

ABOUT THE SPEED OF RANDOM WALKS ON SOLVABLE GROUPS

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ABSTRACT. We show that for each $\lambda \in [\frac{1}{2}, 1]$, there exists a solvable group and a finitely supported measure such that the associated random walk has upper speed exponent λ .

1. INTRODUCTION

The speed of a random walk measures the average distance between the starting point and the position of the particle. When the step distribution has finite support, or more generally finite first moment, the speed reflects some geometric features of the space. For instance, the speed of a random walk on a group is related to the return probability (see Kesten [11] and for recent developments Gournay [7] and Saloff-Coste-Zheng [15]), to the entropy of the random walk (Kaimanovich-Vershik [10]) and to the Hilbert compression (Naor-Peres [13]). In fact, linearity of the speed on a group is equivalent to non-triviality of the Poisson boundary (Kaimanovich-Vershik [10]). This naturally raises the question to understand what are the possible other, i.e. sublinear, behaviors of speed.

The central limit theorem essentially asserts that the speed of symmetric random walks on abelian groups is comparable, up to multiplicative constants, to \sqrt{n} . This behavior, called diffusive, is obtained for a large class of groups, including nilpotent groups (Hebisch-Saloff-Coste[8]), the lamplighter group on cyclic base with finite lamps, solvable Baumslag-Solitar groups and many polycyclic groups (Revelle [14], Thompson [16]). Wreath products, also called lamplighters, and iterated wreath products of abelian groups provide a countable infinity of other behaviors (Erschler [3] [4]). Precisely, for $\Gamma_1 = \mathbb{Z}$ and $\Gamma_{i+1} = \mathbb{Z} \wr \Gamma_i$, the speed of some random walk is comparable with $n^{1-\frac{1}{2^i}}$, and if \mathbb{Z} is replaced by \mathbb{Z}^2 in the inductive definition, the speed is comparable with n over iterated logarithms. Erschler also showed that when the support of the step distribution generates an infinite group, the speed is at least diffusive (Lee-Peres [12]). Moreover, the speed can be arbitrarily close to n (Erschler [5]), it can oscillate between two different powers of n (Erschler [6]) and the amplitude of oscillations can be maximal between diffusive and linear (Brieussel [2]).

A very large class of speed behaviors has been obtained by Amir-Virag [1]. Namely for any sublinear function not less than $n^{\frac{3}{4}}$ (and sufficiently regular in a mild sense), there exists a group with a finitely supported measure for which the

speed of the random walk is comparable with this function. Amir and Virag also conjecture the existence of speed functions comparable to n^λ for any λ between $\frac{1}{2}$ and $\frac{3}{4}$. The main result of this paper is in the direction of this conjecture. Recall that the upper (resp. lower) speed exponent of a random walk is the quantity $\limsup \frac{\log \mathbb{E}|Z_n|}{\log n}$ (resp. $\liminf \frac{\log \mathbb{E}|Z_n|}{\log n}$).

Theorem 1.1. *For any $\lambda \in [\frac{1}{2}, 1]$, there exists a finitely generated solvable group with a finitely supported symmetric measure such that the upper speed exponent of the associated random walk is λ .*

More precisely, let λ belong to the interval $[1 - \frac{1}{2^i}, 1 - \frac{1}{2^{i+1}}]$ for $i \geq 1$ integer, and let $\varepsilon(n)$ be a sequence tending to infinity. Then there exists a finitely generated $i+2$ -solvable group with a finitely supported symmetric measure such that the speed $\mathbb{E}(Z_n)$ of the associated random walk oscillates between $n^{1-\frac{1}{2^i}}$ and n^λ up to multiplicative factor $\varepsilon(n)$ in the sense that $n^{1-\frac{1}{2^i}} \leq \mathbb{E}(Z_n) \leq n^\lambda$ for all n large enough, $\mathbb{E}(Z_n) \geq \frac{n^\lambda}{\varepsilon(n)}$ for infinitely many n and $\mathbb{E}(Z_n) \leq \varepsilon(n)n^{1-\frac{1}{2^i}}$ for infinitely many n .

Combined with [12], we deduce the

Corollary 1.2. *A real number λ is the upper speed exponent of a random walk on a (solvable) group with finite support step distribution if and only if $\lambda \in \{0\} \cup [\frac{1}{2}, 1]$.*

There is nothing specific with the function n^λ , and the theorem is relevant for any function neglectable against $1 - \frac{1}{2^{i+1}}$, for instance $n^\lambda (\log n)^\mu$ when $\varepsilon(n) = \log \log n$. On the other hand, it is not clear if the lower exponent for solvable groups can be prescribed to a value other than $1 - \frac{1}{2^i}$ (see Remark 4.3).

