}[M]} \dim(\mathbf{Y}_{C_{\mathbf{B}}^\circ(Q)}^{C_{\mathbf{G}}^\circ(Q)})$.

Lemma 3.8. — *If* Q *normalizes a pair* $(\mathbf{T}\subset\mathbf{B})$ *where* $\mathbf{T}$ *is an* F*-stable maximal torus and* $\mathbf{B}$ *a Borel subgroup of* $\mathbf{G}$*, then* $\mathscr{Y}[M] \neq \varnothing$*. Moreover, given* $(\mathbf{T},\mathbf{B}) \in \mathscr{Y}[M]$ *such that* $d[M] = \dim(\mathbf{Y}_{C_{\mathbf{B}}^\circ(Q)})$*, the degree* i *term of the complex* $\big(\mathrm{Res}_{\mathbf{G}^F\times Q^{\mathrm{opp}}}^{\mathbf{G}^F\times\mathbf{T}^{F\mathrm{opp}}} \mathrm{G}\Gamma_c(\mathbf{Y}_{\mathbf{B}},\Lambda)\big)^{\mathrm{red}}$ *has no direct summand isomorphic to* M *if* $i \neq d[M]$.

Proof. — Note that $N_{\mathbf{G}^F\times Q^{\mathrm{opp}}}(\Delta Q) = (C_{\mathbf{G}}(Q)^F \times 1)\Delta Q$, and we identify $C_{\mathbf{G}}(Q)^F$ with $N_{\mathbf{G}^F\times Q^{\mathrm{opp}}}(\Delta Q)/\Delta Q$ via the first projection. Let V be an indecomposable projective $kC_{\mathbf{G}}(Q)^F$-module such that $\mathrm{M}(\Delta Q, V) \simeq M$ (cf Appendix), and let L be the simple quotient of V.

Now, let $\mathbf{B}_Q$ be a Borel subgroup of $C_{\mathbf{G}}(Q)$ admitting an F-stable maximal torus $\mathbf{T}_Q$. Let $\mathbf{B}$ be a Borel subgroup of $\mathbf{G}$ containing $\mathbf{B}_Q$ (then $\mathbf{B}_Q = C_{\mathbf{B}}^\circ(Q)$).

We set $D = \big(\mathrm{Res}_{C_{\mathbf{G}}(Q)^F\times 1}^{C_{\mathbf{G}}(Q)^F\times\mathbf{T}_Q^{F\mathrm{opp}}} \mathrm{G}\Gamma_c(\mathbf{Y}_{\mathbf{B}_Q}^{C_{\mathbf{G}}(Q)},k)\big)^{\mathrm{red}}$. By Proposition 3.4(e), we have

$$\mathrm{Br}_{\Delta Q}(\mathrm{G}\Gamma_c(\mathbf{Y}_{\mathbf{B}}^{\mathbf{G}},\Lambda)) \simeq \mathrm{G}\Gamma_c\big((\mathbf{Y}_{\mathbf{B}}^{\mathbf{G}})^{\Delta Q},k\big) \simeq \mathrm{G}\Gamma_c(\mathbf{Y}_{\mathbf{B}_Q}^{C_{\mathbf{G}}(Q)},k) \simeq D$$

in $\mathrm{Ho}^b(kC_{\mathbf{G}}(Q)^F)$. It follows from Lemma A.2 that M is a direct summand of the i-th term of $\left(\mathrm{Res}^{\mathbf{G}^F\times\mathbf{T}_Q^{F\mathrm{opp}}}_{\mathbf{G}^F\times Q^{\mathrm{opp}}}\mathrm{G}\Gamma_c(\mathbf{Y}^{\mathbf{G}}_{\mathbf{B}},\Lambda)\right)^{\mathrm{red}}$ if and only if V is a direct summand of D^i. So the result follows from Theorem 3.6. Note that $d[M]=d[V]=d(L)=\dim\mathbf{Y}^{C_{\mathbf{G}}(Q)}_{\mathbf{B}_Q}$. $\square$

Let $\mathscr{A}$ be the thick subcategory of $\mathrm{Ho}^b(\Lambda\mathbf{G}^F)$ generated by the complexes of the form

$$\mathrm{G}\Gamma_c(\mathbf{Y}_{\mathbf{B}},\Lambda)\otimes_{\Lambda P}L,$$

where

- $\mathbf{T}$ runs over F-stable maximal tori of $\mathbf{G}$
- $\mathbf{B}$ runs over Borel subgroups of $\mathbf{G}$ containing $\mathbf{T}$
- P is an ℓ-subgroup of $N_{\mathbf{G}^F}(\mathbf{T},\mathbf{B})$
- and L is an ΛP-module, free of rank 1 over Λ.

Let $\mathscr{B}$ be the full subcategory of $\Lambda\mathbf{G}^F$-mod consisting of modules whose indecomposable direct summands have a one-dimensional source and a vertex P which normalizes a pair $(\mathbf{T}\subset\mathbf{B})$ where $\mathbf{T}$ is an F-stable maximal torus and $\mathbf{B}$ a Borel subgroup of $\mathbf{G}$.

Theorem 3.9. — *We have* $\mathscr{A}=\mathrm{Ho}^b(\mathscr{B})$.

Proof. — Given N an indecomposable $\Lambda\mathbf{G}^F$-module with a one-dimensional source L and a vertex P which normalizes a pair $(\mathbf{T}\subset\mathbf{B})$, where $\mathbf{T}$ is an F-stable maximal torus and $\mathbf{B}$ a Borel subgroup, we set $d[N]$ to be the minimum of the numbers $d[M]$, where M runs over the set of indecomposable ℓ-permutation $\Lambda(\mathbf{G}^F\times P^{\mathrm{opp}})$-modules with vertex ΔP and such that N is a direct summand of $M\otimes_{\Lambda P}L$.

Note that if M is an indecomposable ℓ-permutation $\Lambda(\mathbf{G}^F\times P^{\mathrm{opp}})$-module with vertex properly contained in ΔP, then the indecomposable direct summands of $M\otimes_{\Lambda P}L$ have vertices of size $<|P|$. Since the $\Lambda(\mathbf{G}^F\times P^{\mathrm{opp}})$-module ΛG is a direct sum of indecomposable modules with vertices contained in ΔP, we deduce that there is an indecomposable ℓ-permutation $\Lambda(\mathbf{G}^F\times P^{\mathrm{opp}})$-module M with vertex ΔP and such that N is a direct summand of $M\otimes_{\Lambda P}L$.

We now proceed by induction on the pair $(|P|,d[N])$ (ordered lexicographically) to show that $N\in\mathscr{A}$. Fix M an indecomposable ℓ-permutation $\Lambda(\mathbf{G}^F\times P^{\mathrm{opp}})$-module M with vertex ΔP and such that N is a direct summand of $M\otimes_{\Lambda P}L$, with $d[N]=d[M]$. Let $(\mathbf{T},\mathbf{B})\in\mathscr{Y}[M]$ be such that $\dim(\mathbf{Y}_{\mathbf{B}})=d[M]$ and let $D=\left(\mathrm{Res}^{\mathbf{G}^F\times N_{\mathbf{G}}(\mathbf{T},\mathbf{B})^{F\mathrm{opp}}}_{\mathbf{G}^F\times P^{\mathrm{opp}}}\mathrm{G}\Gamma_c(\mathbf{Y}^{\mathbf{G}}_{\mathbf{V}},\Lambda)\right)^{\mathrm{red}}$.

If $i\neq d[M]$, then Lemma 3.8 shows that the indecomposable direct summands M' of D^i have vertices of size $<|P|$, or have vertex ΔP and satisfy $d[M']<d[M]$. Therefore, the indecomposable direct summands N' of $D^i\otimes_{\Lambda P}L$ have vertices of size $<|P|$ or have vertex P and satisfy $d[N']<d[N]$. We deduce from the induction

hypothesis that $D^i \otimes_{\Lambda P} L \in \mathscr{A}$ for $i \neq d[N]$. Since N is a direct summand of $D^{d[N]} \otimes_{\Lambda P} L$ and $D \otimes_{\Lambda P} L \in \mathscr{A}$ by construction, we deduce that $N \in \mathscr{A}$. □

Corollary 3.10. — *Assume that every elementary abelian ℓ-subgroup of $\mathbf{G}^F$ normalizes a pair $(\mathbf{T} \subset \mathbf{B})$ where $\mathbf{T}$ is an F-stable maximal torus and $\mathbf{B}$ a Borel subgroup of $\mathbf{G}$. Then $D^b(\Lambda \mathbf{G}^F)$ is generated, as a triangulated category closed under direct summands, by the complexes $\mathscr{R}^{\mathbf{G}}_{\mathbf{T}\subset\mathbf{B}}(\mathrm{Ind}_Q^{\mathbf{T}^F} L)$, where $\mathbf{T}$ runs over the set of F-stable maximal tori of $\mathbf{G}$, $\mathbf{B}$ runs over the set of Borel subgroups of $\mathbf{G}$ containing $\mathbf{T}$, Q runs over the set of ℓ-subgroups of $\mathbf{T}^F$ and L runs over the set of (isomorphism classes) of ΛQ-modules which are free of rank 1 over Λ.*

Proof. — Since the category $\mathrm{D}^b(\Lambda \mathbf{G}^F)$ is generated, as a triangulated category closed under taking direct summands, by indecomposable modules with elementary abelian vertices and one-dimensional source [**Rou3**, Corollary 2.3], the statement follows from Theorem 3.9. □

Remark 3.11. — It is easy to show conversely that if $D^b(\Lambda \mathbf{G}^F)$ is generated by the complexes $\mathscr{R}^{\mathbf{G}}_{\mathbf{T}\subset\mathbf{B}}(\mathrm{Ind}_Q^{\mathbf{T}^F} L)$ as in Corollary 3.10, then $\mathrm{D}^b(\Lambda \mathbf{G}^F)$ is generated by indecomposable modules with a one-dimensional source and an elementary abelian vertex that normalizes a pair $(\mathbf{T} \subset \mathbf{B})$ where $\mathbf{T}$ is an F-stable maximal torus and $\mathbf{B}$ a Borel subgroup

In particular, the generation assumption for $\Lambda = k$ implies that all elementary abelian ℓ-subgroups of $\mathbf{G}^F$ are contained in maximal tori.

The particular case $\mathbf{G}^F = \mathbf{GL}_n(\mathbb{F}_q)$ (for arbitrary n) is enough to ensure that $\mathrm{D}^b(H)$ is generated by indecomposable modules with elementary abelian vertices and one-dimensional source, for any finite group H — this fact is a straightforward consequence of Serre's product of Bockstein's Theorem, but we know of no other proof. It would be interesting to find a direct proof of that result for $\mathbf{GL}_n(\mathbb{F}_q)$.

Recall that an element of $G_0(\Lambda \mathbf{G}^F)$ is *uniform* if it is in the image of $\sum_{\mathbf{T}} R^{\mathbf{G}}_{\mathbf{T}}(G_0(\Lambda \mathbf{T}^F))$, where $\mathbf{T}$ runs over the set of F-stable maximal tori of $\mathbf{G}$.

One can actually describe exactly which complexes are "uniform".

Corollary 3.12. — *Let $\mathscr{T}$ be the full triangulated subcategory of $D^b(\Lambda \mathbf{G}^F)$ generated by the complexes $\mathscr{R}^{\mathbf{G}}_{\mathbf{T}\subset\mathbf{B}}(M)$, where $\mathbf{T}$ runs over the set of F-stable maximal tori of $\mathbf{G}$, $\mathbf{B}$ runs over the set of Borel subgroups of $\mathbf{G}$ containing $\mathbf{T}$ and M runs over the set of (isomorphism classes) of $\Lambda \mathbf{T}^F$-modules. Assume that every elementary abelian ℓ-subgroup of $\mathbf{G}^F$ normalizes a pair $(\mathbf{T} \subset \mathbf{B})$ where $\mathbf{T}$ is an F-stable maximal torus and $\mathbf{B}$ a Borel subgroup of $\mathbf{G}$.*

An object C of $\mathrm{D}^b(\Lambda \mathbf{G}^F)$ is in $\mathscr{T}$ if and only if $[C] \in G_0(\Lambda \mathbf{G}^F)$ is uniform.

Proof. — The statement follows from Corollary 3.10 and from Thomason's classification of full triangulated dense subcategories [**Tho**, Theorem 2.1]. □

Remark 3.13. — Note that Corollary 3.12 holds also for $\Lambda = K$: in the proof, Corollary 3.10 is replaced by Theorem 3.7.

Examples 3.14. — (1) If $\mathbf{G} = \mathbf{GL}_n(\mathbb{F})$ or $\mathbf{SL}_n(\mathbb{F})$, then all abelian subgroups consisting of semisimple elements are contained in maximal tori. This just amounts to the classical result in linear algebra which says that a family of commuting semisimple elements always admits a basis of common eigenvectors.

(2) Assume $\mathbf{G}$ is connected. Let π_1 (respectively π_2) denote the set of prime numbers which divide $|\mathrm{Z}(\mathbf{G}^*)/\mathrm{Z}(\mathbf{G}^*)^\circ|$ (respectively which are *bad* for $\mathbf{G}$), cf [**BoRo1**, End of §11]. We set $\pi = \pi_1 \cup \pi_2$. Now, if $\ell \notin \pi$ and is t is an ℓ-element of $\mathbf{G}^F$, then $C_{\mathbf{G}}(t)$ is connected and is a Levi subgroup of $\mathbf{G}$. An induction argument then allows to show the following fact.

(3.15) *If $\ell \notin \pi$, then all abelian ℓ-subgroups of $\mathbf{G}^F$ are contained in maximal tori.*

So Corollary 3.10 can be applied if $\ell \notin \pi$. This generalises (1).

Counter-example 3.16. — Assume here, and only here, that $\ell = 2$ (so that $p \neq 2$) and that $\mathbf{G} = \mathbf{PGL}_2(\mathbb{F})$. Let t (respectively t') denote the class of the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (respectively $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$) in $\mathbf{G}$. Then $\langle t, t' \rangle$ is an elementary abelian 2-subgroup of $\mathbf{G}$ which is not contained in any maximal torus of $\mathbf{G}$ (indeed, since $\mathbf{G}$ has rank 1, all finite subgroups of maximal tori of $\mathbf{G}$ are cyclic).

4. Rational series

4.A. Rational series in connected groups. — We assume in this section §4.A that $\mathbf{G}$ is connected.

Let $(\mathbf{T}, \theta) \in \nabla(\mathbf{G}, F)$ and let Φ (respectively $\Phi^\vee$) denote the root (respectively coroot) system of $\mathbf{G}$ relative to $\mathbf{T}$.

Let d be a positive integer divisible by δ and such that $(wF)^d(t) = t^{q^{d/\delta}}$ for all $t \in \mathbf{T}$ and $w \in N_{\mathbf{G}}(\mathbf{T})$. Let ζ be a generator of $\mathbb{F}_{q^{d/\delta}}^\times$. Recall that the map

$$\begin{array}{rccl} \mathrm{N}: & Y(\mathbf{T}) & \longrightarrow & \mathbf{T}^F \\ & \lambda & \longmapsto & N_{F^d/F}(\lambda(\zeta)) = \lambda(\zeta)\,{}^F(\lambda(\zeta))\cdots{}^{F^{d-1}}(\lambda(\zeta)) \end{array}$$

is surjective. We set $\theta^Y = \theta \circ \mathrm{N} : Y(\mathbf{T}) \to K^\times$ and

$$\Phi^\vee(\theta) = \Phi^\vee \cap \mathrm{Ker}(\theta^Y).$$

Note that $\Phi^\vee(\theta)$ is closed and symmetric, hence it defines a root system. We denote by $W^\circ_{\mathbf{G}}(\mathbf{T},\theta)$ its Weyl group. It is a subgroup of the Weyl group $N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ and it is contained in the stabilizer $W_{\mathbf{G}}(\mathbf{T},\theta)$ of θ^Y.

This can be translated as follows in the dual group. Let $(\mathbf{G}^*,\mathbf{T}^*,F^*)$ be a triple dual to $(\mathbf{G},\mathbf{T},F)$ [**DeLu**, Definition 5.21]. Let $s \in \mathbf{T}^{*F^*}$ be the element corresponding to θ. This provides an identification of the coroot system $\Phi^\vee$ with the root system of $\mathbf{G}^*$ and, through this identification,

$$\Phi^\vee(\theta) = \{\alpha^\vee \in \Phi^\vee \mid \alpha^\vee(s) = 1\}.$$

The group $W^\circ_{\mathbf{G}}(\mathbf{T},\theta)$ is identified with the Weyl group $W^\circ(\mathbf{T}^*,s)$ of $C^\circ_{\mathbf{G}^*}(s)$ relative to $\mathbf{T}^*$ while $W_{\mathbf{G}}(\mathbf{T},\theta)$ is identified with the Weyl group $W(\mathbf{T}^*,s)$ of $C_{\mathbf{G}^*}(s)$.

Recall that $(\mathbf{T}_1,\theta_1)$ and $(\mathbf{T}_2,\theta_2)$ are in the same *geometric series* if there exists $x \in \mathbf{G}$ such that $(\mathbf{T}_2,\theta_2^Y) = {}^x(\mathbf{T}_1,\theta_1^Y)$ and $x^{-1}F(x)\mathbf{T}_1 \in W(\mathbf{T}_1,\theta_1)$. The pairs are in the same *rational series* if in addition the element $s_2 \in \mathbf{T}_1^{*F^*}$ corresponding to ${}^{x^{-1}}\theta_2$ is $\mathbf{G}^{*F^*}$-conjugate to s_1. We have now a direct description of rational series.

Proposition 4.1. — *The pairs $(\mathbf{T}_1,\theta_1)$ and $(\mathbf{T}_2,\theta_2)$ are in the same rational series if and only if there exists $x \in \mathbf{G}$ such that $(\mathbf{T}_2,\theta_2^Y) = {}^x(\mathbf{T}_1,\theta_1^Y)$ and $x^{-1}F(x)\mathbf{T}_1 \in W^\circ_{\mathbf{G}}(\mathbf{T}_1,\theta_1)$.*

Proof. — Note that given $x \in \mathbf{G}$ such that ${}^x\mathbf{T}_1$ is F-stable, then $x^{-1}F(x) \in N_{\mathbf{G}_1}(\mathbf{T}_1)$.

Let $\mathbf{T}_i^*$ be an F^*-stable maximal torus of $\mathbf{G}^*$ and let $s_i \in \mathbf{T}_i^{*F^*}$ be such that the $\mathbf{G}^{*F^*}$-orbit of $(\mathbf{T}_i^*,s_i)$ corresponds to the $\mathbf{G}^F$-orbit of $(\mathbf{T}_i,\theta_i)$. Then the statement of the proposition is equivalent to the following:

($*$) *s_1 and s_2 are $\mathbf{G}^{*F^*}$-conjugate if and only if there exists $x \in \mathbf{G}^*$ such that $(\mathbf{T}_2^*,s_2) = {}^x(\mathbf{T}_1^*,s_1)$ and $x^{-1}F^*(x)\mathbf{T}_1^* \in W^\circ(\mathbf{T}_1^*,s_1)$.*

So let us prove ($*$).

First, if s_1 and s_2 are $\mathbf{G}^{*F^*}$-conjugate, then there exists $x \in \mathbf{G}^{*F^*}$ such that $s_2 = x s_1 x^{-1}$. Then $\mathbf{T}_1^*$ and $x^{-1}\mathbf{T}_2^* x$ are two maximal tori of $C^\circ_{\mathbf{G}^*}(s_1)$, so there exists $y \in C^\circ_{\mathbf{G}^*}(s_1)$ such that $y\mathbf{T}_1^*y^{-1} = x^{-1}\mathbf{T}_2^*x$. Then $(\mathbf{T}_2^*,s_2) = {}^{xy}(\mathbf{T}_1^*,s_1)$ and

$$(xy)^{-1}F^*(xy) = y^{-1}F^*(y) \in C^\circ_{\mathbf{G}^*}(s_1),$$

as desired.

Conversely, assume that there exists $x \in \mathbf{G}^*$ such that $(\mathbf{T}_2^*,s_2) = {}^x(\mathbf{T}_1^*,s_1)$ and $x^{-1}F^*(x)\mathbf{T}_1^* \in W^\circ(\mathbf{T}_1^*,s_1)$. By Lang's Theorem applied to the connected group $C^\circ_{\mathbf{G}^*}(s_1)$, there exists $y \in C^\circ_{\mathbf{G}^*}(s_1)$ such that $x^{-1}F^*(x) = y^{-1}F^*(y)$. Then $xy^{-1} \in \mathbf{G}^{*F^*}$ and $s_2 = xy^{-1}s_1yx^{-1}$. The proof of ($*$) is complete. □

We can now translate the properties of regularity and super-regularity defined in [**BoRo1**, §11.4]. Let **P** be a parabolic subgroup of **G** and let **L** be a Levi subgroup of **P**. We assume that **L** is F-stable. Let $\mathscr{X} \subset \nabla(\mathbf{L}, F)$ be a rational series.

Proposition 4.2. — *The rational series $\mathscr{X}$ is $(\mathbf{G},\mathbf{L})$-regular (respectively $(\mathbf{G},\mathbf{L})$-super-regular) if and only if $W^\circ_{\mathbf{G}}(\mathbf{T},\theta) \subset \mathbf{L}$ (respectively $W_{\mathbf{G}}(\mathbf{T},\theta) \subset \mathbf{L}$) for some (or any) pair $(\mathbf{T},\theta) \in \mathscr{X}$.*

Proof. — This follows immediately from [**BoRo1**, Lemma 11.6]. □

4.B. Coroots of fixed points subgroups. — We consider now again a non-necessarily connected reductive group **G**.

We fix an element $g \in \mathbf{G}$ which stabilizes a pair $(\mathbf{T},\mathbf{B})$ where **B** is a Borel subgroup of **G** and **T** is a maximal torus of **B**. Such an element is called *quasi-semisimple* in [**DigMi2**] and [**DigMi3**]. For instance, any semisimple element of **G** is quasi-semisimple. Recall from [**DigMi2**, Theorem 1.8] that $(\mathbf{G}^g)^\circ$ is a reductive group, that $(\mathbf{B}^g)^\circ = \mathbf{B}^g \cap (\mathbf{G}^g)^\circ$ is a Borel subgroup of $\mathbf{G}^g$ and that $(\mathbf{T}^g)^\circ = \mathbf{T} \cap (\mathbf{G}^g)^\circ$ is a maximal torus of $\mathbf{B}^g$. We shall be interested in determining the coroot system of the fixed points subgroup $(\mathbf{G}^g)^\circ$.

Let Φ (respectively $\Phi^\vee$) be the root (respectively coroot) system of $\mathbf{G}^\circ$ relative to **T**. Let $\Phi(g)$ (respectively $\Phi^\vee(g)$) denote the root (respectively coroot) system of $(\mathbf{G}^g)^\circ$ relative to $(\mathbf{T}^g)^\circ$. If Ω is a g-orbit in Φ, we denote by $c_\Omega \in \mathbb{F}^\times$ the scalar by which $g^{|\Omega|}$ acts on the one-parameter unipotent subgroup associated with α (through any identification of this one-parameter subgroup with the additive group $\mathbb{F}$). We denote by $(\Phi/g)^a$ the set of g-orbits Ω in Φ such that there exist $\alpha, \beta \in \Omega$ such that $\alpha+\beta \in \Phi$. We denote by $(\Phi/g)^b$ the set of other orbits. We set

$$\Phi[g] = \{\Omega \in (\Phi/g)^a \mid c_\Omega = \pm 1\} \cup \{\Omega \in (\Phi/g)^b \mid c_\Omega = 1\}.$$

Finally, if $\Omega \in (\Phi/g)^a$ (respectively $\Omega \in (\Phi/g)^b$), let $\overline{\Omega}^\vee = 2\sum_{\alpha\in\Omega} \alpha^\vee$ (respectively $\overline{\Omega}^\vee = \sum_{\alpha\in\Omega} \alpha^\vee$). Note that $\overline{\Omega}^\vee$ is g-invariant, so it belongs to $Y(\mathbf{T})^g = Y((\mathbf{T}^g)^\circ)$.

Proposition 4.3. — $\Phi^\vee(g) = \{\overline{\Omega}^\vee \mid \Omega \in \Phi[g]\}$.

Proof. — The statement depends only on the automorphism induced by g on $\mathbf{G}^\circ$ and can be proved with assuming that $\mathbf{G}^\circ$ is semisimple. Since this automorphism can then be lifted uniquely to the simply-connected covering of $\mathbf{G}^\circ$ (see [**St**, 9.16]), we may also assume that $\mathbf{G}^\circ$ is simply-connected. Therefore, g permutes the irreducible components of $\mathbf{G}^\circ$ so an easy reduction argument shows that we may assume that $\mathbf{G}^\circ$ is quasi-simple. Let **U** denote the unipotent radical of **B**, $\mathbf{U}^-$ the unipotent radical of the opposite Borel subgroup and, if $\alpha \in \Phi$, let $\mathbf{U}_\alpha$ denote the corresponding one-parameter unipotent subgroup. We also denote by $\mathbf{G}_\alpha$ the subgroup generated by $\mathbf{U}_\alpha$ and $\mathbf{U}_{-\alpha}$: it is isomorphic to $\mathbf{SL}_2(\mathbb{F})$ because $\mathbf{G}^\circ$ is simply-connected.

Let us first assume that $(\Phi/g)^a = \varnothing$. Then, if $\Omega \in \Phi/g$, the subvariety $\mathbf{U}_\Omega = \prod_{\alpha\in\Omega} \mathbf{U}_\alpha$ of $\mathbf{U}$ or $\mathbf{U}^-$ does not depend on the order chosen on Ω to compute this product. Moreover, it is an abelian g-stable algebraic group and $\mathbf{U}_\Omega^g \neq 1$ if and only if $c_\Omega = 1$. The subgroup $\langle \mathbf{U}_\Omega, \mathbf{U}_{-\Omega}\rangle$ is a direct product of groups isomorphic to $\mathbf{SL}_2(\mathbb{F})$ which are permuted by g. It then follows that the coroot corresponding to the one-parameter subgroup $\mathbf{U}_\Omega^g$ (if $c_\Omega = 1$) is equal to $\overline{\Omega}^\vee$. The fact that these $\mathbf{U}_\Omega^g$ are all the one-parameter unipotent subgroups of $(\mathbf{G}^g)^\circ$ follows from [**St**, Proof of Theorem 8.2].

Let us now assume that $(\Phi/g)^a \neq \varnothing$. Then it follows from the classification of simple root systems that $\mathbf{G}^\circ$ is of type A_{2n} and g acts on $\mathbf{T}$ by the automorphism $t \mapsto {}^{w_0}t^{-1}$, where w_0 is the longest element of the Weyl group of $\mathbf{G}^\circ$ relative to $\mathbf{T}$. A straightforward computation gives the result in this case. □

Remark 4.4. — If $\Omega \in (\Phi/g)^a$, then it follows from the classification that $|\Omega|$ is even, and so the order of g is even.

