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A review of Hybrid High-Order methods: formulations, computational aspects, comparison with other methods

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Abstract

Hybrid High-Order (HHO) methods are formulated in terms of discrete unknowns attached to mesh faces and cells (hence, the term hybrid), and these unknowns are polynomials of arbitrary order $k \geqslant 0$ (hence, the term high-order). HHO methods are devised from local reconstruction operators and a local stabilization term. The discrete problem is assembled cellwise, and cell-based unknowns can be eliminated locally by static condensation. HHO methods support general meshes, are locally conservative, and allow for a robust treatment of physical parameters in various situations, e.g., heterogeneous/anisotropic diffusion, quasi-incompressible linear elasticity, and advection-dominated transport. This paper reviews HHO methods for a variable-diffusion model problem with nonhomogeneous, mixed Dirichlet–Neumann boundary conditions, including both primal and mixed formulations. Links with other discretization methods from the literature are discussed.

1 Introduction

Over the last few years, a significant effort has been devoted to devising and analyzing discretization methods for elliptic PDEs on general meshes including nonmatching interfaces and polytopal cells. Such meshes are encountered, e.g., in the context of subsurface flow simulations in saline aquifers and petroleum basins, where polyhedral elements and nonmatching interfaces appear to account for eroded layers and fractures. In petroleum reservoir modeling, polyhedral elements can also appear in the near-wellbore regions, where radial meshes are usually employed to account for the (qualitative) features of the solution. A more recent and original application of meshes composed of polyhedral elements is adaptive mesh coarsening [2, 7], where a coarse mesh is obtained by element agglomeration from a fine mesh accounting for the geometric details of the domain.

Polytopal discretization methods were first investigated in the framework of lowest-order schemes. In the context of Finite Volume methods, several families of polytopal methods

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have resulted from the effort to circumvent the superadmissible mesh condition required for the consistency of the classical two-point scheme; cf., in particular, [38, Definition 9.1]. Interestingly, most of these methods possess local conservation properties on the primal mesh and exhibit numerical fluxes without resorting to local reconstructions. We can mention here, e.g., the Mixed and Hybrid Finite Volume (MHFV) schemes of [34, 39] and the Discrete Duality Finite Volume (DDFV) method of [33].

Other families of lowest-order polytopal discretization methods have been obtained by reproducing at the discrete level salient features of the continuous problem. Mimetic Finite Difference (MFD) methods were originally derived by mimicking the Stokes theorem in a discrete setting to formulate discrete counterparts of the usual first-order differential operators combined with constitutive relations and of L^2-products; cf. [14, 15] and also [9] for an overview. Another viewpoint starts from the seminal ideas of Tonti [44] and Bossavit [13] hinging on differential geometry and algebraic topology. Related schemes include the so-called Discrete Geometric Approach (DGA) [22], and more generally, the Compatible Discrete Operator (CDO) framework of [11,12], cf. also [10], where the building blocks are metric-free discrete differential operators combined with a discrete Hodge operator approximating constitutive relations. Another approach consists in reproducing classical properties of nonconforming and penalized methods on general meshes, as in the Cell-Centered Galerkin (CCG) method [23] and the generalized Crouzeix–Raviart method [32]. The idea is to formulate the method in terms of (possibly incomplete) polynomial spaces so as to re-deploy classical (nonconforming) Finite Element analysis tools.

Recent works have led to unifying frameworks that capture the links among (some of) the above methods. The close relation between MHFV and MFD methods has been investigated in [35], where equivalence at the algebraic level is demonstrated. A unifying viewpoint that encompasses the above and other classical methods has been proposed under the name of Gradient Schemes [36]. Another unifying viewpoint (closely related to Gradient Schemes) is provided by the CDO framework which encompasses vertex-based schemes (such as first-order Lagrange finite elements and nodal MFD) and cell-based schemes (such as MHFV and MFD).

In parallel, high-order polytopal discretization methods have received significant attention over the last few years. Increasing the approximation order can significantly speed up convergence when the solution exhibits sufficient (local) regularity. When this is not the case, the better convergence properties of high-order methods can be recovered using mesh adaption (by local refinement or coarsening). High-order polytopal discretization methods can be obtained by fully nonconforming approaches such as the Discontinuous Galerkin (DG) method; cf. [4] and also [5, 16] for a unified presentation for the Poisson problem, [37] for Friedrichs' systems, [18] for an *hp*-version, and [26] for a comprehensive introduction. An interesting class of DG methods is that of Hybridizable Discontinuous Galerkin (HDG) methods [21] (cf. also [19]). Such methods were originally devised as discrete versions of a characterization of the exact solution in terms of solutions of local problems globally matched through transmission conditions. Another approach is the Weak Galerkin (WG) method [45, 46] based on so-called weak gradient operators that act on generalized functions defined as couples of standard functions in the interior and on the boundary of the mesh elements.

Very recent works have developed other viewpoints to achieve high-order polytopal discretizations. A salient example is the Virtual Element (VE) method introduced in [8, 17]. The H^1-conforming VE method takes the steps from the nodal MFD method recast in a Finite

Element framework, and can be viewed as a generalization of conforming (Lagrange, Hermite) Finite Element methods. The main idea is to define a local space of basis functions for which only the values of degrees of freedom are known (i.e., no analytical expression is available). Starting from these degrees of freedom, one devises a computable projection onto a polynomial space so as to formulate the local contributions to the discrete problem.

Our focus is here on the Hybrid High-Order (HHO) method introduced in [29, 31]. The term hybrid refers to the fact that the method is originally formulated using discrete unknowns attached to mesh faces and cells. These discrete unknowns are polynomial functions, and the cell-based ones can be eliminated locally by static condensation. The term high-order refers to the fact that the order of the polynomial functions can be an arbitrary integer $k \geqslant 0$. The main idea of HHO methods consists in locally reconstructing high-order differential operators acting on the face- and cell-based unkowns. The guideline underpinning such reconstructions is an integration by parts formula. These reconstructions are then used to formulate the elementwise contributions to the discrete problem including a high-order stabilization term exhibiting a rich structure coupling locally the face- and cell-based unkowns. Local contributions are conceived so that the only globally coupled unknowns after static condensation are discontinuous polynomials on the mesh skeleton. This is a distinctive feature with respect to the VE method, where H^1-conforming reconstructions are present in the background. A study of the relations between HHO and HDG methods can be found in [20], which also fits into the HHO framework (up to equivalent stabilization) the recent high-order MFD method of [6, 43] (also referred to as nonconforming VE method in subsequent publications). We also mention that HHO methods for polynomial order $k = 0$ are closely related to MHFV (and so to lowest-order MFD); cf., in particular, [31, Section 2.5] and [25, Section 5.4].

HHO methods offer several assets. Besides supporting general meshes, their construction is dimension-independent, and they are locally conservative [28]. Moreover, they allow for a natural treatment of physical parameters [30], and lead to discretizations that are robust over the entire range of variation of physical parameters in various situations, e.g., heterogeneous/anisotropic diffusion [30], quasi-incompressible linear elasticity [29] and advection-dominated transport [25]. When compared to interior penalty DG methods, HHO methods are also appealing in terms of computational cost. To achieve an order of convergence of $(k+1)$ in the energy norm for a pure diffusion problem in three space dimensions, the globally coupled degrees of freedom for DG grow as $\frac{1}{6}k^3 N_{\mathrm{E}}$ with N_{E} the number of mesh elements, whereas for HHO they only grow as $\frac{1}{2}k^2 N_{\mathrm{F}}$ with N_{F} the number of mesh faces (only leading-order terms are considered in the above computations).

The goal of this paper is to provide an up-to-date review of HHO methods, with a particular focus on the various possible formulations and computational aspects. For the sake of simplicity, we focus on a model elliptic problem with possibly heterogeneous/anisotropic diffusion tensor. Most of the results contained herein can be derived from relatively straightforward adaptations of the proofs contained in previous works [1, 20, 25, 27–31]; for the sake of conciseness, we provide bibliographic references for the most technical proofs, while some details are included for those proofs that allow us to highlight the more practical aspects of the method. One novel aspect is that we treat nonhomogeneous mixed Dirichlet–Neumann boundary conditions, while previous work has focused on homogeneous, pure Dirichlet boundary conditions. Another novelty is that we detail the main implementation aspects under the viewpoint of an offline/online decomposition.

The material is organized as follows. Section 2 describes the continuous and discrete settings, including the model problem, the notion of admissible mesh sequence, and the assumptions on the data. Section 3 is devoted to the presentation and analysis of the HHO method in primal form, while Section 4 is concerned with the mixed form of the HHO method. Finally, the links between both forms are studied in Section 5, while Section 6 contains some concluding remarks and perspectives.

2 Continuous and discrete settings

This section presents the model problem, the key definitions and notation concerning the mesh, and the assumptions on the data of the model problem.

2.1 Model problem

Let $\Omega \subset \mathbb{R}^d$, $d \geqslant 2$, be an open, connected, bounded polytopal domain, with boundary Γ and unit outward normal $\boldsymbol{n}$. We assume that there exists a partition of Γ such that $\Gamma := \Gamma_{\mathrm{d}} \cup \Gamma_{\mathrm{n}}$, with $\Gamma_{\mathrm{d}} \cap \Gamma_{\mathrm{n}} = \varnothing$, and such that the measure of Γ_{d} is nonzero. For any connected subset $X \subset \overline{\Omega}$ with nonzero Lebesgue measure, the inner product and norm of the Lebesgue space $L^2(X)$ are denoted by $(\cdot,\cdot)_X$ and $\|\cdot\|_X$, respectively, with the convention that the index is omitted if $X = \Omega$.

We consider a variable-diffusion model problem with tensor-valued diffusivity $\mathbb{M}$. Throughout the paper, $\mathbb{M}$ is assumed to be symmetric, piecewise Lipschitz on a polytopal partition P_Ω of Ω, and uniformly elliptic, in the sense that, for a.e. $\boldsymbol{x} \in \Omega$,

$$0 < \mu_\flat \leqslant \mathbb{M}(\boldsymbol{x})\boldsymbol{\xi}{\cdot}\boldsymbol{\xi} \leqslant \mu_\sharp < +\infty, \qquad \forall \boldsymbol{\xi} \in \mathbb{R}^d \text{ such that } |\boldsymbol{\xi}| = 1.$$

The model problem reads: Find $u : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} -\mathrm{div}(\mathbb{M}\boldsymbol{\nabla} u) &= f && \text{in } \Omega,\\ u &= \psi_\partial && \text{on } \Gamma_{\mathrm{d}},\\ \mathbb{M}\boldsymbol{\nabla} u{\cdot}\boldsymbol{n} &= \phi_\partial && \text{on } \Gamma_{\mathrm{n}}, \end{aligned} \tag{1}$$

where $f \in L^2(\Omega)$, $\psi_\partial = (u_\partial)_{|\Gamma_{\mathrm{d}}}$ with $u_\partial \in H^1(\Omega)$, and $\phi_\partial \in L^2(\Gamma_{\mathrm{n}})$ (whenever the measure of Γ_{n} is nonzero). Henceforth, u is termed the potential. Owing to the nonzero assumption on the measure of Γ_{d}, we do not consider pure Neumann boundary conditions; the results presented in what follows can be adapted to this case, up to minor modifications. The pure Dirichlet case, corresponding to a $(d-1)$-dimensional zero-measure set Γ_{n}, is included in the present setting.

2.2 Admissible mesh sequences

Denoting by $\mathcal{H} \subset \mathbb{R}_*^+$ a countable set of meshsizes having 0 as its unique accumulation point, we consider mesh sequences $(\mathcal{T}_h)_{h\in\mathcal{H}}$ where, for all $h \in \mathcal{H}$, $\mathcal{T}_h = \{T\}$ is a finite collection of nonempty disjoint open polytopes (polygons/polyhedra) T, called *elements* or *cells*, such that

$\overline{\Omega} = \bigcup_{T\in\mathcal{T}_h} \overline{T}$ and $h = \max_{T\in\mathcal{T}_h} h_T$ (where h_T stands for the diameter of the element T). Recall that polytopes in $\mathbb{R}^d$ have flat sides.

A hyperplanar closed connected subset F of $\overline{\Omega}$ is called a *face* (for $d > 3$, these geometric objects are also called facets) if it has positive $(d-1)$-dimensional Lebesgue measure and if either (i) there exist $T_1, T_2 \in \mathcal{T}_h$ such that $F = \partial T_1 \cap \partial T_2$ or $F \subset \partial T_1 \cap \partial T_2$ and F is a side of both T_1 and T_2 (and F is termed *interface*), or (ii) there exists $T \in \mathcal{T}_h$ such that $F = \partial T \cap \partial\Omega$ or $F \subset \partial T \cap \partial\Omega$ and F is a side of T (and F is termed *boundary face*). Interfaces are collected in the set $\mathcal{F}_h^{\mathrm{i}}$, boundary faces in $\mathcal{F}_h^{\mathrm{b}}$, and we let $\mathcal{F}_h := \mathcal{F}_h^{\mathrm{i}} \cup \mathcal{F}_h^{\mathrm{b}}$. The diameter of a face $F \in \mathcal{F}_h$ is denoted h_F. For all $T \in \mathcal{T}_h$, $\mathcal{F}_T := \{F \in \mathcal{F}_h \mid F \subset \partial T\}$ denotes the set of faces lying on the boundary of T and, symmetrically, for all $F \in \mathcal{F}_h$, $\mathcal{T}_F := \{T \in \mathcal{T}_h \mid F \subset \partial T\}$ denotes the set gathering the one (if F is a boundary face) or two (if F is an interface) element(s) sharing F. For all $F \in \mathcal{F}_T$, we let $\boldsymbol{n}_{T,F}$ be the unit normal vector to F pointing out of T. Finally, for every interface $F \in \mathcal{F}_h^{\mathrm{i}}$, an orientation is fixed once and for all by means of a unit normal vector $\boldsymbol{n}_F$.

We adopt the following notion of admissible mesh sequence, cf. [26, Section 1.4].

Definition 2.1 (Admissible mesh sequence)**.** *The mesh sequence* $(\mathcal{T}_h)_{h\in\mathcal{H}}$ *is* admissible *if, for all* $h \in \mathcal{H}$, $\mathcal{T}_h$ *admits a matching simplicial submesh* $\mathfrak{T}_h$ *such that there exists a real number* $\gamma > 0$, *called* mesh regularity parameter, *independent of* h *and such that, for all* $h \in \mathcal{H}$,

(i) for all simplex $S \in \mathfrak{T}_h$ *of diameter* h_S *and inradius* r_S, $\gamma h_S \leqslant r_S$;

(ii) for all $T \in \mathcal{T}_h$, *and all* $S \in \mathfrak{T}_T := \{S \in \mathfrak{T}_h \mid S \subseteq T\}$, $\gamma h_T \leqslant h_S$.

Consequences of Definition 2.1 are that (i) the quantity $\max_{T\in\mathcal{T}_h} \mathrm{card}(\mathcal{F}_T)$ is uniformly bounded with respect to the meshsize, and that (ii) mesh faces have a comparable diameter to that of the cells they belong to; cf. [26, Lemmas 1.41 and 1.42]. We add the following notion of compatibility, in order to deal with the partitions associated with the diffusion tensor and with the boundary conditions.

