Klt singularities of horospherical pairs

Boris Pasquier

September 30, 2015

Abstract

Let X be a horospherical G-variety and let D be an effective $\mathbb{Q}$-divisor of X that is stable under the action of a Borel subgroup B of G and such that $D+K_X$ is $\mathbb{Q}$-Cartier. We prove, using Bott-Samelson resolutions, that the pair (X,D) is klt if and only if $\lfloor D\rfloor = 0$.

1 Introduction

Let X be a normal algebraic variety over $\mathbb{C}$ and let D be an effective $\mathbb{Q}$-divisor such that $D+K_X$ is $\mathbb{Q}$-Cartier. If the pair (X,D) has klt singularities (see Definition 2.1) then $\lfloor D\rfloor = 0$ (ie $D=\sum_{D_i \text{ irreducible}} a_iD_i$ with $a_i\in[0,1[$). The inverse implication is false in general. In [AB04], V. Alexeev and M. Brion proved that, if X is a spherical G-variety and D be an effective $\mathbb{Q}$-divisor of X such that $D+K_X$ is $\mathbb{Q}$-Cartier, $\lfloor D\rfloor = 0$ and $D=D_G+D_B$ where D_G is G-stable and D_B is stable under the action of a Borel subgroup B of G, then $(X,D_G+D'_B)$ has klt singularities for general D'_B in $|D_B|$.

Here, we prove that, if X is a horospherical G-variety and D be an effective $\mathbb{Q}$-divisor of X such that $D+K_X$ is $\mathbb{Q}$-Cartier, $\lfloor D\rfloor = 0$ and D is stable under the action of a Borel subgroup B of G, then the pair (X,D) has klt singularities.

The strategy of the proof is the following. In section 3, we recall the definitions and some properties of Bott-Samelson resolutions of any flag variety G/P. In particular, they are log resolutions and the klt singularity condition in the case of flag varieties becomes equivalent to some inequalities on the root systems of G and $P\subset G$, which we prove in section 5. And in section 4, we deduce the horospherical case from the case of flag varieties, using that any horospherical variety admits a desingularization that is a toric fibration over a flag variety (ie a fibration over a flag variety whose fiber is a smooth toric variety).

2 Notations and definitions

In all the paper, varieties are algebraic varieties over $\mathbb{C}$.

We first recall the definition of klt singularities.

Definition 2.1. Let X be a normal variety and let D be an effective $\mathbb{Q}$-divisor such that K_X+D is $\mathbb{Q}$-Cartier. The pair (X,D) is said to be klt (Kawamata log terminal) if for any resolution $f:V\longrightarrow X$ of X such that $K_V=f^*(K_X+D)+\sum_{i\in\mathcal{E}}a_iE_i$ where the E_i's are distinct irreducible divisors, we have $a_i>-1$ for any $i\in\mathcal{E}$.

Remark 2.2. 1. In fact, it is enough to check the above property for one log-resolution to say that a pair (X,D) is klt. A log-resolution of (X,D) is a resolution f such that, the exceptional locus Exc(f) of f is of pure codimension one and the divisor $f_*^{-1}(D)+\sum_{E\subset \mathrm{Exc(f)}}E$ has simple normal crossings (where $f_*^{-1}(D)$ is the strict transform of D by f).

2. The condition "$a_i > -1$ for any $i \in \mathcal{E}$" can be replaced by: $\lfloor D \rfloor = 0$ and for any $i \in \mathcal{E}$ such that E_i is exceptional for f, $a_i > -1$.

In all the paper, G denotes a connected reductive algebraic group over $\mathbb{C}$.

Let T be a maximal torus in G and let B be a Borel subgroup of G containing T. We denote by $\mathcal{R}$ the root system of (G, B, T), by $\mathcal{R}^+$ the set of positive roots and by $\mathcal{S}$ the set of simple roots. For any simple root $\alpha \in \mathcal{S}$ we denote by s_α the corresponding simple reflection of the Weyl group $W = N_G(T)/T$. By abuse of notation, for any w in W, we still denote by w one of its representative in G. We denote by w_0 the longest element of W.