4.C. Centralizers and rational series. — Let $g \in \mathbf{G}^F$ be a quasi-semisimple element of $\mathbf{G}$. Let $(\mathbf{S},\theta) \in \nabla(C^\circ_{\mathbf{G}}(g), F)$. We then set $\mathbf{S}^+ = C_{\mathbf{G}^\circ}(\mathbf{S})$. It follows from [**DigMi2**, Theorem 1.8(iv)] that $\mathbf{S}^+$ is a maximal torus of $\mathbf{G}^\circ$ (containing $\mathbf{S}$). It is stable under the action of g, so we have a map $\mathscr{L}_g : \mathbf{S}^+ \to \mathbf{S}^+$, $t \mapsto t^{-1}gtg^{-1} = [g,t]$ (which is a morphism of groups because $\mathbf{S}^+$ is abelian). If $t = \mathscr{L}_g(s)$, then $t\,{}^g t\,{}^{g^2} t \cdots {}^{g^{m-1}} t = \mathscr{L}_{g^m}(s)$. In particular, if $t \in (\mathbf{S}^+)^g = \operatorname{Ker}\mathscr{L}_g$, then $t^m = \mathscr{L}_{g^m}(s)$. This shows that any element of $(\mathbf{S}^+)^g \cap \mathscr{L}_g(\mathbf{S}^+)$ has order dividing the order of g. Consequently, since $((\mathbf{S}^+)^g)^\circ = \mathbf{S}$ (see [**DigMi2**, Theorem 1.8(iii)]), we get

(4.5) $\quad \mathbf{S}^+ = \mathbf{S}\cdot\mathscr{L}_g(\mathbf{S}^+)$ and $\mathbf{S}\cap\mathscr{L}_g(\mathbf{S}^+)$ is finite of exponent dividing the order of g.

Now, if H is a g-stable finite subgroup of $\mathbf{S}^+$ of order prime to the order of g then $H^g \subset \mathbf{S}^\circ$ (because $(\mathbf{S}^+)^g/\mathbf{S}$ is of order dividing the order of g by [**DigMi2**, Proposition 1.28]) and

$$H = H^g \times \mathscr{L}_g(H). \tag{4.6}$$

So, *if the linear character θ has order prime to the order of g*, then it can be extended canonically to a linear character θ^+ of $\mathbf{S}^{+F}$ as follows: θ^+ is trivial on $\mathscr{L}_g(\mathbf{S}^{+F})$, is trivial on elements of $\mathbf{S}^{+F}$ of order prime to the order of θ and coincides with θ on $\mathbf{S}^F$. The fact that θ^+ is trivial on $\mathscr{L}_g(\mathbf{S}^{+F})$ is equivalent to

$$\theta^+ \textit{ is } g\textit{-stable.} \tag{4.7}$$

Note that, since $\mathbf{S}^+ \cap C^\circ_{\mathbf{G}}(g) = \mathbf{S}$ by [**DigMi2**, Theorem 1.8], we may identify the Weyl group of $C^\circ_{\mathbf{G}}(g)$ relative to $\mathbf{S}$ to a subgroup of the Weyl group of $\mathbf{G}^\circ$ relative to $\mathbf{S}^+$. Through this identification, we get:

Lemma 4.8. — *If the order of θ is prime to the order of g, then $W_{C^\circ_{\mathbf{G}}(g)}(\mathbf{S},\theta) \subset W_{\mathbf{G}^\circ}(\mathbf{S}^+,\theta^+)$ and $W^\circ_{C^\circ_{\mathbf{G}}(g)}(\mathbf{S},\theta) \subset W^\circ_{\mathbf{G}^\circ}(\mathbf{S}^+,\theta^+)$.*

Proof. — Let $w \in W_{C^\circ_{\mathbf{G}}(g)}(\mathbf{S},\theta)$. Then w stabilizes $\mathbf{S}^+ = C_{\mathbf{G}^\circ}(\mathbf{S})$ and its action on $\mathbf{S}$ commutes with the action of g. So it follows from the construction of θ^+ that w stabilizes θ^+.

Let us now prove the second statement. Let $\alpha^\vee$ be a coroot of $C^\circ_{\mathbf{G}}(g)$ relative to $\mathbf{S}$ such that $\theta^Y(\alpha^\vee)=1$. Let $s_{g,\alpha}$ denote the corresponding reflection in $W^\circ_{C^\circ_{\mathbf{G}}(g)}(\mathbf{S},\theta)$. It is sufficient to prove that $s_{g,\alpha} \in W^\circ_{\mathbf{G}^\circ}(\mathbf{S}^+,\theta^+)$. Then it follows from Proposition 4.3 that there exists a coroot $\beta^\vee$ of $\mathbf{G}^\circ$ relative to $\mathbf{S}$ and $m \in \{1,2\}$ such that

$$\alpha^\vee = m\sum_{i=0}^{r-1} g^i(\beta^\vee),$$

where $r \geqslant 1$ is minimal such that $g^r(\beta^\vee)=\beta^\vee$. It follows from Remark 4.4 that, if $m=2$, then g has even order. Now,

$$1=\theta^{+Y}(\alpha^\vee)=\prod_{i=1}^{r-1}\theta^+(g^i(\beta^\vee))^m=\theta^{+Y}(\beta^\vee)^{mr},$$

because θ^+ is g-stable. Since m and r divide the order of g, mr is prime to the order of θ^+, so this implies that $\theta^+(\beta^\vee)=1$. In particular,

$$s_\beta, s_{g(\beta)}, \ldots, s_{g^{r-1}(\beta)} \in W^\circ_{\mathbf{G}^\circ}(\mathbf{S}^+,\theta^+).$$

It follows from [**St**, Proof of Theorem 8.2, Statement (2‴)] that then $s_{g,\alpha}$ belongs to the subgroup generated by s_β, $s_{g(\beta)}, \ldots, s_{g^{r-1}(\beta)}$. □

Let $(\mathbf{T}_1,\theta_1),(\mathbf{T}_2,\theta_2) \in \nabla(\mathbf{G},F)$. We say that $(\mathbf{T}_1,\theta_1)$ and $(\mathbf{T}_2,\theta_2)$ are *geometrically conjugate* (resp. in the same *rational series*) if there is $t \in N_{\mathbf{G}^F}(\mathbf{T}_1)$ such that $(\mathbf{T}_1, {}^t\theta_1)$ and $(\mathbf{T}_2,\theta_2)$ are geometrically conjugate (resp. in the same rational series) for $\nabla(\mathbf{G}^\circ,F)$. We denote by $\nabla(\mathbf{G},F)/\equiv$ the set of rational series.

Let Q be the subgroup of $\mathbf{G}$ generated by g.

Corollary 4.9. — *Let $(\mathbf{S}_1,\theta_1),(\mathbf{S}_2,\theta_2) \in \nabla_{|g|'}(N_{\mathbf{G}}(Q),F)$.*

(a) *If $(\mathbf{S}_1,\theta_1)$ and $(\mathbf{S}_2,\theta_2)$ are geometrically conjugate in $N_{\mathbf{G}}(Q)$, then $(\mathbf{S}_1^+,\theta_1^+)$ and $(\mathbf{S}_2^+,\theta_2^+)$ are geometrically conjugate in $\mathbf{G}$.*

(b) *If $(\mathbf{S}_1,\theta_1)$ and $(\mathbf{S}_2,\theta_2)$ are in the same rational series of $N_{\mathbf{G}}(Q)$, then $(\mathbf{S}_1^+,\theta_1^+)$ and $(\mathbf{S}_2^+,\theta_2^+)$ are in the same rational series of $\mathbf{G}$.*

So, the injective map $\nabla_{|g|'}(N_{\mathbf{G}}(Q),F) \to \nabla_{|g|'}(\mathbf{G},F)$, $(\mathbf{S},\theta) \mapsto (\mathbf{S}^+,\theta^+)$ *induces a map*

$$i_Q^{\mathbf{G}} : \nabla_{|g|'}(N_{\mathbf{G}}(Q),F)/\equiv \;\longrightarrow\; \nabla_{|g|'}(\mathbf{G},F)/\equiv .$$

Proof. — (a) If $(\mathbf{S}_1,\theta_1)$ and $(\mathbf{S}_2,\theta_2)$ are geometrically conjugate in $N_{\mathbf{G}}(Q)^\circ = C^\circ_{\mathbf{G}}(g)$ then, by definition, there exists $x \in C^\circ_{\mathbf{G}}(g)$ such that $\mathbf{S}_2 = {}^x\mathbf{S}_1$ and $\theta_2^Y = {}^x\theta_1^Y = {}^{F(x)}\theta_1^Y$ (as linear characters of $Y(\mathbf{S}_2)$). Since x commutes with g, it sends $\mathscr{L}_g(\mathbf{S}_1^+)$ to $\mathscr{L}_g(\mathbf{S}_2^+)$, so it is immediately checked that $\theta_2^{+Y} = {}^x\theta_1^{+Y} = {}^{F(x)}\theta_1^{+Y}$. The case of geometric conjugacy in $N_{\mathbf{G}}(Q)$ and $\mathbf{G}$ follows immediately.

(b) If $(\mathbf{S}_1,\theta_1)$ and $(\mathbf{S}_2,\theta_2)$ are in the same rational series of $C^\circ_{\mathbf{G}}(g)$, then, by Proposition 4.1, there exists $x \in C^\circ_{\mathbf{G}}(g)$ such that $\mathbf{T}_2 = {}^x\mathbf{T}_1$, $\theta_2^Y = {}^x\theta_1^Y$ (as linear characters of $Y(\mathbf{S}_2)$) and $x^{-1}F(x) \in W^\circ_{C^\circ_{\mathbf{G}}(g)}(\mathbf{S}_1,\theta_1)$. So the result follows from (a) and from Propositions 4.1 and 4.3. The case of rational series in $N_{\mathbf{G}}(Q)$ and $\mathbf{G}$ follows immediately. □

Let $\mathbf{L}$ be an F-stable Levi complement of a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ containing g. Then $C_{\mathbf{L}}(g)$ is an F-stable Levi complement of $C_{\mathbf{P}}(g)$.

Corollary 4.10. — *Let* $\mathscr{X} \in \nabla_{|g|'}(C^\circ_{\mathbf{L}}(g),F)/\equiv$ *be a rational series. If* $\mathscr{X}^+$ *is* $(\mathbf{G}^\circ,\mathbf{L}^\circ)$*-regular (respectively* $(\mathbf{G}^\circ,\mathbf{L}^\circ)$*-super regular), then* $\mathscr{X}$ *is* $(C^\circ_{\mathbf{G}}(g),C^\circ_{\mathbf{L}}(g))$*-regular (respectively* $(C^\circ_{\mathbf{G}}(g),C^\circ_{\mathbf{L}}(g))$*-super regular).*

Proof. — This follows from Proposition 4.2 and Lemma 4.8. □

The results above extend by induction to general nilpotent p'-subgroups. Let Q be a nilpotent subgroup of $\mathbf{G}^F$ of order prime to p. Fix a sequence $1 = Q_0 \subset Q_1 \subset \cdots \subset Q_r = Q$ of normal subgroups of Q such that Q_i/Q_{i-1} is cyclic for $1 \leqslant i \leqslant r$. Let $\mathbf{G}_i = N_{\mathbf{G}}(Q_1 \subset \cdots \subset Q_i)$.

The construction above defines a map

$$\nabla_{|Q|'}(\mathbf{G}_{i+1}/Q_i,F) = \nabla_{|Q|'}(N_{\mathbf{G}_i/Q_i}(Q_{i+1}/Q_i),F) \to \nabla_{|Q|'}(\mathbf{G}_i/Q_i,F) \tag{4.10}$$

that preserves rational and geometric series.

Fix $0 \leqslant j \leqslant i \leqslant r$. Let $(\mathbf{T},\theta) \in \nabla_{|Q|'}(\mathbf{G}_i,F)$. Since θ is trivial on $Q_j \cap \mathbf{T}^F$, it factors through a character θ' of $(\mathbf{T}/(\mathbf{T}\cap Q_j))^F = \mathbf{T}^F/(\mathbf{T}^F \cap Q_j)$. We obtain a pair $(\mathbf{T}/(\mathbf{T}\cap Q_j),\theta') \in \nabla_{|Q|'}(\mathbf{G}_i/Q_j,F)$. This correspondence defines a bijection $\nabla_{|Q|'}(\mathbf{G}_i,F) \xrightarrow{\sim} \nabla_{|Q|'}(\mathbf{G}_i/Q_j,F)$ that preserves rational and geometric series.

Composing those bijections with the map in (4.10), we obtain a map

$$\nabla_{|Q|'}(\mathbf{G}_{i+1},F) \to \nabla_{|Q|'}(\mathbf{G}_i,F)$$

and composing all those maps, we obtain a map

$$\nabla_{|Q|'}(N_{\mathbf{G}}(Q_1 \subset \cdots \subset Q_r),F) \to \nabla_{|Q|'}(\mathbf{G},F).$$

Finally, composing with the canonical map $\nabla_{|Q|'}(C_{\mathbf{G}}(Q),F)\to\nabla_{|Q|'}(N_{\mathbf{G}}(Q_1\subset\cdots\subset Q_r),F)$, we obtain a map

$$\nabla_{|Q|'}(C_{\mathbf{G}}(Q),F)\to\nabla_{|Q|'}(\mathbf{G},F)$$

that preserves rational and geometric series. Note that this map depends not only on Q, but on the filtration $Q_1\subset\cdots\subset Q_r$. Summarizing, we have the following proposition.

Proposition 4.11. — *Let Q be a nilpotent subgroup of $\mathbf{G}^F$ of order prime to p. Fix a sequence $1=Q_0\subset Q_1\subset\cdots\subset Q_r=Q$ of normal subgroups of Q such that Q_i/Q_{i-1} is cyclic for $1\leqslant i\leqslant r$.*

The constructions above define a map

$$i_{Q_\bullet}^{\mathbf{G}}:\nabla_{|Q|'}(C_{\mathbf{G}}(Q),F)/\equiv\;\to\;\nabla_{|Q|'}(\mathbf{G},F)/\equiv$$

Let $\mathbf{L}$ be an F-stable Levi complement of a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ containing Q. Let $\mathscr{X}\in\nabla_{|Q|'}(C_{\mathbf{L}}(Q),F)/\equiv$ be a rational series. Then

- *$C_{\mathbf{L}}(Q)$ is an F-stable Levi complement of $C_{\mathbf{P}}(Q)$*
- *if $i_{Q_\bullet}^{\mathbf{L}}(\mathscr{X})$ is $(\mathbf{G}^\circ,\mathbf{L}^\circ)$-regular then $\mathscr{X}$ is $(C^\circ_{\mathbf{G}}(Q),C^\circ_{\mathbf{L}}(Q))$-regular*
- *if $i_{Q_\bullet}^{\mathbf{L}}(\mathscr{X})$ is $(\mathbf{G}^\circ,\mathbf{L}^\circ)$-super regular then $\mathscr{X}$ is $(C^\circ_{\mathbf{G}}(Q),C^\circ_{\mathbf{L}}(Q))$-super regular.*

The map $i_{Q_\bullet}^{\mathbf{G}}$ is actually independent of the choice of the filtration of Q, cf Remark 4.15.

4.D. Generation and series. — Given $(\mathbf{T},\theta)\in\nabla_\Lambda(\mathbf{G},F)$, we denote by e°_θ the block idempotent of $\Lambda\mathbf{T}^F$ not vanishing on θ.

We have now a generalization of [**BoRo1**, Théorème A].

Given $\mathscr{X}\in\nabla_\Lambda(\mathbf{G},F)/\equiv$, let $\mathscr{C}_{\mathscr{X}}$ be the thick subcategory of $(\Lambda\mathbf{G}^F)$-perf generated by the complexes $\mathrm{R}\Gamma_c(Y_{\mathbf{B}})e^\circ_\theta$ where $(\mathbf{T},\theta)$ runs over $\mathscr{X}$ and $\mathbf{B}$ runs over Borel subgroups of $\mathbf{G}^\circ$ containing $\mathbf{T}$.

Note that we obtain the same thick subcategory by taking instead the complexes $\mathrm{R}\Gamma_c(Y_{\mathbf{B}})e_\theta$ where $e_\theta=\sum_{t\in N_{\mathbf{G}^F}(\mathbf{T},\mathbf{B})/C_{N_{\mathbf{G}^F}(\mathbf{T},\mathbf{B})}(\theta)}e^\circ_{{}^t\theta}$.

Theorem 4.12. — *Let $\mathscr{X}\in\nabla_\Lambda(\mathbf{G},F)/\equiv$. There is a (unique) central idempotent $e_{\mathscr{X}}$ of $\Lambda\mathbf{G}^F$ such that $\mathscr{C}_{\mathscr{X}}=(\Lambda\mathbf{G}^Fe_{\mathscr{X}})$-perf.*

We have a decomposition in central orthogonal idempotents of $\Lambda\mathbf{G}^F$

$$1=\sum_{\mathscr{X}\in\nabla_\Lambda(\mathbf{G},F)/\equiv}e_{\mathscr{X}}.$$

Proof. — Note first that the theorem holds for $\mathbf{G}^\circ$ by [**BoRo1**, Théorème A]. Let $(\mathbf{T}_i,\theta_i)\in\nabla_\Lambda(\mathbf{G},F)$ for $i\in\{1,2\}$. We have

$$\mathrm{Hom}^\bullet_{\Lambda\mathbf{G}^F}(\mathrm{R}\Gamma_c(\mathbf{Y}^{\mathbf{G}}_{\mathbf{B}_1})e^\circ_{\theta_1},\mathrm{R}\Gamma_c(\mathbf{Y}^{\mathbf{G}}_{\mathbf{B}_2})e^\circ_{\theta_2})\simeq\mathrm{Hom}^\bullet_{\Lambda\mathbf{G}^{\circ F}}(\mathrm{R}\Gamma_c(\mathbf{Y}^{\mathbf{G}^\circ}_{\mathbf{B}_1})e^\circ_{\theta_1},\bigoplus_{t\in N_{\mathbf{G}^F}(\mathbf{T}_2,\mathbf{B}_2)/\mathbf{T}_2^F}\mathrm{R}\Gamma_c(\mathbf{Y}^{\mathbf{G}^\circ}_{\mathbf{B}_2})e^\circ_{{}^t\theta_2}).$$

The connected case of the theorem shows this is 0 unless $(\mathbf{T}_1, \theta_1)$ and $(\mathbf{T}_2, {}^t\theta_2)$ are in the same rational series of $(\mathbf{G}^\circ, F)$ for some t.

We have shown that the categories $\mathscr{C}_{\mathscr{X}_1}$ and $\mathscr{C}_{\mathscr{X}_2}$ are othogonal for $\mathscr{X}_1 \neq \mathscr{X}_2$. The theorem follows now from [**BoRo1**, Proposition 9.2] and Theorem 3.7. □

Let $\mathscr{X} \in \nabla_\Lambda(\mathbf{G}, F)/\equiv$. Let $\mathscr{A}_{\mathscr{X}}$ be the thick subcategory of $\mathrm{Ho}^b(\Lambda\mathbf{G}^F)$ generated by the complexes of the form

$$\mathrm{G}\Gamma_c(\mathbf{Y}_\mathbf{B}, \Lambda)e_\theta \otimes_{\Lambda P} L,$$

where

- $(\mathbf{T}, \theta)$ runs over $\mathscr{X}$
- $\mathbf{B}$ runs over Borel subgroups of $\mathbf{G}^\circ$ containing $\mathbf{T}$
- P is an ℓ-subgroup of $N_{\mathbf{G}^F}(\mathbf{T}, \mathbf{B})$
- and L is a ΛP-module, free of rank 1 over Λ.

Let $\mathscr{B}_{\mathscr{X}}$ be the full subcategory of $\Lambda\mathbf{G}^F e_{\mathscr{X}}$-mod consisting of modules whose indecomposable direct summands have a one-dimensional source and a vertex P which normalizes a pair $(\mathbf{T} \subset \mathbf{B})$ where $\mathbf{T}$ is an F-stable maximal torus and $\mathbf{B}$ a Borel subgroup of $\mathbf{G}$.

Theorem 4.13. — *Let $\mathscr{X} \in \nabla_{\ell'}(\mathbf{G}, F)/\equiv$. We have $\mathscr{A}_{\mathscr{X}} = \mathrm{Ho}^b(\mathscr{B}_{\mathscr{X}})$.*

Proof. — By Theorem 4.12, we have $\mathrm{G}\Gamma_c(\mathbf{Y}_\mathbf{V}, \Lambda)e_\theta \otimes_{\Lambda P} L \in \mathrm{Ho}^b(\mathscr{B}_{\mathscr{Y}})$ if $(\mathbf{T}, \theta) \in \mathscr{Y}$. It follows that $\mathscr{A}_{\mathscr{Y}} \subset \mathrm{Ho}^b(\mathscr{B}_{\mathscr{Y}})$ and $\mathscr{A} = \bigoplus_{\mathscr{Y} \in \nabla_{\ell'}(\mathbf{G}, F)/\equiv} \mathscr{A}_{\mathscr{Y}}$. Consequently, the theorem follows from Theorem 3.9. □

4.E. Decomposition map and Deligne-Lusztig induction. — The following result generalizes [**BrMi**, Théorème 3.2] to non-cyclic ℓ-subgroups and to disconnected groups (needed to handle the non-cyclic case by induction).

Theorem 4.14. — *Let Q be an ℓ-subgroup of $\mathbf{G}^F$. The map $i^{\mathbf{G}}_{Q_\bullet}$ (cf Proposition 4.11) is independent of the filtration of Q and we denote it by $i_Q = i^{\mathbf{G}}_Q$.*

Let $\mathscr{X} \in \nabla_{\ell'}(\mathbf{G}, F)/\equiv$. We have

$$\mathrm{br}_Q(e_{\mathscr{X}}) = \sum_{\mathscr{Y} \in i_Q^{-1}(\mathscr{X})} e_{\mathscr{Y}}.$$

Proof. — Assume first that Q is cyclic. Let b be a block idempotent of $kC_{\mathbf{G}^F}(Q)$ such that $\mathrm{br}_Q(e_{\mathscr{X}})b \neq 0$. Let V be a projective indecomposable $b\,k(C_{\mathbf{G}^F}(Q)/Q)$-module and $M = M(Q,V)$. There is a unique block idempotent c of $k\mathbf{G}^F$ such that $\mathrm{br}_Q(c)b \neq 0$. We have $e_{\mathscr{X}}c \neq 0$. Note that $Mc = M$, hence $Me_{\mathscr{X}} = M$. By Theorem 4.13 and its proof, there exists $(\mathbf{T},\theta) \in \mathscr{X}$, $\mathbf{B}$ a Borel subgroup of $\mathbf{G}^\circ$ containing $\mathbf{T}$ with $Q \subset N_{\mathbf{G}^F}(\mathbf{T},\mathbf{B})$ such that V is isomorphic to a direct summand of a component of $\mathrm{Br}_Q(\mathrm{G}\Gamma_c(\mathbf{Y_B},k)e_\theta \otimes_{kQ} k)$. It follows that $b\mathrm{Br}_Q(\mathrm{G}\Gamma_c(\mathbf{Y}_{\mathbf{B}'},k)e_\theta \otimes_{kQ} k) \neq 0$, hence $b\mathrm{br}_Q(e_\theta) \neq 0$. This shows that the rational series $\mathscr{Y}$ of $(C_{\mathbf{G}}(Q),F)$ containing b contains $(\mathbf{T}_Q,\theta')$, where $\mathbf{T}_Q = C^\circ_{\mathbf{T}}(Q)$ and θ' is the restriction to $\mathbf{T}_Q^F$ of a $N_{\mathbf{G}^F}(\mathbf{T},\mathbf{B})$-conjugate of θ. It follows that $i(\mathscr{Y}) = \mathscr{X}$.

So, we have shown that given $\mathscr{Y} \in \nabla_{\ell'}(C_{\mathbf{G}}(Q),F)/\equiv$ with $e_{\mathscr{Y}}\mathrm{br}_Q(e_{\mathscr{X}}) \neq 0$, then $\mathscr{Y} \in i^{-1}(\mathscr{X})$. Since $\sum_{\mathscr{X}\in\nabla_{\ell'}(\mathbf{G},F)/\equiv} \mathrm{br}_Q(e_{\mathscr{X}}) = 1 = \sum_{\mathscr{Y}\in\nabla_{\ell'}(C_{\mathbf{G}}(Q),F)/\equiv} e_{\mathscr{Y}}$, it follows that $\mathrm{br}_Q(e_{\mathscr{X}}) = \sum_{\mathscr{Y}\in i_g^{-1}(\mathscr{X})} e_{\mathscr{Y}}$.

By transitivity of br_Q, we obtain the formula for br_Q for a general Q by induction on $|Q|$, with i_Q replaced by $i_{Q_\bullet}$. This shows that actually $i_{Q_\bullet}$ is independent of the chosen filtration of Q. □

Remark 4.15. — Let $Q = Q' \times Q''$ be a product of two cyclic groups of coprime orders. Fix a filtration $Q_1 = Q'$ and $Q_2 = Q$. We have $i_{Q_\bullet} = i_Q$. It is easy to deduce now from Theorem 4.14 that $i_{Q_\bullet}$ is independent of Q for any nilpotent p'-group Q.

Broué-Michel's proof of Theorem 4.14 for $\mathbf{G}$ connected and Q cyclic relies on the compatibility of Deligne-Lusztig induction with generalized decomposition maps. This does generalize to disconnected groups, as we explain below. A direct approach along the lines of Broué-Michel is possible, based on the results of [**DigMi3**]. While we will not use the results in the remaining part of this section, they might be useful for character theoretic questions.

Let π be a set of prime numbers not containing p. An element of finite order of $\mathbf{G}$ is a π-element (resp. a π'-element) if its order is a product of primes in π (resp. not in π).

Let g be an automorphism of finite order of an algebrac variety $\mathbf{X}$. Write $g = lx = xl$ where l is a π-element and x a π'-element. The following result is an immediate consequence of [**DeLu**, Theorem 3.2]:

$$\sum_{i\geqslant 0}(-1)^i \operatorname{Tr}(g,\mathrm{H}^i_c(\mathbf{X},\overline{\mathbb{Q}}_\ell)) = \sum_{i\geqslant 0}(-1)^i \operatorname{Tr}(x,\mathrm{H}^i_c(\mathbf{X}^l,\overline{\mathbb{Q}}_\ell)). \tag{4.16}$$

Proof. — Write $x = su = us$, where s has order prime to p and u has order a power of p. Then l, s and u commute and have coprime orders. By [**DeLu**, Theorem 3.2], we have

$$\sum_{i\geqslant 0}(-1)^i \operatorname{Tr}(g,\mathrm{H}^i_c(\mathbf{X},\overline{\mathbb{Q}}_\ell)) = \sum_{i\geqslant 0}(-1)^i \operatorname{Tr}(u,\mathrm{H}^i_c(\mathbf{X}^{ls},\overline{\mathbb{Q}}_\ell))$$

and $$\sum_{i\geqslant 0}(-1)^i \operatorname{Tr}(x,\mathrm{H}^i_c(\mathbf{X}^l,\overline{\mathbb{Q}}_\ell)) = \sum_{i\geqslant 0}(-1)^i \operatorname{Tr}(u,\mathrm{H}^i_c((\mathbf{X}^l)^s,\overline{\mathbb{Q}}_\ell)).$$

So the result follows from the fact that $\mathbf{X}^{ls}=(\mathbf{X}^l)^s$ because $\langle ls\rangle = \langle l,s\rangle$. □

Given H a finite group and $h\in H$ a π-element, we have a generalized decomposition map from the vector space of class functions $H\to K$ to the vector space of class functions on π'-elements of $C_H(h)$ given by $d^H_h(f)(u)=f(hu)$ for u a π'-element of $C_H(h)$.