Definition 2.2 (Compatible mesh sequence)**.** *The mesh sequence* $(\mathcal{T}_h)_{h\in\mathcal{H}}$ *is* compatible *if, for all* $h \in \mathcal{H}$,

(i) $\mathcal{T}_h$ *fits the (polytopal) partition* P_Ω *associated with the diffusion tensor* $\mathbb{M}$, *meaning that, for all* $T \in \mathcal{T}_h$, *there is a unique* Ω_i *in* P_Ω *containing* T;

(ii) $\mathcal{T}_h$ *fits the partition* $\Gamma = \Gamma_{\mathrm{d}} \cup \Gamma_{\mathrm{n}}$ *of the boundary, in the sense that we can define two sets,* $\mathcal{F}_h^{\mathrm{d}} := \{F \in \mathcal{F}_h^{\mathrm{b}} \mid F \subseteq \Gamma_{\mathrm{d}}\}$ *and* $\mathcal{F}_h^{\mathrm{n}} := \{F \in \mathcal{F}_h^{\mathrm{b}} \mid F \subseteq \Gamma_{\mathrm{n}}\}$, *such that* $\mathcal{F}_h^{\mathrm{d}} \cup \mathcal{F}_h^{\mathrm{n}} = \mathcal{F}_h^{\mathrm{b}}$.

2.3 Broken polynomial spaces

For integers $k \geqslant 0$, $1 \leqslant l \leqslant d$, we denote by $\mathbb{P}_l^k$ the vector space spanned by l-variate polynomial functions of total degree $\leqslant k$ of dimension

$$N_{k,l} := \binom{k+l}{k}. \tag{2}$$

For all $T \in \mathcal{T}_h$, $\mathbb{P}_d^k(T)$ denotes the restriction to T of functions in $\mathbb{P}_d^k$. We also introduce the broken polynomial space

$$\mathbb{P}_d^k(\mathcal{T}_h) := \{v \in L^2(\Omega) \mid v_{|T} \in \mathbb{P}_d^k(T) \text{ for all } T \in \mathcal{T}_h\}.$$

Broken polynomial spaces are special instances of broken Sobolev spaces, for an integer $m \geqslant 1$:

$$H^m(\mathcal{T}_h) := \{v \in L^2(\Omega) \mid v_{|T} \in H^m(T) \text{ for all } T \in \mathcal{T}_h\}.$$

We use the notation $\boldsymbol{\nabla}_h$ to denote the broken gradient operator acting elementwise on functions from broken Sobolev spaces.

We denote by π_h^k the L^2-orthogonal projector onto $\mathbb{P}_d^k(\mathcal{T}_h)$ such that, for all $v \in L^2(\Omega)$ and all $T \in \mathcal{T}_h$, $(\pi_h^k v)_{|T} := \pi_T^k v_{|T}$, where π_T^k is the L^2-orthogonal projector onto $\mathbb{P}_d^k(T)$. Additionally, for all $F \in \mathcal{F}_h$ and all $v \in L^2(F)$, we denote by $\pi_F^k v$ the L^2-orthogonal projection of v onto $\mathbb{P}_{d-1}^k(F)$, where $\mathbb{P}_{d-1}^k(F)$ is the restriction to F of $\mathbb{P}_{d-1}^k \circ \Xi^{-1}$, with Ξ an affine bijective mapping from $\mathbb{R}^{d-1}$ to the affine hyperplane supporting F.

2.4 Diffusion tensor

We assume, for the sake of simplicity, that $\mathbb{M}$ is piecewise constant on P_Ω, and thus, by Definition 2.2, on $\mathcal{T}_h$ for every $h \in \mathcal{H}$. For $T \in \mathcal{T}_h$, we let $\mathbb{M}_T := \mathbb{M}_{|T}$ (owing to the above assumption, $\mathbb{M}_T$ is a constant matrix), and we denote by $\mu_{\flat,T}$ and $\mu_{\sharp,T}$, respectively, the lowest and largest eigenvalues of $\mathbb{M}_T$. We also introduce the local anisotropy ratio $\rho_T := \mu_{\sharp,T}/\mu_{\flat,T} \geqslant 1$; the global ratio is defined as $\rho := \max_{T\in\mathcal{T}_h} \rho_T$. Finally, for all $T \in \mathcal{T}_h$ and $F \in \mathcal{F}_T$, we set $\mu_{T,F} := \mathbb{M}_T \boldsymbol{n}_F \cdot \boldsymbol{n}_F > 0$.

In what follows, we often abbreviate as $a \lesssim b$ the inequality $a \leqslant Cb$, with $C > 0$ independent of the meshsize h and of the diffusion tensor $\mathbb{M}$, but possibly depending on the mesh regularity parameter γ and on the polynomial degree k.

3 The HHO method in primal form

Let $U := H^1(\Omega)$ and $U_0 := \{v \in U \mid v_{|\Gamma_{\mathrm{d}}} = 0\}$. The starting point of the HHO method in primal form is the following weak formulation of problem (1): Find $u_0 \in U_0$ such that

$$(\mathbb{M}\boldsymbol{\nabla} u_0, \boldsymbol{\nabla} v) = (f, v) - (\mathbb{M}\boldsymbol{\nabla} u_\partial, \boldsymbol{\nabla} v) + (\phi_\partial, v)_{\Gamma_{\mathrm{n}}} \qquad \forall v \in U_0. \tag{3}$$

The solution $u \in U$ is then computed as $u = u_0 + u_\partial$ with u_∂ defined in Section 2.1.

3.1 Discrete setting

Let an integer $k \geqslant 0$ be fixed, and let us consider an admissible and compatible mesh sequence $(\mathcal{T}_h)_{h\in\mathcal{H}}$ in the sense of Definitions 2.1 and 2.2. We further suppose that the assumptions of Section 2.4 concerning the diffusion tensor hold.

3.1.1 Discrete unknowns

We adopt the convention that underlined quantities in roman font (sets, elements from these sets) are hybrid quantities, i.e., quantities featuring both a cell-based and a face-based contribution. We introduce, first locally, then globally, the discrete unknowns associated with the potential.

Local definition For $T \in \mathcal{T}_h$, letting

$$U_T^k := \mathbb{P}_d^k(T), \qquad \mathfrak{U}_F^k := \mathbb{P}_{d-1}^k(F) \text{ for all } F \in \mathcal{F}_T, \tag{4}$$

we define the local set of hybrid potential unknowns, cf. Figure 1, as

$$\underline{\mathrm{U}}_T^k := U_T^k \times \mathfrak{U}_{\partial T}^k, \qquad \mathfrak{U}_{\partial T}^k := \bigtimes_{F \in \mathcal{F}_T} \mathfrak{U}_F^k.$$

In the sequel, any element $\underline{\mathrm{v}}_T \in \underline{\mathrm{U}}_T^k$ is decomposed as $\underline{\mathrm{v}}_T := (v_T \in U_T^k, \mathfrak{v}_{\partial T} \in \mathfrak{U}_{\partial T}^k)$, with $\mathfrak{v}_{\partial T} := (\mathfrak{v}_F \in \mathfrak{U}_F^k)_{F \in \mathcal{F}_T}$. We also introduce the local reduction operator $\underline{\mathrm{I}}_T^k : H^1(T) \to \underline{\mathrm{U}}_T^k$ such that, for all $v \in H^1(T)$, $\underline{\mathrm{I}}_T^k v := \left(\pi_T^k v, (\pi_F^k v)_{F \in \mathcal{F}_T}\right)$.

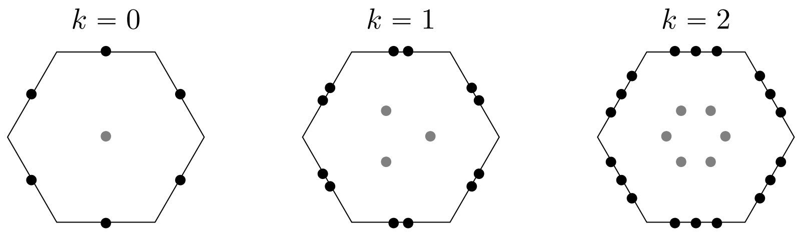


Figure 1: Degrees of freedom associated with hybrid (cell- and face-based) potential discrete unknowns, $d = 2$, $k \in \{0, 1, 2\}$.

Remark 3.1 (Variant on cell-based unknowns)**.** *A variant in the definition of cell-based unknowns is studied in [20], where these unknowns belong to the polynomial space $\mathbb{P}_d^l(T)$ with $l \in \{k-1, k, k+1\}$ (up to some minor adaptations if $k = 0$ and $l = -1$). The choice $l = k-1$ allows one to establish a link (up to equivalent stabilizations) with the high-order MFD method of [6, 43] (in the case $k = 0$, $l = -1$, one can recover the classical Crouzeix–Raviart element on simplices), while the choice $l = k+1$ is related to a variant of the HDG method introduced in [42].*

Global definition We define the global set of hybrid potential unknowns as

$$\underline{\mathrm{U}}_h^k := U_h^k \times \mathfrak{U}_h^k, \tag{5}$$

with

$$U_h^k := \bigtimes_{T \in \mathcal{T}_h} U_T^k, \qquad \mathfrak{U}_h^k := \bigtimes_{F \in \mathcal{F}_h} \mathfrak{U}_F^k.$$

Observe that $U_h^k = \mathbb{P}_d^k(\mathcal{T}_h)$ and that potential unknowns attached to interfaces are single-valued. Given an element $\underline{\mathrm{v}}_h \in \underline{\mathrm{U}}_h^k$, we denote v_h and $\mathfrak{v}_h$ its restrictions to U_h^k and $\mathfrak{U}_h^k$, respectively, while, for any $T \in \mathcal{T}_h$, we denote by $\underline{v}_T = (v_T, \mathfrak{v}_{\partial T}) \in \underline{U}_T^k$ its restriction to the element T. To account for (homogeneous) Dirichlet boundary conditions in a strong manner, we introduce the following subspace of $\underline{U}_h^k$:

$$\underline{U}_{h,0}^k := U_h^k \times \mathfrak{U}_{h,0}^k, \qquad \text{with} \qquad \mathfrak{U}_{h,0}^k := \left\{ \mathfrak{v}_h \in \mathfrak{U}_h^k \mid \mathfrak{v}_F \equiv 0, \forall F \in \mathcal{F}_h^{\mathrm{d}} \right\}.$$

Finally, we introduce the global reduction operator $\underline{I}_h^k : U \to \underline{U}_h^k$ such that, for all $v \in U$, and for all $T \in \mathcal{T}_h$, $(\underline{I}_h^k v)_{|T} := \underline{I}_T^k v_{|T}$. Single-valuedness at interfaces is ensured by the regularity of functions in U.

3.1.2 Potential reconstruction operator

Let $T \in \mathcal{T}_h$. The local potential reconstruction operator $p_T^{k+1} : \underline{U}_T^k \to \mathbb{P}_d^{k+1}(T)$ is defined, for all $\underline{v}_T = (v_T, \mathfrak{v}_{\partial T}) \in \underline{U}_T^k$, as the solution of the well-posed Neumann problem (the usual compatibility condition on the right-hand side is verified)

$$(\mathbb{M}_T \boldsymbol{\nabla} p_T^{k+1} \underline{v}_T, \boldsymbol{\nabla} w)_T = -(v_T, \operatorname{div}(\mathbb{M}_T \boldsymbol{\nabla} w))_T + \sum_{F \in \mathcal{F}_T} (\mathfrak{v}_F, \mathbb{M}_T \boldsymbol{\nabla} w \cdot \boldsymbol{n}_{T,F})_F \quad \forall w \in \mathbb{P}_d^{k+1}(T), \tag{6}$$

which further satisfies $\int_T p_T^{k+1} \underline{v}_T = \int_T v_T$. Computing the operator p_T^{k+1} requires to invert a symmetric positive-definite matrix of size $N_{k+1,d}$, cf. (2), which can be performed effectively via a Cholesky factorization (the cost of such a factorization is roughly $N_{k+1,d}^3/3$ flops). The following result shows that $p_T^{k+1} \underline{I}_T^k$ is the $\mathbb{M}_T$-weighted elliptic projector onto $\mathbb{P}_d^{k+1}(T)$.

Lemma 3.1 (Characterization of $p_T^{k+1} \underline{I}_T^k$ and polynomial consistency)**.** *The following holds for all $v \in H^1(T)$:*

$$(\mathbb{M}_T \boldsymbol{\nabla}(v - p_T^{k+1} \underline{I}_T^k v), \boldsymbol{\nabla} w)_T = 0 \qquad \forall w \in \mathbb{P}_d^{k+1}(T). \tag{7}$$

Consequently, for all $v \in \mathbb{P}_d^{k+1}(T)$, we have

$$p_T^{k+1} \underline{I}_T^k v = v. \tag{8}$$

Proof. For $v \in H^1(T)$, let us plug $\underline{v}_T := \underline{I}_T^k v = \left(\pi_T^k v, (\pi_F^k v)_{F \in \mathcal{F}_T} \right)$ into (6). Since $\mathbb{M}_T$ is a constant tensor and since $w \in \mathbb{P}_d^{k+1}(T)$, we infer that $\operatorname{div}(\mathbb{M}_T \boldsymbol{\nabla} w) \in \mathbb{P}_d^{k-1}(T) \subset \mathbb{P}_d^k(T)$ and that $\mathbb{M}_T \boldsymbol{\nabla} w_{|F} \cdot \boldsymbol{n}_{T,F} \in \mathbb{P}_{d-1}^k(F)$, which means that, for all $w \in \mathbb{P}_d^{k+1}(T)$,

$$(\mathbb{M}_T \boldsymbol{\nabla} p_T^{k+1} \underline{I}_T^k v, \boldsymbol{\nabla} w)_T = -(v, \operatorname{div}(\mathbb{M}_T \boldsymbol{\nabla} w))_T + \sum_{F \in \mathcal{F}_T} (v, \mathbb{M}_T \boldsymbol{\nabla} w \cdot \boldsymbol{n}_{T,F})_F = (\mathbb{M}_T \boldsymbol{\nabla} v, \boldsymbol{\nabla} w)_T,$$

hence concluding the proof of (7). For $v \in \mathbb{P}_d^{k+1}(T)$, we deduce from (7) that $(v - p_T^{k+1} \underline{I}_T^k v) \in \mathbb{P}_d^0(T)$, and we conclude by invoking the relation $\int_T p_T^{k+1} \underline{I}_T^k v = \int_T \pi_T^k v = \int_T v$. $\square$

The next result can be found in [30, Lemma 2.1].

Lemma 3.2 (Approximation)**.** *For all $v \in H^{k+2}(T)$, the following holds:*

$$\begin{aligned} \|v - p_T^{k+1} \underline{I}_T^k v\|_T + h_T^{1/2} \|v - p_T^{k+1} \underline{I}_T^k v\|_{\partial T} + h_T \|\boldsymbol{\nabla}(v - p_T^{k+1} \underline{I}_T^k v)\|_T \\ + h_T^{3/2} \|\boldsymbol{\nabla}(v - p_T^{k+1} \underline{I}_T^k v)\|_{\partial T} \lesssim \rho_T^{1/2} h_T^{k+2} \|v\|_{H^{k+2}(T)}. \end{aligned} \tag{9}$$

In the more general case of a piecewise Lipschitz diffusivity, only approximate polynomial consistency holds, while a factor ρ_T instead of $\rho_T^{1/2}$ appears in the estimate (9) (cf. [30]).

For further use, we define the global potential reconstruction operator

$$p_h^{k+1} : \underline{\mathrm{U}}_h^k \to \mathbb{P}_d^{k+1}(\mathcal{T}_h)$$

such that, for all $\underline{\mathrm{v}}_h \in \underline{\mathrm{U}}_h^k$, and for all $T \in \mathcal{T}_h$, $(p_h^{k+1}\underline{\mathrm{v}}_h)_{|T} := p_T^{k+1}\underline{\mathrm{v}}_T$.