Let P be a parabolic subgroup of G that contains B. Denote by $\mathcal{I}$ the set of simple roots of P (in particular, if $P = B$ we have $\mathcal{I} = \emptyset$ and, if $P = G$ we have $\mathcal{I} = \mathcal{S}$). Denote by W_P the subgroup of W generated by $\{s_\alpha \mid \alpha \in \mathcal{I}\}$. Also denote by W^P the quotient W/W_P and denote by w_0^P the longest element of W^P.

The Bruhat decomposition of G in $B \times B$-orbits gives the following decomposition of G/P:

$$G/P = \bigsqcup_{w \in W^P} BwP/P.$$

Moreover the dimension of a cell BwP/P equals the length of w. In particular, the length of w_0^P is the dimension of G/P and irreducible B-stable divisors of G/P are the closures of the cells $Bs_\alpha w_0^P P/P$ with $\alpha \in \mathcal{S} \backslash \mathcal{I}$. We denote them by D_α.

A horospherical variety X is a normal G-variety with an open G-orbit isomorphic to a torus fibration G/H over a flag variety G/P (ie P/H is a torus). The irreducible divisors of such X that are B-stable but not G-stable, are the closures in X of the inverse images in G/H of the Schubert divisors D_α of G/P defined above. We still denote them by D_α, with $\alpha \in \mathcal{S} \backslash \mathcal{I}$.

If X and Y are varieties such that a parabolic subgroup P have a right action on X and a left action on Y, we denote by $X \times^P Y$ the quotient of the product $X \times Y$ by the following equivalences:

$$\forall (x, y) \in X \times Y, \forall P \in P, \ (x, y) \sim (x \cdot p, p^{-1} \cdot y).$$

3 Bott-Samelson desingularizations and klt pairs of flag varieties

In that section, we prove the following result.

Theorem 3.1. *Let $D = \sum_{\alpha \in \mathcal{S} \backslash \mathcal{I}} d_\alpha D_\alpha$ be a B-stable $\mathbb{Q}$-divisor of G/P such that $\forall \alpha \in \mathcal{S} \backslash \mathcal{I}$, $d_\alpha \in [0, 1[$.*

There exists a B-stable log-resolution $\phi : Z/P \longrightarrow G/P$ of $(G/P, D)$, where Z is a variety with a right action of P and a left action of B, such that the exceptional divisors of ϕ are the quotient by P of irreducible divisors of Z, and such that $K_{Z/P} - \pi^*(K_{G/P} + D) = \sum_{i \in \mathcal{E}} a_i E_i$ where for any $i \in \mathcal{E}$, $a_i > -1$ and E_i is an irreducible divisor of f.

In particular the pair $(G/P, D)$ is klt.

Moreover, for any $i \in \mathcal{E}$, E_i is the quotient of an exceptional $B \times P$-stable divisor F_i of Z by P (left action of B and right action of P).

Remarks 3.2. (i) In general, $\sum_{\alpha \in \mathcal{S} \backslash \mathcal{I}} D_\alpha$ is not a simple normal crossing $\mathbb{Q}$-divisor of G/P. Then, it is not enough to know that G/P is smooth to say that $(G/P, D)$ is klt, when $D \neq 0$.

(ii) Since D is globally generated, then $(G/P, D')$ is klt for a general D' in $|D|$ (consequence of [Laz04, Lemma 9.1.9]). We can generalized this remark to spherical pairs, see [AB04, Theorem 5.3].

To prove Theorem 3.1, we use a Bott-Sameslon resolution of G/P. Bott-Samelson resolution of Schubert varieties of G/B have been introduced by M. Demazure in [Dem74]. Here, we use the easy (and well-known) generalization of his work to G/P. And we choose the equivalent definition of Bott-Samelson resolutions that is now used in almost all papers on the topic.

For any simple root α, we denote by P_α the minimal parabolic subgroup containing B such that α is a simple root of P_α.