The following result generalizes the character formula for $R_{\mathbf{L}}^{\mathbf{G}}$ [**DigMi2**, Proposition 2.6], which corresponds to the case where π is the set of all primes distinct from p,

Proposition 4.17. — *Let* $\mathbf{P}$ *be a parabolic subgroup of* $\mathbf{G}$*, let* $\mathbf{V}$ *be its unipotent radical, let* $\mathbf{L}$ *be a Levi complement of* $\mathbf{P}$ *and assume that* $\mathbf{L}$ *is* F*-stable. Let* $g\in\mathbf{G}^F$ *be a* π*-element. We have*
$$d_g^{\mathbf{G}^F}\circ \mathrm{R}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} = \sum_{\substack{x\in C_{\mathbf{G}}(g)^F\backslash \mathbf{G}^F/\mathbf{L}^F\\ g\in {}^x\mathbf{L}}} \mathrm{R}^{C_{\mathbf{G}}(g)}_{C_{^x\mathbf{L}}(g)\subset C_{^x\mathbf{P}}(g)}\circ d_g^{^x\mathbf{L}^F}\circ x_*.$$

Proof. — Given H a finite group, we denote by H_π (respectively $H_{\pi'}$) the set of π-elements (resp. π'-elements) of H. The proof follows essentially the same argument as the proof of the character formula (see for instance [**DigMi1**, Proposition 12.2]). Let λ be a class function on $\mathbf{L}^F$ and let $u\in C_{\mathbf{G}}(g)^F_{\pi'}$ be an π'-element. By definition of the Deligne-Lusztig induction and by using (4.16), we get
$$\begin{aligned}\mathrm{R}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}(\lambda)(gu) &= \frac{1}{|\mathbf{L}^F|}\sum_{l\in\mathbf{L}^F_\pi}\sum_{v\in C_{\mathbf{L}}(l)^F_{\pi'}}\lambda(lv)\sum_{i\geqslant 0}(-1)^i\operatorname{Tr}((gu,lv),\mathrm{H}^i_c(\mathbf{Y}_{\mathbf{V}},\overline{\mathbb{Q}}_\ell)).\\ &= \frac{1}{|\mathbf{L}^F|}\sum_{l\in\mathbf{L}^F_\pi}\sum_{v\in C_{\mathbf{L}}(l)^F_{\pi'}}\lambda(lv)\sum_{i\geqslant 0}(-1)^i\operatorname{Tr}((u,v),\mathrm{H}^i_c(\mathbf{Y}^{(g,l)}_{\mathbf{V}},\overline{\mathbb{Q}}_\ell)).\end{aligned}$$

But it follows from Lemma 3.5 that $\mathbf{Y}_{\mathbf{V}}^{(g,l)}\neq\varnothing$ if and only if there exists $x\in\mathbf{G}^F$ such that $x^{-1}gx=l$. Moreover, in this case, then $\mathbf{Y}_{\mathbf{V}}^{(g,l)}\simeq \mathbf{Y}^{C_{\mathbf{G}}(g)}_{C_{^x\mathbf{V}}(g)}$ by Proposition 3.4. Therefore,
$$\begin{aligned}\mathrm{R}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}(\lambda)(gu) &= \frac{1}{|\mathbf{L}^F|\cdot|C_{\mathbf{G}}(g)^F|}\sum_{\substack{x\in\mathbf{G}^F\\ g\in{}^x\mathbf{L}}}\sum_{v\in C_{\mathbf{L}}(l)^F_{\pi'}}\lambda(x^{-1}gxv)\sum_{i\geqslant 0}(-1)^i\operatorname{Tr}((u,v),\mathrm{H}^i_c(\mathbf{Y}^{(g,x^{-1}gx)}_{\mathbf{V}},\overline{\mathbb{Q}}_\ell)).\\ &= \frac{1}{|\mathbf{L}^F|\cdot|C_{\mathbf{G}}(g)^F|}\sum_{\substack{x\in\mathbf{G}^F\\ g\in{}^x\mathbf{L}}}\sum_{v\in C_{^x\mathbf{L}}(g)^F_{\pi'}}d_g^{^x\mathbf{L}}(x_*(\lambda))(v)\sum_{i\geqslant 0}(-1)^i\operatorname{Tr}((u,v),\mathrm{H}^i_c(\mathbf{Y}^{C_{\mathbf{G}}(g)}_{C_{^x\mathbf{V}}(g)},\overline{\mathbb{Q}}_\ell)).\end{aligned}$$

Now, if $x \in \mathbf{G}^F$ is such that $g \in {}^x\mathbf{L}$, then

$$|C_{\mathbf{G}}(g)^F\, x\mathbf{L}^F| = \frac{|C_{\mathbf{G}}(g)^F| \cdot |\mathbf{L}^F|}{|C_{{}^x\mathbf{L}}(g)^F|}.$$

So the result follows. □

5. Comparing Y-varieties

From now on, and until the end of this article, we assume $\mathbf{G}$ is connected.

The aim of this section is to prove the preliminary statements necessary for our proof of Theorem 1.3. Roughly speaking, the main result of this section (Theorem 5.16) is almost equivalent to Theorem 1.3 whenever $\mathbf{L}$ is a maximal torus. As Theorem 1.3 will be proved by reduction to this case, Theorem 5.16 may be seen as the crucial step. In the course of the proof, we will also obtain Corollary E as a consequence of some of our geometrical results.

In this section §5, we fix an F-stable torus $\mathbf{T}$ contained in an F-stable Borel subgroup $\mathbf{B}$ and we denote by $\mathbf{U}$ its unipotent radical. We put $W = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$. We denote by Φ the associated root system, by Φ^+ the set of positive roots and by Δ the basis of Φ. Let $\alpha \in \Phi$, we denote by $s_\alpha \in W$ the corresponding reflection and by $\alpha^\vee \in \Phi^\vee$ the corresponding coroot. We put $\mathbf{T}_{\alpha^\vee} = \mathrm{Im}(\alpha^\vee) \subset \mathbf{T}$ and we denote by $\mathbf{U}_\alpha$ the one-parameter subgroup of $\mathbf{G}$ normalized by $\mathbf{T}$ and associated with α. We define $\mathbf{G}_\alpha$ as the subgroup of $\mathbf{G}$ generated by $\mathbf{U}_\alpha$ and $\mathbf{U}_{-\alpha}$.

5.A. Dimension estimates and further. — We fix in this section four parabolic subgroups $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ and $\mathbf{P}_4$ admitting a common Levi complement $\mathbf{L}$. We denote by $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{V}_3$ and $\mathbf{V}_4$ the unipotent radicals of $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ and $\mathbf{P}_4$ respectively.

We define the varieties

$$\mathcal{Y}_{1,2,3} = \{(g_1\mathbf{V}_1, g_2\mathbf{V}_2, g_3\mathbf{V}_3) \in \mathbf{G}/\mathbf{V}_1 \times \mathbf{G}/\mathbf{V}_2 \times \mathbf{G}/\mathbf{V}_3 \mid g_1^{-1}g_2 \in \mathbf{V}_1 \cdot \mathbf{V}_2 \text{ and } g_2^{-1}g_3 \in \mathbf{V}_2 \cdot \mathbf{V}_3\},$$

$$\mathcal{Y}_{1,2,3}^{\mathrm{cl}} = \{(g_1\mathbf{V}_1, g_2\mathbf{V}_2, g_3\mathbf{V}_3) \in \mathcal{Y}_{1,2,3} \mid g_1^{-1}g_3 \in \mathbf{V}_1 \cdot \mathbf{V}_3\}$$

and

$$\mathcal{Y}_{1,3} = \{(g_1\mathbf{V}_1, g_3\mathbf{V}_3) \in \mathbf{G}/\mathbf{V}_1 \times \mathbf{G}/\mathbf{V}_3 \mid g_1^{-1}g_3 \in \mathbf{V}_1 \cdot \mathbf{V}_3\}.$$

We denote by $i_{1,3} : \mathcal{Y}_{1,2,3}^{\mathrm{cl}} \hookrightarrow \mathcal{Y}_{1,2,3}$ the closed immersion and we define

$$\begin{array}{cccc} \pi_{1,3} : & \mathcal{Y}_{1,2,3}^{\mathrm{cl}} & \longrightarrow & \mathcal{Y}_{1,3} \\ & (g_1\mathbf{V}_1, g_2\mathbf{V}_2, g_3\mathbf{V}_3) & \longmapsto & (g_1\mathbf{V}_1, g_3\mathbf{V}_3). \end{array}$$

All these varieties are endowed with a diagonal action of $\mathbf{G}$, and the morphisms $i_{1,3}$ and $\pi_{1,3}$ are $\mathbf{G}$-equivariant.

Proposition 5.1. — *We have:*

(a) $\dim(\mathbf{V}_1) = \dim(\mathbf{V}_2) = \dim(\mathbf{V}_3)$.

(b) $\dim(\mathscr{Y}_{1,2,3}) - \dim(\mathscr{Y}_{1,3}) = \dim(\mathbf{V}_1) + \dim(\mathbf{V}_1 \cap \mathbf{V}_3) - \dim(\mathbf{V}_1 \cap \mathbf{V}_2) - \dim(\mathbf{V}_2 \cap \mathbf{V}_3)$.

(c) $\dim(\mathscr{Y}_{1,2,3}) - \dim(\mathscr{Y}_{1,3}) = 2\big(\dim(\mathbf{V}_1 \cap \mathbf{V}_3) - \dim(\mathbf{V}_1 \cap \mathbf{V}_2 \cap \mathbf{V}_3)\big)$.

Proof. — (a) is well-known. Also,

$$\begin{aligned}\dim(\mathscr{Y}_{1,2,3}) &= \dim(\mathbf{G}/\mathbf{V}_1) + \dim(\mathbf{V}_1 \cdot \mathbf{V}_2/\mathbf{V}_2) + \dim(\mathbf{V}_2 \cdot \mathbf{V}_3/\mathbf{V}_3) \\ &= \dim(\mathbf{G}/\mathbf{V}_1) + \dim(\mathbf{V}_1) - \dim(\mathbf{V}_1 \cap \mathbf{V}_2) + \dim(\mathbf{V}_2) - \dim(\mathbf{V}_2 \cap \mathbf{V}_3)\end{aligned}$$

while

$$\begin{aligned}\dim(\mathscr{Y}_{1,3}) &= \dim(\mathbf{G}/\mathbf{V}_1) + \dim(\mathbf{V}_1 \cdot \mathbf{V}_3/\mathbf{V}_3) \\ &= \dim(\mathbf{G}/\mathbf{V}_1) + \dim(\mathbf{V}_1) - \dim(\mathbf{V}_1 \cap \mathbf{V}_3).\end{aligned}$$

So (b) follows from the two equalities (and from (a)).

Let us now prove (c). For this, we may assume that $\mathbf{T} \subset \mathbf{L}$. Let Φ_i denote the set of roots $\alpha \in \Phi$ such that $\mathbf{U}_\alpha \subset \mathbf{V}_i$. Then $\Phi_1 \cup -\Phi_1 = \Phi_2 \cup -\Phi_2 = \Phi_3 \cup -\Phi_3$. In particular,

$$\Phi_1 \cup -\Phi_1 = (\Phi_1 \cup \Phi_2 \cup \Phi_3) \cup -(\Phi_1 \cap \Phi_2 \cap \Phi_3).$$

Therefore

$$2|\Phi_1| = |\Phi_1 \cup \Phi_2 \cup \Phi_3| + |\Phi_1 \cap \Phi_2 \cap \Phi_3|.$$

On the other hand, by general facts about the cardinality of a union of finite sets,

$$|\Phi_1 \cup \Phi_2 \cup \Phi_3| = |\Phi_1| + |\Phi_2| + |\Phi_3| - |\Phi_1 \cap \Phi_2| - |\Phi_1 \cap \Phi_3| - |\Phi_2 \cap \Phi_3| + |\Phi_1 \cap \Phi_2 \cap \Phi_3|.$$

Hence (c) follows from (a), (b) and from these last two equalities. □

Let $d_{1,3} = \dim(\mathbf{V}_1 \cap \mathbf{V}_3) - \dim(\mathbf{V}_1 \cap \mathbf{V}_2 \cap \mathbf{V}_3)$. By Proposition 5.1, we have

$$d_{1,3} = \frac{1}{2}\big(\dim(\mathscr{Y}_{1,2,3}) - \dim(\mathscr{Y}_{1,3})\big).$$

Let

$$\begin{array}{rccc}\kappa_{1,3}: & \mathbf{G}/(\mathbf{V}_1 \cap \mathbf{V}_3) & \longrightarrow & \mathscr{Y}_{1,3} \\ & g(\mathbf{V}_1 \cap \mathbf{V}_3) & \longmapsto & (g\mathbf{V}_1, g\mathbf{V}_3)\end{array}$$

and

$$\begin{array}{rccc}\kappa_{1,2,3}^{\mathrm{cl}}: & \mathbf{G}/(\mathbf{V}_1 \cap \mathbf{V}_2 \cap \mathbf{V}_3) & \longrightarrow & \mathscr{Y}_{1,2,3}^{\mathrm{cl}} \\ & g(\mathbf{V}_1 \cap \mathbf{V}_2 \cap \mathbf{V}_3) & \longmapsto & (g\mathbf{V}_1, g\mathbf{V}_2, g\mathbf{V}_3).\end{array}$$

Both maps are $\mathbf{G}$-equivariant morphisms of varieties.

Proposition 5.2. — *The maps $\kappa_{1,3}$ and $\kappa_{1,2,3}^{\mathrm{cl}}$ are isomorphisms of varieties.*

Proof. — The fact that $\kappa_{1,3}$ is an isomorphism is clear. It is also clear that $\kappa^{\mathrm{cl}}_{1,2,3}$ is a closed immersion. It is so sufficient to prove that $\kappa^{\mathrm{cl}}_{1,2,3}$ is surjective.

So, let $(g_1\mathbf{V}_1, g_2\mathbf{V}_2, g_3\mathbf{V}_3) \in \mathscr{Y}^{\mathrm{cl}}_{1,2,3}$. Using the $\mathbf{G}$-action and the fact that $\kappa_{1,3}$ is an isomorphism, we may assume that $g_1 = g_3 = 1$. Therefore,

$$g_2 \in (\mathbf{V}_1 \cdot \mathbf{V}_2) \cap (\mathbf{V}_3 \cdot \mathbf{V}_2).$$

By the uniqueness of the factorisation of elements in a big cell, we have

$$(\mathbf{V}_1 \cdot \mathbf{V}_2) \cap (\mathbf{V}_3 \cdot \mathbf{V}_2) = (\mathbf{V}_1 \cap \mathbf{V}_3) \cdot \mathbf{V}_2.$$

So there exists $h \in \mathbf{V}_1 \cap \mathbf{V}_3$ such that $h\mathbf{V}_2 = g_2\mathbf{V}_2$. It is then clear that $(g_1\mathbf{V}_1, g_2\mathbf{V}_2, g_3\mathbf{V}_3) = \kappa^{\mathrm{cl}}_{1,2,3}(h)$, as desired. □

Corollary 5.3. — *The map $\pi_{1,3}$ is a smooth morphism with fibers isomorphic to the affine space of dimension $d_{1,3}$. Moreover,*

$$\dim(\mathscr{Y}_{1,2,3}) - \dim(\mathscr{Y}^{\mathrm{cl}}_{1,2,3}) = \dim(\mathscr{Y}^{\mathrm{cl}}_{1,2,3}) - \dim(\mathscr{Y}_{1,3}) = d_{1,3}.$$

Proof. — Using the isomorphisms $\kappa_{1,3}$ and $\kappa^{\mathrm{cl}}_{1,2,3}$ of Proposition 5.2, the map $\pi_{1,3}$ may be identified with the canonical projection $\mathbf{G}/(\mathbf{V}_1 \cap \mathbf{V}_3) \twoheadrightarrow \mathbf{G}/(\mathbf{V}_1 \cap \mathbf{V}_2 \cap \mathbf{V}_3)$. The corollary follows. □

Let us now define

$$\begin{aligned}\mathscr{Y}^{\mathrm{cl}}_{1,2,3,4} \quad = \quad & \{(g_1\mathbf{V}_1, g_2\mathbf{V}_2, g_3\mathbf{V}_3, g_4\mathbf{V}_4) \in \mathbf{G}/\mathbf{V}_1 \times \mathbf{G}/\mathbf{V}_2 \times \mathbf{G}/\mathbf{V}_3 \times \mathbf{G}/\mathbf{V}_4 \mid \\ & \qquad g_1^{-1}g_2 \in \mathbf{V}_1 \cdot \mathbf{V}_2,\ g_2^{-1}g_3 \in \mathbf{V}_2 \cdot \mathbf{V}_3,\ g_3^{-1}g_4 \in \mathbf{V}_3 \cdot \mathbf{V}_4 \\ & \qquad \text{and } g_1^{-1}g_4 \in \mathbf{V}_1 \cdot \mathbf{V}_4\},\end{aligned}$$

$$\mathscr{Y}^{\mathrm{cl},2}_{1,2,3,4} = \{(g_1\mathbf{V}_1, g_2\mathbf{V}_2, g_3\mathbf{V}_3, g_4\mathbf{V}_4) \in \mathscr{Y}^{\mathrm{cl}}_{1,2,3,4} \mid g_1^{-1}g_3 \in \mathbf{V}_1 \cdot \mathbf{V}_3\},$$

and
$$\mathscr{Y}^{\mathrm{cl},3}_{1,2,3,4} = \{(g_1\mathbf{V}_1, g_2\mathbf{V}_2, g_3\mathbf{V}_3, g_4\mathbf{V}_4) \in \mathscr{Y}^{\mathrm{cl}}_{1,2,3,4} \mid g_2^{-1}g_4 \in \mathbf{V}_2 \cdot \mathbf{V}_4\}.$$

Then:

Corollary 5.4. — *Assume that at least one of the following holds:*

(1) $\mathbf{V}_1 \subset \mathbf{V}_4 \cdot \mathbf{V}_2$.
(2) $\mathbf{V}_2 \subset \mathbf{V}_1 \cdot \mathbf{V}_3$.
(3) $\mathbf{V}_3 \subset \mathbf{V}_2 \cdot \mathbf{V}_4$.
(4) $\mathbf{V}_4 \subset \mathbf{V}_3 \cdot \mathbf{V}_1$.

Then $\mathscr{Y}^{\mathrm{cl},2}_{1,2,3,4} = \mathscr{Y}^{\mathrm{cl},3}_{1,2,3,4}$.

Proof. — Using the fact that the map $(g_1\mathbf{V}_1, g_2\mathbf{V}_2, g_3\mathbf{V}_3, g_4\mathbf{V}_4) \mapsto (g_4\mathbf{V}_4, g_1\mathbf{V}_1, g_2\mathbf{V}_2, g_3\mathbf{V}_3)$ induces an isomorphism $\mathscr{Y}^{\mathrm{cl}}_{1,2,3,4} \simeq \mathscr{Y}^{\mathrm{cl}}_{4,1,2,3}$ (with obvious notation), we see that it is sufficient to prove only one of the statements.

So let us assume that $\mathbf{V}_2 \subset \mathbf{V}_1 \cdot \mathbf{V}_3$. Let $(g_1\mathbf{V}_1, g_2\mathbf{V}_2, g_3\mathbf{V}_3, g_4\mathbf{V}_4) \in \mathscr{Y}^{\mathrm{cl}}_{1,2,3,4}$. Then $g_1^{-1}g_3 = (g_1^{-1}g_2)(g_2^{-1}g_3) \in \mathbf{V}_1 \cdot \mathbf{V}_2 \cdot \mathbf{V}_3 = \mathbf{V}_1 \cdot \mathbf{V}_3$ and so $\mathscr{Y}^{\mathrm{cl},2}_{1,2,3,4} = \mathscr{Y}^{\mathrm{cl}}_{1,2,3,4}$. So it remains to prove that $(g_1\mathbf{V}_1, g_2\mathbf{V}_2, g_3\mathbf{V}_3, g_4\mathbf{V}_4) \in \mathscr{Y}^{\mathrm{cl},3}_{1,2,3,4}$. Using the action of $\mathbf{G}$ and the isomorphism $\kappa^{\mathrm{cl}}_{1,3,4}$ of Proposition 5.2, we may assume that $g_1 = g_3 = g_4 = 1$. But then $g_2^{-1}g_4 = g_2^{-1} \in \mathbf{V}_2 \subset \mathbf{V}_2 \cdot \mathbf{V}_4$, as desired. □

Remark 5.5. — Let w_1, w_2 and w_3 be three elements of W and let us assume here that $\mathbf{V}_1 = \mathbf{U}$, $\mathbf{V}_2 = {}^{w_1}\mathbf{V}_1$, $\mathbf{V}_3 = {}^{w_1 w_2}\mathbf{V}_1$ and $\mathbf{V}_4 = {}^{w_1 w_2 w_3}\mathbf{V}_1$. Then the conditions (1), (2), (3) and (4) of Corollary 5.4 are respectively equivalent to:

(1) $l(w_2 w_3) = l(w_1 w_2 w_3) + l(w_1)$.
(2) $l(w_1 w_2) = l(w_1) + l(w_2)$.
(3) $l(w_2 w_3) = l(w_2) + l(w_3)$.
(4) $l(w_1 w_2) = l(w_1 w_2 w_3) + l(w_3)$.

5.B. Setting. — We fix a positive integer r. Given a family of objects $m_1, \ldots, m_r$ belonging to a structure acted on by F, we put $m_{j+er} = F^e(m_j)$ for $1 \leqslant j \leqslant r$ and $e \geqslant 0$.

Let $\mathbf{n} = (n_1, \ldots, n_r)$ be a sequence of elements of $N_{\mathbf{G}}(\mathbf{T})$. We denote by w_i the image of n_i in W and we put $w = w_1 \cdots w_r$.

We define

$$\mathbf{Y}(\mathbf{n}) = \{(g_1\mathbf{U}, g_2\mathbf{U}, \ldots, g_r\mathbf{U}) \in (\mathbf{G}/\mathbf{U})^r \mid g_j \xrightarrow{n_j} g_{j+1} \;\; \forall 1 \leqslant j \leqslant r\}$$

where $g_j \xrightarrow{n_j} g_{j+1}$ means $g_j^{-1} g_{j+1} \in \mathbf{U} n_j \mathbf{U}$. This variety has a left action by multiplication of $\mathbf{G}^F$ and a right action by multiplication of $\mathbf{T}^{wF}$.

We fix a positive integer m such that $F^m(n_i) = n_i$ for all i. The action of F^m on $(\mathbf{G}/\mathbf{U})^r$ restricts to an action on $\mathbf{Y}(\mathbf{n})$.

Given $\mathbf{Z}$ a variety of pure dimension n, we put $\mathrm{R}\Gamma_{\mathrm{c}}^{\dim}(\mathbf{Z}, \Lambda) = \mathrm{R}\Gamma_{\mathrm{c}}(\mathbf{Z}, \Lambda)[n](n/2)$, where $(n/2)$ denotes a Tate twist.

Given $2 \leqslant j \leqslant r$, we denote by $\mathbf{Y}^{\mathrm{cl}}_j(\mathbf{n})$ the F^m-stable closed subvariety of $\mathbf{Y}(\mathbf{n})$ defined by

$$\mathbf{Y}^{\mathrm{cl}}_j(\mathbf{n}) = \{(g_1\mathbf{U}, g_2\mathbf{U}, \ldots, g_r\mathbf{U}) \in \mathbf{Y}(\mathbf{n}) \mid g_{j-1}^{-1} g_{j+1} \in \mathbf{U} n_{j-1} n_j \mathbf{U}\}$$

and we denote by $\mathbf{Y}_j^{\mathrm{op}}(\mathbf{n})$ its open complement. They are both stable under the action of $\mathbf{G}^F \times \mathbf{T}^{wF}$. We denote by $\pi_j : (\mathbf{G}/\mathbf{U})^r \to (\mathbf{G}/\mathbf{U})^{r-1}$ the morphism of varieties which forgets the j-th component and we set

$$c_j(\mathbf{n}) = (n_1, n_2, \ldots, n_{j-2}, n_{j-1}n_j, n_{j+1}, \ldots, n_r)$$

and
$$d_j(\mathbf{n}) = \frac{l(w_{j-1}) + l(w_j) - l(w_{j-1}w_j)}{2}.$$

Let $i_{\mathbf{n},j} : \mathbf{Y}_j^{\mathrm{cl}}(\mathbf{n}) \hookrightarrow \mathbf{Y}_j(\mathbf{n})$ denote the closed immersion and

$$\pi_{\mathbf{n},j} : \mathbf{Y}_j^{\mathrm{cl}}(\mathbf{n}) \to \mathbf{Y}(c_j(\mathbf{n}))$$

denote the restriction of π_j. Note that $\pi_{\mathbf{n},j}$ is $(\mathbf{G}^F, \mathbf{T}^{wF})$-equivariant and commutes with the action of F^m. The next lemma follows from Corollary 5.3.

Lemma 5.6. — *If $2 \leqslant j \leqslant r$, then $\pi_{\mathbf{n},j}$ is a smooth morphism with fibers isomorphic to an affine space of dimension $d_j(\mathbf{n})$. Moreover, the codimension of $\mathbf{Y}_j^{\mathrm{cl}}(\mathbf{n})$ in $\mathbf{Y}(\mathbf{n})$ is also equal to $d_j(\mathbf{n})$.*

The map $\pi_{\mathbf{n},j}$ induces a quasi-isomorphism of complexes of $(\Lambda\mathbf{G}^F, \Lambda\mathbf{T}^{wF})$-bimodules

$$\mathrm{R}\Gamma_c(\mathbf{Y}_j^{\mathrm{cl}}(\mathbf{n}), \Lambda) \xrightarrow{\sim} \mathrm{R}\Gamma_c(\mathbf{Y}(c_j(\mathbf{n})), \Lambda)[-2d_j(\mathbf{n})](-d_j(\mathbf{n})). \tag{5.7}$$

Composing this isomorphism with the morphism $i_{\mathbf{n},j}^* : \mathrm{R}\Gamma_c(\mathbf{Y}(\mathbf{n}), \Lambda) \to \mathrm{R}\Gamma_c(\mathbf{Y}_j^{\mathrm{cl}}(\mathbf{n}), \Lambda)$, we obtain a morphism of complexes of $(\Lambda\mathbf{G}^F, \Lambda\mathbf{T}^{wF})$-bimodules

$$\Psi_{\mathbf{n},j} : \mathrm{R}\Gamma_c^{\dim}(\mathbf{Y}(\mathbf{n}), \Lambda) \longrightarrow \mathrm{R}\Gamma_c^{\dim}(\mathbf{Y}(c_j(\mathbf{n})), \Lambda)$$

which commutes with the action of F^m, and whose cone is quasi-isomorphic to $\mathrm{R}\Gamma_c^{\dim}(\mathbf{Y}_j^{\mathrm{op}}(\mathbf{n}), \Lambda)[1]$.