3.1.3 Stabilization

For all $T \in \mathcal{T}_h$, we define the stabilization bilinear form $j_T : \underline{\mathrm{U}}_T^k \times \underline{\mathrm{U}}_T^k \to \mathbb{R}$ such that

$$j_T(\underline{\mathrm{u}}_T, \underline{\mathrm{v}}_T) := \sum_{F \in \mathcal{F}_T} \frac{\mu_{T,F}}{h_F} (\pi_F^k(q_T^{k+1}\underline{\mathrm{u}}_T - \mathfrak{u}_F), \pi_F^k(q_T^{k+1}\underline{\mathrm{v}}_T - \mathfrak{v}_F))_F, \tag{10}$$

with $q_T^{k+1} : \underline{\mathrm{U}}_T^k \to \mathbb{P}_d^{k+1}(T)$ such that, for all $\underline{\mathrm{w}}_T \in \underline{\mathrm{U}}_T^k$,

$$q_T^{k+1}\underline{\mathrm{w}}_T := w_T + (p_T^{k+1}\underline{\mathrm{w}}_T - \pi_T^k p_T^{k+1}\underline{\mathrm{w}}_T).$$

Notice that j_T is symmetric, positive semi-definite, and polynomially consistent (as a consequence of (8)) in the sense that, for all $v \in \mathbb{P}_d^{k+1}(T)$,

$$j_T(\underline{\mathrm{I}}_T^k v, \underline{\mathrm{w}}_T) = 0 \qquad \forall \underline{\mathrm{w}}_T \in \underline{\mathrm{U}}_T^k. \tag{11}$$

Another important property of j_T is the following approximation property: For all $v \in H^{k+2}(T)$, the following bound holds:

$$j_T(\underline{\mathrm{I}}_T^k v, \underline{\mathrm{I}}_T^k v)^{1/2} \lesssim \mu_{\sharp,T}^{1/2} \rho_T^{1/2} h_T^{k+1} \|v\|_{H^{k+2}(T)}, \tag{12}$$

showing that j_T matches the approximation properties of the gradient of p_T^{k+1}; cf. Lemma 3.2.

3.2 Discrete problem: formulation and key properties

3.2.1 Formulation

For all $T \in \mathcal{T}_h$, we define the following local bilinear form:

$$a_T : \underline{\mathrm{U}}_T^k \times \underline{\mathrm{U}}_T^k \to \mathbb{R}; \qquad (\underline{\mathrm{u}}_T, \underline{\mathrm{v}}_T) \mapsto (\mathbb{M}_T \boldsymbol{\nabla} p_T^{k+1}\underline{\mathrm{u}}_T, \boldsymbol{\nabla} p_T^{k+1}\underline{\mathrm{v}}_T)_T + j_T(\underline{\mathrm{u}}_T, \underline{\mathrm{v}}_T), \tag{13}$$

with potential reconstruction operator p_T^{k+1} defined by (6) and stabilization bilinear form j_T defined by (10). Introduce now the following global bilinear form obtained by a standard element-by-element assembly procedure:

$$a_h : \underline{\mathrm{U}}_h^k \times \underline{\mathrm{U}}_h^k \to \mathbb{R}; \qquad (\underline{\mathrm{u}}_h, \underline{\mathrm{v}}_h) \mapsto \sum_{T \in \mathcal{T}_h} a_T(\underline{\mathrm{u}}_T, \underline{\mathrm{v}}_T).$$

Then, the (primal) HHO discretization of problem (3) reads: Find $\underline{\mathrm{u}}_{h,0} \in \underline{\mathrm{U}}_{h,0}^k$ such that

$$a_h(\underline{\mathrm{u}}_{h,0}, \underline{\mathrm{v}}_h) = (f, v_h) - a_h(\underline{\mathrm{u}}_{h,\partial}, \underline{\mathrm{v}}_h) + \sum_{F \in \mathcal{F}_h^{\mathrm{n}}} (\phi_\partial, \mathfrak{v}_F)_F \qquad \forall \underline{\mathrm{v}}_h \in \underline{\mathrm{U}}_{h,0}^k, \tag{14}$$

where $\underline{\mathrm{u}}_{h,\partial} := \underline{\mathrm{I}}_h^k u_\partial \in \underline{\mathrm{U}}_h^k$ is the reduction of the continuous lifting u_∂ of ψ_∂. The discrete solution $\underline{\mathrm{u}}_h \in \underline{\mathrm{U}}_h^k$ is finally computed as

$$\underline{\mathrm{u}}_h = \underline{\mathrm{u}}_{h,0} + \underline{\mathrm{u}}_{h,\partial}. \tag{15}$$

Remark 3.2 (Discrete Dirichlet datum)**.** *In practical implementation, the continuous lifting u_∂ of the Dirichlet datum is not needed, and one can simply select $\underline{\mathrm{u}}_{h,\partial}$ such that*

$$u_{T,\partial} \equiv 0 \quad \forall T \in \mathcal{T}_h, \qquad \mathfrak{u}_{F,\partial} = \pi_F^k \psi_\partial \quad \forall F \in \mathcal{F}_h^{\mathrm{d}}, \qquad \mathfrak{u}_{F,\partial} \equiv 0 \quad \forall F \in \mathcal{F}_h \backslash \mathcal{F}_h^{\mathrm{d}}.$$

3.2.2 Stability

Let us introduce, for all $T \in \mathcal{T}_h$, the following diffusion-dependent seminorm on $\underline{\mathrm{U}}_T^k$:

$$\|\underline{\mathrm{v}}_T\|_{U,T}^2 := \rho_T^{-1} \left(\|\mathbb{M}_T^{1/2} \boldsymbol{\nabla} v_T\|_T^2 + \sum_{F \in \mathcal{F}_T} \frac{\mu_{T,F}}{h_F} \|v_T - \mathfrak{v}_F\|_F^2 \right). \tag{16}$$

It can be proved that the map

$$\|\underline{\mathrm{v}}_h\|_{U,h}^2 := \sum_{T \in \mathcal{T}_h} \|\underline{\mathrm{v}}_T\|_{U,T}^2,$$

defines a norm on $\underline{\mathrm{U}}_{h,0}^k$. Stability for problem (14) is expressed by the following result (cf. [30, Lemma 3.1]).

Lemma 3.3 (Stability)**.** *For all $T \in \mathcal{T}_h$ and all $\underline{\mathrm{v}}_T \in \underline{\mathrm{U}}_T^k$, the following holds:*

$$\|\underline{\mathrm{v}}_T\|_{U,T} \lesssim a_T(\underline{\mathrm{v}}_T, \underline{\mathrm{v}}_T)^{1/2} \lesssim \rho_T \|\underline{\mathrm{v}}_T\|_{U,T}. \tag{17}$$

As a consequence, we infer that

$$\|\underline{\mathrm{v}}_h\|_{U,h}^2 \lesssim a_h(\underline{\mathrm{v}}_h, \underline{\mathrm{v}}_h) \quad \forall \underline{\mathrm{v}}_h \in \underline{\mathrm{U}}_h^k, \tag{18}$$

implying that problem (14) is well-posed.

3.2.3 Error estimates

Let $u \in U$ be such that $u = u_0 + u_\partial$, where $u_0 \in U_0$ is the (unique) solution to (3), and $u_\partial \in U$ is defined in Section 2.1. Let $\underline{\mathrm{u}}_h \in \underline{\mathrm{U}}_h^k$ be such that $\underline{\mathrm{u}}_h = \underline{\mathrm{u}}_{h,0} + \underline{\mathrm{u}}_{h,\partial}$, where $\underline{\mathrm{u}}_{h,0} \in \underline{\mathrm{U}}_{h,0}^k$ is the (unique) solution to (14), and $\underline{\mathrm{u}}_{h,\partial} \in \underline{\mathrm{U}}_h^k$ is defined in Section 3.2.1. Finally, let us introduce the notation $\|\cdot\|_h := a_h(\cdot,\cdot)^{1/2}$. Then, we can state the following result, whose proof can be easily adapted from the one of [30, Theorem 4.1] (where the norm $\|\cdot\|_h$ is to be used under the supremum in Eq. (12)). Note that the constants in the error bounds can depend on the polynomial degree following the use of discrete trace and inverse inequalities.

Theorem 3.1 (Energy-norm error estimate)**.** *Assume that u further belongs to $H^{k+2}(P_\Omega)$ (so that, by Definition 2.2, $u \in H^{k+2}(\mathcal{T}_h)$). Then, the following holds:*

$$\|\underline{\mathrm{I}}_h^k u - \underline{\mathrm{u}}_h\|_{U,h} \lesssim \|\underline{\mathrm{I}}_h^k u - \underline{\mathrm{u}}_h\|_h \lesssim \left\{ \sum_{T\in\mathcal{T}_h} \mu_{\sharp,T}\rho_T^2 h_T^{2(k+1)} \|u\|_{H^{k+2}(T)}^2 \right\}^{1/2}, \tag{19}$$

which implies, by an additional use of Lemma 3.2, that

$$\|\mathbb{M}^{1/2}(\boldsymbol{\nabla} u - \boldsymbol{\nabla}_h p_h^{k+1}\underline{\mathrm{u}}_h)\| \lesssim \left\{ \sum_{T\in\mathcal{T}_h} \mu_{\sharp,T}\rho_T^2 h_T^{2(k+1)} \|u\|_{H^{k+2}(T)}^2 \right\}^{1/2}. \tag{20}$$

In the more general case of a piecewise (non-constant) polynomial diffusivity, estimates (19) and (20) still hold with a factor ρ_T^3 instead of ρ_T^2 (cf. [30]). For non-polynomial diffusivity, an additional quadrature error has to be accounted for.

Whenever elliptic regularity holds, a L^2-norm error estimate of order h^{k+2} can be established, as an adaptation of [30, Theorem 4.2] (where the assumption of piecewise constant diffusivity is to be added).

Theorem 3.2 (L^2-norm error estimate)**.** *Assume elliptic regularity for problem (3) under the form $\|z\|_{H^2(P_\Omega)} \lesssim \mu_\flat^{-1}\|g\|$ for all $g \in L^2(\Omega)$ and $z \in U_0$ solving (3) with data g and homogeneous (mixed Dirichlet-Neumann) boundary conditions. Assume $f \in H^{k+\delta}(\Omega), \phi_\partial \in W^{k+\delta,\infty}(\Gamma_{\mathrm{n}})$, with $\delta = 0$ for $k \geqslant 1$ and $\delta = 1$ for $k = 0$. Then, under the same assumption on u as in Theorem 3.1, the following holds:*

$$\mu_\flat \|I_h^k u - u_h\| \lesssim \mu_\sharp^{1/2}\rho h \left\{ \sum_{T\in\mathcal{T}_h} \mu_{\sharp,T}\rho_T^2 h_T^{2(k+1)} \|u\|_{H^{k+2}(T)}^2 \right\}^{1/2} + h^{k+2}\left\{ \|f\|_{H^{k+\delta}(\Omega)} + \|\phi_\partial\|_{W^{k+\delta,\infty}(\Gamma_{\mathrm{n}})} \right\}. \tag{21}$$

3.2.4 Local conservativity

For all $T \in \mathcal{T}_h$, let us first introduce the local bilinear form $\hat{a}_T : \underline{\mathrm{U}}_T^k \times \underline{\mathrm{U}}_T^k \to \mathbb{R}$ such that, for all $\underline{\mathrm{w}}_T, \underline{\mathrm{v}}_T \in \underline{\mathrm{U}}_T^k$,

$$\hat{a}_T(\underline{\mathrm{w}}_T, \underline{\mathrm{v}}_T) := (\mathbb{M}_T \boldsymbol{\nabla} p_T^{k+1}\underline{\mathrm{w}}_T, \boldsymbol{\nabla} p_T^{k+1}\underline{\mathrm{v}}_T)_T + \sum_{F\in\mathcal{F}_T} \frac{\mu_{T,F}}{h_F}(w_T - \mathfrak{w}_F, v_T - \mathfrak{v}_F)_F. \tag{22}$$

Then, we use (22) to define the local isomorphism $\underline{\mathrm{c}}_T^k : \underline{\mathrm{U}}_T^k \to \underline{\mathrm{U}}_T^k$ such that, for all $\underline{\mathrm{w}}_T \in \underline{\mathrm{U}}_T^k$, $\underline{\mathrm{c}}_T^k \underline{\mathrm{w}}_T$ is uniquely defined from the following local problem:

$$\hat{a}_T(\underline{\mathrm{c}}_T^k \underline{\mathrm{w}}_T, \underline{\mathrm{v}}_T) = a_T(\underline{\mathrm{w}}_T, \underline{\mathrm{v}}_T) + \sum_{F\in\mathcal{F}_T} \frac{\mu_{T,F}}{h_F}(w_T - \mathfrak{w}_F, v_T - \mathfrak{v}_F)_F \qquad \forall\, \underline{\mathrm{v}}_T \in \underline{\mathrm{U}}_T^k,$$

and $\int_T c_T^k \underline{\mathrm{w}}_T = \int_T w_T$. Finally, we define the local gradient reconstruction operator $\boldsymbol{G}_T^{k+1} : \underline{\mathrm{U}}_T^k \to \boldsymbol{\nabla}\mathbb{P}_d^{k+1}(T)$ such that

$$\boldsymbol{G}_T^{k+1} := \boldsymbol{\nabla}(p_T^{k+1} \circ \underline{\mathrm{c}}_T^k).$$

Adapting the arguments of [28, Lemmata 2 and 3], one can show the following result.

Lemma 3.4 (Local conservativity). *Let $\underline{\mathrm{u}}_h \in \underline{\mathrm{U}}_h^k$ be defined as in (15) from the solution of problem (14). Then, for all $T \in \mathcal{T}_h$, the following local equilibrium relation holds:*

$$(\mathbb{M}_T \boldsymbol{G}_T^{k+1} \underline{\mathrm{u}}_T, \boldsymbol{\nabla} v_T)_T - \sum_{F \in \mathcal{F}_T} (\Phi_{T,F}(\underline{\mathrm{u}}_T), v_T)_F = (f, v_T)_T \qquad \forall\, v_T \in \mathbb{P}_d^k(T), \tag{23}$$

where the numerical flux operator $\Phi_{T,F} : \underline{\mathrm{U}}_T^k \to \mathbb{P}_{d-1}^k(F)$ is such that, for all $\underline{\mathrm{v}}_T \in \underline{\mathrm{U}}_T^k$,

$$\Phi_{T,F}(\underline{\mathrm{v}}_T) := \mathbb{M}_T \boldsymbol{G}_T^{k+1} \underline{\mathrm{v}}_T \cdot \boldsymbol{n}_{T,F} - \frac{\mu_{T,F}}{h_F} \left[(c_T^k \underline{\mathrm{v}}_T - v_T) - (\mathfrak{c}_F^k \underline{\mathrm{v}}_T - \mathfrak{v}_F) \right]. \tag{24}$$

In addition, the numerical fluxes are equilibrated in the following sense: For all $F \in \mathcal{F}_h^{\mathrm{i}}$ such that $F \subseteq \partial T_1 \cap \partial T_2$,

$$\Phi_{T_1,F}(\underline{\mathrm{u}}_T) + \Phi_{T_2,F}(\underline{\mathrm{u}}_T) = 0, \tag{25}$$

and $\Phi_{T,F}(\underline{\mathrm{u}}_T) = \pi_F^k \phi_\partial$ for all $F \in \mathcal{F}_h^{\mathrm{n}}$ such that $F \subseteq \partial T \cap \partial\Omega$.

Numerical fluxes can thus be computed by local element-by-element post-processing.

3.3 Computational aspects

This section discusses various relevant computational aspects: the elimination of cell-based unknowns by static condensation, the offline/online decomposition of the computations, and the choice of polynomial bases.

3.3.1 Static condensation

Following [20, Section 2.5], we show how cell-based unknowns can be locally eliminated from problem (14), thereby leading to a global system in terms of face-based unknowns only.