Definition 3.3. Let $s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_N}$ be a reduced decomposition of w_0^P with $\alpha_1\dots,\alpha_N$ in $\mathcal{S}$. We define the Bott-Samelson variety BS to be the quotient of $P_{\alpha_1}\times P_{\alpha_2}\times\cdots\times P_{\alpha_N}$ by the right action of B^N given by,

$$(p_1,p_2,\dots,p_N)\cdot(b_1,b_2,\dots,b_N)=(p_1b_1,b_1^{-1}p_2b_2,\dots,b_{N-1}^{-1}p_Nb_N).$$

The map $\phi':\ BS\longrightarrow G/P$ that sends $(p_1,p_2,\dots,p_N)$ to $p_1p_2\cdots p_NP/P$ is well-defined and birational (it is an isomorphism from the quotient of $Bs_{\alpha_1}B\times Bs_{\alpha_2}B\times\cdots\times Bs_{\alpha_N}B$ by the right action of B^N to $\underline{Bw_0^P/P}$). (We can decompose this map by the usual map from V to the Schubert variety $\overline{Bw_0^PB/B}$ of G/B and the projection map from G/B to G/P.)

Hence, to get Z as in Theorem 3.1, we define Z to be the quotient of $P_{\alpha_1}\times\cdots\times P_{\alpha_{N-1}}\times P_{\alpha_N\cup\mathcal{I}}$ by the right action of B^{N-1} given by,

$$(p_1,\dots,p_N)\cdot(b_1,b_2,\dots,b_{N-1})=(p_1b_1,\dots,b_{N-1}^{-1}p_N).$$

Then, since $P_{\alpha_N\cup\mathcal{I}}/P=P_{\alpha_N}P/P\simeq P_{\alpha_N}/B$, the B-varieties Z/P and BS are isomorphic and $\phi:\ Z/P\longrightarrow G/P$ that sends $(p_1,\dots,p_N)$ to $p_1\cdots p_NP/P$ is well-defined and birational.

The lines bundles and divisors Bott-Samelson varieties are well-known, so that we can describe the lines bundles of Z/P, and the divisors of Z/P and Z.

Proposition 3.4. *For any $i\in\{1,\dots,N-1\}$, we define F_i to be the $B\times P$-stable divisor of Z defined by $p_i\in B$; and we define F_N to be the $B\times P$-stable divisor of Z defined by $p_N\in P$.*

Then, we can also define E_i to be the B-stable divisor F_i/P of Z/P. Moreover, the B-stable irreducible divisors of Z/P are the E_i's with $i\in\{1,\dots,N\}$, and the family $(E_i)_{i\in\{1,\dots,N\}}$ is a basis of the cone of effective divisors of Z/P.

First remark that the divisor $\sum_{i=1}^N E_i$ is clearly a simple normal crossing divisor. Also, since G/P is smooth and by [Kol96, VI.1, Theorem 1.5], we know that the exceptional locus of ϕ is of pure codimension one, so it is the union of the E_i's contracted by ϕ.

Now, let λ be a character of P. It defines a line bundle $\mathcal{L}_{G/P}(\lambda)$ on G/P (where P acts on the fiber over P/P by the character λ). And by pull-back by ϕ, it defines a line bundle $\mathcal{L}_{Z/P}(\lambda)$ on Z/P.

The total space of $\mathcal{L}_{Z/P}(\lambda)$ is the quotient of $P_{\alpha_1}\times\cdots\times P_{\alpha_{N-1}}\times P_{\alpha_N\cup\mathcal{I}}\times\mathbb{C}$ by the right action of $B^{N-1}\times P$ given by,

$$(p_1,\dots,p_N,z)\cdot(b_1,\dots,b_{N-1},p)=(p_1b_1,b_1^{-1}p_2b_2,\dots,b_{N-1}^{-1}p_Np,\lambda(p)z).$$

By [Dem74, Section 2.5, Proposition 1] adapted to our notation and by induction on N, we have the following result.