5.C. Preliminaries. — We first recall some results from [**BoRo1**].

We denote by B the braid group of W, and by $\sigma : W \to B$ the unique map (not a group morphism) that is a right inverse to the canonical map $B \to W$ and that preserves lengths. We extend it to sequences of elements of W by $\sigma(w_1, \ldots, w_r) = \sigma(w_1)\cdots\sigma(w_r)$.

We denote by $n \mapsto \bar{n} : N_{\mathbf{G}}(\mathbf{T}) \to W$ the quotient map. We fix $\dot{\sigma} : N_{\mathbf{G}}(\mathbf{T}) \to B \ltimes \mathbf{T}$ a map (not a group morphism) such that $\dot{\sigma}(nt) = \dot{\sigma}(n)t$ for all $t \in \mathbf{T}$ and such that the image of $\dot{\sigma}(n)$ in $B = (B \ltimes \mathbf{T})/\mathbf{T}$ is equal to $\sigma(\bar{n})$. We extend it to sequences of elements of $N_{\mathbf{G}}(\mathbf{T})$ by $\dot{\sigma}(n_1, \ldots, n_r) = \dot{\sigma}(n_1)\cdots\dot{\sigma}(n_r)$.

The following result is [**BoRo1**, Proposition 5.4].

Lemma 5.8. — *Let $\mathbf{n}'$ be a sequence of elements of $N_{\mathbf{G}}(\mathbf{T})$. Then:*

(a) *If* $\dot\sigma(\mathbf{n}) = \dot\sigma(\mathbf{n}')$ *(they are elements of* $B \ltimes \mathbf{T}$*), then the varieties* $\mathbf{Y}(\mathbf{n})$ *and* $\mathbf{Y}(\mathbf{n}')$ *are canonically isomorphic* $\mathbf{G}^F$*-varieties-*$\mathbf{T}^{wF}$*.*

(b) *If* $\sigma(\overline{\mathbf{n}}) = \sigma(\overline{\mathbf{n}}')$ *(they are elements of* B*), then the varieties* $\mathbf{Y}(\mathbf{n})$ *and* $\mathbf{Y}(\mathbf{n}')$ *are (non-canonically) isomorphic* $\mathbf{G}^F$*-varieties-*$\mathbf{T}^{wF}$*.*

Proof. — (a) is proved in [**BoRo1**, 5.5], while (b) is [**BoRo1**, Proposition 5.4]. □

Using Lemma 5.8(a), we shall now write $\mathbf{Y}(\mathbf{n}) = \mathbf{Y}(\mathbf{n}')$ when $\dot\sigma(\mathbf{n}) = \dot\sigma(\mathbf{n}')$. Strictly speaking, Lemma 5.8(a) says that these two varieties are only isomorphic but, since this isomorphism is canonical, we shall use the symbol = to simplify the exposition.

We define the *cyclic shift* $\mathrm{sh}(\mathbf{n})$ of $\mathbf{n}$ by

$$\mathrm{sh}(\mathbf{n}) = (n_2, \ldots, n_r, F(n_1)).$$

The next result is proved in [**DigMiRo**, Proposition 3.1.6] for the varieties $\mathbf{X}(\mathbf{w})$ and $\mathbf{X}(\mathbf{w}')$. The same proof shows the more precise result below.

Lemma 5.9. — *The map*

$$\begin{array}{ccc} \mathbf{Y}(\mathbf{n}) & \longrightarrow & \mathbf{Y}(\mathrm{sh}(\mathbf{n})) \\ (g_1\mathbf{U}, \ldots, g_r\mathbf{U}) & \longmapsto & (g_2\mathbf{U}, \ldots, g_r\mathbf{U}, F(g_1)\mathbf{U}) \end{array}$$

induces an equivalence of étale sites. Moreover, it is a morphism of $\mathbf{G}^F$*-varieties-*$\mathbf{T}^{wF}$*, where* $t \in \mathbf{T}^{wF}$ *acts on* $\mathbf{Y}(\mathrm{sh}(\mathbf{n}))$ *by right multiplication by* $n_1^{-1}\, t\, n_1$*. Consequently, the diagram*

$$\begin{tikzcd} \mathrm{D}^b(\Lambda\mathbf{T}^{w_1^{-1} w F(w_1) F}) \arrow[rr, "n_{1,*}"] \arrow[dr, "\mathscr{R}_{\mathrm{sh}(\mathbf{n})}"'] & & \mathrm{D}^b(\Lambda\mathbf{T}^{wF}) \arrow[dl, "\mathscr{R}_{\mathbf{n}}"] \\ & \mathrm{D}^b(\Lambda\mathbf{G}^F) & \end{tikzcd}$$

is commutative.

Assume in the remaining part of §5.C that $3 \leqslant j \leqslant r$ (in particular, $r \geqslant 3$). Note that $c_{j-1}(c_j(\mathbf{n})) = c_{j-1}(c_{j-1}(\mathbf{n}))$. Consider the diagram

$$\begin{tikzcd} \mathrm{R}\Gamma_c^{\dim}(\mathbf{Y}(\mathbf{n}), \Lambda) \arrow[r, "\Psi_{\mathbf{n},j}"] \arrow[d, "\Psi_{\mathbf{n},j-1}"'] & \mathrm{R}\Gamma_c^{\dim}(\mathbf{Y}(c_j(\mathbf{n})), \Lambda) \arrow[d, "\Psi_{c_j(\mathbf{n}),j-1}"] \\ \mathrm{R}\Gamma_c^{\dim}(\mathbf{Y}(c_{j-1}(\mathbf{n})), \Lambda) \arrow[r, "\Psi_{c_{j-1}(\mathbf{n}),j-1}"'] & \mathrm{R}\Gamma_c^{\dim}(\mathbf{Y}(c_{j-1}(c_j(\mathbf{n}))), \Lambda). \end{tikzcd} \tag{5.10}$$

It does not seem reasonable to expect that the diagram (5.10) is commutative in general. However, it is in some cases.

Let us first define the folowing two varieties:

$$\mathbf{Y}^{\mathrm{cl}}_{j,j-1}(\mathbf{n}) = \mathbf{Y}^{\mathrm{cl}}_{j-1}(c_j(\mathbf{n})) \times_{\mathbf{Y}(c_j(\mathbf{n}))} \mathbf{Y}^{\mathrm{cl}}_{j}(\mathbf{n})$$

and
$$\mathbf{Y}^{\mathrm{cl}}_{j-1,j}(\mathbf{n}) = \mathbf{Y}^{\mathrm{cl}}_{j-1}(c_{j-1}(\mathbf{n})) \times_{\mathbf{Y}(c_{j-1}(\mathbf{n}))} \mathbf{Y}^{\mathrm{cl}}_{j-1}(\mathbf{n}).$$

More concretely, they are the closed subvarieties of $\mathbf{Y}(\mathbf{n})$ defined by

$$\mathbf{Y}^{\mathrm{cl}}_{j,j-1}(\mathbf{n}) = \{(g_1\mathbf{U},\dots,g_r\mathbf{U}) \in \mathbf{Y}(\mathbf{n}) \mid g_{j-2}\mathbf{U} \xrightarrow{\;n_{j-2}n_{j-1}n_j\;} g_{j+1}\mathbf{U} \text{ and } g_{j-1}\mathbf{U} \xrightarrow{\;n_{j-1}n_j\;} g_{j+1}\mathbf{U}\}$$

and
$$\mathbf{Y}^{\mathrm{cl}}_{j-1,j}(\mathbf{n}) = \{(g_1\mathbf{U},\dots,g_r\mathbf{U}) \in \mathbf{Y}(\mathbf{n}) \mid g_{j-2}\mathbf{U} \xrightarrow{\;n_{j-2}n_{j-1}n_j\;} g_{j+1}\mathbf{U} \text{ and } g_{j-2}\mathbf{U} \xrightarrow{\;n_{j-2}n_{j-1}\;} g_{j}\mathbf{U}\}.$$

Lemma 5.11. — *If* $\mathbf{Y}^{\mathrm{cl}}_{j,j-1}(\mathbf{n}) = \mathbf{Y}^{\mathrm{cl}}_{j-1,j}(\mathbf{n})$*, then the diagram (5.10) is commutative.*

Proof. — For the purpose of this proof, we also define

$$\mathbf{INT}_j(\mathbf{n}) = \mathbf{Y}^{\mathrm{cl}}_{j,j-1}(\mathbf{n}) \cap \mathbf{Y}^{\mathrm{cl}}_{j-1,j}(\mathbf{n})$$

and
$$\mathbf{FIB}_j(\mathbf{n}) = \mathbf{Y}^{\mathrm{cl}}_{j-1}(c_{j-1}(\mathbf{n})) \times_{\mathbf{Y}(c_{j-1}(c_j(\mathbf{n})))} \mathbf{Y}^{\mathrm{cl}}_{j-1}(c_j(\mathbf{n})).$$

Again, concretely,

$$\begin{aligned}
\mathbf{FIB}_j(\mathbf{n}) \quad = \quad & \{(g_1\mathbf{U}, g_2\mathbf{U},\dots,g_r\mathbf{U}) \in (\mathbf{G}/\mathbf{U})^r \mid \\
& \qquad (g_1\mathbf{U},\dots,g_{j-2}\mathbf{U}, g_j\mathbf{U},\dots,g_r\mathbf{U}) \in \mathbf{Y}(c_{j-1}(\mathbf{n})), \\
& \qquad (g_1\mathbf{U},\dots,g_{j-1}\mathbf{U}, g_{j+1}\mathbf{U},\dots,g_r\mathbf{U}) \in \mathbf{Y}(c_{j}(\mathbf{n})) \\
& \qquad \text{and } g_{j-2}\mathbf{U} \xrightarrow{\;n_{j-2}n_{j-1}n_j\;} g_{j+1}\mathbf{U}\}
\end{aligned}$$

and
$$\mathbf{INT}_j(\mathbf{n}) = \{(g_1\mathbf{U}, g_2\mathbf{U},\dots,g_r\mathbf{U}) \in \mathbf{FIB}_j(\mathbf{n}) \mid g_{j-1}\mathbf{U} \xrightarrow{\;n_2\;} g_j\mathbf{U}\}.$$

This shows that all these varieties fit into the following commutative diagram, in which all the arrows of the form $\hookrightarrow$ are closed immersions and all the arrows of the form $\twoheadrightarrow$ are smooth morphisms with fibers isomorphic to an affine space:

(5.12)

$$\begin{tikzcd}
 & & & \mathbf{Y}(\mathbf{n}) & & & \\
\mathbf{Y}^{\mathrm{cl}}_{j-1}(\mathbf{n}) \arrow[urrr, hook, "i_{\mathbf{n},j-1}"] \arrow[ddd, two heads, "\pi_{\mathbf{n},j-1}"'] & & & & & & \mathbf{Y}^{\mathrm{cl}}_{j}(\mathbf{n}) \arrow[ulll, hook', "i_{\mathbf{n},j}"'] \arrow[ddd, two heads, "\pi_{\mathbf{n},j}"] \\
 & & \mathbf{Y}^{\mathrm{cl}}_{j-1,j}(\mathbf{n}) \arrow[ull, hook', "i"'] \arrow[ddd, two heads, "\pi"'] & & \mathbf{Y}^{\mathrm{cl}}_{j,j-1}(\mathbf{n}) \arrow[urr, hook, "i'"] \arrow[ddd, two heads, "\pi'"] & & \\
 & \blacksquare & & \mathbf{INT}_j(\mathbf{n}) \arrow[ul, hook'] \arrow[ur, hook] \arrow[d, hook] & & \blacksquare & \\
\mathbf{Y}(c_{j-1}(\mathbf{n})) & & & \mathbf{FIB}_j(\mathbf{n}) \arrow[dl, two heads] \arrow[dr, two heads] & & & \mathbf{Y}(c_j(\mathbf{n})) \\
 & & \mathbf{Y}^{\mathrm{cl}}_{j-1}(c_{j-1}(\mathbf{n})) \arrow[ull, hook', "i_{c_{j-1}(\mathbf{n}),j-1}"] \arrow[dr, two heads, "\pi_{c_{j-1}(\mathbf{n}),j-1}"'] & \blacksquare & \mathbf{Y}^{\mathrm{cl}}_{j-1}(c_j(\mathbf{n})) \arrow[urr, hook, "i_{c_j(\mathbf{n}),j-1}"'] \arrow[dl, two heads, "\pi_{c_j(\mathbf{n}),j-1}"] & & \\
 & & & \mathbf{Y}(c_{j-1}(c_j(\mathbf{n}))) & & &
\end{tikzcd}$$

Note that the three squares marked with the symbol $\blacksquare$ are cartesian. By the proper base change Theorem, the composition $\Psi_{c_{j-1}(\mathbf{n}),j-1} \circ \Psi_{\mathbf{n},j-1}$ is obtained as the composition of $(i_{\mathbf{n},j-1} \circ i)^*$ with the inverse of the isomorphism induced by $(\pi_{c_{j-1}(\mathbf{n}),j-1} \circ \pi)^*$. Similarly, the composition $\Psi_{c_j(\mathbf{n}),j-1} \circ \Psi_{\mathbf{n},j}$ is equal to the composition of $(i_{\mathbf{n},j} \circ i')^*$ with the inverse of the isomorphism induced by $(\pi_{c_j(\mathbf{n}),j-1} \circ \pi')^*$. The lemma follows. □

Lemma 5.13. — *Assume that one of the following holds:*

(1) $l(w_{j-2}w_{j-1}) = l(w_{j-2}) + l(w_{j-1})$.
(2) $l(w_{j-1}w_j) = l(w_{j-1}) + l(w_j)$.
(3) $l(w_{j-2}w_{j-1}) = l(w_{j-2}w_{j-1}w_j) + l(w_j)$.
(4) $l(w_{j-1}w_j) = l(w_{j-2}) + l(w_{j-2}w_{j-1}w_j)$.

Then the diagram (5.10) is commutative.

Proof. — It is sufficient, by Lemma 5.11, to prove that, if (1), (2), (3) or (4) holds, then $\mathbf{Y}^{\mathrm{cl}}_{j,j-1}(\mathbf{n}) = \mathbf{Y}^{\mathrm{cl}}_{j-1,j}(\mathbf{n})$. This follows, after base change, from Corollary 5.4. □

5.D. Comparison of complexes. — We start with the description of varieties of the form $\mathbf{Y}_1^{\mathrm{op}}(\mathbf{n}')$ in a very special case, which will be the fundamental step in the proof of Theorem 5.16.

Let $\mathbf{w} = \bar{\mathbf{n}} = (w_1, \dots, w_r)$. Given $\alpha \in \Delta$, we define a subgroup of $\mathbf{T}^{r+1}$

$$\mathbf{S}(\alpha, \mathbf{w}) = \{(a_1, \dots, a_{r+1}) \in \mathbf{T}^{r+1} \mid a_1^{-1} s_\alpha a_2 s_\alpha^{-1} \in \mathbf{T}_{\alpha^\vee},\ a_i^{-1} w_{i-1} a_{i+1} w_i^{-1} = 1 \text{ for } 2 \leqslant i \leqslant r \\ \text{and } a_{r+1}^{-1} w_r F(a_1) w_r^{-1} = 1\}.$$

Let $x \in \{1, s_\alpha\}$. The group morphism

$$\mathbf{T} \to \mathbf{T}^{r+1},\ a \mapsto (a, xax^{-1}, w_1 xax^{-1} w_1^{-1}, \dots, w_{r-1} \cdots w_1 xax^{-1} w_1^{-1} \cdots w_{r-1}^{-1})$$

restricts to an embedding of $\mathbf{T}^{xwF}$ in $\mathbf{S}(\alpha, \mathbf{w})$.

Given $\mathbf{a} = (a_1, \dots, a_m)$ and $\mathbf{b} = (b_1, \dots, b_n)$ two sequences, we denote the concatenation of the sequences by $\mathbf{a} \bullet \mathbf{b} = (a_1, \dots, a_m, b_1, \dots, b_n)$.

Lemma 5.14. *— Let $\alpha \in \Delta$ and and let $\dot{s}$ be a representative of s_α in $N_{\mathbf{G}}(\mathbf{T}) \cap \mathbf{G}_\alpha$. We assume that $\mathbf{G}_\alpha \simeq \mathbf{SL}_2(\mathbb{F})$. There exists a closed immersion $\mathbf{Y}(\dot{s} \bullet \mathbf{n}) \hookrightarrow \mathbf{Y}_2^{\mathrm{op}}((\dot{s}, \dot{s}^{-1}) \bullet \mathbf{n})$ and an action of $\mathbf{S}(\alpha, \mathbf{w})$ on $\mathbf{Y}_2^{\mathrm{op}}((\dot{s}, \dot{s}^{-1}) \bullet \mathbf{n})$ such that*

$$\mathbf{Y}_2^{\mathrm{op}}((\dot{s}, \dot{s}^{-1}) \bullet \mathbf{n}) \simeq \mathbf{Y}(\dot{s} \bullet \mathbf{n}) \times_{\mathbf{T}^{s_\alpha wF}} \mathbf{S}(\alpha, \mathbf{w}),$$

as $\mathbf{G}^F$-varieties-$\mathbf{T}^{wF}$.

Proof. — Given $i \in \{1, \dots, r\}$, consider a reduced decomposition $w_i = s_{i,1} \cdots s_{i,d_i}$. We put $\tilde{\mathbf{w}} = (s_{1,1}, \dots, s_{1,d_1}, s_{2,1}, \dots, s_{2,d_2}, \dots, s_{r,1}, \dots, s_{r,d_r})$. Note that $\mathbf{S}(\alpha, \mathbf{w})$ is isomorphic to the group $\mathbf{S}(s_\alpha \bullet \tilde{\mathbf{w}}, 1 \bullet \tilde{\mathbf{w}})$ defined in [**BoRo1**, §4.4.3]:

$$\mathbf{S}(s_\alpha \bullet \tilde{\mathbf{w}}, 1 \bullet \tilde{\mathbf{w}}) \xrightarrow{\sim} \mathbf{S}(\alpha, \mathbf{w}),\ (a_1, \dots, a_{1+d_1+\dots+d_r}) \mapsto (a_1, a_2, a_{2+d_1}, a_{2+d_1+d_2}, \dots, a_{2+d_1+\dots+d_{r-1}}).$$

The following computation in $\mathbf{SL}_2(\mathbb{F}) \simeq \mathbf{G}_\alpha$

$$(\#) \qquad \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 - xy & x + z - xyz \\ -y & 1 - yz \end{pmatrix}$$

shows that the map

$$\begin{array}{ccc} \mathbf{U}_\alpha \times (\mathbf{U}_\alpha \setminus \{1\}) \times \mathbf{U}_\alpha & \longrightarrow & \mathbf{U}_\alpha \mathbf{T}_{\alpha^\vee} \dot{s} \mathbf{U}_\alpha \\ (u_1, u_2, u_3) & \longmapsto & u_1 \dot{s} u_2 \dot{s}^{-1} u_3 \end{array}$$

is an isomorphism of varieties. Therefore, one may forget the second coordinate in the definition of the variety $\mathbf{Y}_2^{\mathrm{op}}((\dot{s}, \dot{s}^{-1}) \bullet \mathbf{n})$ and we get

$$(5.15) \qquad \begin{array}{rcl} \mathbf{Y}_2^{\mathrm{op}}((\dot{s}, \dot{s}^{-1}) \bullet \mathbf{n}) & \simeq & \{(g\mathbf{U}, g_1\mathbf{U}, \dots, g_r\mathbf{U}) \mid g^{-1} g_1 \in \mathbf{U}\mathbf{T}_{\alpha^\vee} \dot{s} \mathbf{U} \quad \text{and} \\ & & \qquad g_1\mathbf{U} \xrightarrow{n_1} g_2\mathbf{U} \xrightarrow{n_2} \cdots \xrightarrow{n_{r-1}} g_r\mathbf{U} \xrightarrow{n_r} F(g)\mathbf{U}\}. \end{array}$$

We will use this description of $\mathbf{Y}_2^{\mathrm{op}}((\dot{s}, \dot{s}^{-1}) \bullet \mathbf{n})$ towards the end of this proof.

This description shows that the group $\mathbf{S}(\alpha,\mathbf{w})$ acts on $\mathbf{Y}_2^{\mathrm{op}}((\dot{s},\dot{s}^{-1})\bullet\mathbf{n})$ (as the restriction of the action by right multiplication of $\mathbf{T}^{r+1}$ on $(\mathbf{G}/\mathbf{U})^{r+1}$). Also, as $\mathbf{U}\dot{s}\mathbf{U}$ is closed in $\mathbf{U}\mathbf{T}_{\alpha^\vee}\dot{s}\mathbf{U}$, the natural map $\mathbf{Y}(\dot{s}\bullet\mathbf{n})\hookrightarrow\mathbf{Y}_2^{\mathrm{op}}((\dot{s},\dot{s}^{-1})\bullet\mathbf{n})$ is a closed immersion. We have embeddings $\mathbf{T}^{s_\alpha\mathbf{n}F}\hookrightarrow\mathbf{S}(\alpha,\mathbf{w})$ and $\mathbf{T}^{\mathbf{n}F}\hookrightarrow\mathbf{S}(\alpha,\mathbf{w})$ and

$$\mathbf{S}(\alpha,\mathbf{w})=\mathbf{T}^{s_\alpha wF}\cdot\mathbf{S}(\alpha,\mathbf{w})^\circ=\mathbf{S}(\alpha,\mathbf{w})^\circ\cdot\mathbf{T}^{wF}$$

(see [**BoRo1**, Proposition 4.11]). The stabilizer of the closed subvariety $\mathbf{Y}(\dot{s}\bullet\mathbf{n})$ under this action is $\mathbf{T}^{s_\alpha wF}$ so it is readily checked that this induces an isomorphism of $\mathbf{G}^F$-varieties-$\mathbf{T}^{wF}$

$$\mathbf{Y}(\dot{s}\bullet\mathbf{n})\times_{\mathbf{T}^{s_\alpha wF}}\mathbf{S}(\alpha,\mathbf{w})\xrightarrow{\sim}\mathbf{Y}_2^{\mathrm{op}}((\dot{s},\dot{s}^{-1})\bullet\mathbf{n}),$$

as desired. □

The next theorem is the main result of this section. It provides a sufficient condition for $\Psi_{\mathbf{n},j}$ to induce a quasi-isomorphism $\mathrm{R}\Gamma_c^{\dim}(\mathbf{Y}(\mathbf{n}),\Lambda)e_\theta\xrightarrow{\sim}\mathrm{R}\Gamma_c^{\dim}(\mathbf{Y}(c_j(\mathbf{n})),\Lambda)e_\theta$.

Given $x,y\in W$, we put

$$\Phi^+(x,y)=\{\alpha\in\Phi^+\mid x^{-1}(\alpha)\in-\Phi^+\text{ and }(xy)^{-1}(\alpha)\in\Phi^+\}.$$

We define $N_w:Y(\mathbf{T})\to\mathbf{T}^{wF},\ \lambda\mapsto N_{F^d/wF}(\lambda(\zeta))$ (cf §4.A).

Theorem 5.16. — *Let $\theta:\mathbf{T}^{wF}\to\Lambda^\times$ be a character. Let $j\in\{2,3,\dots,r\}$ and assume that $\theta(N_w(w_1\cdots w_{j-2}(\alpha^\vee)))\neq 1$ for all $\alpha\in\Phi^+(w_{j-1},w_j)$. We have $\mathrm{R}\Gamma_c(\mathbf{Y}_j^{\mathrm{op}}(\mathbf{n}),\Lambda)e_\theta=0$ and*

$$\Psi_{\mathbf{n},j,\theta}:\mathrm{R}\Gamma_c^{\dim}(\mathbf{Y}(\mathbf{n}),\Lambda)e_\theta\xrightarrow{\sim}\mathrm{R}\Gamma_c^{\dim}(\mathbf{Y}(c_j(\mathbf{n})),\Lambda)e_\theta$$

is a quasi-isomorphism of complexes of $(\Lambda\mathbf{G}^F,\Lambda\mathbf{T}^{wF})$-bimodules commuting with the action of F^m.

Proof. — If $2\leqslant j\leqslant r$, we denote by $\mathscr{P}(\mathbf{n},j,\theta)$ the following property:

$\mathscr{P}(\mathbf{n},j,\theta)$ For all $\alpha\in\Phi^+(w_{j-1},w_j)$, we have $\theta(N_w(w_1\cdots w_{j-2}(\alpha^\vee)))\neq 1$.

We want to prove that $\mathscr{P}(\mathbf{n},j,\theta)$ implies that $\mathrm{R}\Gamma_c(\mathbf{Y}_j^{\mathrm{op}}(\mathbf{n}),\Lambda)e_\theta=0$. By [**BoRo1**, Proposition 5.19 and Remark 5.21], it is sufficient to prove it whenever $\mathbf{G}=\hat{\mathbf{G}}$, so we will assume that $\mathbf{G}=\hat{\mathbf{G}}$.

So assume from now that $\mathscr{P}(\mathbf{n},j,\theta)$ holds. We will prove by induction on $l(w_{j-1})$ that $\mathrm{R}\Gamma_c(\mathbf{Y}_j^{\mathrm{op}}(\mathbf{n}),\Lambda)e_\theta=0$. Note that the induction hypothesis does not depend on r. But, first, note that, if $j\geqslant 3$, then $\mathscr{P}(\mathbf{n},j,\theta)$ is equivalent to $\mathscr{P}(\mathrm{sh}(\mathbf{n}),j-1,\theta\circ n_1)$ and that the morphism constructed in Lemma 5.9 sends $\mathbf{Y}_j^{\mathrm{op}}(\mathbf{n})$ to $\mathbf{Y}_{j-1}^{\mathrm{op}}(\mathrm{sh}(\mathbf{n}))$. Thus $\mathrm{R}\Gamma_c(\mathbf{Y}_j^{\mathrm{op}}(\mathbf{n}),\Lambda)e_\theta=0$ is equivalent to $\mathrm{R}\Gamma_c(\mathbf{Y}_{j-1}^{\mathrm{op}}(\mathrm{sh}(\mathbf{n})),\Lambda)e_{\theta\circ n_1}=0$. By successive applications of this remark, this shows that we may assume that $j=2$.