Introducing the notation $f_T := f_{|T}$ for all $T \in \mathcal{T}_h$, we begin by observing that problem (14) can be equivalently rewritten using (15) as follows:

$$a_T((u_T, 0), (v_T, 0)) = (f_T, v_T)_T - a_T((0, \mathfrak{u}_{\partial T}), (v_T, 0)) \qquad \forall v_T \in U_T^k,\ \forall T \in \mathcal{T}_h, \tag{26a}$$

$$a_h(\underline{\mathrm{u}}_h, (0, \mathfrak{v}_h)) = \sum_{F \in \mathcal{F}_h^{\mathrm{n}}} (\phi_\partial, \mathfrak{v}_F)_F \qquad \forall \mathfrak{v}_h \in \mathfrak{U}_{h,0}^k, \tag{26b}$$

that is to say, problem (14) can be split into $\mathrm{card}(\mathcal{T}_h)$ local problems (26a) that allow one to express, for all $T \in \mathcal{T}_h$, u_T in terms of $\mathfrak{u}_{\partial T}$ and f_T, and one global problem (26b) written in terms of face-based unknowns only.

We now introduce two local cell-based potential lifting operators:

- a trace-based lifting $t_T^k : \mathfrak{U}_{\partial T}^k \to U_T^k$ such that, for all $\mathfrak{w}_{\partial T} \in \mathfrak{U}_{\partial T}^k$, $t_T^k \mathfrak{w}_{\partial T} \in U_T^k$ solves

$$a_T((t_T^k \mathfrak{w}_{\partial T}, 0), (v_T, 0)) = -a_T((0, \mathfrak{w}_{\partial T}), (v_T, 0)) \qquad \forall v_T \in U_T^k; \tag{27}$$

- a datum-based lifting $d_T^k : L^2(T) \to U_T^k$ such that, for all $\varphi_T \in L^2(T)$, $d_T^k \varphi_T \in U_T^k$ solves

$$a_T((d_T^k \varphi_T, 0), (v_T, 0)) = (\varphi_T, v_T)_T \qquad \forall v_T \in U_T^k. \tag{28}$$

Problems (27) and (28) are well-posed owing to the first inequality in (17) and the fact that $\|\cdot\|_{U,T}$ is a norm on the zero-trace subspace of $\underline{\mathrm{U}}_T^k$, cf. (16). Problem (27) can be rewritten as

$$a_T((t_T^k \mathfrak{w}_{\partial T}, \mathfrak{w}_{\partial T}), (v_T, 0)) = 0 \qquad \forall v_T \in U_T^k. \tag{29}$$

Using (26a), (29), and (28), we infer that

$$\underline{\mathrm{u}}_T = (t_T^k \mathfrak{u}_{\partial T} + d_T^k f_T, \mathfrak{u}_{\partial T}). \tag{30}$$

Introducing the global operators $t_h^k : \mathfrak{U}_h^k \to U_h^k$ and $d_h^k : L^2(\Omega) \to U_h^k$ such that, for all $\mathfrak{w}_h \in \mathfrak{U}_h^k$, all $\varphi \in L^2(\Omega)$, and all $T \in \mathcal{T}_h$, $(t_h^k \mathfrak{w}_h)_{|T} := t_T^k \mathfrak{w}_{\partial T}$ and $(d_h^k \varphi)_{|T} := d_T^k \varphi_{|T}$, we can rewrite (30) globally as follows:

$$\underline{\mathrm{u}}_h = (t_h^k \mathfrak{u}_h + d_h^k f, \mathfrak{u}_h). \tag{31}$$

Finally, we reformulate the global problem (26b) under an equivalent form. We remark, using (31), that

$$\begin{aligned}
a_h(\underline{\mathrm{u}}_h, (0, \mathfrak{v}_h)) &= a_h(\underline{\mathrm{u}}_h, (t_h^k \mathfrak{v}_h, \mathfrak{v}_h)) - a_h(\underline{\mathrm{u}}_h, (t_h^k \mathfrak{v}_h, 0)) \\
&= a_h((t_h^k \mathfrak{u}_h, \mathfrak{u}_h), (t_h^k \mathfrak{v}_h, \mathfrak{v}_h)) + a_h((d_h^k f, 0), (t_h^k \mathfrak{v}_h, \mathfrak{v}_h)) \\
&\quad - a_h((t_h^k \mathfrak{u}_h, \mathfrak{u}_h), (t_h^k \mathfrak{v}_h, 0)) - a_h((d_h^k f, 0), (t_h^k \mathfrak{v}_h, 0)) \\
&:= \mathfrak{T}_1 + \mathfrak{T}_2 - \mathfrak{T}_3 - \mathfrak{T}_4,
\end{aligned}$$

where $\mathfrak{T}_2 = \mathfrak{T}_3 = 0$ owing to (29) and to the symmetry of a_h, while $\mathfrak{T}_4 = (f, t_h^k \mathfrak{v}_h)$ owing to (28). Introducing for all $\mathfrak{w}_h \in \mathfrak{U}_h^k$ the notation $\underline{t}_h^k \mathfrak{w}_h := (t_h^k \mathfrak{w}_h, \mathfrak{w}_h)$ and the decomposition $\mathfrak{u}_h = \mathfrak{u}_{h,0} + \mathfrak{u}_{h,\partial}$ for the face-based unknowns, the previous relation enables us to rewrite the global problem (26b) as follows: Find $\mathfrak{u}_{h,0} \in \mathfrak{U}_{h,0}^k$ such that

$$a_h(\underline{t}_h^k \mathfrak{u}_{h,0}, \underline{t}_h^k \mathfrak{v}_h) = (f, t_h^k \mathfrak{v}_h) - a_h(\underline{t}_h^k \mathfrak{u}_{h,\partial}, \underline{t}_h^k \mathfrak{v}_h) + \sum_{F \in \mathcal{F}_h^{\mathrm{n}}} (\phi_\partial, \mathfrak{v}_F)_F \qquad \forall \mathfrak{v}_h \in \mathfrak{U}_{h,0}^k. \tag{32}$$

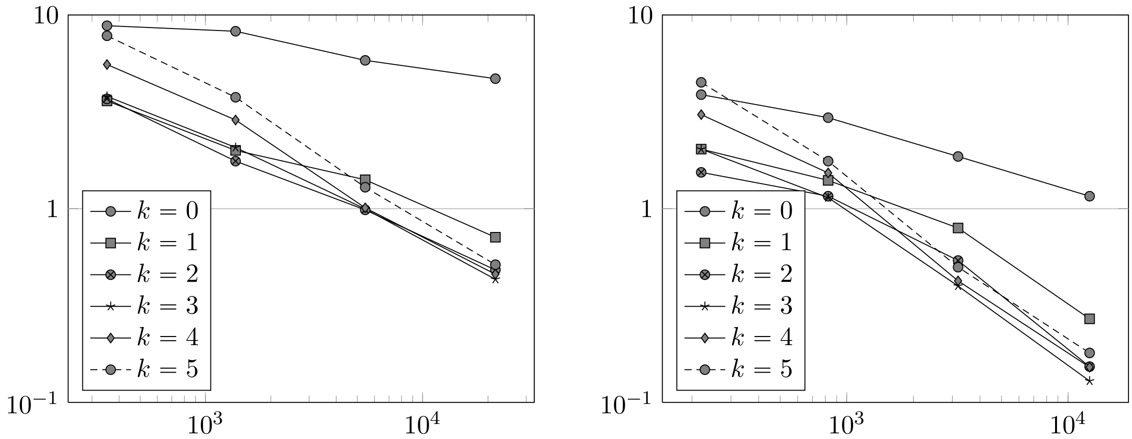
Problem (32) is well-posed owing to (18) and to the fact that $\|\cdot\|_{U,h}$ defines a norm on $\underline{\mathrm{U}}_{h,0}^k$. The following proposition summarizes the above considerations.

Proposition 3.1 (Characterization of the approximate solution)**.** *The solution $\underline{\mathrm{U}}_h^k \ni \underline{\mathrm{u}}_h = \underline{\mathrm{u}}_{h,0} + \underline{\mathrm{u}}_{h,\partial}$ with $\underline{\mathrm{u}}_{h,0} \in \underline{\mathrm{U}}_{h,0}^k$ solving (14) can be expressed as (31), where the operator t_h^k and the vector of cell-based unknowns $d_h^k f$ are defined cell-wise as the solutions of the local problems (27) and (28), respectively, and where $\mathfrak{u}_h \in \mathfrak{U}_h^k$ is such that $\mathfrak{u}_h = \mathfrak{u}_{h,0} + \mathfrak{u}_{h,\partial}$ with $\mathfrak{u}_{h,0} \in \mathfrak{U}_{h,0}^k$ the unique solution of the global problem (32).*

3.3.2 Offline/online solution strategy

Static condensation naturally points to an offline/online decomposition of the computations.

In the offline step, we begin by solving, for all $T \in \mathcal{T}_h$, the local problems (6), in order to compute the operator p_h^{k+1}. This first substep essentially requires to invert $\mathrm{card}(\mathcal{T}_h)$ symmetric positive-definite matrices of size $N_{k+1,d}$. This can be done effectively using Cholesky factorization. Then, for all $T \in \mathcal{T}_h$, we solve the local problems (27) and (28). As both problems involve the same matrix, this second substep essentially requires the inversion of



(a) Triangular mesh family

(b) Hexagonal mesh family

Figure 2: Assembly time divided by the solution time as a function of card($\mathcal{F}_h$) for a triangular mesh family (left panel) and a (predominantly) hexagonal mesh family (right panel); the symbols indicate in both panels the polynomial degree that is being used.

card($\mathcal{T}_h$) symmetric positive-definite matrices of size $N_{k,d}$. Note that both substeps are fully parallelizable. At the end of the offline step, one has computed the trace-based lifting t_h^k, and the restriction of the datum-based lifting d_h^k to $U_h^k = \mathbb{P}_d^k(\mathcal{T}_h)$. This fully determines d_h^k since the right-hand side of (28) only requires the projection of the datum onto U_h^k.

In the online step, given a right-hand side $f \in L^2(\Omega)$, we compute its L^2-orthogonal projection onto U_h^k, and we solve the global problem (32); the size of this problem is approximately equal to card$(\mathcal{F}_h) \times N_{k,d-1}$. The approximate solution is finally computed applying (31). A modification of the right-hand side (or of the boundary conditions) only requires to perform again the online step.

The offline/online solution strategy is particularly attractive in a multi-query context where one wants to compute the solution of problem (14) for a large number of right-hand sides $f \in L^2(\Omega)$.

3.3.3 Implementation

An important step in the implementation consists in selecting bases for the polynomial spaces on elements and faces that appear in the construction (cf. (6), (27), (28), (32)). For $T \in \mathcal{T}_h$, we denote by $\boldsymbol{x}_T$ a point in T (typically the barycenter of T). One possibility leading to a hierarchical basis for $\mathbb{P}_d^l(T)$, $l \in \{k, k+1\}$, is to choose the following family of monomial functions:

$$\left\{ \prod_{i=1}^{d} \xi_{T,i}^{\alpha_i} \mid \xi_{T,i} := \frac{x_i - x_{T,i}}{h_T} \; \forall\, 1 \leqslant i \leqslant d,\; \boldsymbol{\alpha} = (\alpha_i)_{1 \leqslant i \leqslant d} \in \mathbb{N}^d,\; \|\boldsymbol{\alpha}\|_{l^1} \leqslant l \right\}.$$

Similarly, for all $F \in \mathcal{F}_h$, we can define a basis for $\mathbb{P}_{d-1}^k(F)$ spanned by monomials with respect to a local frame scaled using the face diameter and, say, the barycenter of F.

3.3.4 Cost assessment

Another important question linked to implementation is the scaling of the time devoted to the assembly (computation of the local contributions, static condensation, and matrix/right-hand side assembly) with respect to the time devoted to the solution (solving of the global problem), and how this scaling depends on the meshsize and on the order of approximation. Let us assume a naive implementation that does not exploit parallelism, and let us focus on problem (14) for a given right-hand side in two space dimensions. On Figure 2, we plot, for polynomial degrees up to 5, the assembly/solution time ratio as a function of the number of mesh faces for two families of meshes corresponding, respectively, to the triangular (first) mesh family of the FVCA5 benchmark [41] and to the (predominantly) hexagonal mesh family introduced in [32, Section 4.2.3]. The global system is solved using the sparse direct solver of Eigen v3. This way, both the assembly and solution times are only marginally influenced by the problem data (right-hand side, boundary conditions). As illustrated in Figure 2, the overall cost of the assembly time becomes quickly negligible in comparison with the solution time with mesh refinement (except for $k = 0$). This can be dramatically improved, e.g., using thread-based parallelism to solve the (independent) local problems for both the computation of the potential reconstructions and the static condensation inside each element.

4 The HHO method in mixed form

In this section, we study the HHO method in mixed formulation. The starting point is the following mixed form of the model problem (1): Find $\boldsymbol{s} : \Omega \to \mathbb{R}^d$, $u : \Omega \to \mathbb{R}$, such that

$$\begin{aligned} \boldsymbol{s} &= \mathbb{M}\boldsymbol{\nabla} u && \text{in } \Omega, \\ -\mathrm{div}\boldsymbol{s} &= f && \text{in } \Omega, \\ u &= \psi_\partial && \text{on } \Gamma_{\mathrm{d}}, \\ \boldsymbol{s}{\cdot}\boldsymbol{n} &= \phi_\partial && \text{on } \Gamma_{\mathrm{n}}. \end{aligned} \tag{33}$$

To write this problem in weak form, we introduce the functional spaces

$$\boldsymbol{S} := \boldsymbol{H}(\mathrm{div},\Omega), \qquad \boldsymbol{S}_0 := \left\{\boldsymbol{t} \in \boldsymbol{S} \mid \boldsymbol{t}{\cdot}\boldsymbol{n}_{|\Gamma_{\mathrm{n}}} = 0\right\}, \qquad V := L^2(\Omega),$$

so that the weak problem reads: Find $(\boldsymbol{s}_0, u) \in \boldsymbol{S}_0 \times V$ such that

$$\begin{aligned} (\mathbb{M}^{-1}\boldsymbol{s}_0, \boldsymbol{t}) + (u, \mathrm{div}\,\boldsymbol{t}) &= \langle \boldsymbol{t}{\cdot}\boldsymbol{n}, (u_\partial)_{|\Gamma}\rangle_\Gamma - (\mathbb{M}^{-1}\boldsymbol{s}_\partial, \boldsymbol{t}) && \forall \boldsymbol{t} \in \boldsymbol{S}_0, \\ -(\mathrm{div}\boldsymbol{s}_0, v) &= (f, v) + (\mathrm{div}\boldsymbol{s}_\partial, v) && \forall v \in V, \end{aligned} \tag{34}$$

where $\boldsymbol{s}_\partial \in \boldsymbol{S}$ is a lifting of the Neumann datum such that $(\boldsymbol{s}_\partial{\cdot}\boldsymbol{n})_{|\Gamma_{\mathrm{n}}} = \phi_\partial$ (which can be taken to be $\boldsymbol{s}_\partial = \boldsymbol{\nabla}\theta$ where $\theta \in H^1(\Omega)$ solves $\theta - \Delta\theta = 0$ in Ω with $\boldsymbol{\nabla}\theta{\cdot}\boldsymbol{n} = \phi_\Gamma$ on Γ where ϕ_Γ is the zero-extension of ϕ_∂ to Γ), and $\langle\cdot,\cdot\rangle_\Gamma$ denotes the duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$ (note that, owing to the fact that $\boldsymbol{t} \in \boldsymbol{S}_0$, $\langle \boldsymbol{t}{\cdot}\boldsymbol{n}, (u_\partial)_{|\Gamma}\rangle_\Gamma$ does not depend on the choice of the lifting u_∂ of ψ_∂). The solution $(\boldsymbol{s}, u) \in \boldsymbol{S} \times V$ is then computed as $(\boldsymbol{s}, u) = (\boldsymbol{s}_0 + \boldsymbol{s}_\partial, u)$.