Proposition 3.5. *Let λ be a character of P. Then $\mathcal{L}_{G/P}(\lambda)$ is the line bundle associated to the B-stable divisor $D_\lambda:=\sum_{\alpha\in\mathcal{S}\backslash\mathcal{I}}\langle\lambda,\alpha^\vee\rangle D_\alpha$.*

Moreover, $\phi^*(D_\lambda)=\sum_{i=1}^N\langle\lambda,\beta_i^\vee\rangle E_i$, where for any $i\in\{1,\dots,N\}$, $\beta_i=s_{\alpha_1}\cdots s_{\alpha_i-1}(\alpha_i)$.

If $\mathcal{I} \subset \mathcal{S}$, we denote by $\mathcal{R}_{\mathcal{I}}^+$ the set of positive roots generated by simple roots of $\mathcal{I}$. Then we define ρ to be the half sum of positive roots, and ρ^P to be the half sum of positive roots that are not in $\mathcal{R}_{\mathcal{I}}^+$ (in particular, $\rho^B = \rho$).

It is well known that an anticanonical divisor of G/P is $D_{2\rho^P}$. Anticanonical divisors of Bott-Sameslon resolutions are also well-known.

Proposition 3.6. *([Ram85, Proposition 2]) An anticanonical divisor of Z/P is $\phi^*(D_\rho) + \sum_{i=1}^N E_i$.*

Corollary 3.7. *The pair $(G/P, D)$ (with $\lfloor D \rfloor = 0$ as in Theorem 3.1) is klt if and only if for any β in $\mathcal{R}^+ \backslash \mathcal{R}_{\mathcal{I}}^+$,*

$$\langle 2\rho^P - \rho - \sum_{\alpha \in \mathcal{S} \backslash \mathcal{I}} d_\alpha \varpi_\alpha, \beta^\vee \rangle > 0.$$

Proof. By Propositions 3.5 and 3.6, we get

$$\begin{aligned} K_{Z/P} - \phi^*(K_{G/P} + D) &= -\phi^*(D_\rho) - \textstyle\sum_{i=1}^N E_i + \phi^*(D_{2\rho^P}) - \phi^*(D) \\ &= \phi^*(D_{2\rho^P - \rho - \sum_{\alpha \in \mathcal{S} \backslash \mathcal{I}} d_\alpha \varpi_\alpha}) - \textstyle\sum_{i=1}^N E_i \\ &= \textstyle\sum_{i=1}^N (\langle 2\rho^P - \rho - \sum_{\alpha \in \mathcal{S} \backslash \mathcal{I}} d_\alpha \varpi_\alpha, \beta_i^\vee \rangle - 1) E_i. \end{aligned}$$

We conclude by remarking that, since $s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_N}$ is a reduced expression of w_0^P, the set $\{\beta_i \mid i = 1 \cdots N\}$ is $\mathcal{R}^+ \backslash \mathcal{R}_{\mathcal{I}}^+$. □

The condition of Corollary 3.7 is always satisfied by Proposition 5.1 and the hypothesis that $\lfloor D \rfloor = 0$. Then Theorem 3.1 is proved.

4 Horospherical pairs

From the classification of horospherical G-varieties, the description of G-equivariant morphisms between horospherical G-varieties, the description of B-stable Cartier divisor of horospherical G-varieties and the description of a B-stable anticanonical divisor of horospherical G-varieties (see for example [Pas08]), we have the following result.

Proposition 4.1. *Let X be a horospherical G-variety with open G-orbit isomorphic to G/H, torus fibration over the flag variety G/P. Then, there exists a smooth toric P/H-variety Y and a G-equivariant birational morphism f from the smooth horospherical G-variety $V := G \times^P Y$ to X, such that the exceptional locus of f is of pure codimension one.*

Let D be a B-stable effective $\mathbb{Q}$-divisor of X such that $\lfloor D \rfloor = 0$. Write $K_V - f^*(K_X + D) = -f_*^{-1}(D) + \sum_{i \in \mathcal{E}} a_i V_i$. Then, for any $i \in \mathcal{E}$, V_i is exceptional and G-stable, in particular there exists a P-stable divisor Y_i of Y such that $V_i = G \times^P Y_i$. Moreover, $a_i > -1$ for any $i \in \mathcal{E}$.