First case: Assume that $l(w_1)=0$. This means that $n_1 \in \mathbf{T}$ and it follows from Lemma 5.6 (or Lemma 5.8(a)) that $\mathbf{Y}_2^{\mathrm{op}}(\mathbf{n})=\varnothing$. So the result follows in this case.

Second case: Assume that $l(w_1)=1$ and $n_1 n_2=1$. Let $\alpha \in \Delta$ be such that $w_1=s_\alpha$ and we may assume that $n_1=\dot{s}$ is a representative of s_α lying in $\mathbf{G}_\alpha$. Note also that, since we assume that $\hat{\mathbf{G}}=\mathbf{G}$, we have $\mathbf{G}_\alpha \simeq \mathbf{SL}_2(\mathbb{F})$. Define $\mathbf{S}=\mathbf{S}(\alpha,(w_3,\dots,w_r))$. Lemma 5.14 shows that

$$\mathrm{R}\Gamma_c(\mathbf{Y}_2^{\mathrm{op}}(\mathbf{n}),\Lambda)e_\theta=\mathrm{R}\Gamma_c(\mathbf{Y}(\dot{s},n_3,\dots,n_r),\Lambda)\otimes_{\Lambda\mathbf{T}^{s_\alpha wF}}\mathrm{R}\Gamma_c(\mathbf{S},\Lambda)e_\theta.$$

But $\Phi^+(w_1,w_2)=\Phi^+(s_\alpha,s_\alpha)=\{\alpha\}$, so $\theta(N_w(\alpha^\vee))\neq 1$ by hypothesis. Note also that $\mathbf{T}^{wF}\cap\mathbf{S}^\circ$ acts trivially on the cohomology groups of the complex $\mathrm{R}\Gamma_c(\mathbf{S}^\circ)$, as its action extends to the connected group $\mathbf{S}^\circ$. Since $N_w(\alpha^\vee)\in\mathbf{S}^\circ$ (see [**BoRo1**, Proof of Proposition 4.11, Equality (a)]), this proves that $\mathrm{R}\Gamma_c(\mathbf{S},\Lambda)e_\theta=0$ and so $\mathrm{R}\Gamma_c(\mathbf{Y}_2^{\mathrm{op}}(\mathbf{n}),\Lambda)e_\theta=0$, as desired.

Last case: Assume that $l(w_1)\geqslant 1$. So let us assume now that $l(w_1)\geqslant 1$. Let $\alpha\in\Delta$ be such that $w_1=s_\alpha w_1'$, with $l(w_1')=l(w_1)-1$. Let $\dot{s}$ be a representative of s_α in $\mathbf{G}_\alpha$ and let $n_1'=\dot{s}^{-1}n_1$. We will write $\mathbf{n}'=(n_1',n_2,\dots,n_r)$. Then n_1' is a representative of w_1' and, by Lemma 5.8(a), we have $\mathbf{Y}(\mathbf{n})=\mathbf{Y}(\dot{s}\bullet\mathbf{n}')$ (see also the remark following Lemma 5.8).

It is well-known that $\Phi^+(w_1,1)=\{\alpha\}\coprod s_\alpha(\Phi^+(w_1',1))$. Therefore

$$\Phi^+(w_1,w_2)=\begin{cases} s_\alpha\big(\Phi^+(w_1',w_2)\big) & \text{if } l(w_1'w_2)<l(w_1w_2),\\ \{\alpha\}\coprod s_\alpha\big(\Phi^+(w_1',w_2)\big) & \text{if } l(w_1'w_2)>l(w_1w_2).\end{cases} \tag{\#}$$

Let us now consider the diagram (5.12) with $\mathbf{n}$ replaced by $\dot{s}\bullet\mathbf{n}'$ and j is replaced by 3. We can apply Lemma 5.13 (since the hypothesis (1) holds), so we only need to prove that $\mathrm{R}\Gamma_c(\mathbf{Y}_3^{\mathrm{op}}(\dot{s}\bullet\mathbf{n}'),\Lambda)e_\theta=\mathrm{R}\Gamma_c(\mathscr{U},\Lambda)e_\theta=0$, where $\mathscr{U}$ is the complement of $\mathbf{Y}_{3,2}^{\mathrm{cl}}(\dot{s}\bullet\mathbf{n}')$ in $\mathbf{Y}_3^{\mathrm{cl}}(\dot{s}\bullet\mathbf{n}')$. But the fact that $\mathrm{R}\Gamma_c(\mathbf{Y}_3^{\mathrm{op}}(\dot{s}\bullet\mathbf{n}'),\Lambda)e_\theta=0$ follows from the induction hypothesis (indeed, by (#), the Property $\mathscr{P}(\dot{s}\bullet\mathbf{n}',3,\theta)$ is fulfiled. So it remains to show that $\mathrm{R}\Gamma_c(\mathscr{U},\Lambda)e_\theta=0$. Using the fact that the square $(i',\pi',\pi_{\dot{s}\bullet\mathbf{n}',3},i_{c_3(\dot{s}\bullet\mathbf{n}'),3})$ of the diagram (5.12) is cartesian, and using also Lemma 5.6, it amounts to prove that $\mathrm{R}\Gamma_c(\mathbf{Y}_2^{\mathrm{op}}(c_3(\dot{s}\bullet\mathbf{n}'),\Lambda)e_\theta=0$. Note that $c_3(\dot{s}\bullet\mathbf{n}')=\dot{s}\bullet c_2(\mathbf{n}')$. Two cases may occur:

• Assume first that $l(w_1'w_2)<l(w_1w_2)$. Then $\mathbf{Y}_2^{\mathrm{op}}(c_3(\dot{s}\bullet\mathbf{n}')=\varnothing$, and the result follows.

• Assume now that $l(w_1'w_2)>l(w_1w_2)$. Then, again by Lemma 5.8(a), we have $\mathbf{Y}(\dot{s}\bullet c_2(\mathbf{n}'))=\mathbf{Y}((\dot{s},\dot{s}^{-1})\bullet c_2(\mathbf{n}))$ and, through this identification, $\mathbf{Y}_2^{\mathrm{op}}(c_3(\dot{s}\bullet\mathbf{n}'))$ is identified with $\mathbf{Y}_2^{\mathrm{op}}((\dot{s},\dot{s}^{-1})\bullet c_2(\mathbf{n}))$. So the result now follows from the *second case* (thanks to (#)). □

Remark 5.17. — Theorem 5.16 provides a comparison of modules, together with the Frobenius action. Consider the case $\Lambda = K$. We have an isomorphism of $K\mathbf{G}^F$-modules, compatible with the Frobenius action

$$H_c^i(\mathbf{Y}(\mathbf{n}), K) \otimes_{K\mathbf{T}^{wF}} K_\theta \simeq H_c^{i-2r}(\mathbf{Y}(c_j(\mathbf{n})), K) \otimes_{K\mathbf{T}^{wF}} K_\theta(-r).$$

where $r = d_j(\mathbf{n})$.

Following the same lines as in the proof of Theorem 5.16, we obtain a new proof of the following classical result.

Theorem 5.18. — *If Λ is a field, then $\mathrm{R}_{\mathbf{n}} = \mathrm{R}_w$.*

Proof. — By [**BoRo1**, Proposition 5.19 and Remark 5.21], it is sufficient to prove the Theorem whenever $\mathbf{G} = \hat{\mathbf{G}}$, so we will assume that $\mathbf{G} = \hat{\mathbf{G}}$. Also, by proceeding step-by-step, it is enough to prove that $\mathrm{R}_{\mathbf{n}} = \mathrm{R}_{c_j(\mathbf{n})}$. For this, let $\mathrm{R}\Gamma_{\mathbf{n},j}^{\mathrm{op}}$ denote the class of the complex $\mathrm{R}\Gamma_c(\mathbf{Y}_j^{\mathrm{op}}(\mathbf{n}), \Lambda)$ in $G_0(\Lambda\mathbf{G}^F \otimes \Lambda\mathbf{T}^{wF})$. We only need to prove that $\mathrm{R}\Gamma_{\mathbf{n},j}^{\mathrm{op}} = 0$.

Proceeding by induction on $l(w_j)$ as in the proof of Theorem 5.16, and following the same strategy and arguments, we see that it is enough to prove Theorem 5.18 whenever $j = 1$, $n_1 = \dot{s} = n_2^{-1}$, where $\dot{s}$ is a representative in $\mathbf{G}_\alpha$ of s_α (for some $\alpha \in \Delta$). By Lemma 5.14, it is sufficient to prove that the class $\mathrm{R}\Gamma_{\alpha,\mathbf{w}}$ of the complex $\mathrm{R}\Gamma_c(\mathbf{S}(\alpha, \mathbf{w}), \Lambda)$ in $G_0(\Lambda\mathbf{T}^{s_\alpha wF} \otimes \Lambda\mathbf{T}^{wF})$ is equal to 0.

Now, let T denote the subgroup of $\mathbf{T}^{s_\alpha wF} \times \mathbf{T}^{wF}$ consisting of pairs (t_1, t_2) such that $t_1 t_2 \in \mathbf{S}(\alpha, \mathbf{w})^\circ$ and let $\mathrm{R}\Gamma^\circ$ denote the class of the complex $\mathrm{R}\Gamma_c(\mathbf{S}(\alpha, \mathbf{w})^\circ)$ in $G_0(\Lambda T)$. Then $\mathrm{R}\Gamma_{\alpha,\mathbf{w}} = \mathrm{Ind}_T^{\mathbf{T}^{s_\alpha wF} \times \mathbf{T}^{wF}} \mathrm{R}^\circ$. But the action of T on $\mathbf{S}(\alpha, \mathbf{w})$ extends to an action of a connected group, hence T acts trivially on the cohomology groups of $\mathbf{S}(\alpha, \mathbf{w})^\circ$. Since the Euler characteristic of a torus is equal to 0, this gives $\mathrm{R}\Gamma^\circ = 0$, and consequently $\mathrm{R}\Gamma_{\alpha,\mathbf{w}} = 0$, as desired. □

Corollary 5.19. — *Let $\mathbf{n}' = (n'_1, n'_2, \ldots, n'_{r'})$ be a sequence of elements of $N_{\mathbf{G}}(\mathbf{T})$, let $x \in W$ and let w' denote the image of $n'_1 n'_2 \cdots n'_{r'}$ in W. We assume that Λ is a field and that $w' = x^{-1} w F(x)$. Then the diagram*

$$\begin{tikzcd}
G_0(\Lambda\mathbf{T}^{\mathbf{n}'F}) \arrow[rr, "x_*"] \arrow[dr, "\mathrm{R}_{\mathbf{n}'}"'] & & G_0(\Lambda\mathbf{T}^{\mathbf{n}F}) \arrow[dl, "\mathrm{R}_{\mathbf{n}}"] \\
& G_0(\Lambda\mathbf{G}^F) &
\end{tikzcd}$$

is commutative.

Proof. — Let $\mathbf{n}'' = (\dot{x}^{-1}, n_1, n_2, \dots, n_r, F(\dot{x}))$. Then, by Lemma 5.9,

$$\mathrm{R}_{\mathbf{n}''} = \mathrm{R}_{\mathbf{n}\bullet(F(\dot{x}),F(\dot{x})^{-1})} \circ x_*.$$

But, by Theorem 5.18, $\mathrm{R}_{\mathbf{n}''} = \mathrm{R}_{w'} = \mathrm{R}_{\mathbf{n}'}$ and $\mathrm{R}_{\mathbf{n}\bullet(F(\dot{x}),F(\dot{x})^{-1})} = \mathrm{R}_w = \mathrm{R}_{\mathbf{n}}$. □

Corollary 5.20. — *Let $\mathbf{T}'$ be an F-stable maximal torus of $\mathbf{G}$ and let $\mathbf{B}'$ and $\mathbf{B}''$ be two Borel subgroups of $\mathbf{G}$ containing $\mathbf{T}'$. Then $\mathrm{R}^{\mathbf{G}}_{\mathbf{T}'\subset\mathbf{B}'}(\theta') = \mathrm{R}^{\mathbf{G}}_{\mathbf{T}'\subset\mathbf{B}''}(\theta')$ for all $\theta' \in \mathrm{Irr}(\mathbf{T}'^F)$.*

Proof. — Let g' and g'' be two elements of $\mathbf{G}$ such that $\mathbf{B}' = {}^{g'}\mathbf{B}$, $\mathbf{B}'' = {}^{g''}\mathbf{B}$ and $\mathbf{T}' = {}^{g'}\mathbf{T} = {}^{g''}\mathbf{T}$. Let $n' = g'^{-1}F(g')$ and $n'' = g''^{-1}F(g'')$, and $n = g'^{-1}g''$. Then n, n', $n'' \in N_{\mathbf{G}}(\mathbf{T})$ and $n'' = n^{-1}n'F(n)$. The result follows now from Corollary 5.19 and [**BoRo1**, §11.1]. □

Remark 5.21. — Corollary 5.20 is well-known. In [**DeLu**, Corollary 4.3], this result is first proved "geometrically" for $\theta' = 1$ [**DeLu**, Theorem 1.6] by relating the varieties $\mathbf{X}^{\mathbf{G}}_{\mathbf{B}'}$ and $\mathbf{X}^{\mathbf{G}}_{\mathbf{B}''}$, and extended to the general case using the *character formula* [**DeLu**, Theorem 4.2]. Note that this result is then used in [**DeLu**, Theorem 6.8] to deduce the *Mackey formula* for Deligne-Lusztig induction functors.

In [**Lu2**], Lusztig proposed another argument: the Mackey formula is proved "geometrically" and *a priori* [**Lu2**, Theorem 2.3], and Corollary 5.20 follows [**Lu2**, Corollary 2.4].

Our argument relies neither on the Mackey formula nor on the character formula: we lift Deligne-Lusztig's comparison of $\mathbf{X}^{\mathbf{G}}_{\mathbf{B}'}$ and $\mathbf{X}^{\mathbf{G}}_{\mathbf{B}''}$ to a relation between the varieties $\mathbf{Y}^{\mathbf{G}}_{\mathbf{U}'}$ and $\mathbf{Y}^{\mathbf{G}}_{\mathbf{U}''}$ (here $\mathbf{U}'$ and $\mathbf{U}''$ are the unipotent radicals of $\mathbf{B}'$ and $\mathbf{B}''$ respectively).

Remark 5.22. — Some of the results in [**BoRo2**] (Lemma 4.3, Proposition 4.5 and Theorem 4.6) rely on a disjointness result used in [**BoRo2**, Line 16 of Page 30]. This disjointness result was "proved" using the isomorphism in [**BoRo2**, Line 18 of Page 30]: it has been pointed out to the attention of the authors by H. Wang that this equality is false. However, Wang provided a complete proof of this disjointness result [**Wa**, Proposition 3.4.3], so [**BoRo2**, Lemma 4.3, Proposition 4.5 and Theorem 4.6] remain valid.

Another proof of this disjointness result has been obtained independently by Nguyen [**Ng**] (with slightly different methods). Using a version of Remark 5.17, Wang and Nguyen have been able to keep track of the Frobenius eigenvalues.

6. Independence on the parabolic subgroup

We assume in this section §6 that $\mathbf{G}$ is connected. We fix an F-stable maximal torus $\mathbf{T}$ of $\mathbf{G}$ and we denote by $(\mathbf{G}^*, \mathbf{T}^*, F^*)$ a triple dual to $(\mathbf{G}, \mathbf{T}, F)$.

We fix a family of parabolic subgroups $\mathbf{P}_1$, $\mathbf{P}_2, \dots$, $\mathbf{P}_r$ admitting $\mathbf{L}$ as a Levi complement. Given $1 \leqslant j \leqslant r$, we denote by $\mathbf{V}_j$ the unipotent radical of $\mathbf{P}_j$. We denote by $\mathbf{V}_\bullet$ the sequence $(\mathbf{V}_1, \dots, \mathbf{V}_r)$.

The identification of the root system of $\mathbf{G}$ with the coroot system of $\mathbf{G}^*$ allows us to define parabolic subgroups $\mathbf{P}_1^*$, $\mathbf{P}_2^*, \dots$, $\mathbf{P}_r^*$, admitting a common F^*-stable Levi complement $\mathbf{L}^*$ and such that $\mathbf{L}^*$ and $\mathbf{P}_j^*$ and are dual to $\mathbf{L}$ and $\mathbf{P}_j$ respectively. We denote by $\mathbf{V}_j$ and $\mathbf{V}_j^*$ the unipotent radicals of $\mathbf{P}_j$ and $\mathbf{P}_j^*$ respectively.

Finally, we fix a semisimple element $s \in \mathbf{L}^{*F^*}$ whose order is invertible in Λ.

6.A. Isomorphisms. — As announced in the introduction, the isomorphism of functors described in Theorem 1.3 is *canonical*. So, before giving the proof, we will explain how it is realized. For this, let us define

$$\mathbf{Y}_{\mathbf{V}_\bullet} = \{(g_1\mathbf{V}_1, \dots, g_r\mathbf{V}_r) \in \mathbf{G}/\mathbf{V}_1 \times \cdots \times \mathbf{G}/\mathbf{V}_r \mid \forall\ j \in \{1, 2, \dots, r\},\ g_j^{-1} g_{j+1} \in \mathbf{V}_j \cdot \mathbf{V}_{j+1}\}$$

and, if $2 \leqslant j \leqslant r$, we set

$$\mathbf{Y}^{\mathrm{cl}}_{\mathbf{V}_\bullet, j} = \{(g_1\mathbf{V}_1, g_2\mathbf{V}_2, \dots, g_r\mathbf{V}_r) \in \mathbf{Y}_{\mathbf{V}_\bullet} \mid g_{j-1}^{-1} g_j \in \mathbf{V}_{i-1} \cdot \mathbf{V}_{j+1}\}.$$

It is a closed subvariety of $\mathbf{Y}_{\mathbf{V}_\bullet}$ and we denote by $i_{\mathbf{V}_\bullet, j} : \mathbf{Y}^{\mathrm{cl}}_{\mathbf{V}_\bullet, j} \hookrightarrow \mathbf{Y}_{\mathbf{V}_\bullet}$ the closed immersion. Let $\mathbf{Y}^{\mathrm{op}}_{\mathbf{V}_\bullet, j}$ denote its open complement. We define the sequence $c_j(\mathbf{V}_\bullet)$ as obtained from the sequence $\mathbf{V}_\bullet$ by removing the j-th component. We then define

$$\pi_{\mathbf{V}_\bullet, j} : \mathbf{Y}^{\mathrm{cl}}_{\mathbf{V}_\bullet, j} \longrightarrow \mathbf{Y}_{c_j(\mathbf{V}_\bullet)}$$

as the map which forgets the j-th component and we set

$$d_j(\mathbf{V}_\bullet) = \dim(\mathbf{V}_{j-1} \cap \mathbf{V}_{j+1}) - \dim(\mathbf{V}_{j-1} \cap \mathbf{V}_j \cap \mathbf{V}_{j+1}).$$

Note that $\mathbf{G}^F$ acts diagonally on $\mathbf{Y}_{\mathbf{V}_\bullet}$ by left translation, that $\mathbf{L}^F$ acts diagonally by right translation, and that this endows $\mathbf{Y}_{\mathbf{V}_\bullet}$ with a structure of $\mathbf{G}^F$-variety-$\mathbf{L}^F$. The varieties $\mathbf{Y}^{\mathrm{cl}}_{\mathbf{V}_\bullet}$ and $\mathbf{Y}^{\mathrm{op}}_{\mathbf{V}_\bullet}$ are stable under these actions, and the morphisms $i_{\mathbf{V}_\bullet, j}$ and $\pi_{\mathbf{V}_\bullet, j}$ are equivariant. As for their analogues $i_{\mathbf{n}, j}$ and $\pi_{\mathbf{n}, j}$ defined in §5.B, we have the following properties, which follow from Corollary 5.3 by base change.

Lemma 6.1. — *The map $\pi_{\mathbf{V}_\bullet, j}$ is smooth with fibers are isomorphic to an affine space of dimension $d_j(\mathbf{V}_\bullet)$. The codimension of $\mathbf{Y}^{\mathrm{cl}}_{\mathbf{V}_\bullet}$ in $\mathbf{Y}_{\mathbf{V}_\bullet}$ is also equal to $d_j(\mathbf{V}_\bullet)$.*

We deduce that $\pi_{\mathbf{V}_\bullet,j}$ induces a quasi-isomorphism of complexes of $(\Lambda\mathbf{G}^F,\Lambda\mathbf{L}^F)$-bimodules

$$\mathrm{R}\Gamma_c(\mathbf{Y}^{\mathrm{cl}}_{\mathbf{V}_\bullet,j},\Lambda)\simeq \mathrm{R}\Gamma_c(\mathbf{Y}_{c_j(\mathbf{V}_\bullet)},\Lambda)[-2d_j(\mathbf{V}_\bullet)](-d_j(\mathbf{V}_\bullet)).$$

The closed immersion $i_{\mathbf{V}_\bullet,j}:\mathbf{Y}^{\mathrm{cl}}_{\mathbf{V}_\bullet,j}\hookrightarrow \mathbf{Y}_{\mathbf{V},\mathbf{V}'}$ induces a morphism of complexes of $(\Lambda\mathbf{G}^F,\Lambda\mathbf{L}^F)$-bimodules

$$i^*_{\mathbf{V}_\bullet,j}:\mathrm{R}\Gamma_c(\mathbf{Y}_{\mathbf{V}_\bullet,j},\Lambda)\longrightarrow \mathrm{R}\Gamma_c(\mathbf{Y}^{\mathrm{cl}}_{\mathbf{V}_\bullet,j},\Lambda)$$

which, composed with the previous isomorphism, induces a morphism

$$\Psi_{\mathbf{V}_\bullet,j}:\mathrm{R}\Gamma_c^{\dim}(\mathbf{Y}_{\mathbf{V}_\bullet},\Lambda)\longrightarrow \mathrm{R}\Gamma_c^{\dim}(\mathbf{Y}_{c_j(\mathbf{V}_\bullet)},\Lambda).$$

The main result of this section is the following.

Theorem 6.2. — *Let $j\in\{2,3,\dots,r\}$ such that $C_{\mathbf{V}^*_{j-1}\cap\mathbf{V}^*_{j+1}}(s)\subset C_{\mathbf{V}^*_j}(s)$. We have*

$$\mathrm{R}\Gamma_c(\mathbf{Y}^{\mathrm{op}}_{\mathbf{V}_\bullet,j},\Lambda)e_s^{\mathbf{L}^F}=0,$$

hence $\Psi_{\mathbf{V}_\bullet,j}$ induces a quasi-isomorphism of complexes of $(\Lambda\mathbf{G}^F,\Lambda\mathbf{L}^F)$-bimodules

$$\Psi_{\mathbf{V}_\bullet,j,s}:\mathrm{R}\Gamma_c^{\dim}(\mathbf{Y}_{\mathbf{V}_\bullet,j},\Lambda)e_s^{\mathbf{L}^F}\xrightarrow{\sim}\mathrm{R}\Gamma_c^{\dim}(\mathbf{Y}_{c_j(\mathbf{V}_\bullet)},\Lambda)e_s^{\mathbf{L}^F}.$$

Proof. — The proof will proceed in two steps. We first prove the theorem when $\mathbf{L}$ is a maximal torus: in fact, it will be shown that it is a consequence of Theorem 5.16. We then use [**BoRo1**, Theorem A′] to deduce the general case from this particular one.

First step: Assume here that $\mathbf{L}$ *is a maximal torus.* Let $a_1,\dots,a_r$ be elements of $\mathbf{G}$ such that $(\mathbf{L},\mathbf{P}_i)={}^{a_i}(\mathbf{T},\mathbf{B})$ for all $i\in\{1,2,\dots,r\}$). As usual, we set $a_{r+1}=F(a_r)$. Now, let $n_i=a_i^{-1}a_{i+1}$. It follows from the definition of the a_i's that $n_i\in N_{\mathbf{G}}(\mathbf{T})$. We set $\mathbf{n}=(n_1,\dots,n_r)$. Note that $n_1n_2\cdots n_r=a_1^{-1}F(a_1)$. We denote by w_i the image of n_i in W and we set $w=w_1w_2\cdots w_r$. It is then easily checked that the map

$$(g_1\mathbf{V}_1,\dots,g_r\mathbf{V}_r)\longmapsto(g_1\mathbf{V}_1a_1,\dots,g_r\mathbf{V}_ra_r)$$

induces an isomorphism of varieties

$$\mathbf{Y}_{\mathbf{V}_\bullet}\xrightarrow{\sim}\mathbf{Y}(\mathbf{n})$$

which sends $\mathbf{Y}^{\mathrm{cl}}_{\mathbf{V}_\bullet,j}$ to $\mathbf{Y}^{\mathrm{cl}}_j(\mathbf{n})$. Moreover, conjugacy by a_1 induces an isomorphism $\mathbf{T}^{wF}\simeq\mathbf{L}^F$ and it is easily checked that the above isomorphism is $(\mathbf{G}^F,\mathbf{L}^F)$-equivariant through this idenfication. Now, to s is associated a linear character of $\mathbf{L}^F$ which, through the identification $\mathbf{T}^{wF}\simeq\mathbf{L}^F$, defines a linear character $\theta:\mathbf{T}^{wF}\to\Lambda^\times$.

By Theorem 5.16, we only need to prove that Condition $C_{\mathbf{V}^*_{j-1}\cap\mathbf{V}^*_{j+1}}(s)\subset C_{\mathbf{V}^*_j}(s)$ is equivalent to $\mathscr{P}(\mathbf{n},j,\theta)$. So let us prove this last fact. The property $\mathscr{P}(\mathbf{n},j,\theta)$ can be rewritten as follows:

Property $\mathscr{P}(\mathbf{n},j,\theta)$. *If* $\alpha \in \Phi^+$ *is such that* $\theta(N_w(w_1\cdots w_{j-2}(\alpha^\vee)))=1$ *and* $(w_{j-1}w_j)^{-1}(\alpha)\in\Phi^+$, *then* $w_{j-1}^{-1}(\alpha)\in\Phi^+$.

If we set $s'=a_1^{-1}sa_1\in\mathbf{T}^{wF}$, then $\mathscr{P}(\mathbf{n},j,\theta)$ becomes equivalent to

$$C_{^{w_1\cdots w_{j-2}}\mathbf{U}^*}(s')\cap {}^{w_1\cdots w_j}\mathbf{U}^*\subset {}^{w_1\cdots w_{j-1}}\mathbf{U}^*.$$

By conjugating by a_1, and since ${}^{a_1n_1\cdots n_i}\mathbf{U}^*=\mathbf{V}_i^*$, we get that $\mathscr{P}(\mathbf{n},j,\theta)$ is equivalent to $C_{\mathbf{V}_{j-1}^*}(s)\cap\mathbf{V}_{j+1}^*\subset\mathbf{V}_j^*$, as desired.