4.1 Discrete setting

Let us fix an integer $k \geqslant 0$ and consider an admissible and compatible mesh sequence $(\mathcal{T}_h)_{h\in\mathcal{H}}$ in the sense of Definitions 2.1 and 2.2. We suppose that the assumptions of Section 2.4 concerning the diffusivity hold.

4.1.1 Discrete unknowns

We adopt the same notation as in Section 3.1.1, to which we add the use of boldface to denote vector-valued quantities. We introduce, first locally then globally, the discrete unknowns associated with the flux and with the potential. For the flux, we consider hybrid unknowns, in the sense that they consist of both cell- and face-based contributions. The cell-based flux unknowns are vector-valued while the face-based ones are scalar-valued. For the potential, we consider scalar-valued cell-based unknowns.

Local definition Let $T \in \mathcal{T}_h$. Setting

$$\boldsymbol{S}_T^k := \mathbb{M}_T \boldsymbol{\nabla} \mathbb{P}_d^k(T), \qquad \mathfrak{S}_F^k := \mathbb{P}_{d-1}^k(F) \text{ for all } F \in \mathcal{F}_T,$$

we define the local set of hybrid flux unknowns, cf. Figure 3, as

$$\underline{\mathbf{S}}_T^k := \boldsymbol{S}_T^k \times \mathfrak{S}_{\partial T}^k, \qquad \text{where } \mathfrak{S}_{\partial T}^k := \bigtimes_{F\in\mathcal{F}_T} \mathfrak{S}_F^k.$$

In the lowest-order case $k = 0$, cell-based flux unknowns are unnecessary and $\boldsymbol{S}_T^k$ has dimension zero. Any element $\underline{\mathsf{t}}_T \in \underline{\mathbf{S}}_T^k$ can be decomposed as $\underline{\mathsf{t}}_T := (\boldsymbol{t}_T \in \boldsymbol{S}_T^k, \mathsf{t}_{\partial T} \in \mathfrak{S}_{\partial T}^k)$, with $\mathsf{t}_{\partial T} := (\mathsf{t}_F \in \mathfrak{S}_F^k)_{F\in\mathcal{F}_T}$.

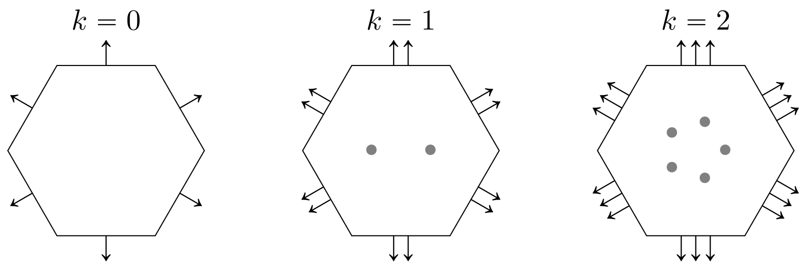


Figure 3: Degrees of freedom associated with hybrid flux discrete unknowns, $d = 2$, $k \in \{0, 1, 2\}$.

Letting, for $q > 2$,

$$\boldsymbol{S}^+(T) := \{\boldsymbol{t} \in \boldsymbol{L}^q(T) \mid \operatorname{div} \boldsymbol{t} \in L^2(T)\},$$

and recalling that functions in this space have integrable normal component on all faces of T, we introduce the local reduction operator $\underline{\mathbf{I}}_T^k : \boldsymbol{S}^+(T) \to \underline{\mathbf{S}}_T^k$ such that, for all $\boldsymbol{t} \in \boldsymbol{S}^+(T)$,

$$\underline{\mathbf{I}}_T^k \boldsymbol{t} := \left(\mathbb{M}_T \boldsymbol{\nabla} y, (\pi_F^k(\boldsymbol{t}\cdot\boldsymbol{n}_F))_{F\in\mathcal{F}_T} \right),$$

where $y \in \mathbb{P}_d^k(T)$ is a solution (defined up to an additive constant) of the Neumann problem

$$(\mathbb{M}_T \boldsymbol{\nabla} y, \boldsymbol{\nabla} w)_T = (\boldsymbol{t}, \boldsymbol{\nabla} w)_T \qquad \forall w \in \mathbb{P}_d^k(T), \tag{35}$$

observing that the required compatibility condition on the right-hand side is verified.

As far as the potential is concerned, we let U_T^k, introduced in (4), be the associated local set of (cell-based) discrete unknowns.

Global definition We define the global set of hybrid flux unknowns as

$$\underline{\mathbf{S}}_h^k := \boldsymbol{S}_h^k \times \left\{ \bigtimes_{F \in \mathcal{F}_h} \mathfrak{S}_F^k \right\},$$

where $\boldsymbol{S}_h^k := \bigtimes_{T \in \mathcal{T}_h} \boldsymbol{S}_T^k$. Observe that the flux unknowns attached to interfaces are single-valued. Given an element $\underline{\mathrm{t}}_h \in \underline{\mathbf{S}}_h^k$, for any $T \in \mathcal{T}_h$, we denote by $\underline{\mathrm{t}}_T = (\boldsymbol{t}_T, \mathrm{t}_{\partial T}) \in \underline{\mathbf{S}}_T^k$ its restriction to the element T. We introduce the following subspace of $\underline{\mathbf{S}}_h^k$, that allows one to account for (homogeneous) Neumann boundary conditions in a strong manner:

$$\underline{\mathbf{S}}_{h,0}^k := \left\{ \underline{\mathrm{t}}_h \in \underline{\mathbf{S}}_h^k \mid \mathrm{t}_F \equiv 0, \forall F \in \mathcal{F}_h^{\mathrm{n}} \right\}.$$

We also define the global reduction operator $\underline{\mathbf{I}}_h^k : \boldsymbol{S} \cap \boldsymbol{S}^+(\mathcal{T}_h) \to \underline{\mathbf{S}}_h^k$ such that, for all $\boldsymbol{t} \in \boldsymbol{S} \cap \boldsymbol{S}^+(\mathcal{T}_h)$, and for all $T \in \mathcal{T}_h$, $(\underline{\mathbf{I}}_h^k \boldsymbol{t})_{|T} := \underline{\mathbf{I}}_T^k \boldsymbol{t}_{|T}$. Single-valuedness at interfaces is ensured by the regularity of functions in $\boldsymbol{S} \cap \boldsymbol{S}^+(\mathcal{T}_h)$.

We finally define U_h^k, cf. (5), as the global set of discrete (cell-based) potential unknowns, and we denote by $v_T \in U_T^k$ the restriction of any $v_h \in U_h^k$ to the element $T \in \mathcal{T}_h$.

4.1.2 Divergence reconstruction operator

Let $T \in \mathcal{T}_h$. We define the local divergence reconstruction operator $D_T^k : \underline{\mathbf{S}}_T^k \to U_T^k$ as the operator such that, for all $\underline{\mathrm{t}}_T = (\boldsymbol{t}_T, \mathrm{t}_{\partial T}) \in \underline{\mathbf{S}}_T^k$,

$$(D_T^k \underline{\mathrm{t}}_T, v_T)_T = -(\boldsymbol{t}_T, \boldsymbol{\nabla} v_T)_T + \sum_{F \in \mathcal{F}_T} (\mathrm{t}_F \varepsilon_{T,F}, v_T)_F \qquad \forall v_T \in U_T^k, \tag{36}$$

where $\varepsilon_{T,F} := \boldsymbol{n}_F \cdot \boldsymbol{n}_{T,F}$ for all $T \in \mathcal{T}_h$ and $F \in \mathcal{F}_T$. This definition reproduces at the discrete level an integration by parts formula, that brings into action the local hybrid flux unknowns. The following property is crucial for inf-sup stability, cf. [27, Lemmas 2 and 5].

Lemma 4.1 (Commuting property)**.** *The following holds for all $\boldsymbol{t} \in \boldsymbol{S}^+(T)$:*

$$D_T^k \underline{\mathbf{I}}_T^k \boldsymbol{t} = \pi_T^k(\operatorname{div} \boldsymbol{t}). \tag{37}$$

Proof. For $\boldsymbol{t} \in \boldsymbol{S}^{+}(T)$, let us plug the quantity $\underline{\mathbf{t}}_T := \underline{\mathbf{I}}_T^k \boldsymbol{t} = \left(\mathbb{M}_T \boldsymbol{\nabla} y, (\pi_F^k(\boldsymbol{t}\cdot\boldsymbol{n}_F))_{F\in\mathcal{F}_T}\right)$ into (36), where $y \in \mathbb{P}_d^k(T)$ is a solution to (35). Let $v_T \in U_T^k$, and observe that $v_T \in \mathbb{P}_d^k(T)$ and $v_{T|F} \in \mathbb{P}_{d-1}^k(F)$. Hence,

$$(D_T^k \underline{\mathbf{I}}_T^k \boldsymbol{t}, v_T)_T = -(\boldsymbol{t}, \boldsymbol{\nabla} v_T)_T + \sum_{F\in\mathcal{F}_T} (\boldsymbol{t}\cdot\boldsymbol{n}_{T,F}, v_T)_F = (\operatorname{div} \boldsymbol{t}, v_T)_T,$$

which concludes the proof. □

For further use, we introduce the global divergence reconstruction operator $D_h^k : \underline{\mathbf{S}}_h^k \to U_h^k$ such that, for all $\underline{\mathbf{t}}_h \in \underline{\mathbf{S}}_h^k$, and all $T \in \mathcal{T}_h$, $(D_h^k \underline{\mathbf{t}}_h)_{|T} := D_T^k \underline{\mathbf{t}}_T$.

4.1.3 Flux reconstruction operator

Let $T \in \mathcal{T}_h$. The local flux reconstruction operator $\boldsymbol{F}_T^{k+1} : \underline{\mathbf{S}}_T^k \to \boldsymbol{S}_T^{k+1}$ is defined, for all $\underline{\mathbf{t}}_T = (\boldsymbol{t}_T, \mathrm{t}_{\partial T}) \in \underline{\mathbf{S}}_T^k$, as $\boldsymbol{F}_T^{k+1}\underline{\mathbf{t}}_T := \mathbb{M}_T \boldsymbol{\nabla} z$, where $z \in \mathbb{P}_d^{k+1}(T)$ is a solution (defined up to an additive constant) of the Neumann problem

$$(\mathbb{M}_T \boldsymbol{\nabla} z, \boldsymbol{\nabla} w)_T = (\boldsymbol{t}_T, \boldsymbol{\nabla}\pi_T^k w)_T + \sum_{F\in\mathcal{F}_T} (\mathrm{t}_F \varepsilon_{T,F}, \pi_F^k w - \pi_T^k w)_F \qquad \forall w \in \mathbb{P}_d^{k+1}(T), \tag{38}$$

observing that the required compatibility condition on the right-hand side is verified. The definition of $\boldsymbol{F}_T^{k+1}\underline{\mathbf{t}}_T$ is motivated by the following link between $\boldsymbol{F}_T^{k+1}\underline{\mathbf{t}}_T$ and the divergence reconstruction operator defined in (36): For all $\underline{\mathbf{t}}_T = (\boldsymbol{t}_T, \mathrm{t}_{\partial T}) \in \underline{\mathbf{S}}_T^k$,

$$(\boldsymbol{F}_T^{k+1}\underline{\mathbf{t}}_T, \boldsymbol{\nabla} w)_T = -(D_T^k \underline{\mathbf{t}}_T, w)_T + \sum_{F\in\mathcal{F}_T} (\mathrm{t}_F \varepsilon_{T,F}, w)_F \qquad \forall w \in \mathbb{P}_d^{k+1}(T). \tag{39}$$

As in Section 3.1.2, computing the operator $\boldsymbol{F}_T^{k+1}$ using (38) or (39) requires to invert a symmetric positive-definite matrix of size $N_{k+1,d}$, cf. (2), which can be performed effectively via Cholesky factorization. The following result can be found in [27, Lemma 3] (and requires, as its primal counterpart (8), that the diffusion tensor be piecewise constant).

Lemma 4.2 (Polynomial consistency)**.** *The following holds for all* $\boldsymbol{t} \in \boldsymbol{S}_T^{k+1}$*:*

$$\boldsymbol{F}_T^{k+1} \underline{\mathbf{I}}_T^k \boldsymbol{t} = \boldsymbol{t}. \tag{40}$$

Proof. Let $\boldsymbol{t} \in \boldsymbol{S}_T^{k+1}$ and plug $\underline{\mathbf{t}}_T := \underline{\mathbf{I}}_T^k \boldsymbol{t}$ into (39). Using the commuting property (37) leads to $D_T^k \underline{\mathbf{I}}_T^k \boldsymbol{t} = \pi_T^k(\operatorname{div} \boldsymbol{t}) = \operatorname{div} \boldsymbol{t}$ since $\boldsymbol{t} \in \boldsymbol{S}_T^{k+1} \subset \mathbb{P}_d^k(T)$ ($\mathbb{M}_T$ is a constant tensor), which combined with the fact that $\pi_F^k(\boldsymbol{t}\cdot\boldsymbol{n}_F) = \boldsymbol{t}\cdot\boldsymbol{n}_F$ (since faces are planar), allows us to infer that, for all $w \in \mathbb{P}_d^{k+1}(T)$,

$$(\boldsymbol{F}_T^{k+1}\underline{\mathbf{I}}_T^k \boldsymbol{t}, \boldsymbol{\nabla} w)_T = -(\operatorname{div} \boldsymbol{t}, w)_T + \sum_{F\in\mathcal{F}_T} (\boldsymbol{t}\cdot\boldsymbol{n}_{T,F}, w)_F = (\boldsymbol{t}, \boldsymbol{\nabla} w)_T.$$

This last relation proves (40) since $(\boldsymbol{F}_T^{k+1}\underline{\mathbf{I}}_T^k \boldsymbol{t} - \boldsymbol{t}) \in \boldsymbol{S}_T^{k+1} = \mathbb{M}_T \boldsymbol{\nabla}\mathbb{P}_d^{k+1}(T)$. □

The next result is adapted from [27, Lemma 9], and is related, in the light of Lemma 5.1 below, to Lemma 3.2.

Lemma 4.3 (Approximation). *For all $v \in H^{k+2}(T)$, letting $\boldsymbol{t} := \mathbb{M}_T \boldsymbol{\nabla} v$, the following holds for all $F \in \mathcal{F}_T$:*

$$\|\mathbb{M}_T^{-1/2}(\boldsymbol{t} - \boldsymbol{F}_T^{k+1}\underline{\mathbf{I}}_T^k \boldsymbol{t})\|_T + h_F^{1/2}\mu_{T,F}^{-1/2}\|(\boldsymbol{t} - \boldsymbol{F}_T^{k+1}\underline{\mathbf{I}}_T^k \boldsymbol{t})\cdot \boldsymbol{n}_F\|_F \lesssim \rho_T^{1/2}\mu_{\sharp,T}^{1/2} h_T^{k+1}\|v\|_{H^{k+2}(T)}. \tag{41}$$

For further use, we define the global flux reconstruction operator $\boldsymbol{F}_h^{k+1} : \underline{\mathbf{S}}_h^k \to \boldsymbol{S}_h^{k+1}$ such that, for all $\underline{\mathbf{t}}_h \in \underline{\mathbf{S}}_h^k$, and all $T \in \mathcal{T}_h$, $(\boldsymbol{F}_h^{k+1}\underline{\mathbf{t}}_h)_{|T} := \boldsymbol{F}_T^{k+1}\underline{\mathbf{t}}_T$.