We do not want here to recall the long description and theory of horospherical varieties. To get more details, see for example [Pas08] or [Pas15].

Proof. With the description in terms of colored fans of horospherical G-varieties and G-equivariant morphisms between them, Y can be chosen as the toric P/H-variety associated to a smooth subdivision $\mathbb{F}_Y$ of the fan associated to the colored fan $\mathbb{F}_X$ of X. Then we clearly have that $V := G \times^P Y$ is smooth and associated to the fan $\mathbb{F}_Y$ considered as a colored fan without color. In particular, there exists a G-equivariant morphism from $V := G \times^P Y$ to X.

Moreover, we can choose $\mathbb{F}_Y$ such that:

- each image of a color of $\mathbb{F}_X$ is in an edge of $\mathbb{F}_Y$ and,
- each cone of $\mathbb{F}_Y$ that is not a cone of $\mathbb{F}_X$ contains an edge that is in $\mathbb{F}_Y$ but not in $\mathbb{F}_X$.

These two conditions implies that the exceptional locus of f is of pure codimension one.

Any exceptional divisor V_i of f is G-stable and of the form $G \times^P Y_i$ where Y_i is a P-stable divisor of Y.

It remains to prove that $a_i > -1$ for any $i \in \mathcal{E}$. We use that $-K_X = \sum_{i=1}^m X_i + \sum_{\alpha\in\mathcal{S}\backslash\mathcal{I}} a_\alpha D_\alpha$ where the X_i's are the G-stable irreducible divisors of X and the a_α are positive integers. Similarly, with our notation, $K_V = -\sum_{i=1}^m X_i - \sum_{i\in\mathcal{E}} f_*^{-1}(X_i) - \sum_{\alpha\in\mathcal{S}\backslash\mathcal{I}} a_\alpha D_\alpha$. In particular, by hypothesis on D, we remark that the divisor $-K_X - D$ is strictly effective (ie,$\sum_{i=1}^m b_i X_i + \sum_{\alpha\in\mathcal{S}\backslash\mathcal{I}} b_\alpha D_\alpha$, with $b_i > 0$ for any $i \in \{1,\dots,m\}$ and $b_\alpha > 0$ for any $\alpha \in \mathcal{S}\backslash\mathcal{I}$) and then, by the description of pull-backs of B-stable divisors of horospherical varieties, $f^*(-K_X - D)$ is also strictly effective. Hence, we have $a_i > -1$ for any $i \in \mathcal{E}$. □

Theorem 4.2. *Let X be a horospherical G-variety. Let D be any B-stable $\mathbb{Q}$-divisor D of X such that $\lfloor D \rfloor = 0$, then (X, D) has klt singularities.*

Proof. Let f be as in Proposition 4.1 and let Z be as in Theorem 3.1. Define $V' := Z \times^P Y$ and let $\pi : V' \longrightarrow V$ the natural B-equivariant morphism defined from ϕ.

We first prove that the B-equivariant morphism $f \circ \pi : V' \longrightarrow X$ is a log resolution of (X, D). By composition, it is clearly a birational morphism and its exceptional locus is the union of the inverse images $Z \times^P Y_i$ of the exceptional divisors of f and the exceptional divisors $F_i \times^P Y$ of π (the exceptional locus of π is of pure codimension one because V is smooth).

The divisor $(f\circ\pi)_*^{-1}(D) + \sum_{E\in \mathrm{Exc(f\circ\pi)}} E$ is a B-stable divisor of V' and then has simple normal crossings. Indeed, a B-stable irreducible divisor of V' is either $F_i \times^P Y$ where F_i is one of the B-stable irreducible divisors of Z described in Proposition 3.4, or $Z \times^P Y_i$ where Y_i is a P-stable divisor of Y. (Recall that, any divisor of a smooth toric variety that is stable under the action of the torus has simple normal crossings, because such a variety is everywhere locally isomorphic to $\mathbb{C}^n$ with the natural action of $(\mathbb{C}^*)^n$.)