Second step: The general case. Let us now come back to the general case: we no longer assume that $\mathbf{L}$ is a maximal torus. Since $\mathrm{R}\Gamma_c(\mathbf{Y}^{\mathrm{op}}_{\mathbf{V}_\bullet,j},\Lambda)e_s^{\mathbf{L}^F}=\mathrm{R}\Gamma_c(\mathbf{Y}^{\mathrm{op}}_{\mathbf{V}_\bullet,j},\Lambda)\otimes_{\Lambda\mathbf{L}^F}\Lambda\mathbf{L}^Fe_s^{\mathbf{L}^F}$, and since $\Lambda\mathbf{L}^Fe_s^{\mathbf{L}^F}$ lives in the category generated by the complexes $\mathscr{R}^{\mathbf{L}}_{\mathbf{T}'\subset\mathbf{B}'}(\Lambda\mathbf{T}'^Fe_s^{\mathbf{T}'^F})$, where $\mathbf{B}'$ runs over the set of Borel subgroups of $\mathbf{L}$ admitting an F-stable maximal torus $\mathbf{T}'$ whose dual torus contains s (see [**BoRo1**, Theorem A′]), it is sufficient to prove that

$$\mathrm{R}\Gamma_c(\mathbf{Y}^{\mathrm{op}}_{\mathbf{V}_\bullet},\Lambda)\otimes_{\Lambda\mathbf{L}^F}\mathscr{R}^{\mathbf{L}}_{\mathbf{T}'\subset\mathbf{B}'}(\Lambda\mathbf{T}'^Fe_s^{\mathbf{T}'^F})=0. \tag{?}$$

So let $(\mathbf{T}',\mathbf{B}')$ be a pair as above. Let $\mathbf{U}'$ denote the unipotent radical of $\mathbf{B}'$, let $\mathbf{T}'^*$ be an F^*-stable maximal torus of $\mathbf{L}^*$, containing s and dual to $\mathbf{T}'$ and let $\mathbf{B}'^*$ be a Borel subgroup of $\mathbf{L}^*$ containing $\mathbf{T}'^*$ and dual to $\mathbf{B}'$. It is immediately checked that

$$\mathbf{Y}_{\mathbf{V}_\bullet}\times_{\mathbf{L}^F}\mathbf{Y}^{\mathbf{L}}_{\mathbf{U}'}\simeq\mathbf{Y}_{\mathbf{U}'\mathbf{V}_\bullet},$$

(as $\mathbf{G}^F$-varieties-$\mathbf{T}'^F$). Here, we have set $\mathbf{U}'\mathbf{V}_\bullet=(\mathbf{U}'\mathbf{V}_1,\ldots,\mathbf{U}'\mathbf{V}_r)$. Moreover, through this isomorphism, $\mathbf{Y}^{\mathrm{op}}_{\mathbf{V}_\bullet,j}\times_{\mathbf{L}^F}\mathbf{Y}^{\mathbf{L}}_{\mathbf{U}'}$ is sent to $\mathbf{Y}^{\mathrm{op}}_{\mathbf{U}'\mathbf{V}_\bullet,j}$ hence, by applying the first step of this proof, we only need to prove that $C_{\mathbf{U}'^*\mathbf{V}^*_{j-1}\cap\mathbf{U}'^*\mathbf{V}^*_{j+1}}(s)\subset C_{\mathbf{U}'^*\mathbf{V}^*_j}(s)$. This follows directly from the assumption. □

Remark 6.3. — Theorem 6.2 provides a comparison of modules, together with the Frobenius action. We have an isomorphism of $(\Lambda\mathbf{G}^F,\Lambda\mathbf{L}^F)$-bimodules compatible with the Frobenius action

$$H^i_c(\mathbf{Y}_{\mathbf{V},j},\Lambda)e_s^{\mathbf{L}^F}\simeq H^{i-2r}_c(\mathbf{Y}_{c_j(\mathbf{V}_\bullet)},\Lambda)e_s^{\mathbf{L}^F}(-r).$$

where $r=d_j(\mathbf{V}_\bullet)$.

Let $\mathrm{sh}(\mathbf{V}_\bullet)=(\mathbf{V}_2,\ldots,\mathbf{V}_r,{}^F\mathbf{V}_1)$. The map

$$\begin{array}{rccc}\mathrm{sh}_{\mathbf{V}_\bullet}: & \mathbf{Y}_{\mathbf{V}_\bullet} & \longrightarrow & \mathbf{Y}_{\mathrm{sh}(\mathbf{V}_\bullet)}\\ & (g_1\mathbf{V}_1,\ldots,g_r\mathbf{V}_r) & \longmapsto & (g_2\mathbf{V}_2,\ldots,g_r\mathbf{V}_r,F(g_1\mathbf{V}_1))\end{array}$$

is $(\mathbf{G}^F,\mathbf{L}^F)$-equivariant and induces an equivalence of étale sites. Therefore, it induces a quasi-isomorphism of complexes of bimodules

$$\mathrm{sh}^*_{\mathbf{V}_\bullet}:\mathrm{R}\Gamma_c(\mathbf{Y}_{\mathrm{sh}(\mathbf{V}_\bullet)},\Lambda)\xrightarrow{\sim}\mathrm{R}\Gamma_c(\mathbf{Y}_{\mathbf{V}_\bullet},\Lambda).$$

Applying twice Theorem 6.2, we obtain the following result.

Corollary 6.4. — *Let* $j \in \{2,\dots,r\}$ *and assume that*

$$C_{\mathbf{V}^*_{j-1}\cap\mathbf{V}^*_{j+1}}(s) \subset C_{\mathbf{V}^*_j}(s) \quad \textit{and} \quad C_{\mathbf{V}^*_j\cap\mathbf{V}^*_{j+2}}(s) \subset C_{\mathbf{V}^*_{j+1}}(s).$$

The map $\Psi_{\mathbf{V}_\bullet,j,s} \circ \mathrm{sh}^*_{\mathbf{V}_\bullet} \circ \Psi^{-1}_{\mathrm{sh}(\mathbf{V}_\bullet),j,s}$ *is a quasi-isomorphism of complexes of* $(\Lambda\mathbf{G}^F,\Lambda\mathbf{L}^F)$*-bimodules*

$$\mathrm{R}\Gamma^{\dim}_c(\mathbf{Y}_{c_j(\mathrm{sh}(\mathbf{V}_\bullet))},\Lambda)e_s^{\mathbf{L}^F} \xrightarrow{\sim} \mathrm{R}\Gamma^{\dim}_c(\mathbf{Y}_{c_j(\mathbf{V}_\bullet)},\Lambda)e_s^{\mathbf{L}^F}.$$

In the case $r=2$, Corollary 6.4 becomes the following result.

Corollary 6.5. — *Assume*

$$C_{\mathbf{V}^*_1\cap {}^{F^*}\mathbf{V}^*_1}(s) \subset C_{\mathbf{V}^*_2}(s) \quad \textit{and} \quad C_{\mathbf{V}^*_2\cap {}^{F^*}\mathbf{V}^*_2}(s) \subset C_{{}^{F^*}\mathbf{V}^*_1}(s).$$

The map $\Psi_{\mathbf{V}_1,\mathbf{V}_2,2,s} \circ \mathrm{sh}^*_{\mathbf{V}_1,\mathbf{V}_2} \circ \Psi^{-1}_{\mathbf{V}_2,F(\mathbf{V}_1),2,s}$ *is a quasi-isomorphism of complexes of* $(\Lambda\mathbf{G}^F,\Lambda\mathbf{L}^F)$*-bimodules*

$$\mathrm{R}\Gamma^{\dim}_c(\mathbf{Y}_{\mathbf{V}_2},\Lambda)e_s^{\mathbf{L}^F} \xrightarrow{\sim} \mathrm{R}\Gamma^{\dim}_c(\mathbf{Y}_{\mathbf{V}_1},\Lambda)e_s^{\mathbf{L}^F}.$$

As a consequence, we obtain a quasi-isomorphism of functors between

$$\mathscr{R}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}_1}[\dim(\mathbf{Y}_{\mathbf{V}_1})] : \mathrm{D}^b(\Lambda\mathbf{L}^F e_s^{\mathbf{L}^F}) \longrightarrow \mathrm{D}^b(\Lambda\mathbf{G}^F e_s^{\mathbf{G}^F})$$

and

$$\mathscr{R}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}_2}[\dim(\mathbf{Y}_{\mathbf{V}_2})] : \mathrm{D}^b(\Lambda\mathbf{L}^F e_s^{\mathbf{L}^F}) \longrightarrow \mathrm{D}^b(\Lambda\mathbf{G}^F e_s^{\mathbf{G}^F}).$$

Remark 6.6. — The isomorphism of functors of Corollary 6.5 comes with a Tate twist. Keeping track of this twist has important applications **[Wa]**, **[Ng]**.

Remark 6.7. — Let us make here some comments about the condition

$$(\mathscr{C}_{\mathbf{V}_1,\mathbf{V}_2}) \qquad C_{\mathbf{V}^*_1\cap {}^{F^*}\mathbf{V}^*_1}(s) \subset C_{\mathbf{V}^*_2}(s) \quad \textit{and} \quad C_{\mathbf{V}^*_2\cap {}^{F^*}\mathbf{V}^*_2}(s) \subset C_{{}^{F^*}\mathbf{V}^*_1}(s).$$

Note that if $C_{\mathbf{V}^*_1}(s) = C_{\mathbf{V}^*_2}(s)$, then Condition $(\mathscr{C}_{\mathbf{V}_1,\mathbf{V}_2})$ is satisfied.

Since $C_{\mathbf{V}^*_i}(s)$ is connected, it follows that if $C^\circ_{\mathbf{G}^*}(s) \subset \mathbf{L}^*$, then Condition $(\mathscr{C}_{\mathbf{V}_1,\mathbf{V}_2})$ is satisfied.

Example 6.8. — Of course, Condition $(\mathscr{C}_{\mathbf{V}_1,\mathbf{V}_1})$ is fulfiled for all s. Gluing the quasi-isomorphisms obtained from Corollary 6.5, we get a quasi-isomorphism of complexes of bimodules

$$\Theta_{\mathbf{V}_1,\mathbf{V}_1} : \mathrm{R}\Gamma_c(\mathbf{Y}_{\mathbf{V}_1},\Lambda) \xrightarrow{\sim} \mathrm{R}\Gamma_c(\mathbf{Y}_{\mathbf{V}_1},\Lambda).$$

But, since $\mathbf{Y}^{\mathrm{op}}_{\mathbf{V}_1,\mathbf{V}_1} = \varnothing$, it is readily checked that $\Theta_{\mathbf{V}_1,\mathbf{V}_1} = \mathrm{Id}_{\mathrm{R}\Gamma_c(\mathbf{Y}_{\mathbf{V}_1},\Lambda)}$.

Example 6.9. — Similarly, Condition $(\mathscr{C}_{\mathbf{V}_1,F(\mathbf{V}_1)})$ is fulfiled for all s. Gluing the quasi-isomorphisms obtained from Corollary 6.5, we obtain a quasi-isomorphism of complexes of bimodules

$$\Theta_{\mathbf{V}_1,F(\mathbf{V}_1)} : \mathrm{R}\Gamma_c(\mathbf{Y}_{\mathbf{V}_1},\Lambda) \xrightarrow{\sim} \mathrm{R}\Gamma_c(\mathbf{Y}_{F(\mathbf{V}_1)},\Lambda).$$

But, since $\mathbf{Y}^{\mathrm{op}}_{\mathbf{V}_1,F(\mathbf{V}_1)} = \varnothing$, it is readily checked that $\Theta_{\mathbf{V}_1,F(\mathbf{V}_1)} = F$.

Remark 6.10. — If $(\mathscr{C}_{\mathbf{V}_1,\mathbf{V}_2})$ holds, we denote by

$$\Theta_{\mathbf{V}_1,\mathbf{V}_2,s} : \mathrm{R}\Gamma_c^{\dim}(\mathbf{Y}_{\mathbf{V}_2},\Lambda)e_s^{\mathbf{L}^F} \xrightarrow{\sim} \mathrm{R}\Gamma_c^{\dim}(\mathbf{Y}_{\mathbf{V}_1},\Lambda)e_s^{\mathbf{L}^F}$$

the quasi-isomorphism defined by $\Theta_{\mathbf{V}_1,\mathbf{V}_2,s} = \Psi_{\mathbf{V}_1,\mathbf{V}_2,2,s} \circ \mathrm{sh}^*_{\mathbf{V}_1,\mathbf{V}_2} \circ \Psi^{-1}_{\mathbf{V}_2,F(\mathbf{V}_1),2,s}$. Assume moreover that $(\mathscr{C}_{\mathbf{V}_1,\mathbf{V}_3})$ and $(\mathscr{C}_{\mathbf{V}_2,\mathbf{V}_3})$ hold, so that the quasi-isomorphisms of complexes $\Theta_{\mathbf{V}_1,\mathbf{V}_3,s}$ and $\Theta_{\mathbf{V}_2,\mathbf{V}_3,s}$ are also well-defined. It is natural to ask the following

Question. *When does the equality* $\Theta_{\mathbf{V}_1,\mathbf{V}_3,s} = \Theta_{\mathbf{V}_1,\mathbf{V}_2,s} \circ \Theta_{\mathbf{V}_2,\mathbf{V}_3,s}$ *hold?*

For instance, taking Example 6.8 into account, when does the equality $\Theta^{-1}_{\mathbf{V}_1,\mathbf{V}_2,s} = \Theta_{\mathbf{V}_2,\mathbf{V}_1,s}$ hold?

We do not know the answer to this question, but we can just say that the equality does not always hold. Indeed, if m is minimal such that $F^m(\mathbf{V}_1) = \mathbf{V}_1$, then the isomorphisms $\Theta_{\mathbf{V}_1,F(\mathbf{V}_1),s}$, $\Theta_{F(\mathbf{V}_1),F^2(\mathbf{V}_1),s}, \dots, \Theta_{F^{r-1}(\mathbf{V}_1),\mathbf{V}_1}$ are well-defined and all coincide with the Frobenius endomorphism F (see Example 6.9), and so

$$\Theta_{\mathbf{V}_1,F(\mathbf{V}_1),s} \circ \Theta_{F(\mathbf{V}_1),F^2(\mathbf{V}_1),s} \circ \cdots \circ \Theta_{F^{r-1}(\mathbf{V}_1),\mathbf{V}_1} = F^r \neq \mathrm{Id} = \Theta_{\mathbf{V}_1,\mathbf{V}_1,s}$$

(see Example 6.8).

Example 6.11. — Let $\mathbf{P}_0$ be a parabolic subgroup admitting an F-stable Levi subgroup $\mathbf{L}_0$ containing $\mathbf{L}$. We denote by $\mathbf{V}_0$ the unipotent radical of $\mathbf{P}_0$ and $\mathbf{L}_0^*$ the corresponding Levi subgroup of a parabolic subgroup of $\mathbf{G}^*$ containing $\mathbf{L}^*$, which is dual to $\mathbf{L}_0$. We assume in this example that $C^\circ_{\mathbf{G}^*}(s) \subset \mathbf{L}_0^*$. Then it follows from [**BoRo1**, Theorem 11.7], Corollary 6.5 and Remark 6.7 that we have an isomorphism of $(\Lambda\mathbf{G}^F, \Lambda\mathbf{L}^F)$-bimodules

$$\mathrm{H}_c^{d_0}(\mathbf{Y}_{\mathbf{V}_0},\Lambda) \otimes_{\Lambda\mathbf{L}_0^F} \mathrm{R}\Gamma_c^{\dim}(\mathbf{Y}^{\mathbf{L}_0}_{\mathbf{V}\cap\mathbf{L}_0},\Lambda)e_s^{\mathbf{L}^F} \simeq \mathrm{R}\Gamma_c^{\dim}(\mathbf{Y}_{\mathbf{V}},\Lambda)e_s^{\mathbf{L}^F},$$

where $d_0 = \dim(\mathbf{Y}_{\mathbf{V}_0})$.

Remark 6.12. — Let us consider the Harish-Chandra case: assume that $\mathbf{V}_1$ and $\mathbf{V}_2$ are F-stable. The functors $\mathscr{R}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}_1}$ and $\mathscr{R}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}_2}$ are isomorphic without truncating by any series [**DipDu, HoLe**].

6.B. Transitivity. — We will provide here an analogue to Lemma 5.13 in the more general context of this section. Assume in this subsection, and only in this subsection, that $3 \leqslant j \leqslant r$ (in particular, $r \geqslant 3$). Since $c_{j-1}(c_j(\mathbf{V}_\bullet)) = c_{j-1}(c_{j-1}(\mathbf{V}_\bullet))$, we can build a diagram

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{c}}^{\dim}(\mathbf{Y}_{\mathbf{V}_\bullet},\Lambda) \arrow[r, "\Psi_{\mathbf{V}_\bullet,j}"] \arrow[d, "\Psi_{\mathbf{V}_\bullet,j-1}"'] & \mathrm{R}\Gamma_{\mathrm{c}}^{\dim}(\mathbf{Y}_{c_j(\mathbf{V}_\bullet)},\Lambda) \arrow[d, "\Psi_{c_j(\mathbf{V}_\bullet),j-1}"'] \\
\mathrm{R}\Gamma_{\mathrm{c}}^{\dim}(\mathbf{Y}_{c_{j-1}(\mathbf{V}_\bullet)},\Lambda) \arrow[r, "\Psi_{c_{j-1}(\mathbf{V}_\bullet),j-1}"] & \mathrm{R}\Gamma_{\mathrm{c}}^{\dim}(\mathbf{Y}_{c_{j-1}(c_j(\mathbf{V}_\bullet)},\Lambda).
\end{tikzcd} \tag{6.13}$$

It does not seem reasonable to expect that the diagram (6.13) is commutative in general. However, we have the following result, obtained from the results of section § 5.A below by copying the proof of Lemma 5.13.

Lemma 6.14. — *Assume that one of the following holds:*

(1) $\mathbf{V}_{j-2} \subset \mathbf{V}_{j+1} \cdot \mathbf{V}_{j-1}$.
(2) $\mathbf{V}_{j-1} \subset \mathbf{V}_{j-2} \cdot \mathbf{V}_{j}$.
(3) $\mathbf{V}_{j} \subset \mathbf{V}_{j-1} \cdot \mathbf{V}_{j+1}$.
(4) $\mathbf{V}_{j+1} \subset \mathbf{V}_{j} \cdot \mathbf{V}_{j-2}$.

Then the diagram (6.13) is commutative.

7. Jordan decomposition and quasi-isolated blocks

In this section, we assume $\mathbf{G}$ is connected. We fix an F-stable maximal torus $\mathbf{T}$ of $\mathbf{G}$ and we denote by $(\mathbf{G}^*, \mathbf{T}^*, F^*)$ a triple dual to $(\mathbf{G}, \mathbf{T}, F)$.

We start in §7.A with a recollection of some of the results of **[BoRo1]** on the vanishing on the truncated cohomology of certain Deligne-Lusztig varieties outside the middle degree. We fix an F-stable Levi subgroup $\mathbf{L}$ and consider $s \in \mathbf{G}^{*F^*}$ of order invertible in Λ such that $C^\circ_{\mathbf{G}^*}(s) \subset \mathbf{L}^*$ (and we take $\mathbf{L}$ minimal with that property). We show that the corresponding middle degree $(\Lambda\mathbf{G}^F, \Lambda\mathbf{L}^F)$-bimodule $\mathrm{H}_c^{\dim(\mathbf{Y_P})}(\mathbf{Y_P}, \Lambda)e_s^{\mathbf{L}^F}$ does not depend on the choice of the parabolic subgroup $\mathbf{P}$, up to isomorphism, thanks to the results of §6. In particular, it is stable under the action of the stabilizer N of $e_s^{\mathbf{L}^F}$ in $N_{\mathbf{G}^F}(\mathbf{L})$.

Section §7.B develops some Clifford theory tools in order to extend the action of $\Lambda\mathbf{L}^F$ on $\mathrm{H}_c^{\dim(\mathbf{Y_P})}(\mathbf{Y_P}, \Lambda)e_s^{\mathbf{L}^F}$ to an action of N. We apply this in §7.C by embedding $\mathbf{G}$

in a group $\tilde{\mathbf{G}}$ with connected center. This provides a Morita equivalence, extending the main result of [**BoRo1**] to the quasi-isolated case.

In section §7.D, we show that the action of $\mathbf{L}^F$ on the complex of cohomology $C = G\Gamma_c(\mathbf{Y_P}, \Lambda)e_s^{\mathbf{L}^F}$ also extends to N, and the resulting complex provides a splendid Rickard equivalence. This relies on checking that given Q an ℓ-subgroup of $\mathbf{L}^F$, the complex $\mathrm{Br}_{\Delta Q}(C)$ arises in a Jordan decomposition setting for $C_{\mathbf{G}}(Q)$, and then applying the results of the Appendix. The main difficulty is to prove that $\mathrm{br}_Q(e_s^{\mathbf{L}^F})$ is a sum of idempotents associated to a Jordan decomposition setting for $C_{\mathbf{G}}(Q)$. An added difficulty is that the group $C_{\mathbf{G}}(Q)$ need not be connected.

7.A. Quasi-isolated setting. — We fix a semisimple element $s \in \mathbf{G}^{*F^*}$ whose order is invertible in Λ. Let $\mathbf{L}^* = C_{\mathbf{G}^*}\big(\mathrm{Z}(C^\circ_{\mathbf{G}^*}(s))^\circ\big)$, an F^*-stable Levi complement of some parabolic subgroup $\mathbf{P}^*$ of $\mathbf{G}^*$. Note that $\mathbf{L}^*$ is a minimal Levi subgroup with respect to the property of containing $C^\circ_{\mathbf{G}^*}(s)$ and $\ell \nmid [C_{\mathbf{G}^*}(s) : C^\circ_{\mathbf{G}^*}(s)]$.

We denote by $(\mathbf{L}, \mathbf{P})$ a pair dual to $(\mathbf{L}^*, \mathbf{P}^*)$. Note that $\mathbf{P}$ is a parabolic subgroup of $\mathbf{G}$ admitting $\mathbf{L}$ as an F-stable Levi complement. The unipotent radical of $\mathbf{P}$ will be denoted by $\mathbf{V}$. We put $d = \dim(\mathbf{Y_V})$.

The group $C_{\mathbf{G}^*}(s)$ normalises $\mathbf{L}^*$ and we set $\mathbf{N}^* = C_{\mathbf{G}^*}(s)^{F^*} \cdot \mathbf{L}^*$: it is a subgroup of $N_{\mathbf{G}^*}(\mathbf{L}^*)$ containing $\mathbf{L}^*$. Via the canonical isomorphism between $N_{\mathbf{G}^*}(\mathbf{L}^*)/\mathbf{L}^*$ and $N_{\mathbf{G}}(\mathbf{L})/\mathbf{L}$, we define the subgroup $\mathbf{N}$ of $N_{\mathbf{G}}(\mathbf{L})$ containing $\mathbf{L}$ such that $\mathbf{N}/\mathbf{L}$ corresponds to $\mathbf{N}^*/\mathbf{L}^*$. Note that $\mathbf{N}^*$ is F^*-stable and so $\mathbf{N}$ is F-stable.

Let us first derive some consequences of these assumptions. Note that $\mathbf{N}^*/\mathbf{L}^* = (\mathbf{N}^*/\mathbf{L}^*)^{F^*} = \mathbf{N}^{*F^*}/\mathbf{L}^{*F^*}$, so that $\mathbf{N}/\mathbf{L} = (\mathbf{N}/\mathbf{L})^F = \mathbf{N}^F/\mathbf{L}^F$. Also, $\mathbf{N}^{*F^*}$ is the stabilizer, in $N_{\mathbf{G}^{*F^*}}(\mathbf{L}^*)$, of the $\mathbf{L}^{*F^*}$-conjugacy class of s. Therefore

(7.1) $\mathbf{N}^F$ *is the stabilizer of* $e_s^{\mathbf{L}^F}$ *in* $N_{\mathbf{G}^F}(\mathbf{L})$.

It follows that $e_s^{\mathbf{L}^F}$ is a central idempotent of $\Lambda\mathbf{N}^F$. By [**BoRo1**, Theorem 11.7], we have

$$\mathrm{H}^i_c(\mathbf{Y_V}, \Lambda)e_s^{\mathbf{L}^F} = 0 \text{ for } i \neq d.$$

Our first result on the Jordan decomposition is the independence of the choice of parabolic subgroups.

Theorem 7.2. — *Given* $\mathbf{P}'$ *a parabolic subgroup of* $\mathbf{G}$ *with Levi complement* $\mathbf{L}$ *and unipotent radical* $\mathbf{V}'$, *then* $\mathrm{H}_c^{\dim(\mathbf{Y_V})}(\mathbf{Y_V}, \Lambda)e_s^{\mathbf{L}^F} \simeq \mathrm{H}_c^{\dim(\mathbf{Y_{V'}})}(\mathbf{Y_{V'}}, \Lambda)e_s^{\mathbf{L}^F}$ *as* $(\Lambda\mathbf{G}^F, \Lambda\mathbf{L}^F)$*-bimodules.*

The $(\Lambda\mathbf{G}^F, \Lambda\mathbf{L}^F)$*-bimodule* $\mathrm{H}^d_c(\mathbf{Y_V}, \Lambda)e_s^{\mathbf{L}^F}$ *is* $\mathbf{N}^F$*-stable.*

Proof. — The first result follows from Remark 6.7 and Corollary 6.5.

Let $n \in \mathbf{N}^F$. The isomorphism of varieties $\mathbf{G}/\mathbf{V} \xrightarrow{\sim} \mathbf{G}/{}^n\mathbf{V}$, $g\mathbf{V} \mapsto g\mathbf{V}n^{-1}$ induces an isomorphism of varieties $\mathbf{Y_V} \xrightarrow{\sim} \mathbf{Y}_{{}^n\mathbf{V}}$. As a consequence, we have an isomorphism of $(\Lambda\mathbf{G}^F, \Lambda\mathbf{L}^F)$-bimodules

$$\mathrm{H}^d_c(\mathbf{Y_V}, \Lambda) \simeq n_*\big(\mathrm{H}^d_c(\mathbf{Y}_{{}^n\mathbf{V}}, \Lambda)\big),$$

where $n_*\big(\mathrm{H}^d_c(\mathbf{Y}_{{}^n\mathbf{V}}, \Lambda)\big) = \mathrm{H}^d_c(\mathbf{Y}_{{}^n\mathbf{V}}, \Lambda)$ as a left $\Lambda\mathbf{G}^F$-module and the right action of $a \in \Lambda\mathbf{L}^F$ on $n_*\big(\mathrm{H}^d_c(\mathbf{Y}_{{}^n\mathbf{V}}, \Lambda)\big)$ is given by the right action of nan^{-1} on $\mathrm{H}^d_c(\mathbf{Y}_{{}^n\mathbf{V}}, \Lambda)$.