4.1.4 Stabilization

For all $T \in \mathcal{T}_h$, we define the stabilization bilinear form $J_T : \underline{\mathbf{S}}_T^k \times \underline{\mathbf{S}}_T^k \to \mathbb{R}$ such that

$$J_T(\underline{\mathbf{s}}_T, \underline{\mathbf{t}}_T) := \sum_{F \in \mathcal{F}_T} \frac{h_F}{\mu_{T,F}}((\boldsymbol{F}_T^{k+1}\underline{\mathbf{s}}_T)\cdot \boldsymbol{n}_F - \mathfrak{s}_F, (\boldsymbol{F}_T^{k+1}\underline{\mathbf{t}}_T)\cdot \boldsymbol{n}_F - \mathrm{t}_F)_F.$$

Notice that J_T is symmetric, positive semi-definite, and polynomially consistent (as a consequence of Lemma 4.2) in the sense that, for all $\boldsymbol{t} \in \boldsymbol{S}_T^{k+1}$,

$$J_T(\underline{\mathbf{I}}_T^k \boldsymbol{t}, \underline{\mathbf{r}}_T) = 0 \qquad \forall \underline{\mathbf{r}}_T \in \underline{\mathbf{S}}_T^k. \tag{42}$$

This result can be found in [27, Eq. (18)]. Another important property of J_T is the following approximation property (see [27, Lemma 9] and Lemma 4.3 above): For all $v \in H^{k+2}(T)$, the following holds with $\boldsymbol{t} := \mathbb{M}_T \boldsymbol{\nabla} v$:

$$J_T(\underline{\mathbf{I}}_T^k \boldsymbol{t}, \underline{\mathbf{I}}_T^k \boldsymbol{t})^{1/2} \lesssim \rho_T^{1/2}\mu_{\sharp,T}^{1/2} h_T^{k+1}\|v\|_{H^{k+2}(T)}. \tag{43}$$

4.2 Discrete problem: formulation and key properties

4.2.1 Formulation

For all $T \in \mathcal{T}_h$, we define the following local bilinear form:

$$H_T : \underline{\mathbf{S}}_T^k \times \underline{\mathbf{S}}_T^k \to \mathbb{R}; \qquad (\underline{\mathbf{s}}_T, \underline{\mathbf{t}}_T) \mapsto (\mathbb{M}_T^{-1}\boldsymbol{F}_T^{k+1}\underline{\mathbf{s}}_T, \boldsymbol{F}_T^{k+1}\underline{\mathbf{t}}_T)_T + J_T(\underline{\mathbf{s}}_T, \underline{\mathbf{t}}_T), \tag{44}$$

where the notation H_T is reminiscent of the similarity with the discrete Hodge operator considered in the CDO framework in the lowest-order case [11]. Introduce now the following global bilinear form:

$$H_h : \underline{\mathbf{S}}_h^k \times \underline{\mathbf{S}}_h^k \to \mathbb{R}; \qquad (\underline{\mathbf{s}}_h, \underline{\mathbf{t}}_h) \mapsto \sum_{T \in \mathcal{T}_h} H_T(\underline{\mathbf{s}}_T, \underline{\mathbf{t}}_T). \tag{45}$$

The mixed form of the HHO method for problem (34) reads: Find $(\underline{\mathbf{s}}_{h,0}, u_h) \in \underline{\mathbf{S}}_{h,0}^k \times U_h^k$ such that

$$\begin{aligned} H_h(\underline{\mathbf{s}}_{h,0}, \underline{\mathbf{t}}_h) + (u_h, D_h^k \underline{\mathbf{t}}_h) &= \sum_{F \in \mathcal{F}_h^{\mathrm{d}}} (\mathrm{t}_F, \psi_\partial)_F - H_h(\underline{\mathbf{s}}_{h,\partial}, \underline{\mathbf{t}}_h) \quad && \forall \underline{\mathbf{t}}_h \in \underline{\mathbf{S}}_{h,0}^k, \\ -(D_h^k \underline{\mathbf{s}}_{h,0}, v_h) &= (f, v_h) + (D_h^k \underline{\mathbf{s}}_{h,\partial}, v_h) && \forall v_h \in U_h^k, \end{aligned} \tag{46}$$

where $\underline{\mathbf{s}}_{h,\partial} := \underline{\mathbf{I}}_h^k \boldsymbol{s}_\partial \in \underline{\mathbf{S}}_h^k$ is the reduction of the lifting $\boldsymbol{s}_\partial$ of the Neumann datum ϕ_∂. The discrete solution $(\underline{\mathbf{s}}_h, u_h) \in \underline{\mathbf{S}}_h^k \times U_h^k$ is finally computed as

$$(\underline{\mathbf{s}}_h, u_h) = (\underline{\mathbf{s}}_{h,0} + \underline{\mathbf{s}}_{h,\partial}, u_h). \tag{47}$$

Remark 4.1 (Discrete Neumann datum)**.** *Similarly to Remark 3.2, the discrete lifting $\underline{\mathfrak{s}}_{h,\partial}$ of the Neumann datum can be obtained without explicitly knowing $\boldsymbol{s}_\partial$ by setting*

$$\boldsymbol{s}_{T,\partial} \equiv \mathbf{0} \quad \forall T \in \mathcal{T}_h, \qquad \mathfrak{s}_{F,\partial} = \pi_F^k \phi_\partial \quad \forall F \in \mathcal{F}_h^{\mathrm{n}}, \qquad \mathfrak{s}_{F,\partial} \equiv 0 \quad \forall F \in \mathcal{F}_h \backslash \mathcal{F}_h^{\mathrm{n}}.$$

4.2.2 Stability

Let us introduce, for all $T \in \mathcal{T}_h$, the following norm on $\underline{\mathbf{S}}_T^k$:

$$\|\underline{\mathbf{t}}_T\|_{\boldsymbol{S},T}^2 := \mu_{\sharp,T}^{-1} \left(\|\boldsymbol{t}_T\|_T^2 + \sum_{F \in \mathcal{F}_T} h_F \|\mathfrak{t}_F\|_F^2 \right). \tag{48}$$

Setting $\|\underline{\mathbf{t}}_h\|_{\boldsymbol{S},h}^2 := \sum_{T \in \mathcal{T}_h} \|\underline{\mathbf{t}}_T\|_{\boldsymbol{S},T}^2$ for all $\underline{\mathbf{t}}_h \in \underline{\mathbf{S}}_h^k$, it follows that $\|\cdot\|_{\boldsymbol{S},h}$ defines a norm on $\underline{\mathbf{S}}_h^k$. The coercivity of H_h can be expressed in terms of this norm, cf. [27, Lemma 4].

Lemma 4.4 (Stability for H_h)**.** *For all $T \in \mathcal{T}_h$, and for all $\underline{\mathbf{t}}_T \in \underline{\mathbf{S}}_T^k$, the following holds:*

$$\|\underline{\mathbf{t}}_T\|_{\boldsymbol{S},T} \lesssim H_T(\underline{\mathbf{t}}_T, \underline{\mathbf{t}}_T)^{1/2} \lesssim \rho_T^{1/2} \|\underline{\mathbf{t}}_T\|_{\boldsymbol{S},T}. \tag{49}$$

Consequently, we infer that

$$\|\underline{\mathbf{t}}_h\|_{\boldsymbol{S},h}^2 \lesssim H_h(\underline{\mathbf{t}}_h, \underline{\mathbf{t}}_h) \qquad \forall \underline{\mathbf{t}}_h \in \underline{\mathbf{S}}_h^k. \tag{50}$$

We can then state the following result, whose proof hinges on Lemma 4.1, and which is a slightly modified version of [27, Lemma 5].

Lemma 4.5 (Well-posedness of (46))**.** *For all $v_h \in U_h^k$, the following holds:*

$$\mu_\flat^{1/2} \|v_h\| \lesssim \sup_{\underline{\mathbf{t}}_h \in \underline{\mathbf{S}}_{h,0}^k, \|\underline{\mathbf{t}}_h\|_{\boldsymbol{S},h}=1} (D_h^k \underline{\mathbf{t}}_h, v_h). \tag{51}$$

Combining (51) with Lemma 4.4, we infer that problem (46) is well-posed.

4.2.3 Error estimates

Let $(\boldsymbol{s}, u) \in \boldsymbol{S} \times V$ be such that $(\boldsymbol{s}, u) = (\boldsymbol{s}_0 + \boldsymbol{s}_\partial, u)$, where $(\boldsymbol{s}_0, u) \in \boldsymbol{S}_0 \times V$ is the (unique) solution to (34), and $\boldsymbol{s}_\partial \in \boldsymbol{S}$ is defined above. We further assume that $\boldsymbol{s} \in \boldsymbol{S}$ fulfills the additional regularity $\boldsymbol{s} \in \boldsymbol{S}^+(\mathcal{T}_h)$. Similarly, let $(\underline{\mathbf{s}}_h, u_h) \in \underline{\mathbf{S}}_h^k \times U_h^k$ be such that $(\underline{\mathbf{s}}_h, u_h) = (\underline{\mathbf{s}}_{h,0} + \underline{\mathbf{s}}_{h,\partial}, u_h)$, where $(\underline{\mathbf{s}}_{h,0}, u_h) \in \underline{\mathbf{S}}_{h,0}^k \times U_h^k$ is the (unique) solution to (46), and $\underline{\mathbf{s}}_{h,\partial} \in \underline{\mathbf{S}}_h^k$ is defined in Section 4.2.1. Finally, let us introduce the notation $\|\cdot\|_h := H_h(\cdot,\cdot)^{1/2}$. Then, we can state the following result, whose proof can be easily adapted from the one of [27, Theorem 6]. Note that, here again, the constants in the error bounds can depend on the polynomial degree following the use of discrete trace and inverse inequalities.

Theorem 4.1 (Error estimate for the flux)**.** *Assume the additional regularity $u \in H^{k+2}(P_\Omega)$ (so that, by Definition 2.2, $u \in H^{k+2}(\mathcal{T}_h)$). Then, the following holds:*

$$\|\underline{\mathbf{I}}_h^k \boldsymbol{s} - \underline{\mathbf{s}}_h\|_{\boldsymbol{S},h} \lesssim \|\underline{\mathbf{I}}_h^k \boldsymbol{s} - \underline{\mathbf{s}}_h\|_h \lesssim \left\{ \sum_{T \in \mathcal{T}_h} \mu_{\sharp,T} \rho_T h_T^{2(k+1)} \|u\|_{H^{k+2}(T)}^2 \right\}^{1/2}, \tag{52}$$

which implies, by an additional use of Lemma 4.3,

$$\|\mathbb{M}^{-1/2}(\boldsymbol{s} - \boldsymbol{F}_h^{k+1}\underline{\mathsf{s}}_h)\| \lesssim \left\{ \sum_{T \in \mathcal{T}_h} \mu_{\sharp,T}\rho_T h_T^{2(k+1)} \|u\|_{H^{k+2}(T)}^2 \right\}^{1/2}. \tag{53}$$

Whenever elliptic regularity holds, a supercloseness result for the potential can be established, as an adaptation of [27, Theorem 7].

Theorem 4.2 (Supercloseness of the potential)**.** *Assume elliptic regularity for problem (3) under the form $\|z\|_{H^2(P_\Omega)} \lesssim \mu_\flat^{-1}\|g\|$ for all $g \in L^2(\Omega)$ and $z \in U_0$ solving (3) with data g and homogeneous (mixed Dirichlet-Neumann) boundary conditions. Assume $f \in H^{k+\delta}(\Omega), \phi_\partial \in W^{k+\delta,\infty}(\Gamma_{\mathrm{n}})$, with $\delta = 0$ for $k \geqslant 1$ and $\delta = 1$ for $k = 0$. Then, under the same assumption on u as in Theorem 4.1, the following holds:*

$$\mu_\flat\|I_h^k u - u_h\| \lesssim \mu_\sharp^{1/2}\rho^{1/2} h \left\{ \sum_{T \in \mathcal{T}_h} \mu_{\sharp,T}\rho_T h_T^{2(k+1)} \|u\|_{H^{k+2}(T)}^2 \right\}^{1/2} + h^{k+2}\left\{ \|f\|_{H^{k+\delta}(\Omega)} + \|\phi_\partial\|_{W^{k+\delta,\infty}(\Gamma_{\mathrm{n}})} \right\}. \tag{54}$$

4.3 Static condensation

There are two ways of reducing the size of the discrete problem (46).

First, as exposed in [27, Section 3.4], it is possible to eliminate locally the cell-based flux unknowns and the potential unknowns, up to one constant value per element. Thus, the global system to solve only writes in terms of the face-based flux unknowns and of the mean value of the potential in each element. For all $T \in \mathcal{T}_h$, let $U_T^{k,0}$ be the space of d-variate polynomials of degree at most k having zero mean value in T, so that $U_T^k = U_T^0 \oplus U_T^{k,0}$. Hence, any function $v_T \in U_T^k$ can be written $v_T = v_T^0 + \hat{v}_T$ with $v_T^0 \in U_T^0$ and $\hat{v}_T \in U_T^{k,0}$. Then, we infer from (46) that, for all $T \in \mathcal{T}_h$, $(\boldsymbol{s}_T, \hat{u}_T) \in \boldsymbol{S}_T^k \times U_T^{k,0}$ can be eliminated locally by solving the following saddle point problem with right-hand side depending on $\mathfrak{s}_{\partial T} \in \mathfrak{S}_{\partial T}^k$ and $f_T := f_{|T}$:

$$\begin{aligned} \hat{H}_T(\boldsymbol{s}_T, \boldsymbol{t}_T) - (\boldsymbol{t}_T, \boldsymbol{\nabla}\hat{u}_T)_T &= -H_T((\boldsymbol{0}, \mathfrak{s}_{\partial T}), (\boldsymbol{t}_T, 0)) && \forall\, \boldsymbol{t}_T \in \boldsymbol{S}_T^k, \\ (\boldsymbol{s}_T, \boldsymbol{\nabla}\hat{v}_T)_T &= (f_T, \hat{v}_T)_T + \sum_{F \in \mathcal{F}_T} (\mathfrak{s}_F \varepsilon_{T,F}, \hat{v}_T)_F && \forall\, \hat{v}_T \in U_T^{k,0}, \end{aligned} \tag{55}$$

where $\hat{H}_T(\boldsymbol{s}_T, \boldsymbol{t}_T) := H_T((\boldsymbol{s}_T, 0), (\boldsymbol{t}_T, 0))$. Problem (55) is the counterpart in a mixed context of problem (26a) obtained in the primal context; the further splitting of (55) leading to datum- and trace-based lifting operators is omitted for brevity. Problem (55) is well-posed, since, according to (49) and (48), $\hat{H}_T(\boldsymbol{t}_T, \boldsymbol{t}_T)$ is uniformly equivalent to $\|\boldsymbol{t}_T\|_T^2$ and the inf-sup condition holds. The global (saddle point) problem resulting from the local elimination (55) has the same size and structure as that obtained with the Multiscale Hybrid-Mixed (MHM) method derived in [3, 40] on simplicial meshes.

The second static condensation approach is based on a reformulation of the mixed problem (46) into a primal problem. Following [1, Section 3.3], the reformulation is based on the introduction of Lagrange multipliers that enforce the continuity of interface-based flux unknowns and that can be interpreted as potential traces on mesh faces. One can eliminate the cell- and face-based flux unknowns, and, once the reformulation has been performed, one can adapt the arguments of Section 3.3.1 to further eliminate locally the cell-based potential unknowns, ending up with a global system only depending on the Lagrange multipliers (face-based potential unknowns). This static condensation approach has the double advantage that it requires to solve local coercive problems (as opposed to local saddle point problems) and that it yields a coercive global problem. For this reason, we discuss it in more detail in Section 5.