Since D is B-stable, we have $D = \sum_{i=1}^m d_i X_i + \sum_{\alpha\in\mathcal{S}\backslash\mathcal{I}} d_\alpha D_\alpha$ where the X_i's are the G-stable irreducible divisors of X. We denote by D_B the B-stable but not G-stable part $\sum_{\alpha\in\mathcal{S}\backslash\mathcal{I}} d_\alpha D_\alpha$ of D. Then we decompose $K_{V'} - (f\circ\pi)^*(K_X + D)$ as follows:

$$(K_{V'} - \pi^*(K_V + f_*^{-1}(D_B))) + \pi^*(K_V - f^*(K_X + D) + f_*^{-1}(D_B)).$$

By Proposition 4.1, $K_V - f^*(K_X + D) + f_*^{-1}(D_B) = \sum_{i\in\mathcal{E}} a_i V_i$, where for any $i \in \mathcal{E}$, $a_i > -1$ and $V_i = G \times^P Y_i$ with some P-stable irreducible divisor Y_i of Y. We remark that the inverse image of V_i by π is the irreducible divisor $Z \times^P V_i$ so that $\pi^*(V_i) = Z \times^P Y_i$. Hence, $\pi^*(K_V - f^*(K_X + D) + f_*^{-1}(D)) = \sum_{i\in\mathcal{E}} a_i Z \times^P Y_i$.

To compute $K_{V'} - \pi^*(K_V + f_*^{-1}(D_B))$, we use the fibrations $p : V = G \times^P Y \longrightarrow G/P$ and $p' : V' = Z \times^P Y \longrightarrow Z/P$, which have the same fiber. To summarize, we get the following commutative diagram.

$$\begin{array}{ccccc} V' = Z \times^P Y & \xrightarrow{\pi} & V = G \times^P Y & \xrightarrow{f} & X \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} & & \\ Z/P & \xrightarrow{\phi} & G/P & & \end{array}$$

In particular, we have $K_V = p^*(K_{G/P}) + K_p$ and $K_{V'} = p^*(K_{Z/P}) + K_{p'}$. Moreover, the relative canonical divisors $K_{p'}$ and K_p satisfy $K_{p'} = \pi^*(K_p)$.

Moreover, for any B-stable irreducible divisor D of V that is not G-stable, D is the pull-back by p of a Schubert divisor of G/P, in particular $D = p^*(p_*(D))$.

Hence, we get

$$\begin{aligned} K_{V'} - \pi^*(K_V + f_*^{-1}(D_B)) &= p'^*(K_{Z/P}) + K_{p'} - \pi^* p^*(K_{G/P}) - \pi^*(K_p) - \pi^*(f_*^{-1}(D_B)) \\ &= p'^*(K_{Z/P}) + \pi^*(K_p) - p'^*\phi^*(K_{G/P}) - \pi^*(K_p) \\ &\qquad - \pi^*(p^* p_*(f_*^{-1}(D_B))) \\ &= p'^*(K_{Z/P} - \phi^*(K_{G/P} + p_*(f_*^{-1}(D_B))). \end{aligned}$$

Remark that $\lfloor p_*(f_*^{-1}(D_B)\rfloor = \lfloor D_B \rfloor$, so that by Theorem 3.1, we get $K_{Z/P} - \phi^*(K_{G/P} + p_*(f_*^{-1}(D_B)) = \sum_{i\in\mathcal{E}'} a_i F_i/P$, where for any $i \in \mathcal{E}'$, we have $a_i > -1$ and F_i is a $B \times P$-stable irreducible divisor of Z.

Hence, we have $K_{V'} - \pi^*(K_V + f_*^{-1}(D_B)) = \sum_{i\in\mathcal{E}'} a_i F_i \times^P Y$.