Since n fixes $e_s^{\mathbf{L}^F}$, we deduce that

$$\mathrm{H}^d_c(\mathbf{Y_V}, \Lambda)e_s^{\mathbf{L}^F} \simeq n_*\big(\mathrm{H}^d_c(\mathbf{Y}_{{}^n\mathbf{V}}, \Lambda)e_s^{\mathbf{L}^F}\big).$$

On the other hand, the first part of the theorem shows that

$$\mathrm{H}^d_c(\mathbf{Y_V}, \Lambda)e_s^{\mathbf{L}^F} \simeq \mathrm{H}^d_c(\mathbf{Y}_{{}^n\mathbf{V}}, \Lambda)e_s^{\mathbf{L}^F}.$$

It follows that $\mathrm{H}^d_c(\mathbf{Y_V}, \Lambda)e_s^{\mathbf{L}^F} \simeq n_*\big(\mathrm{H}^d_c(\mathbf{Y_V}, \Lambda)e_s^{\mathbf{L}^F}\big)$. □

Recall that, if $\mathbf{N}^F = \mathbf{L}^F$ (that is, if $C_{\mathbf{G}^*}(s)^{F^*} \subset \mathbf{L}^*$), then $\mathrm{H}^d_c(\mathbf{Y_V}, \Lambda)e_s^{\mathbf{L}^F}$ induces a Morita equivalence between $\Lambda\mathbf{G}^F e_s^{\mathbf{G}^F}$ and $\Lambda\mathbf{L}^F e_s^{\mathbf{L}^F}$ by [**BoRo1**, Theorem B′]. Note that the assumption in [**BoRo1**, Theorem B′] is $C_{\mathbf{G}^*}(s) \subset \mathbf{L}^*$, but it can easily be seen that the proof requires only the assumption $C_{\mathbf{G}^*}(s)^{F^*} \subset \mathbf{L}^*$. Theorem 7.2 shows that this Morita equivalence does not depend on the choice of a parabolic subgroup.

We will generalize the Morita equivalence to our situation. The main difficulty is to extend the action of $\mathbf{L}^F$ on $\mathrm{H}^d_c(\mathbf{Y_V}, \Lambda)e_s^{\mathbf{L}^F}$ to $\mathbf{N}^F$.

7.B. Clifford theory. — Let us recall some basic facts of Clifford theory. Let $\mathbf{k}$ be field. Let Y be a finite group and X a normal subgroup of Y. Let M be a finitely generated $\mathbf{k}X$-module that is Y-stable.

Given $y \in Y$, let N_y be the set of $\phi \in \mathrm{End}_{\mathbf{k}}(M)^\times$ such that $\phi(xm) = yxy^{-1}\phi(m)$ for all $x \in X$ and $m \in M$. Note that $N_yN_{y'} = N_{yy'}$ for all $y, y' \in Y$.

Let $N = \bigcup_{y \in Y} N_y$, a subgroup of $\mathrm{End}_{\mathbf{k}}(M)^\times$ containing $N_1 = \mathrm{End}_{\mathbf{k}X}(M)^\times$ as a normal subgroup. The action of $x \in X$ on M defines an element of N_x, and this gives a morphism $X \to N$. The Y-stability of M gives a surjective morphism of groups $Y \to N/N_1$, $y \mapsto N_y$.

Let $\hat{Y} = Y \times_{N/N_1} N$. There is a diagonal embedding of X as a normal subgroup of $\hat{Y}$. There is a commutative diagram whose horizontal and vertical sequences are exact:

$$\begin{tikzcd}
 & & 1 \arrow[d] & 1 \arrow[d] & \\
 & & X \arrow[r, equal] \arrow[d] & X \arrow[d] & \\
1 \arrow[r] & \mathrm{End}_{\mathbf{k}X}(M)^\times \arrow[r] \arrow[d, equal] & \hat{Y} \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathrm{End}_{\mathbf{k}X}(M)^\times \arrow[r] & \hat{Y}/X \arrow[r] \arrow[d] & Y/X \arrow[r] \arrow[d] & 1 \\
 & & 1 & 1 &
\end{tikzcd}$$

The action of X on M extends to an action of Y if and only if the canonical morphism of groups $\hat{Y} \to Y$ has a splitting that is the identity on X. This is equivalent to the fact that the canonical morphism of groups $\hat{Y}/X \to Y/X$ is a split surjection.

There is a split extension of groups

$$1 \to 1 + J(\mathrm{End}_{\mathbf{k}X}(M)) \to \mathrm{End}_{\mathbf{k}X}(M)^\times \to \mathrm{End}_{\mathbf{k}X}(M)^\times/(1 + J(\mathrm{End}_{\mathbf{k}X}(M))) \to 1.$$

If $\ell \nmid [Y : X]$, then every group extension $1 \to 1 + J(\mathrm{End}_{\mathbf{k}X}(M)) \to Z \to Y/X \to 1$ splits, since $1 + J(\mathrm{End}_{\mathbf{k}X}(M)$ is the finite extension of abelian groups

$$(1 + J(\mathrm{End}_{\mathbf{k}X}(M))^i)/(1 + J(\mathrm{End}_{\mathbf{k}X}(M))^{i+1}) \simeq J(\mathrm{End}_{\mathbf{k}X}(M))^i / J(\mathrm{End}_{\mathbf{k}X}(M))^{i+1},$$

and those are $\mathbf{k}(Y/X)$-modules. Consequently, if $[Y : X] \in \mathbf{k}^\times$, then the action of X on M extends to an action of Y if and only if the extension

$$1 \to \mathrm{End}_{\mathbf{k}X}(M)^\times/(1 + J(\mathrm{End}_{\mathbf{k}X}(M))) \to \hat{Y}/X(1 + J(\mathrm{End}_{\mathbf{k}X}(M))) \to Y/X \to 1$$

splits.

Consider now $\tilde{Y}$ a finite group with Y and $\tilde{X}$ two normal subgroups such that $X = Y \cap \tilde{X}$ and $\tilde{Y} = Y\tilde{X}$. Let $\tilde{M} = \mathrm{Ind}_X^{\tilde{X}}(M)$. We define $\tilde{N}_y$, $\tilde{N}$ and $\hat{\tilde{Y}}$ as above, replacing M by $\tilde{M}$.

We define a map $\rho : N_y \to \tilde{N}_y$, $\phi \mapsto (a \otimes m \mapsto yay^{-1} \otimes \phi(m))$ for $a \in \mathbf{k}\tilde{X}$ and $m \in M$. This gives a morphism of groups $N \to \tilde{N}$ extending the canonical morphism $\mathrm{End}_{\mathbf{k}X}(M) \to \mathrm{End}_{\mathbf{k}\tilde{X}}(\tilde{M})$ and a morphism of groups $\hat{Y}/X \to \hat{\tilde{Y}}/\tilde{X}$ giving a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{End}_{\mathbf{k}X}(M)^\times \arrow[r] \arrow[d, hook] & \hat{Y}/X \arrow[r] \arrow[d] & Y/X \arrow[r] \arrow[d, "\sim"] & 1 \\
1 \arrow[r] & \mathrm{End}_{\mathbf{k}\tilde{X}}(\tilde{M})^\times \arrow[r] & \hat{\tilde{Y}}/\tilde{X} \arrow[r] & \tilde{Y}/\tilde{X} \arrow[r] & 1
\end{tikzcd}$$

It induces a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{End}_{\mathbf{k}X}(M)^\times/(1+J(\mathrm{End}_{\mathbf{k}X}(M))) \arrow[r] \arrow[d, hook] & \hat{Y}/X(1+J(\mathrm{End}_{\mathbf{k}X}(M))) \arrow[r] \arrow[d] & Y/X \arrow[r] \arrow[d, "\sim"] & 1 \\
1 \arrow[r] & \mathrm{End}_{\mathbf{k}\tilde{X}}(\tilde{M})^\times/(1+J(\mathrm{End}_{\mathbf{k}\tilde{X}}(\tilde{M}))) \arrow[r] & \hat{\tilde{Y}}/\tilde{X}(1+J(\mathrm{End}_{\mathbf{k}\tilde{X}}(\tilde{M}))) \arrow[r] & \tilde{Y}/\tilde{X} \arrow[r] & 1
\end{tikzcd}$$

Assume the inclusion

$$\mathrm{End}_{\mathbf{k}X}(M)^\times/(1+J(\mathrm{End}_{\mathbf{k}X}(M))) \hookrightarrow \mathrm{End}_{\mathbf{k}\tilde{X}}(\tilde{M})^\times/(1+J(\mathrm{End}_{\mathbf{k}\tilde{X}}(\tilde{M})))$$

splits (this happens for example if $\mathrm{End}_{\mathbf{k}\tilde{X}}(\mathrm{Ind}_X^{\tilde{X}}(M))/J\big(\mathrm{End}_{\mathbf{k}\tilde{X}}(\mathrm{Ind}_X^{\tilde{X}}(M))\big) \simeq \mathbf{k}^n$ for some n). If the surjection $\hat{\tilde{Y}}/\tilde{X}(1+J(\mathrm{End}_{\mathbf{k}\tilde{X}}(\tilde{M}))) \to \tilde{Y}/\tilde{X}$ splits, then the surjection $\hat{Y}/X(1+J(\mathrm{End}_{\mathbf{k}X}(M))) \to Y/X$ splits.

As a consequence, we have the following proposition.

Proposition 7.3. — *Let $\tilde{Y}$ be a finite group and Y, $\tilde{X}$ be two normal subgroups of $\tilde{Y}$. Let $X = Y \cap \tilde{X}$. We assume $\tilde{Y} = Y\tilde{X}$. Let $\mathbf{k}$ be a field with $[Y : X] \in \mathbf{k}^\times$.*

Let M be a finitely generated $\mathbf{k}X$-module that is Y-stable. We assume that

$$\mathrm{End}_{\mathbf{k}\tilde{X}}(\mathrm{Ind}_X^{\tilde{X}}(M))/J\big(\mathrm{End}_{\mathbf{k}\tilde{X}}(\mathrm{Ind}_X^{\tilde{X}}(M))\big) \simeq \mathbf{k}^n \text{ for some } n.$$

If $\mathrm{Ind}_X^{\tilde{X}}(M)$ extends to $\tilde{Y}$, then M extends to Y.

7.C. Embedding in a group with connected center and Morita equivalence. — We fix a connected reductive algebraic group $\tilde{\mathbf{G}}$ containing $\mathbf{G}$ as a closed subgroup, with an extension of F to an endomorphism of $\tilde{\mathbf{G}}$ such that F^δ is a Frobenius endomorphism of $\tilde{\mathbf{G}}$ defining an $\mathbb{F}_q$-structure, and such that $\tilde{\mathbf{G}} = \mathbf{G} \cdot Z(\tilde{\mathbf{G}})$ and $Z(\tilde{\mathbf{G}})$ is connected (cf [**DeLu**, proof of Corollary 5.18]).

Let $\tilde{\mathbf{T}} = \mathbf{T} \cdot Z(\tilde{\mathbf{G}})$, an F-stable maximal torus of $\tilde{\mathbf{G}}$. Fix a triple $(\tilde{\mathbf{G}}^*, \tilde{\mathbf{T}}^*, F^*)$ dual to $(\tilde{\mathbf{G}}, \tilde{\mathbf{T}}, F)$. The inclusion $i : \mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ induces a surjection $i^* : \tilde{\mathbf{G}}^* \twoheadrightarrow \mathbf{G}^*$. Let $\tilde{\mathbf{L}} = \mathbf{L} \cdot Z(\tilde{\mathbf{G}})$, so that $\tilde{\mathbf{L}}^* = (i^*)^{-1}(\mathbf{L}^*)$. Let $\tilde{\mathbf{N}} = \mathbf{N}\tilde{\mathbf{L}}$.

Let J be a set of representatives of conjugacy classes of ℓ'-elements $\tilde{t} \in \tilde{\mathbf{G}}^{*F^*}$ such that $i^*(\tilde{t}) = s$. Note that $J \subset \tilde{\mathbf{L}}^{*F^*}$.

Lemma 7.4. — *We have $e_s^{\mathbf{G}^F} = \sum_{\tilde{t} \in J} e_{\tilde{t}}^{\tilde{\mathbf{G}}^F}$ and $e_s^{\mathbf{L}^F} = \sum_{n \in \mathbf{N}^F/\mathbf{L}^F} \sum_{\tilde{t} \in J} n e_{\tilde{t}}^{\tilde{\mathbf{L}}^F} n^{-1}$.*

Proof. — The first statement is a classical translation from $\mathbf{G}^*$ to $\mathbf{G}$, cf for instance [**Bo3**, Proposition 11.7].

Let $\tilde{s}$ be a semisimple element of $\tilde{\mathbf{G}}^{*F^*}$ such that $i^*(\tilde{s}) = s$. If $\Lambda \neq K$, we will assume that $\tilde{s}$ has order prime to ℓ (this is always possible as we may replace $\tilde{s}$ by its ℓ'-part if necessary). Note that $\tilde{s} \in \tilde{\mathbf{L}}^{*F^*}$.

Let $n \in \tilde{\mathbf{N}}^{*F^*}$ such that $n\tilde{s}n^{-1}$ is $\tilde{\mathbf{L}}^{*F^*}$-conjugate to $\tilde{s}$. Then $n \in \tilde{\mathbf{L}}^{*F^*} \cdot C_{\tilde{\mathbf{G}}^*}(\tilde{s})$. Since $i^*(C_{\tilde{\mathbf{G}}^*}(\tilde{s})) \subset C^\circ_{\mathbf{G}^*}(s) \subset \mathbf{L}^*$, it follows that $i^*(n) \in \mathbf{L}^{*F^*}$. We have $\mathbf{N}^{*F^*}/\mathbf{L}^{*F^*} = \tilde{\mathbf{N}}^{*F^*}/\tilde{\mathbf{L}}^{*F^*}$, hence $n \in \tilde{\mathbf{L}}^{*F^*}$.

It follows that $\mathbf{N}^{*F^*}/\mathbf{L}^{*F^*}$ acts freely on the set of conjugacy classes of $\tilde{\mathbf{L}}^{*F^*}$ whose image under i^* is the $\mathbf{L}^{*F^*}$-conjugacy class of s. Through the identification of $\mathbf{N}^{*F^*}/\mathbf{L}^{*F^*}$ with $\mathbf{N}^F/\mathbf{L}^F$, this shows that given $\tilde{t} \in J$, the stabilizer in $\mathbf{N}^F$ of $e_{\tilde{t}}^{\tilde{L}^F}$ is $\mathbf{L}^F$. □

Theorem 7.5. — *The action of $k\mathbf{G}^F e_s^{\mathbf{G}^F} \otimes (k\mathbf{L}^F e_s^{\mathbf{L}^F})^{\mathrm{opp}}$ on $\mathrm{H}^d_c(\mathbf{Y_V}, k)e_s^{\mathbf{L}^F}$ extends to an action of $k\mathbf{G}^F e_s^{\mathbf{G}^F} \otimes (k\mathbf{N}^F e_s^{\mathbf{L}^F})^{\mathrm{opp}}$. The resulting $(k\mathbf{G}^F e_s^{\mathbf{G}^F}, k\mathbf{N}^F e_s^{\mathbf{L}^F})$-bimodule induces a Morita equivalence.*

Proof. — Let $\tilde{\mathbf{P}} = \mathbf{P}\cdot Z(\tilde{\mathbf{G}})$ and let $\tilde{\mathbf{P}}^* = i^{*-1}(\mathbf{P}^*)$. Note that $\tilde{\mathbf{L}}$ (resp. $\tilde{\mathbf{L}}^*$) is a Levi complement of $\tilde{\mathbf{P}}$ (resp. $\tilde{\mathbf{P}}^*$) and it is F-stable (resp. F^*-stable) and the pair $(\tilde{\mathbf{L}}^*, \tilde{\mathbf{P}}^*)$ is dual to $(\tilde{\mathbf{L}}, \tilde{\mathbf{P}})$.

We put

$$X = (\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}})\cdot \Delta\tilde{\mathbf{L}}^F,\ \ Y = (\mathbf{G}^F \times (\mathbf{N}^F)^{\mathrm{opp}})\cdot \Delta\tilde{\mathbf{N}}^F,$$

$$\tilde{X} = \tilde{\mathbf{G}}^F \times (\tilde{\mathbf{L}}^F)^{\mathrm{opp}} \text{ and } \tilde{Y} = \tilde{\mathbf{G}}^F \times (\tilde{\mathbf{N}}^F)^{\mathrm{opp}}.$$

Let $\tilde{\mathbf{Y}}_\mathbf{V} = \mathbf{Y}_\mathbf{V}^{\tilde{\mathbf{G}}}$. The stabilizer in $\tilde{X}$ of the subvariety $\mathbf{Y_V}$ of $\tilde{\mathbf{Y}}_\mathbf{V}$ is X, hence we have an isomorphism of $\tilde{X}$-varieties $\mathrm{Ind}_X^{\tilde{X}} \mathbf{Y_V} \xrightarrow{\sim} \tilde{\mathbf{Y}}_\mathbf{V}$.

Let $M = \mathrm{H}^d_c(\mathbf{Y_V}, k)e_s^{\mathbf{L}^F}$, a $(kX(e_s^{\mathbf{G}^F} \otimes e_s^{\mathbf{L}^F}))$-module. Let $\tilde{M} = \mathrm{Ind}_X^{\tilde{X}} M$, a $(k\tilde{X}(e_s^{\mathbf{G}^F} \otimes e_s^{\mathbf{L}^F}))$-module. We have an isomorphism of $(k\tilde{X}(e_s^{\mathbf{G}^F} \otimes e_s^{\mathbf{L}^F}))$-modules $\tilde{M} \xrightarrow{\sim} \mathrm{H}^d_c(\tilde{\mathbf{Y}}_\mathbf{V}, k)e_s^{\mathbf{L}^F}$.

We put $e = \sum_{\tilde{t}\in J} e_{\tilde{t}}^{\tilde{\mathbf{L}}^F}$. We have $e_s^{\mathbf{L}^F} = \sum_{n\in\tilde{\mathbf{N}}^F/\tilde{\mathbf{L}}^F} n e n^{-1}$ and e is a central idempotent of $k\tilde{\mathbf{L}}^F$ (Lemma 7.4).

The kX-module M is $\mathbf{N}^F$-stable (Theorem 7.2), hence the $k\tilde{X}$-module $\tilde{M}$ is $\mathbf{N}^F$-stable as well. It follows that given $\tilde{t} \in J$ and $n \in \mathbf{N}^F$, we have $n_*(\mathrm{H}^d_c(\tilde{\mathbf{Y}})e_{\tilde{t}}^{\tilde{L}^F}) \simeq \mathrm{H}^d_c(\tilde{\mathbf{Y}})e_{n\tilde{t}n^{-1}}^{\tilde{L}^F}$ as $k\tilde{X}$-modules. The classical Mackey formula for induction and restriction in finite groups shows now that

$$\mathrm{H}^d_c(\tilde{\mathbf{Y}}_\mathbf{V}, k)\Big(\sum_{n\in\mathbf{N}^F/\mathbf{L}^F} e_{n\tilde{t}n^{-1}}^{\tilde{L}^F}\Big) \simeq \mathrm{Res}_{\tilde{X}}^{\tilde{Y}} \mathrm{Ind}_{\tilde{X}}^{\tilde{Y}}\big(\mathrm{H}^d_c(\tilde{\mathbf{Y}})e_{\tilde{t}}^{\tilde{L}^F}\big),$$

hence

$$\tilde{M} \simeq \mathrm{Res}_{\tilde{X}}^{\tilde{Y}} \mathrm{Ind}_{\tilde{X}}^{\tilde{Y}}\big(\tilde{M}e\big).$$

Lemma 7.4 shows that $\tilde{M}e$ induces a Morita equivalence between $k\tilde{\mathbf{G}}^F e_s^{\mathbf{G}^F}$ and $k\tilde{\mathbf{L}}^F e$ (cf [**BoRo1**, Theorem B′]). In particular, it is a direct sum of indecomposable modules no two of which are isomorphic.

Since $ek\tilde{\mathbf{N}}^F$ induces a Morita equivalence between $k\tilde{\mathbf{L}}^F e$ and $k\tilde{\mathbf{N}}^F e_s^{\mathbf{L}^F}$, we deduce that the right action of $\tilde{\mathbf{L}}^F$ on $\tilde{M} \simeq \tilde{M}e \otimes_{k\tilde{\mathbf{L}}e} ek\tilde{\mathbf{N}}^F$ extends to an action of $\tilde{\mathbf{N}}^F$ commuting with the left action of $\tilde{\mathbf{G}}^F$ and the extended bimodule $\tilde{M}'$ induces a Morita equivalence between $k\tilde{\mathbf{G}}^F e_s^{\mathbf{G}^F}$ and $k\tilde{\mathbf{N}}^F e_s^{\mathbf{L}^F}$. It follows that

$$\mathrm{End}_{k\tilde{X}}(\tilde{M}) \simeq \mathrm{End}_{k(\tilde{\mathbf{N}}^F \times (\tilde{\mathbf{L}}^F)^{\mathrm{opp}})}(k\tilde{\mathbf{N}}^F e_s^{\mathbf{L}^F}).$$

Given $n_1, n_2 \in \tilde{\mathbf{N}}^F$ with $n_1 \notin n_2 \tilde{\mathbf{L}}^F$, the central idempotents $n_1 e n_1^{-1}$ and $n_2 e n_2^{-1}$ of $k\tilde{\mathbf{L}}^F$ are orthogonal. It follows that

$$\mathrm{End}_{k(\tilde{\mathbf{N}}^F \times (\tilde{\mathbf{L}}^F)^{\mathrm{opp}})}(k\tilde{\mathbf{N}}^F e_s^{\mathbf{L}^F}) \simeq \prod_{n \in \mathbf{N}^F/\mathbf{L}^F} \mathrm{End}_{k(\tilde{\mathbf{N}}^F \times (\tilde{\mathbf{L}}^F)^{\mathrm{opp}})}(k\tilde{\mathbf{N}}^F n e n^{-1}) \simeq \left(Z(k\tilde{\mathbf{L}}^F e)\right)^{[\mathbf{N}^F/\mathbf{L}^F]}.$$

We deduce that $\mathrm{End}_{k\tilde{X}}(\tilde{M})^\times / \left(1 + J(\mathrm{End}_{k\tilde{X}}(\tilde{M}))\right) \simeq (k^\times)^r$ for some r. Since $[Y : X] = [\mathbf{N} : \mathbf{L}]$ is prime to ℓ, it follows from Proposition 7.3 that the action of X on M extends to an action of Y. Denote by M' the extended module. We have $\mathrm{Res}_{\tilde{X}}^{\tilde{Y}} \mathrm{Ind}_Y^{\tilde{Y}}(M')e \simeq \tilde{M}e \simeq \mathrm{Res}_{\tilde{X}}^{\tilde{Y}}(\tilde{M}')e$, hence $\mathrm{Ind}_Y^{\tilde{Y}}(M') \simeq \tilde{M}'$. It follows that $\mathrm{Ind}_Y^{\tilde{Y}}(M')$ induces a Morita equivalence between $k\tilde{\mathbf{G}}^F e_s^{\mathbf{G}^F}$ and $k\tilde{\mathbf{N}}^F e_s^{\mathbf{L}^F}$. We have

$$\mathrm{End}_{k\tilde{\mathbf{G}}^F}(\mathrm{Ind}_Y^{\tilde{Y}}(M')) \simeq \mathrm{End}_{k\tilde{\mathbf{G}}^F}(k\tilde{\mathbf{G}}^F \otimes_{k\mathbf{G}^F} M) \simeq \mathrm{Hom}_{k\mathbf{G}^F}(M, M \otimes_{k\mathbf{N}^F} k\tilde{\mathbf{N}}^F) \simeq \mathrm{End}_{k\mathbf{G}^F}(M) \otimes_{k\mathbf{N}^F} k\tilde{\mathbf{N}}^F.$$

The canonical map $k\tilde{\mathbf{N}}^F e_s^{\mathbf{L}^F} \to \mathrm{End}_{k\tilde{\mathbf{G}}^F}(\mathrm{Ind}_Y^{\tilde{Y}} M')$ is an isomorphism, hence the canonical map $k\mathbf{N}^F e_s^{\mathbf{L}^F} \to \mathrm{End}_{k\mathbf{G}^F}(M')$ is an isomorphism as well. Also, M is a faithful $k\mathbf{G}^F e_s^{\mathbf{G}^F}$-module, since $\tilde{M} = \mathrm{Ind}_{\mathbf{G}^F}^{\tilde{\mathbf{G}}^F} M$ is a faithful $k\tilde{\mathbf{G}}^F e_s^{\mathbf{G}^F}$-module. We deduce that M' induces a Morita equivalence between $k\mathbf{G}^F e_s^{\mathbf{G}^F}$ and $k\mathbf{N}^F e_s^{\mathbf{L}^F}$. □

7.D. Splendid Rickard equivalence and local structure. —

Theorem 7.6. — *The action of $k\mathbf{G}^F e_s^{\mathbf{G}^F} \otimes (k\mathbf{L}^F e_s^{\mathbf{L}^F})^{\mathrm{opp}}$ on $G\Gamma_c(\mathbf{Y_V}, k)e_s^{\mathbf{L}^F}$ extends to an action of $k\mathbf{G}^F e_s^{\mathbf{G}^F} \otimes (k\mathbf{N}^F e_s^{\mathbf{L}^F})^{\mathrm{opp}}$. The resulting complex induces a splendid Rickard equivalence between $k\mathbf{G}^F e_s^{\mathbf{G}^F}$ and $k\mathbf{N}^F e_s^{\mathbf{L}^F}$.*

Proof. — • *Step 1*: Identification of $\mathrm{End}^\bullet_{k\mathbf{G}^F}(G\Gamma_c(\mathbf{Y_V}, k)e_s^{\mathbf{L}^F})$ in $\mathrm{Ho}^b(k(\mathbf{L}^F \times (\mathbf{L}^F)^{\mathrm{opp}}))$.