5 Bridging the primal and mixed forms of the HHO method

The goal of this section is to bridge the primal and mixed forms of the HHO method studied in Sections 3 and 4, respectively. As discussed in the previous section, this can be exploited in practice to implement the mixed form of the HHO method in terms of a coercive problem posed on the Lagrange multipliers only.

5.1 Unpatching interface-based flux unknowns

We introduce a global set of hybrid flux unknowns where interface-based unknowns are two-valued; we refer to these unknowns as unpatched. The unpatched global set of hybrid flux unknowns is defined as

$$\check{\underline{\mathbf{S}}}_h^k := \bigtimes_{T\in\mathcal{T}_h} \underline{\mathbf{S}}_T^k,$$

with subset

$$\check{\underline{\mathbf{S}}}_{h,0}^k := \left\{ \check{\underline{\mathsf{t}}}_h \in \check{\underline{\mathbf{S}}}_h^k \mid \check{\mathsf{t}}_F \equiv 0, \forall F \in \mathcal{F}_h^{\mathrm{n}} \right\}. \tag{56}$$

Given an element $\check{\underline{\mathsf{t}}}_h \in \check{\underline{\mathbf{S}}}_h^k$, for any $T \in \mathcal{T}_h$, we denote by $\check{\underline{\mathsf{t}}}_T := (\check{\boldsymbol{t}}_T, (\check{\mathsf{t}}_{T,F})_{F\in\mathcal{F}_T}) \in \underline{\mathbf{S}}_T^k$ its restriction to the element T. For boundary faces $F \in \mathcal{F}_h^{\mathrm{b}}$, the subscript T in $\check{\mathsf{t}}_{T,F}$ can be omitted, and we simply write $\check{\mathsf{t}}_F$, as we already did in (56).

Let us introduce the following subspace of $\check{\underline{\mathbf{S}}}_h^k$ (respectively, $\check{\underline{\mathbf{S}}}_{h,0}^k$):

$$\check{\underline{\mathbf{Z}}}_{h(,0)}^k := \left\{ \check{\underline{\mathsf{t}}}_h \in \check{\underline{\mathbf{S}}}_{h(,0)}^k \mid \sum_{T\in\mathcal{T}_F} \check{\mathsf{t}}_{T,F} = 0, \forall F \in \mathcal{F}_h^{\mathrm{i}} \right\}.$$

It can be easily seen that there exists a natural isomorphism $\underline{\mathbf{J}}_h^k$ from $\check{\underline{\mathbf{Z}}}_h^k$ onto the space $\underline{\mathbf{S}}_h^k$. Note that the restriction of $\underline{\mathbf{J}}_h^k$ to $\check{\underline{\mathbf{Z}}}_{h,0}^k$ defines an isomorphism onto $\underline{\mathbf{S}}_{h,0}^k$.

5.2 Unpatched mixed formulation

We begin by extending to $\check{\underline{\mathbf{S}}}_h^k$ the definitions, respectively built from (36) and (38), of the divergence reconstruction operator D_h^k and of the flux reconstruction operator $\boldsymbol{F}_h^{k+1}$, for which we keep the same notation (locally, the definitions are unchanged up to the replacement of

$\mathfrak{t}_F\varepsilon_{T,F}$ by $\check{\mathfrak{t}}_{T,F}$ in face terms). We can then naturally extend the bilinear form H_h, defined in (45) and built from (44), to the product space $\check{\underline{\mathbf{S}}}_h^k \times \check{\underline{\mathbf{S}}}_h^k$.

We next introduce, for all $T \in \mathcal{T}_h$, the additional bilinear form

$$B_T : \underline{\mathbf{S}}_T^k \times \underline{\mathrm{U}}_T^k \to \mathbb{R}; \qquad (\check{\underline{\mathbf{t}}}_T, \underline{\mathrm{v}}_T) \mapsto (v_T, D_T^k \check{\underline{\mathbf{t}}}_T)_T - \sum_{F \in \mathcal{F}_T \cap \mathcal{F}_h^{\mathrm{i}}} (\mathfrak{v}_F, \check{\mathfrak{t}}_{T,F})_F, \tag{57}$$

whose global version is as usual obtained by element-by-element assembly:

$$B_h : \check{\underline{\mathbf{S}}}_h^k \times \underline{\mathrm{U}}_h^k \to \mathbb{R}; \qquad (\check{\underline{\mathbf{t}}}_h, \underline{\mathrm{v}}_h) \mapsto \sum_{T \in \mathcal{T}_h} B_T(\check{\underline{\mathbf{t}}}_T, \underline{\mathrm{v}}_T).$$

This bilinear form includes interface terms that enforce the single-valuedness constraints for interface-based flux unknowns. In that vision, the face-based potential unknowns can be seen as Lagrange multipliers.

The unpatched (mixed) HHO discretization of problem (34) then reads: Find $(\check{\underline{\mathbf{s}}}_{h,0}, \check{\underline{\mathrm{u}}}_{h,0}) \in \check{\underline{\mathbf{S}}}_{h,0}^k \times \underline{\mathrm{U}}_{h,0}^k$ such that, for all $(\check{\underline{\mathbf{t}}}_h, \underline{\mathrm{v}}_h) \in \check{\underline{\mathbf{S}}}_{h,0}^k \times \underline{\mathrm{U}}_{h,0}^k$,

$$H_h(\check{\underline{\mathbf{s}}}_{h,0}, \check{\underline{\mathbf{t}}}_h) + B_h(\check{\underline{\mathbf{t}}}_h, \check{\underline{\mathrm{u}}}_{h,0}) = \sum_{F \in \mathcal{F}_h^{\mathrm{d}}} (\check{\mathfrak{t}}_F, \psi_\partial)_F - H_h(\check{\underline{\mathbf{s}}}_{h,\partial}, \check{\underline{\mathbf{t}}}_h) - B_h(\check{\underline{\mathbf{t}}}_h, \underline{\mathrm{u}}_{h,\partial}), \tag{58a}$$

$$-B_h(\check{\underline{\mathbf{s}}}_{h,0}, \underline{\mathrm{v}}_h) = (f, v_h) + B_h(\check{\underline{\mathbf{s}}}_{h,\partial}, \underline{\mathrm{v}}_h), \tag{58b}$$

where $\check{\underline{\mathbf{s}}}_{h,\partial} := (\underline{\mathbf{J}}_h^k)^{-1}(\underline{\mathbf{s}}_{h,\partial}) \in \check{\underline{\mathbf{Z}}}_h^k$ is such that, for all $T \in \mathcal{T}_h$, $\check{\underline{\mathbf{s}}}_{T,\partial} = (\boldsymbol{s}_{T,\partial}, (\mathfrak{s}_{F,\partial}\varepsilon_{T,F})_{F \in \mathcal{F}_T})$, with $\underline{\mathbf{s}}_{h,\partial} \in \underline{\mathbf{S}}_h^k$ defined in Section 4.2.1, and where $\underline{\mathrm{u}}_{h,\partial}$ is defined in Section 3.2.1. Finally, we define

$$(\check{\underline{\mathbf{s}}}_h, \check{\underline{\mathrm{u}}}_h) := (\check{\underline{\mathbf{s}}}_{h,0} + \check{\underline{\mathbf{s}}}_{h,\partial}, \check{\underline{\mathrm{u}}}_{h,0} + \underline{\mathrm{u}}_{h,\partial}) \in \check{\underline{\mathbf{S}}}_h^k \times \underline{\mathrm{U}}_h^k. \tag{59}$$

5.3 Equivalence between primal and mixed formulations

The bridge between primal- and mixed-form HHO methods is built in two steps: first, we prove the equivalence between the mixed and unpatched mixed formulations; then, we prove that the unpatched mixed formulation can be recast into a primal formulation.

The following result is an adaptation of [1, Lemma 3.3].

Theorem 5.1 (Equivalence (46)-(58))**.** *Denote by $(\underline{\mathbf{s}}_{h,0}, u_h) \in \underline{\mathbf{S}}_{h,0}^k \times U_h^k$ and $(\check{\underline{\mathbf{s}}}_{h,0}, \check{\underline{\mathrm{u}}}_{h,0}) \in \check{\underline{\mathbf{S}}}_{h,0}^k \times \underline{\mathrm{U}}_{h,0}^k$ the solutions to (46) and (58), respectively. Then, $\check{\underline{\mathbf{s}}}_{h,0} \in \check{\underline{\mathbf{Z}}}_{h,0}^k$ and $\underline{\mathbf{s}}_{h,0} = \underline{\mathbf{J}}_h^k(\check{\underline{\mathbf{s}}}_{h,0})$, so that $\underline{\mathbf{s}}_h = \underline{\mathbf{J}}_h^k(\check{\underline{\mathbf{s}}}_h)$ ($\underline{\mathbf{s}}_h$ and $\check{\underline{\mathbf{s}}}_h$ are defined in (47) and (59), respectively); furthermore, $u_h = \check{u}_h$ (recall that $\check{u}_h$ denotes the cell-based part of $\check{\underline{\mathrm{u}}}_h$, defined in (59)).*

Following [1, Section 3.3], let us now introduce, for all $T \in \mathcal{T}_h$, the local potential-to-flux mapping operator $\underline{\boldsymbol{\varsigma}}_T^k : \underline{\mathrm{U}}_T^k \to \underline{\mathbf{S}}_T^k$ such that, for all $\underline{\mathrm{v}}_T \in \underline{\mathrm{U}}_T^k$,

$$H_T(\underline{\boldsymbol{\varsigma}}_T^k \underline{\mathrm{v}}_T, \check{\underline{\mathbf{t}}}_T) = -B_T(\check{\underline{\mathbf{t}}}_T, \underline{\mathrm{v}}_T) + \sum_{F \in \mathcal{F}_T \cap \mathcal{F}_h^{\mathrm{b}}} (\check{\mathfrak{t}}_F, \mathfrak{v}_F)_F \qquad \forall \check{\underline{\mathbf{t}}}_T \in \underline{\mathbf{S}}_T^k. \tag{60}$$

This yields a well-posed problem owing to the first inequality in (49). Defining next another local flux reconstruction operator $\check{\boldsymbol{F}}_T^{k+1} : \underline{\mathrm{U}}_T^k \to \boldsymbol{S}_T^{k+1}$ such that

$$\check{\boldsymbol{F}}_T^{k+1} := \boldsymbol{F}_T^{k+1} \circ \underline{\check{\varsigma}}_T^k, \tag{61}$$

one can prove the following result.

Lemma 5.1 (Link between $\boldsymbol{F}_T^{k+1}$ and p_T^{k+1})**.** *For all $\underline{\mathrm{v}}_T \in \underline{\mathrm{U}}_T^k$, the following holds:*

$$\check{\boldsymbol{F}}_T^{k+1} \underline{\mathrm{v}}_T = \mathbb{M}_T \boldsymbol{\nabla} p_T^{k+1} \underline{\mathrm{v}}_T. \tag{62}$$

Proof. Let $\underline{\mathrm{v}}_T \in \underline{\mathrm{U}}_T^k$, and let us plug, for $w \in \mathbb{P}_d^{k+1}(T)$, $\underline{\check{\mathfrak{t}}}_T := \underline{\mathbf{I}}_T^k(\mathbb{M}_T \boldsymbol{\nabla} w)$ into (60). Using (57), (36), the polynomial consistency property of Lemma 4.2 coupled to (61), and the one of (42), we get

$$(\check{\boldsymbol{F}}_T^{k+1} \underline{\mathrm{v}}_T, \boldsymbol{\nabla} w)_T = (\boldsymbol{\nabla} v_T, \mathbb{M}_T \boldsymbol{\nabla} w)_T + \sum_{F \in \mathcal{F}_T} (\mathfrak{v}_F - v_T, \mathbb{M}_T \boldsymbol{\nabla} w \cdot \boldsymbol{n}_{T,F})_F, \tag{63}$$

where we have used that $(\check{\boldsymbol{t}}_T, \boldsymbol{\nabla} v_T)_T = (\mathbb{M}_T \boldsymbol{\nabla} w, \boldsymbol{\nabla} v_T)_T$ and $\check{\mathfrak{t}}_{T,F} = \mathbb{M}_T \boldsymbol{\nabla} w \cdot \boldsymbol{n}_{T,F}$, owing to (35) and to the fact that $w \in \mathbb{P}_d^{k+1}(T)$. Finally, performing a last integration by parts in (63), and comparing to the definition (6) of p_T^{k+1}, we prove (62). $\square$

Now, defining $\underline{\check{\varsigma}}_h^k : \underline{\mathrm{U}}_h^k \to \underline{\check{\mathbf{S}}}_h^k$ such that, for all $\underline{\mathrm{v}}_h \in \underline{\mathrm{U}}_h^k$, and for all $T \in \mathcal{T}_h$, $(\underline{\check{\varsigma}}_h^k \underline{\mathrm{v}}_h)_{|T} := \underline{\check{\varsigma}}_T^k \underline{\mathrm{v}}_T$, we infer from (60) that

$$H_h(\underline{\check{\varsigma}}_h^k \underline{\check{\mathrm{u}}}_h, \underline{\check{\mathfrak{t}}}_h) = -B_h(\underline{\check{\mathfrak{t}}}_h, \underline{\check{\mathrm{u}}}_h) + \sum_{F \in \mathcal{F}_h^{\mathrm{d}}} (\check{\mathfrak{t}}_F, \psi_\partial)_F \qquad \forall\, \underline{\check{\mathfrak{t}}}_h \in \underline{\check{\mathbf{S}}}_{h,0}^k, \tag{64}$$

where we have used the fact that $\check{\mathfrak{t}}_F \equiv 0$ for all $F \in \mathcal{F}_h^{\mathrm{n}}$ and that $\check{\mathrm{u}}_F = \pi_F^k \psi_\partial$ for all $F \in \mathcal{F}_h^{\mathrm{d}}$. Comparing (64) with (58a), it is readily inferred that $\underline{\check{\mathfrak{s}}}_h = \underline{\check{\varsigma}}_h^k \underline{\check{\mathrm{u}}}_h$. Plugging this relation into (58b), we get that

$$-B_h(\underline{\check{\varsigma}}_h^k \underline{\check{\mathrm{u}}}_h, \underline{\mathrm{v}}_h) = (f, v_h) \qquad \forall \underline{\mathrm{v}}_h \in \underline{\mathrm{U}}_{h,0}^k. \tag{65}$$

Using again (60), we additionally prove that

$$H_h(\underline{\check{\varsigma}}_h^k \underline{\mathrm{v}}_h, \underline{\check{\varsigma}}_h^k \underline{\check{\mathrm{u}}}_h) = -B_h(\underline{\check{\varsigma}}_h^k \underline{\check{\mathrm{u}}}_h, \underline{\mathrm{v}}_h) + \sum_{F \in \mathcal{F}_h^{\mathrm{n}}} (\check{\mathfrak{s}}_F, \mathfrak{v}_F)_F \qquad \forall\, \underline{\mathrm{v}}_h \in \underline{\mathrm{U}}_{h,0}^k, \tag{66}$$

where we have used the fact that $\mathfrak{v}_F \equiv 0$ for all $F \in \mathcal{F}_h^{\mathrm{d}}$. Plugging (66) into (65), using the symmetry of H_h, the decomposition $\underline{\check{\mathrm{u}}}_h = \underline{\check{\mathrm{u}}}_{h,0} + \underline{\mathrm{u}}_{h,\partial}$, and the fact that $\check{\mathfrak{s}}_F = \pi_F^k \phi_\partial$ for all $F \in \mathcal{F}_h^{\mathrm{n}}$, we obtain

$$H_h(\underline{\check{\varsigma}}_h^k \underline{\check{\mathrm{u}}}_{h,0}, \underline{\check{\varsigma}}_h^k \underline{\mathrm{v}}_h) = (f, v_h) - H_h(\underline{\check{\varsigma}}_h^k \underline{\mathrm{u}}_{h,\partial}, \underline{\check{\varsigma}}_h^k \underline{\mathrm{v}}_h) + \sum_{F \in \mathcal{F}_h^{\mathrm{n}}} (\phi_\partial, \mathfrak{v}_F)_F \qquad \forall\, \underline{\mathrm{v}}_h \in \underline{\mathrm{U}}_{h,0}^k. \tag{67}$$

Finally, introducing the global bilinear form $A_h : \underline{\mathrm{U}}_h^k \times \underline{\mathrm{U}}_h^k \to \mathbb{R}$ such that $A_h(\underline{\mathrm{u}}_h, \underline{\mathrm{v}}_h) := H_h(\underline{\check{\varsigma}}_h^k \underline{\mathrm{u}}_h, \underline{\check{\varsigma}}_h^k \underline{\mathrm{v}}_h)$, problem (67) can be rewritten under the form

$$A_h(\underline{\check{\mathrm{u}}}_{h,0}, \underline{\mathrm{v}}_h) = (f, v_h) - A_h(\underline{\mathrm{u}}_{h,\partial}, \underline{\mathrm{v}}_h) + \sum_{F \in \mathcal{F}_h^{\mathrm{n}}} (\phi_\partial, \mathfrak{v}_F)_F \qquad \forall\, \underline{\mathrm{v}}_h \in \underline{\mathrm{U}}_{h,0}^k. \tag{68}$$

Using (45), (44), (61), and (62), we also infer that

$$A_h(\underline{\mathrm{u}}_h, \underline{\mathrm{v}}_h) = \sum_{T\in\mathcal{T}_h} (\mathbb{M}_T \boldsymbol{\nabla} p_T^{k+1}\underline{\mathrm{u}}_T, \boldsymbol{\nabla} p_T^{k+1}\underline{\mathrm{v}}_T)_T + \sum_{T\in\mathcal{T}_h} J_T(\check{\underline{\boldsymbol{\varsigma}}}_T^k\underline{\mathrm{u}}_T, \check{\underline{\boldsymbol{\varsigma}}}_T^k\underline{\mathrm{v}}_T). \tag{69}$$

Finally, owing to (69), the comparison of problem (68) to problem (14) allows to infer the following result, cf. [1, Section 3.3.4].