And finally, we have

$$K_{V'} - (f \circ \pi)^*(K_X + D) = \sum_{i\in\mathcal{E}'} a_i F_i \times^P Y + \sum_{i\in\mathcal{E}} a_i Z \times^P Y_i,$$

with, for any $i \in \mathcal{E}' \cup \mathcal{E}$, $a_i > -1$. □

5 A result on root systems

In that independent section, we prove the result that permits to deduce Theorem 3.1 from Corollary 3.7. We keep notations of section 2 and we recall that, if $\mathcal{I} \subset \mathcal{S}$, we denote by $\mathcal{R}_\mathcal{I}^+$ the set of positive roots generated by simple roots of $\mathcal{I}$, ρ denotes the half sum of positive roots, and ρ^P denotes the half sum of positive roots that are not in $\mathcal{R}_\mathcal{I}^+$.

Proposition 5.1. *For any (proper) parabolic subgroup P of G containing B, and for any β in $\mathcal{R}^+ \backslash \mathcal{R}_\mathcal{I}^+$,*

$$\langle 2\rho^P - \rho - \sum_{\alpha\in\mathcal{S}\backslash\mathcal{I}} \varpi_\alpha, \beta^\vee \rangle \geq 0. \tag{5.1.1}$$

Note that $\rho = \sum_{\alpha\in\mathcal{S}} \varpi_\alpha$ and that $2\rho^P = 2\rho - \sum_{\gamma\in\mathcal{R}_\mathcal{I}^+} \gamma = 2\sum_{\alpha\in\mathcal{S}} \varpi_\alpha - \sum_{\gamma\in\mathcal{R}_\mathcal{I}^+} \gamma$. Hence, equation 5.1.1 is equivalent to

$$\langle \sum_{\alpha\in\mathcal{I}} \varpi_\alpha - \sum_{\gamma\in\mathcal{R}_\mathcal{I}^+} \gamma, \beta^\vee \rangle \geq 0. \tag{5.1.2}$$

Remark 5.2. (i) If $\mathcal{I} = \emptyset$ (ie if $P = B$), equations 5.1.1 and 5.1.2 are trivially satisfied.

(ii) If $\beta \in \mathcal{R}_\mathcal{I}^+$ then $\langle \sum_{\alpha\in\mathcal{I}} \varpi_\alpha - \sum_{\gamma\in\mathcal{R}_\mathcal{I}^+} \gamma, \beta^\vee \rangle = -\langle \sum_{\alpha\in\mathcal{I}} \varpi_\alpha, \beta^\vee \rangle$ and is negative.

Lemma 5.3. *Denote by $w_{0,P}$ the longest element of W_P. Then, we have*

$$w_{0,P}(\sum_{\alpha\in\mathcal{I}} \varpi_\alpha) = \sum_{\alpha\in\mathcal{I}} \varpi_\alpha - \sum_{\gamma\in\mathcal{R}_\mathcal{I}^+} \gamma.$$

Proof. First note that, for any character λ of T and for any $w \in W_P$, $w(\lambda) - \lambda$ is in the lattice $\bigoplus_{\alpha\in\mathcal{I}} \mathbb{Z}\alpha$ (it can be easily proved by induction on the length of w).

Moreover, $\rho_P := \sum_{\gamma\in\mathcal{R}_\mathcal{I}^+} \gamma$ satisfies $\langle \rho_P, \alpha^\vee \rangle = 2$ for any $\alpha \in \mathcal{I}$ (same result as the well-known result: $\langle \rho, \alpha^\vee \rangle = 2$ for any $\alpha \in \mathcal{S}$). And, since $w_{0,P}$ is the longest element of W_P, we that for any $\alpha \in \mathcal{I}$, the root $w_{0,P}(\alpha)$ is the opposite of a simple root in $\mathcal{I}$.