Let $C = (G\Gamma_c(\mathbf{Y_V}, k)e_s^{\mathbf{L}^F})^{\mathrm{red}}$. The vertices of the indecomposable direct summands of components of C are contained in $\Delta\mathbf{L}^F$ by Lemma 3.5. Let Q be an ℓ-subgroup of $\mathbf{L}^F$. We have $\mathrm{Br}_{\Delta Q}(C) \simeq G\Gamma_c(\mathbf{Y}_{C_{\mathbf{V}}(Q)}^{C_{\mathbf{G}}(Q)}, k)\mathrm{br}_Q(e_s^{\mathbf{L}^F})$ in $\mathrm{Ho}^b(k(C_{\mathbf{G}^F}(Q) \times C_{\mathbf{L}^F}(Q)^{\mathrm{opp}}))$ by Proposition 3.4. Let $\mathscr{X}$ be the rational series of $(\mathbf{L}, F)$ corresponding to s, so that $e_s^{\mathbf{L}^F} = e_{\mathscr{X}}$. Theorem 4.14 shows that

$$\mathrm{br}_Q(e_{\mathscr{X}}) = \sum_{\mathscr{Y} \in (i_Q^{\mathbf{L}})^{-1}(\mathscr{X})} e_{\mathscr{Y}}.$$

Let $\mathscr{X}'$ be the rational series of $(\mathbf{G}, F)$ corresponding to s. We have $\mathscr{X} \subset \mathscr{X}'$. Given $\mathscr{Y} \in (i_Q^{\mathbf{L}})^{-1}(\mathscr{X})$, let $\mathscr{Y}'$ be the rational series of $(C_{\mathbf{G}}^\circ(Q), F)$ containing $\mathscr{Y}$. We have $i_Q^{\mathbf{G}}(\mathscr{Y}') = \mathscr{X}'$ and Proposition 4.11 shows that $\mathscr{Y}$ is $(C_{\mathbf{G}}^\circ(Q), C_{\mathbf{L}}^\circ(Q))$-regular. It follows from [**BoRo1**, Theorem 11.7] that $\mathrm{H}_c^i(\mathbf{Y}_{C_{\mathbf{V}}(Q)}^{C_{\mathbf{G}}^\circ(Q)}, k)e_{\mathscr{Y}} = 0$ for $i \neq \dim \mathbf{Y}_{C_{\mathbf{V}}(Q)}^{C_{\mathbf{G}}^\circ(Q)}$, hence $\mathrm{H}_c^i(\mathbf{Y}_{C_{\mathbf{V}}(Q)}^{C_{\mathbf{G}}(Q)}, k)e_{\mathscr{Y}} = 0$ for $i \neq \dim \mathbf{Y}_{C_{\mathbf{V}}(Q)}^{C_{\mathbf{G}}(Q)}$. We have shown that the cohomology of $\mathrm{Br}_{\Delta Q}(C)$ is concentrated in a single degree. Note that $\mathrm{Res}_{kC_{\mathbf{G}^F}(Q)}(\mathrm{Br}_{\Delta Q}(C))$ is a perfect complex, hence its homology is projective as a $kC_{\mathbf{G}^F}(Q)$-module. We deduce from Theorem A.4 that

$$\mathrm{End}^\bullet_{k\mathbf{G}^F}(C) \simeq \mathrm{End}_{D^b(k\mathbf{G}^F)}(C) \text{ in } \mathrm{Ho}^b(k(\mathbf{L}^F \times (\mathbf{L}^F)^{\mathrm{opp}})).$$

• *Step 2*: Study of $\mathrm{End}_{\mathrm{Ho}^b(k(\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}))}(\mathrm{Ind}_{\mathbf{G}^F\times(\mathbf{L}^F)^{\mathrm{opp}}}^{\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}} \mathrm{G}\Gamma_c(\mathbf{Y_V},k)e_s^{\mathbf{L}^F})$.

Let $C' = \mathrm{Ind}_{\mathbf{G}^F\times(\mathbf{L}^F)^{\mathrm{opp}}}^{\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}} C$. Let P be a complex of $k(\mathbf{N}^F \times (\mathbf{N}^F)^{\mathrm{opp}})$-proj with $P^i = 0$ for $i > 0$, together with a quasi-isomorphism $P \to k\mathbf{N}^F$ of $k(\mathbf{N}^F \times (\mathbf{N}^F)^{\mathrm{opp}})$-modules. As the terms of C' are projective $k\mathbf{G}^F$-modules, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{End}_{\mathrm{Ho}^b(k(\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}))}(C') \arrow[r] \arrow[d, "\sim"'] & \mathrm{End}_{D^b(k(\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}))}(C') \arrow[d, "\sim"] \\
\mathrm{Hom}_{\mathrm{Ho}^b(k(\mathbf{N}^F\times(\mathbf{N}^F)^{\mathrm{opp}}))}(k\mathbf{N}^F, \mathrm{End}^\bullet_{k\mathbf{G}^F}(C')) \arrow[r] & \mathrm{Hom}_{\mathrm{Ho}^b(k(\mathbf{N}^F\times(\mathbf{N}^F)^{\mathrm{opp}}))}(P, \mathrm{End}^\bullet_{k\mathbf{G}^F}(C'))
\end{tikzcd}$$

Using the isomorphisms of complexes in $\mathrm{Ho}^b(k(\mathbf{N}^F \times (\mathbf{N}^F)^{\mathrm{opp}}))$

$$\mathrm{End}^\bullet_{k\mathbf{G}^F}(C') \simeq \mathrm{Ind}_{\mathbf{L}^F\times(\mathbf{L}^F)^{\mathrm{opp}}}^{\mathbf{N}^F\times(\mathbf{N}^F)^{\mathrm{opp}}}(\mathrm{End}^\bullet_{k\mathbf{G}^F}(C))$$

and

$$\mathrm{End}_{D^b(k\mathbf{G}^F)}(C') \simeq \mathrm{Ind}_{\mathbf{L}^F\times(\mathbf{L}^F)^{\mathrm{opp}}}^{\mathbf{N}^F\times(\mathbf{N}^F)^{\mathrm{opp}}}(\mathrm{End}_{D^b(k\mathbf{G}^F)}(C)),$$

we deduce that

$$\mathrm{End}^\bullet_{k\mathbf{G}^F}(C') \simeq \mathrm{End}_{D^b(k\mathbf{G}^F)}(C') \text{ in } \mathrm{Ho}^b(k(\mathbf{N}^F \times (\mathbf{N}^F)^{\mathrm{opp}})).$$

Now, the canonical map

$$\mathrm{Hom}_{\mathrm{Ho}^b(k(\mathbf{N}^F\times(\mathbf{N}^F)^{\mathrm{opp}}))}(k\mathbf{N}^F, \mathrm{End}_{D^b(k\mathbf{G}^F)}(C')) \to \mathrm{Hom}_{\mathrm{Ho}^b(k(\mathbf{N}^F\times(\mathbf{N}^F)^{\mathrm{opp}}))}(P, \mathrm{End}_{D^b(k\mathbf{G}^F)}(C'))$$

is an isomorphism, since $\mathrm{End}_{D^b(k\mathbf{G}^F)}(C')$ is a complex concentrated in degree 0. It follows that the top horizontal map in the commutative diagram above is an isomorphism, hence we have canonical isomorphisms

$$\mathrm{End}_{\mathrm{Ho}^b(k(\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}))}(C') \xrightarrow{\sim} \mathrm{End}_{D^b(k(\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}))}(C') \xrightarrow{\sim} \mathrm{End}_{k(\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}})}(\mathrm{Ind}_{\mathbf{G}^F\times(\mathbf{L}^F)^{\mathrm{opp}}}^{\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}} \mathrm{H}_c^d(\mathbf{Y_V},k)).$$

• *Step 3*: Construction of a direct summand $\tilde{C}$ of $\mathrm{Ind}_{\mathbf{G}^F\times(\mathbf{L}^F)^{\mathrm{opp}}}^{\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}}(\mathrm{G}\Gamma_c(\mathbf{Y_V},k)e_s^{\mathbf{L}^F})$.

We have shown (Theorem 7.5) that there is a direct summand M' of $\mathrm{Ind}_{\mathbf{G}^F\times(\mathbf{L}^F)^{\mathrm{opp}}}^{\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}} \mathrm{H}_c^d(\mathbf{Y_V},k)$ whose restriction to $\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}$ is isomorphic to $\mathrm{H}_c^d(\mathbf{Y_V},k)$. Let i be the corresponding idempotent of $\mathrm{End}_{k(\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}})}(\mathrm{Ind}_{\mathbf{G}^F\times(\mathbf{L}^F)^{\mathrm{opp}}}^{\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}} \mathrm{H}_c(\mathbf{Y_V},k))$ and j its inverse image in $\mathrm{End}_{\mathrm{Ho}^b(k(\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}))}(C')$ via the isomorphisms above. We have a surjective homomorphism of finite-dimensional k-algebras

$$\mathrm{End}_{\mathrm{Comp}(k(\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}))}(C') \twoheadrightarrow \mathrm{End}_{\mathrm{Ho}^b(k(\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}))}(C').$$

Consequently, j lifts to an idempotent j' of $\mathrm{End}_{\mathrm{Comp}(k(\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}))}(C')$. It corresponds to a direct summand $\tilde{C}$ of C' quasi-isomorphic to M' and $\mathrm{Res}_{\mathbf{G}^F\times(\mathbf{L}^F)^{\mathrm{opp}}}^{\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}}(\tilde{C})$ is a direct summand of $\mathrm{Res}_{\mathbf{G}^F\times(\mathbf{L}^F)^{\mathrm{opp}}}^{\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}}(C') \simeq C^{\oplus[\mathbf{N}^F:\mathbf{L}^F]}$.

• *Step 4*: $\tilde{C}$ lifts $G\Gamma_c(\mathbf{Y_V}, k)e_s^{\mathbf{L}^F}$.

Let $C = \bigoplus_{1 \leqslant j \leqslant n} C_j$ be a decomposition into a direct sum of indecomposable objects of $\mathrm{Ho}^b(k(\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}})$. This induces a decomposition $M = \bigoplus_{1 \leqslant j \leqslant n} M_j$, where $M_j = H^d(C_j)$ and M_j and $M_{j'}$ have no isomorphic indecomposable summands for $j \neq j'$ (cf proof of Theorem 7.5). We have $\mathrm{Res}^{\mathbf{G}^F \times (\mathbf{N}^F)^{\mathrm{opp}}}_{\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}}(\tilde{C}) \simeq \bigoplus_{1 \leqslant j \leqslant n} C_j^{\oplus a_j}$ in $\mathrm{Ho}^b(k(\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}})$ for some integers $a_j \geqslant 0$ and $\bigoplus_{1 \leqslant j \leqslant n} H^d(C_j)^{\oplus a_j} \simeq M$. It follows that $a_j = 1$ for all j, hence $\mathrm{Res}^{\mathbf{G}^F \times (\mathbf{N}^F)^{\mathrm{opp}}}_{\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}}(\tilde{C}) \simeq C$ in $\mathrm{Ho}^b(k(\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}})$. This shows the first statement.

• *Step 5*: Rickard equivalence.

We have shown above that $\mathrm{End}^\bullet_{k\mathbf{G}^F}(\tilde{C}) \simeq \mathrm{End}_{D^b(k\mathbf{G}^F)}(\tilde{C})$ in $\mathrm{Ho}^b(k(\mathbf{N}^F \times (\mathbf{N}^F)^{\mathrm{opp}}))$. On the other hand, $\mathrm{End}_{D^b(k\mathbf{G}^F)}(\tilde{C}) \simeq \mathrm{End}_{k\mathbf{G}^F}(M') \simeq k\mathbf{N}^F e_s^{\mathbf{L}^F}$. It follows from Corollary A.5 that $\tilde{C}$ induces a splendid Rickard equivalence. □

We now summarize and complete the description of the Jordan decomposition of blocks.

Theorem 7.7. — *The complex of* $(\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}, \mathcal{O}\mathbf{L}^F e_s^{\mathbf{L}^F})$*-bimodules* $G\Gamma_c(\mathbf{Y_V}, \mathcal{O})e_s^{\mathbf{L}^F}$ *extends to a complex* C *of* $(\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}, \mathcal{O}\mathbf{N}^F e_s^{\mathbf{L}^F})$*-bimodules. The complex* C *induces a splendid Rickard equivalence between* $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$ *and* $\mathcal{O}\mathbf{N}^F e_s^{\mathbf{L}^F}$.

There is a (unique) bijection $b \mapsto b'$ *between blocks of* $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$ *and* $\mathcal{O}\mathbf{N}^F e_s^{\mathbf{L}^F}$ *such that* $bC \simeq Cb'$.

Given b *a block of* $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$*, then:*

- *the bimodule* $H^{\dim \mathbf{Y_V}}(bCb')$ *induces a Morita equivalence between* $\mathcal{O}\mathbf{G}^F b$ *and* $\mathcal{O}\mathbf{N}^F b'$
- *the complex* bCb' *induces a splendid Rickard equivalence between* $\mathcal{O}\mathbf{G}^F b$ *and* $\mathcal{O}\mathbf{N}^F b'$
- *there is a (unique) equivalence* $(Q, b'_Q) \mapsto (Q, b_Q)$ *from the category of* b'*-subpairs to the category of* b*-subpairs such that* $b_Q \mathrm{Br}_{\Delta Q}(C) = \mathrm{Br}_{\Delta Q}(C) b'_Q$. *In particular, if* D *is a defect group of* b'*, then* D *is a defect group of* b.

Proof. — Theorem 7.6 provides a complex C' of $(k\mathbf{G}^F e_s^{\mathbf{G}^F} \otimes (k\mathbf{N}^F e_s^{\mathbf{L}^F})^{\mathrm{opp}})$-modules inducing a splendid Rickard equivalence. By Rickard's lifting Theorem [**Ri2**, Theorem 5.2], there is a splendid complex C of $(\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F} \otimes (\mathcal{O}\mathbf{N}^F e_s^{\mathbf{L}^F})^{\mathrm{opp}})$-modules, unique up to isomorphism in $\mathrm{Comp}(\mathcal{O}(\mathbf{G}^F \times (\mathbf{N}^F)^{\mathrm{opp}}))$, such that $kC \simeq C'$. Also, [**Ri2**, proof of Theorem 5.2] shows that $G\Gamma_c(\mathbf{Y_V}, \mathcal{O})e_s^{\mathbf{L}^F}$ is the unique splendid complex that lifts $G\Gamma_c(\mathbf{Y_V}, k)e_s^{\mathbf{L}^F}$. As a consequence,

$$\mathrm{Res}^{\mathbf{G}^F \times (\mathbf{N}^F)^{\mathrm{opp}}}_{\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}}(C) \simeq G\Gamma_c(\mathbf{Y_V}, \mathcal{O})e_s^{\mathbf{L}^F}.$$

By [**Ri2**, Theorem 5.2], the complex C induces a splendid Rickard equivalence.

Since $H^d(bkCb')$ induces a Morita equivalence, it follows that $H^d(bCb')$ induces a Morita equivalence (cf e.g. [**Ri2**, proof of Theorem 5.2]).

The existence of the bijection between blocks follows from the isomorphism of algebras $Z(\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}) \xrightarrow{\sim} Z(\mathcal{O}\mathbf{N}^F e_s^{\mathbf{L}^F})$ induced by the Morita equivalence, and the blockwise statements on Morita and Rickard equivalence are clear.

By [**Pu**, Theorem 19.7], it follows that the Brauer categories of $k\mathbf{G}^F b$ and $k\mathbf{N}^F b'$ are equivalent, and in particular, $k\mathbf{G}^F b$ and $k\mathbf{N}^F b'$ have isomorphic defect groups. □

Remark 7.8. — Assume $C_{\mathbf{G}^*}(s) \subset \mathbf{L}^*$. Fix a block b of $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$. Kessar and Malle have proven that if either b or b' has a defect group that is abelian modulo the ℓ-center of $\mathbf{G}^F$, then b and b' have isomorphic defect groups [**KeMa1**, Theorem 1.3].

Example 7.9. — Assume in this example that $C^\circ_{\mathbf{G}^*}(s) = \mathbf{L}^*$ and that $(C_{\mathbf{G}^*}(s)/C^\circ_{\mathbf{G}^*}(s))^{F^*}$ is cyclic. The element s defines a linear character $\hat{s} : \mathbf{L}^F \to \mathcal{O}^\times$ which induces an isomorphism of algebra $\mathcal{O}\mathbf{L}^F e_s^{\mathbf{L}^F} \simeq \mathcal{O}\mathbf{L}^F e_1^{\mathbf{L}^F}$. The linear character $\hat{s}$ is stable under the action of $\mathbf{N}^F$ so, since $\mathbf{N}^F/\mathbf{L}^F$ is cyclic, it extends to a linear character $\hat{s}^+ : \mathbf{N}^F \to \mathcal{O}^\times$. Again, $\hat{s}^+$ induces an isomorphism of algebra $\mathcal{O}\mathbf{N}^F e_s^{\mathbf{L}^F} \simeq \mathcal{O}\mathbf{N}^F e_1^{\mathbf{L}^F}$. Combined with this, Theorem 7.5 provides a Morita equivalence between $\mathcal{O}\mathbf{N}^F e_1^{\mathbf{L}^F}$ and $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$.

Example 7.10 (Type A). — Assume in this example that all the simple components of $\mathbf{G}$ are of type A (no assumption is made on the action of F). Then $C^\circ_{\mathbf{G}^*}(s) = \mathbf{L}^*$ and $C_{\mathbf{G}^*}(s)/C^\circ_{\mathbf{G}^*}(s)$ is cyclic. Therefore, Example 7.9 can be applied to provide a Morita equivalence between $\mathcal{O}\mathbf{N}^F e_1^{\mathbf{L}^F}$ and $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$.

Remark 7.11. — This article was announced at the end of the introduction of [**BoRo1**]. Unfortunately, we have not been able to settle the problem of finiteness of source algebras. On the other hand, in addition to what was announced in [**BoRo1**], we have provided an extension of the Jordan decomposition to the quasi-isolated case.

Appendix A
About ℓ-permutation modules

Let us recall here some results of Broué and Puig, cf [**Br1**, §3.6]. Let G be a finite group and P an ℓ-subgroup of G. First, an indecomposable ℓ-permutation $\mathcal{O}G$-module M has a vertex containing P if and only if $\mathrm{Br}_P(M) \neq 0$. Also, given V an indecomposable projective $k[N_G(P)/P]$-module (it is then an ℓ-permutation kG-module), there exists a unique indecomposable ℓ-permutation $\mathcal{O}G$-module $\mathrm{M}(P,V)$ such that $\mathrm{Br}_P\mathrm{M}(P,V) \simeq V$. Moreover, every indecomposable ℓ-permutation $\mathcal{O}G$-module with vertex P is isomorphic to such an $\mathrm{M}(P,V)$.

The following lemma is a variant of [**Bou**, Proposition 6.4].

Lemma A.1. — *Let M and N be ℓ-permutation kG-modules and let $\psi \in \mathrm{Hom}_{kG}(M,N)$. Assume that all indecomposable summands of M (respectively N) have a vertex contained in P (respectively equal to P), and that $\mathrm{Br}_P(\psi)$ is a split surjection. Then ψ is a split surjection.*

Proof. — Proceeding by induction on the dimension of N, we can assume that N is indecomposable. Fix a decomposition $M = \bigoplus_{i\in I} M_i$ where M_i is indecomposable for all $i \in I$ and let $\psi_i : M_i \to N$ denote the restriction of ψ. Since $\mathrm{Br}_P(\psi)$ is a split surjection and $\mathrm{Br}_P(N)$ is indecomposable, we deduce that $\mathrm{Br}_P(\psi_i) : \mathrm{Br}_P(M_i) \to \mathrm{Br}_P(N)$ is an isomorphism for some $i \in I$ (because $\mathrm{Br}_P(M_i)$ is equal to zero or is indecomposable).

By the above classification of indecomposable ℓ-permutation $\mathcal{O}G$-modules, this forces M_i to be isomorphic to N. Let $\psi' : N \xrightarrow{\sim} M_i$ be an isomorphism. Then $\mathrm{Br}_P(\psi_i\psi') = \mathrm{Br}_P(\psi_i)\mathrm{Br}_P(\psi')$ is an isomorphism, so it is not nilpotent. Therefore, $\psi_i\psi'$ does not belong to the radical of $\mathrm{End}_{\mathcal{O}G}(N)$, hence it is invertible (because $\mathrm{End}_{\mathcal{O}G}(N)$ is a local ring). So ψ_i is an isomorphism, as desired. □

Lemma A.2. — *Let C be a bounded complex of ℓ-permutation kG-modules, all of whose indecomposable summands have a vertex contained in P. Let D be a bounded complex of finitely generated projective $k[N_G(P)/P]$-modules. We assume that $\mathrm{Br}_P(C)$ is homotopy equivalent to D.*

Then there exists a bounded complex C' of ℓ-permutation kG-modules, all of whose direct summands have a vertex contained in P, such that C' is homotopy equivalent to C and $\mathrm{Br}_P(C')$ is isomorphic (in $\mathrm{Comp}^b(kG)$) to D.

Proof. — Up to isomorphism in $\mathrm{Ho}^b(kG)$, we may assume that $C = C^{\mathrm{red}}$. We write $C = (C^\bullet, d^\bullet)$. We will first show by induction on the length of C that $\mathrm{Br}_P(C) = \mathrm{Br}_P(C)^{\mathrm{red}}$. So let n be maximal such that $C^{n+1} \neq 0$. We fix a decomposition $C^{n+1} = \bigoplus_{i\in I} M_i$ where M_i is indecomposable for all $i \in I$ and we denote by $p_i : C^{n+1} \to M_i$ the projection.

First, assume that the composition

$$\mathrm{Br}_P(C^n) \xrightarrow{\mathrm{Br}_P(d^n)} \mathrm{Br}_P(C^{n+1})) \xrightarrow{\mathrm{Br}_P(p_i)} \mathrm{Br}_P(M_i)$$

is a split surjection for some i such that $\mathrm{Br}_P(M_i) \neq 0$. Then it follows from Lemma A.1 that $p_i d^n : C^n \to M_i$ is a split surjection: this contradicts the fact that $C = C^{\mathrm{red}}$. So the complex

$$0 \longrightarrow \mathrm{Br}_P(C^n) \xrightarrow{\mathrm{Br}_P(d^n)} \mathrm{Br}_P(C^{n+1}) \longrightarrow 0$$

has no non-zero direct summand that is homotopy equivalent to 0. By the induction hypothesis, the complex

$$\cdots \longrightarrow \mathrm{Br}_P(C^{n-1}) \xrightarrow{\mathrm{Br}_P(d^{n-1})} \mathrm{Br}_P(C^n) \longrightarrow 0$$

has no non-zero direct summand that is homotopy equivalent to 0. It follows that $\mathrm{Br}_P(C) = \mathrm{Br}_P(C)^{\mathrm{red}}$.

We deduce from this that $D \simeq \mathrm{Br}_P(C) \oplus D'$, where D' is homotopy equivalent to 0. So M is a sum of complexes of the form $0 \to M \xrightarrow{\mathrm{Id}_M} M \to 0$ (up to a shift), hence there is a bounded complex C' of ℓ-permutation kG-modules, all of whose indecomposable summands have vertex P, such that $\mathrm{Br}_P(C') \simeq D'$. Therefore $\mathrm{Br}_P(C \oplus C') \simeq D$, as desired. □

The following lemma is close to [**Bou**, Proposition 7.9].

Lemma A.3. — *Let G be a finite group and C a bounded complex of ℓ-permutation kG-modules. Assume $H^i(\mathrm{Br}_Q(C)) = 0$ for all $i \neq 0$ and all ℓ-subgroups Q of G.*

Then $C \simeq H^0(C)$ in $\mathrm{Ho}^b(kG)$.

Proof. — Replacing C by C^{red}, we can and will assume that C has no nonzero direct summands that are homotopy equivalent to 0.

Let $i > 0$ be maximal such that $C^i \neq 0$. The map $d^{i-1}_{\mathrm{Br}_Q(C)} = \mathrm{Br}_Q(d^{i-1}_C) : \mathrm{Br}_Q(C^{i-1}) \to \mathrm{Br}_Q(C^i)$ is surjective for all ℓ-subgroups Q. It follows from [**Bou**, Proposition 6.4] that d^{i-1}_C is a split surjection: this contradicts our assumption on C. So $C^i = 0$ for $i > 0$. Replacing C by C^*, we obtain similarly that $C^i_Q = 0$ for $i < 0$. The lemma follows. □

The following theorem is a variant of [**Rou2**, Theorem 5.6].

Theorem A.4. — *Let G be a finite group, H a subgroup of G and P a Sylow ℓ-subgroup of H. Let C be a bounded complex of ℓ-permutation $k(G\times H^{\mathrm{opp}})$-modules all of whose indecomposable summands have a vertex contained in ΔH.*

Assume $\mathrm{Hom}_{D^b(kC_G(Q))}(\mathrm{Br}_{\Delta Q}(C),\mathrm{Br}_{\Delta Q}(C)[i])=0$ *for all $i\neq 0$ and all ℓ-subgroups Q of H.*

Then $\mathrm{End}^\bullet_{kG}(C)$ *is isomorphic to* $\mathrm{End}_{D^b(kG)}(C)$ *in* $\mathrm{Ho}^b(k(H\times H^{\mathrm{opp}}))$.

Proof. — Let R be an ℓ-subgroup of $H\times H^{\mathrm{opp}}$. By **[Ri2**, proof of Theorem 4.1], we have $\mathrm{Br}_R(\mathrm{End}^\bullet_{kG}(C))=0$ if R is not conjugate to a subgroup of ΔH, and given $Q\leqslant H$ an ℓ-subgroup, we have

$$\mathrm{Br}_{\Delta Q}(\mathrm{End}^\bullet_{kG}(C))\simeq \mathrm{End}^\bullet_{kC_G(Q)}(\mathrm{Br}_{\Delta Q}(C))$$

in $\mathrm{Comp}(k(C_H(Q)\times C_H(Q)^{\mathrm{opp}}))$.

Note that the terms of $\mathrm{Br}_{\Delta Q}(C)$ are projective for $kC_G(Q)$, hence

$$H^i(\mathrm{End}^\bullet_{kC_G(Q)}(\mathrm{Br}_{\Delta Q}(C)))\simeq \mathrm{Hom}_{D^b(kC_G(Q))}(\mathrm{Br}_{\Delta Q}(C),\mathrm{Br}_{\Delta Q}(C)[i])$$

and this vanishes for $i\neq 0$. Consequently,

$$\mathrm{Br}_{\Delta Q}(\mathrm{End}^\bullet_{kG}(C))\simeq \mathrm{End}_{D^b(kC_G(Q))}(\mathrm{Br}_{\Delta Q}(C))$$

in $D^b(k(C_H(Q)\times C_H(Q)^{\mathrm{opp}}))$.

The conclusion of the theorem follows now from Lemma A.3 applied to the complex $\mathrm{End}^\bullet_{kG}(C)$. □

Corollary A.5. — *Let G be a finite group, H a subgroup of G, b a block idempotent of $\mathcal{O}G$, c a block idempotent of ΛH. Let C be a bounded complex of ℓ-permutation $(\Lambda Gb,\Lambda Hc)$-bimodules all of whose indecomposable summands have a vertex contained in ΔH. Assume* $\mathrm{Hom}_{D^b(kC_G(Q))}(\mathrm{Br}_{\Delta Q}(C),\mathrm{Br}_{\Delta Q}(C)[i])=0$ *for all $i\neq 0$ and all ℓ-subgroups Q of H and the canonical map $kHc\to \mathrm{End}_{D^b(kG)}(kC)$ is an isomorphism.*

Then C induces a splendid Rickard equivalence between ΛGb and ΛHc.

Proof. — Theorem A.4 shows that the canonical map $kHc\to \mathrm{End}^\bullet_{kG}(kC)$ is an isomorphism in $\mathrm{Ho}^b(k(H\times H^{\mathrm{opp}}))$. It follows from **[Ri2**, Theorem 2.1] that kC induces a Rickard equivalence between kGb and kHc. The result follows now from **[Ri2**, proof of Theorem 5.2]. □

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