Theorem 5.2 (Equivalence (14)-(58))**.** *Let us denote by $\underline{\mathrm{u}}_{h,0} \in \underline{\mathrm{U}}_{h,0}^k$ and $(\check{\underline{\mathbf{s}}}_{h,0}, \check{\underline{\mathrm{u}}}_{h,0}) \in \check{\underline{\mathbf{S}}}_{h,0}^k \times \underline{\mathrm{U}}_{h,0}^k$ the solutions to (14) and (58), respectively. Then, up to a choice of stabilization $j_T(\cdot,\cdot) := J_T(\check{\underline{\boldsymbol{\varsigma}}}_T^k\cdot, \check{\underline{\boldsymbol{\varsigma}}}_T^k\cdot)$ in (13) for problem (14), $\underline{\mathrm{u}}_{h,0} = \check{\underline{\mathrm{u}}}_{h,0}$, so that $\underline{\mathrm{u}}_h = \check{\underline{\mathrm{u}}}_h$ ($\underline{\mathrm{u}}_h$ and $\check{\underline{\mathrm{u}}}_h$ are defined in (15) and (59), respectively).*

The combination of Theorems 5.1 and 5.2 states the equivalence between primal- and mixed-form HHO methods, up to an appropriate choice of stabilization.

From a practical point of view, to compute the solution $(\underline{\mathbf{s}}_{h,0}, u_h)$ of the mixed problem (46), it suffices to solve the coercive global problem (68) (once the operator $\check{\underline{\boldsymbol{\varsigma}}}_h^k$ has been computed solving (60) locally in each element) and to use the relation $(\underline{\mathbf{s}}_{h,0} + \underline{\mathbf{s}}_{h,\partial}, u_h) = (\underline{\mathbf{J}}_h^k(\check{\underline{\boldsymbol{\varsigma}}}_h^k\check{\underline{\mathrm{u}}}_h), \check{u}_h)$ combined with $\check{\underline{\mathrm{u}}}_h = \check{\underline{\mathrm{u}}}_{h,0} + \underline{\mathrm{u}}_{h,\partial}$. Adapting the arguments of Section 3.3.1, static condensation can be performed on problem (68), hence leading to a global problem expressed in terms of Lagrange multipliers (face-based potential unknowns) only.

6 Conclusion and perspectives

HHO methods are very recent polytopal discretization methods which, by now, rest on a firm theoretical basis for elliptic PDEs in primal and mixed forms. Advantages offered by HHO methods are a dimension-independent construction, local conservativity, the possibility to consider an arbitrary polynomial order, a natural treatment of variable diffusion coefficients, and tight computational costs in particular owing to static condensation and an offline/online decomposition of the solution procedure. The price to pay is, on the one hand, the need to solve local problems in the assembly phase (numerical experiments indicate, however, that the relative cost with respect to solving the global problem swiftly decreases as mesh resolution increases). On the other hand, HHO methods are essentially nonconforming (as DG methods) so that some post-processing of the discrete solution may be useful when visualizing the solution on coarse meshes (on fine meshes, the jumps swiftly converge to zero). Note, however, that contrary to interior penalty DG methods, the stabilization does not require user-dependent parameters that must be large enough. Expanding the HHO methodology to systems of quasi-linear or even nonlinear PDEs poses new challenges. Encouraging results (in the linear case) include the robustness with respect to Péclet number in case of advection-diffusion and with respect to incompressibility in linear elasticity, while a nonlinear Leray–Lions problem is addressed in [24]. Another attractive potential application of HHO methods is in the context of multiscale problems, where adequate local problems that take into account the small scales of the problem can be coupled through a global problem posed on a coarse mesh.

References

[1] J. Aghili, S. Boyaval, and D. A. Di Pietro. Hybridization of mixed high-order methods on general meshes and application to the Stokes equations. *Comput. Meth. Appl. Math.*, 15(2):111–134, 2015.

[2] P. F. Antonietti, S. Giani, and P. Houston. *hp*-version composite discontinuous Galerkin methods for elliptic problems on complicated domains. *SIAM J. Sci. Comput*, 35(3):A1417–A1439, 2013.

[3] R. Araya, C. Harder, D. Paredes, and F. Valentin. Multiscale hybrid-mixed method. *SIAM J. Numer. Anal.*, 51(6):3505–3531, 2013.

[4] D. N. Arnold. An interior penalty finite element method with discontinuous elements. *SIAM J. Numer. Anal.*, 19:742–760, 1982.

[5] D. N. Arnold, F. Brezzi, B. Cockburn, and L. D. Marini. Unified analysis of discontinuous Galerkin methods for elliptic problems. *SIAM J. Numer. Anal.*, 39(5):1749–1779, 2002.

[6] B. Ayuso de Dios, K. Lipnikov, and G. Manzini. The nonconforming virtual element method. Submitted. Preprint arXiv:1405.3741, May 2014.

[7] F. Bassi, L. Botti, A. Colombo, D. A. Di Pietro, and P. Tesini. On the flexibility of agglomeration based physical space discontinuous Galerkin discretizations. *J. Comput. Phys.*, 231(1):45–65, 2012.

[8] L. Beirão da Veiga, F. Brezzi, and L.D. Marini. Virtual elements for linear elasticity problems. *SIAM J. Numer. Anal.*, 51(2):794–812, 2013.

[9] L. Beirão da Veiga, K. Lipnikov, and G. Manzini. *The Mimetic Finite Difference Method for Elliptic Problems*, volume 11 of *MS&A*. Springer, New York, 2014.

[10] J. Bonelle, D. A. Di Pietro, and A. Ern. Low-order reconstruction operators on polyhedral meshes: Application to Compatible Discrete Operator schemes. *Computer Aided Geometric Design*, 35–36:27–41, 2015.

[11] J. Bonelle and A. Ern. Analysis of compatible discrete operator schemes for elliptic problems on polyhedral meshes. *M2AN Math. Model. Numer. Anal.*, 48:553–581, 2014.

[12] J. Bonelle and A. Ern. Analysis of Compatible Discrete Operator schemes for the Stokes equations on polyhedral meshes. *IMA J. Numer. Anal.*, 2014. Published online. DOI: 10.1093/imanum/dru051.

[13] A. Bossavit. Computational electromagnetism and geometry. *J. Japan Soc. Appl. Electromagn. & Mech.*, 7-8:150–9 (no 1), 294–301 (no 2), 401–8 (no 3), 102–9 (no 4), 203–9 (no 5), 372–7 (no 6), 1999-2000.

[14] F. Brezzi, K. Lipnikov, and M. Shashkov. Convergence of the mimetic finite difference method for diffusion problems on polyhedral meshes. *SIAM J. Numer. Anal.*, 43(5):1872–1896, 2005.

[15] F. Brezzi, K. Lipnikov, and V. Simoncini. A family of mimetic finite difference methods on polygonal and polyhedral meshes. *M3AS Math. Models Methods Appl. Sci.*, 15(10):1533–1551, 2005.

[16] F. Brezzi, G. Manzini, L. D. Marini, P. Pietra, and A. Russo. Discontinuous Galerkin approximations for elliptic problems. *Numer. Methods Partial Differential Equations*, 16(4):365–378, 2000.

[17] F. Brezzi and L. D. Marini. Virtual elements for plate bending problems. *Comput. Methods Appl. Mech. Engrg.*, 253:455–462, 2013.

[18] A. Cangiani, E. H. Georgoulis, and P. Houston. *hp*-Version discontinuous Galerkin methods on polygonal and polyhedral meshes. *Math. Models Methods Appl. Sci. (M3AS)*, 24(10):2009–2041, 2014.

[19] P. Castillo, B. Cockburn, I. Perugia, and D. Schötzau. An a priori error analysis of the local discontinuous Galerkin method for elliptic problems. *SIAM J. Numer. Anal.*, 38:1676–1706, 2000.

[20] B. Cockburn, D. A. Di Pietro, and A. Ern. Bridging the Hybrid High-Order and Hybridizable Discontinuous Galerkin methods. *ESAIM: Math. Model Numer. Anal. (M2AN)*, 2015. Published online. DOI: 10.1051/m2an/2015051.

[21] B. Cockburn, J. Gopalakrishnan, and R. Lazarov. Unified hybridization of discontinuous Galerkin, mixed, and continuous Galerkin methods for second-order elliptic problems. *SIAM J. Numer. Anal.*, 47(2):1319–1365, 2009.

[22] L. Codecasa, R. Specogna, and F. Trevisan. A new set of basis functions for the discrete geometric approach. *J. Comput. Phys.*, 19(299):7401–7410, 2010.

[23] D. A. Di Pietro. Cell-centered Galerkin methods for diffusive problems. *M2AN Math. Model. Numer. Anal.*, 46(1):111–144, 2012.

[24] D. A. Di Pietro and J. Droniou. A Hybrid High-Order method for Leray–Lions elliptic equations on general meshes. Submitted. Preprint arXiv:1508.01918, August 2015.

[25] D. A. Di Pietro, J. Droniou, and A. Ern. A discontinuous-skeletal method for advection-diffusion-reaction on general meshes. *SIAM J. Numer. Anal.*, 2015. Published online. DOI: 10.1137/140993971.

[26] D. A. Di Pietro and A. Ern. *Mathematical Aspects of Discontinuous Galerkin Methods*, volume 69 of *Mathématiques & Applications*. Springer, Berlin Heidelberg, 2012.

[27] D. A. Di Pietro and A. Ern. Arbitrary-order mixed methods for heterogeneous anisotropic diffusion on general meshes, 2013. Preprint hal-00918482.

[28] D. A. Di Pietro and A. Ern. Equilibrated tractions for the Hybrid High-Order method. *C. R. Acad. Sci. Paris, Ser. I*, 353:279–282, 2015.

[29] D. A. Di Pietro and A. Ern. A Hybrid High-Order locking-free method for linear elasticity on general meshes. *Comput. Meth. Appl. Mech. Engrg.*, 283:1–21, 2015.

[30] D. A. Di Pietro and A. Ern. Hybrid High-Order methods for variable-diffusion problems on general meshes. *C. R. Acad. Sci Paris, Ser. I*, 353:31–34, 2015.

[31] D. A. Di Pietro, A. Ern, and S. Lemaire. An arbitrary-order and compact-stencil discretization of diffusion on general meshes based on local reconstruction operators. *Comput. Methods Appl. Math.*, 14(4):461–472, 2014.

[32] D. A. Di Pietro and S. Lemaire. An extension of the Crouzeix–Raviart space to general meshes with application to quasi-incompressible linear elasticity and Stokes flow. *Math. Comp.*, 84:1–31, 2015.

[33] K. Domelevo and P. Omnes. A finite volume method for the Laplace equation on almost arbitrary two-dimensional grids. *ESAIM: Mathematical Modelling and Numerical Analysis*, 39(6):1203–1249, 2005.

[34] J. Droniou and R. Eymard. A mixed finite volume scheme for anisotropic diffusion problems on any grid. *Numer. Math.*, 105:35–71, 2006.

[35] J. Droniou, R. Eymard, T. Gallouët, and R. Herbin. A unified approach to mimetic finite difference, hybrid finite volume and mixed finite volume methods. *M3AS Mathematical Models and Methods in Applied Sciences*, 20(2):1–31, 2010.

[36] J. Droniou, R. Eymard, T. Gallouët, and R. Herbin. Gradient schemes: a generic framework for the discretisation of linear, nonlinear and nonlocal elliptic and parabolic equations. *Math. Models Methods Appl. Sci. (M3AS)*, 23:2395–2432, 2013.

[37] A. Ern and J.-L. Guermond. Discontinuous Galerkin methods for Friedrichs' systems. I. General theory. *SIAM J. Numer. Anal.*, 44(2):753–778, 2006.

[38] R. Eymard, T. Gallouët, and R. Herbin. Finite volume methods. In P. G. Ciarlet and J.-L. Lions, editors, *Handbook of Numerical Analysis, Vol. 7 - Techniques of Scientific Computing (Part III)*, pages 713–1020. 2000.

[39] R. Eymard, T. Gallouët, and R. Herbin. Discretization of heterogeneous and anisotropic diffusion problems on general nonconforming meshes. SUSHI: a scheme using stabilization and hybrid interfaces. *IMA J. Numer. Anal.*, 30(4):1009–1043, 2010.

[40] C. Harder, D. Paredes, and F. Valentin. A family of multiscale hybrid-mixed finite element methods for the Darcy equation with rough coefficients. *J. Comput. Phys.*, 245:107–130, 2013.

[41] R. Herbin and F. Hubert. Benchmark on discretization schemes for anisotropic diffusion problems on general grids. In R. Eymard and J.-M. Hérard, editors, *Finite Volumes for Complex Applications V*, pages 659–692. John Wiley & Sons, 2008.

[42] C. Lehrenfeld. *Hybrid Discontinuous Galerkin methods for solving incompressible flow problems*. PhD thesis, Rheinisch-Westfälischen Technischen Hochschule Aachen, 2010.

[43] K. Lipnikov and G. Manzini. A high-order mimetic method on unstructured polyhedral meshes for the diffusion equation. *J. Comput. Phys.*, 272:360–385, 2014.

[44] E. Tonti. On the formal structure of physical theories. *Quaderni dei Gruppi di Ricerca Matematica del CNR*, 1975.

[45] J. Wang and X. Ye. A weak Galerkin element method for second-order elliptic problems. *J. Comput. Appl. Math.*, 241:103–115, 2013.

[46] J. Wang and X. Ye. A weak Galerkin mixed finite element method for second-order elliptic problems. *Math. Comp.*, 83(289):2101–2126, 2014.