Contrary to the groups in [1] [2] [5] and [6], the groups of Theorem 1.1 are solvable. This shows that there are more behaviors of speed among solvable groups than suggested by the previous examples [3] [4] [14] [16]. In particular, the speed of random walks on solvable groups can oscillate, and there are uncountably many pairwise non-comparable behaviors. It would be interesting to know if solvability implies some specific constraints on the speed function.

The heart of Theorem 1.1 is the case $i = 1$ when $\lambda \in [\frac{1}{2}, \frac{3}{4}]$. These groups are obtained by "interpolating" between $\mathbb{Z} \wr D_2$ and $\mathbb{Z} \wr D_\infty$, where D_l is the dihedral group of size $2l$. With usual "switch-walk-switch" step distribution, the first one has speed \sqrt{n} and the second $n^{\frac{3}{4}}$ by Erschler [4]. The idea is that the switch generators a and b of order two of the dihedral lamp-group in $\mathbb{Z} \wr D_l$ should be placed at positions far apart. This guarantees that this group locally "looks like" $\mathbb{Z} \wr D_2$. It is also asymptotically equivalent to it, hence has diffusive speed when l is finite. However, letting l tend to infinity, the speed tends to $n^{\frac{3}{4}}$ pointwise. By an intermediate value argument, we can find some l with a scale where the speed is roughly n^λ . The argument is repeated infinitely many times by means of a diagonal product.

This construction is reminiscent of the technics of perturbation of automata groups used by Erschler [5], Kassabov-Pak [9] and Brioussel [2]. Theorem 1.1 could be obtained in the setting of section 7 in [2] for the directed infinite dihedral automata group. However, the presentation given here avoids any reference to this theory, preferring the group \mathbb{Z} rather than the action of D_∞ on a tree.

The notion of diagonal product and local properties of marked groups are presented in section 2. Some properties of random walks on lamplighter groups $\mathbb{Z} \wr D_l$ are studied in section 3. Theorem 1.1 is proved in section 4, dealing first with the main case $\frac{1}{2} \leq \lambda \leq \frac{3}{4}$.

2. LOCAL PROPERTIES OF MARKED GROUPS

2.1. Diagonal product of marked groups. Let $\Gamma = \langle \gamma_1, \dots, \gamma_k \rangle$ and $F = \langle f_1, \dots, f_k \rangle$ be two k -generated groups. We consider the groups together with their fixed and ordered generating set. (Equivalently, we regard them as two quotients of the same free group \mathbb{F}_k of rank k). Their diagonal product is the subgroup Δ of $\Gamma \times F$ generated by the diagonal generating set :

$$\Delta = \langle (\gamma_1, f_1), \dots, (\gamma_k, f_k) \rangle < \Gamma \times F.$$

For each word in \mathbb{F}_k , we denote by w^Γ (respectively w^F , w^Δ) its evaluation in Γ (respectively F , Δ).

Recall that Γ is a marked quotient of Γ' if the application $\Gamma' \rightarrow \Gamma$ sending γ'_i to γ_i induces a surjective group homomorphism (equivalently, if $w^{\Gamma'} = e$ implies $w^\Gamma = e$ for any w in \mathbb{F}_k), and that the marked balls of radius R in Γ and Γ' are equivalent when $w^{\Gamma'} = e$ is equivalent to $w^\Gamma = e$ for any w of length $\leq 2R$ in \mathbb{F}_k .

Lemma 2.1. *With the above notations, assume F is a finite group. Then there exists $C_1, C_2 > 0$ such that for any word w in \mathbb{F}_k*

$$|w^\Gamma| \leq |w^\Delta| \leq C_1 |w^\Gamma| + C_2,$$

where norms are with respect to the above generating sets.

Moreover, the constants C_1, C_2 depend only on the finite group F and locally on Γ , in the sense that there exists $R > 0$ such that if $\Gamma' = \langle \gamma'_1, \dots, \gamma'_k \rangle$ is another marked k -generated group with Γ a marked quotient of Γ' and $B_{\Gamma'}(R) \simeq B_\Gamma(R)$, then for any word w in \mathbb{F}_k , we also have

$$|w^{\Gamma'}| \leq |w^{\Delta'}| \leq C_1 |w^{\Gamma'}| + C_2,$$

where Δ' is the diagonal product of Γ' and F .

Proof. The first inequality is easy. For $w^\Delta \in \Delta$, there exists a shortest word W in \mathbb{F}_k such that $l(W) = |w^\Delta|$ and $W^\Delta = w^\Delta$. As the generating set is diagonal, we deduce $W^\Gamma = w^\Gamma$, so $|w^\Gamma| \leq l(W) = |w^\Delta|$.