Hence, if we denote by λ the character $w_{0,P}(\sum_{\alpha\in\mathcal{I}}\varpi_\alpha)-\sum_{\alpha\in\mathcal{I}}\varpi_\alpha+\sum_{\gamma\in\mathcal{R}_\mathcal{I}^+}\gamma$, we get $\lambda\in\bigoplus_{\alpha\in\mathcal{I}}\mathbb{Z}\alpha$ and, for any $\alpha\in\mathcal{I}$, we have

$$\langle\lambda,\alpha^\vee\rangle=\langle\sum_{\alpha\in\mathcal{I}}\varpi_\alpha,w_{0,P}(\alpha^\vee)\rangle-\langle\sum_{\alpha\in\mathcal{I}}\varpi_\alpha,\alpha^\vee\rangle+\langle\rho_P,\alpha^\vee\rangle=-1-1+2=0.$$

Then, we deduce that $\lambda=0$, which proves the lemma. □

Proof of Proposition 5.1. By Lemma 5.3, we get for any $\beta\in\mathcal{R}^+\backslash\mathcal{R}_\mathcal{I}^+$,

$$\langle\sum_{\alpha\in\mathcal{I}}\varpi_\alpha-\sum_{\gamma\in\mathcal{R}_\mathcal{I}^+}\gamma,\beta^\vee\rangle=\langle\sum_{\alpha\in\mathcal{I}}\varpi_\alpha,w_{0,P}(\beta^\vee)\rangle.$$

But, the positive roots that are send to negative roots by $w_{0,P}$ are exactly the roots in $\mathcal{R}_\mathcal{I}^+$. In particular, for any $\beta\in\mathcal{R}^+\backslash\mathcal{R}_\mathcal{I}^+$, $w_{0,P}(\beta^\vee)$ is a positive coroot, and

$$\langle\sum_{\alpha\in\mathcal{I}}\varpi_\alpha,w_{0,P}(\beta^\vee)\rangle\geq 0.$$

□

Remark 5.4. Let $\beta\in\mathcal{R}^+\backslash\mathcal{R}_\mathcal{I}^+$. Then $\langle\sum_{\alpha\in\mathcal{I}}\varpi_\alpha-\sum_{\gamma\in\mathcal{R}_\mathcal{I}^+}\gamma,\beta^\vee\rangle=0$ if and only if $w_{0,P}(\beta)\in\mathcal{R}_{\mathcal{S}\backslash\mathcal{I}}^+$. In particular, if $\mathcal{I}\neq\mathcal{S}$, there exists $\beta\in\mathcal{R}^+\backslash\mathcal{R}_\mathcal{I}^+$ such that $\langle\sum_{\alpha\in\mathcal{I}}\varpi_\alpha-\sum_{\gamma\in\mathcal{R}_\mathcal{I}^+}\gamma,\beta^\vee\rangle=0$.

Acknowledgement: I would like to thank Cédric Bonnafé who points me how to give a shorter proof of Proposition 5.1.

References

[AB04] Valery Alexeev and Michel Brion, *Stable reductive varieties. II. Projective case*, Adv. Math. **184** (2004), no. 2, 380–408.

[Bou75] N. Bourbaki, *Éléments de mathématique*, Hermann, Paris, 1975, Fasc. XXXVIII: Groupes et algèbres de Lie. Chapitre VII: Sous-algèbres de Cartan, éléments réguliers. Chapitre VIII: Algèbres de Lie semi-simples déployées, Actualités Scientifiques et Industrielles, No. 1364.

[Dem74] Michel Demazure, *Désingularisation des variétés de Schubert généralisées*, Ann. Sci. École Norm. Sup. (4) **7** (1974), 53–88, Collection of articles dedicated to Henri Cartan on the occasion of his 70th birthday, I. MR 0354697 (50 #7174)

[Kol96] János Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 32, Springer-Verlag, Berlin, 1996.

[Laz04] Robert Lazarsfeld, *Positivity in algebraic geometry. II*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 49, Springer-Verlag, Berlin, 2004, Positivity for vector bundles, and multiplier ideals.

[Pas08] Boris Pasquier, *Variétés horosphériques de Fano*, Bull. Soc. Math. France **136** (2008), no. 2, 195–225.

[Pas15] ———, *A survey on the singularities of spherical varieties*, to appear soon in arXiv (2015).

[Ram85] A. Ramanathan, *Schubert varieties are arithmetically Cohen-Macaulay*, Invent. Math. **80** (1985), no. 2, 283–294. MR 788411 (87d:14044)