In order to get the second inequality, we introduce the generating set $S = \{(\gamma_1, e), \dots, (\gamma_k, e), (e, f_1), \dots, (e, f_k)\}$ of $\Gamma \times F$. Obviously, for any $(\gamma, f) \in \Gamma \times F$, we have

$$|\gamma| \leq |(\gamma, f)|_S \leq |\gamma| + \text{diam}(F).$$

The projection of Δ on the first factor Γ is onto, so Δ is a finite index subgroup of $\Gamma \times F$, and we can find a transversal of the quotient with representatives of the form $a_i = (e, f_i)$, that is f_1, \dots, f_r in F such that $\Delta(e, f_1) \sqcup \dots \sqcup \Delta(e, f_r) = \Gamma \times F$.

Now the set T of products of the form $a_i s a_j^{-1}$ with s in S which belong to Δ is a generating set of Δ and we have $|\delta|_T \leq |\delta|_S$ for any δ in Δ . (Indeed, for

$\delta = s_1 \dots s_n$, let $a_{i_0} = e$ and Δa_{i_j} be the representative of $\Delta a_{i_{j-1}} s_j$, then $\delta = a_{i_0} s_1 a_{i_1}^{-1} \dots a_{i_{n-1}} s_n a_{i_n}^{-1} a_{i_n}$, with $a_{i_n} = e$ when δ belongs to Δ .)

So each word in this generating set T can be written as a word in the diagonal generating set $D = ((\gamma_1, f_1), \dots, (\gamma_k, f_k))$. Thus $|\delta|_D \leq C_1 |\delta|_T$ for any δ in Δ where C_1 is the maximal length of the shortest words in D representing the generators in T . We conclude that for any $\delta = (\gamma, f)$ in Δ , we have $|\delta|_D \leq C_1(|\gamma| + \text{diam}F)$.

To see that the constants are local (among groups admitting Γ as quotient), first observe that in order to find the transversal, we need only to know which couples (e, f) in $\Gamma \times F$ belong to Δ . If (e, f) belongs to Δ , then there exists $w_{(e,f)}$ in \mathbb{F}_k with $w_{(e,f)}^\Delta = (e, f)$, that is $w_{(e,f)}^\Gamma = e$ and $w_{(e,f)}^F = f$. If (e, f) does not belong to Δ , then whenever Γ' admits Γ as marked quotient, there is no word such that $w^{\Gamma' \times F} = (e, f)$ (otherwise, we would also get $w^{\Gamma \times F} = (e, f)$ by taking quotient).

Secondly, to describe the generating set T , we need only to know which products $t = a_i s a_j^{-1}$ (which are now seen as words in S) belong to Δ . If t belongs to Δ , then there is a word w_t with $w_t^\Delta = t$. If t does not belong to Δ , then whenever Γ' admits Γ as a quotient, there is no word w such that $w^{\Gamma' \times F} = t$ (otherwise, we would get $w^{\Gamma \times F} = t$ by taking quotient).

Finally, the lemma holds when R is greater than the length of the words $w_{(e,f)}^\Delta$ and w_t^Δ . Indeed, these relations and the fact that Γ' admits Γ as quotient guarantee that the transversal and the generating set T are not changed (as set of words in S) when we replace Γ by Γ' . And any generator in T is still expressed by the same word in D , preserving C_1 . \square

2.2. Local coincidence of lamplighter groups. Let l belong to $\mathbb{Z}_{\geq 1} \cup \{\infty\}$, we denote $D_l = \langle a, b | a^2 = b^2 = (ab)^l = 1 \rangle$ the dihedral group of size $2l$.

We denote $B \wr L = B \times \sum_B L$ the lamplighter group (or wreath product) with base-group B and lamp-group L , where $\sum_B L$ is the set of functions from B to L neutral almost everywhere, with the shift action of B on it. Its elements are couples (g, f) with g in B and $f : B \rightarrow L$ of finite support. The function identically neutral on B is denoted $\mathbb{1}$, and the function taking value l at g and neutral elsewhere is denoted $l\delta_g$.

For integers $k \geq 0$, $l \geq 2$ and $m \geq 1$, denote $\Gamma = \Gamma(k, l, m)$ the group $\mathbb{Z}/m\mathbb{Z} \wr D_l$ with marking generating set $(+1, \mathbb{1}), (0, a\delta_0), (0, b\delta_k)$. The value ∞ is allowed for l and m . For instance, the group $\Gamma(0, 2, \infty)$ is the lamplighter group $\mathbb{Z} \wr D_2$ with usual generating set.

Lemma 2.2. *For $l \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$ and $2k < m$, the connected component containing the identity of the subset $\{(n, f) | |n| < k/2\}$ of the groups $\Gamma(k, l, m)$ and $\Gamma(0, 2, \infty)$. In particular, the balls of radius $\frac{k-1}{2}$ in these two marked groups coincide.*

Proof. Any word in the generators can be written in the form

$$\begin{aligned} w &= (i_1, \mathbb{1})(0, a^{\varepsilon_1} \delta_0)(0, b^{\eta_1} \delta_k) \dots (i_s, \mathbb{1})(0, a^{\varepsilon_s} \delta_0)(0, b^{\eta_s} \delta_k) \\ &= (i_1 + \dots + i_s, a^{\varepsilon_1} \delta_{i_1} b^{\eta_1} \delta_{i_1+k} \dots a^{\varepsilon_s} \delta_{i_1+\dots+i_s} b^{\eta_s} \delta_{i_1+\dots+i_s+k}), \end{aligned}$$

using the relation $(i, \mathbb{1})(0, b\delta_k)(-i, \mathbb{1}) = (0, b\delta_{i+k})$. The word w represents an element (i, f) of the wreath product. We obtain a word describing $f(x)$ by keeping in the product above only the factors a^{ε_j} and $b^{\eta_{j'}}$ where $i_1 + \dots + i_j = x$ and $i_1 + \dots + i_{j'} + k = x$. The word w represents the identity if and only if $i = i_1 + \dots + i_s = 0$ and $f(x) = e$ for all x in \mathbb{Z} .

As $D_2 \simeq \langle a \rangle \times \langle b \rangle$ is a product of two groups of order 2, the a -lamps and the b -lamps are independant in $\mathbb{Z} \wr D_2$. Moreover in $\mathbb{Z}/m\mathbb{Z} \wr D_l$, as long as $|i_1 + \dots + i_j| < k/2$ for all j , then for every x in \mathbb{Z} , the lamp $f(x)$ takes values in $\{e, a, b\}$. Thus, the lamp function of w in $\mathbb{Z} \wr D_2$ can be recovered from that in $\mathbb{Z}/m\mathbb{Z} \wr D_l$ and vice-versa, just by shifting the b -valued part of the function. \square

Remark 2.3. The above proof also shows that for each $k \geq 0$ there is an isomorphism of marked groups between $\Gamma(k, 2, \infty)$ and $\Gamma(0, 2, \infty)$.

2.3. An infinite diagonal product of lamplighter groups with dihedral lamps. Let $k = (k_s)$, $l = (l_s)$ and $m = (m_s)$ be three sequences of integers (the sequences l and m are allowed to take value ∞) such that $2k_s < m_s$ and denote $\Delta = \Delta(k, l, m)$ the infinite diagonal product of the groups $\Gamma(k_s, l_s, m_s)$, that is the subgroup of $\prod_s \mathbb{Z}/m_s\mathbb{Z} \wr D_{l_s}$, generated by the three sequences $\tau = (\tau_s)$, $\alpha = (\alpha_s)$ and $\beta = (\beta_s)$ where $\tau_s = (+1, \mathbb{1})$, $\alpha_s = (0, a_s\delta_0)$ and $\beta_s = (0, b_s\delta_{k_s})$.

Lemma 2.4. *Let $\Delta = \Delta(k, l, m)$ be as above, and let $R < \frac{\min k - 1}{2}$, then the balls of radius R in Δ and $\Gamma(0, 2, \infty)$ coincide. Moreover, if k is unbounded, $\Gamma(0, 2, \infty)$ is a quotient of Δ .*

Proof. The first statement follows directly from Lemma 2.2. It also implies that if $w^{\Gamma(0, 2, \infty)} \neq e$, then for k_s large enough, $w^{\Gamma(k_s, l_s, m_s)} \neq e$, hence $w^\Delta \neq e$, proving the second statement. \square

3. RANDOM WALKS ON LAMPLIGHTER GROUPS

To the generating set $\tau = (+1, \mathbb{1})$, $\alpha = (0, a\delta_0)$, $\beta = (0, b\delta_k)$ of the wreath product $\Gamma = \Gamma(k, l, m) = \mathbb{Z} \wr D_l$ between the infinite cyclic group and a dihedral group, we associate the "switch-walk-switch" generating set consisting of all words $u_1\tau^\eta u_2$ with u_i in $\{\alpha^{\varepsilon_1}\beta^{\varepsilon_2}, \beta^{\varepsilon_1}\alpha^{\varepsilon_2}\}$, and we denote μ the measure obtained on it by equidistribution and independence of ε_j taking value 0 or 1 and η taking value -1 or 1 . We denote Z_n , or Z_n^l when we want to stress the order of the dihedral group, the random walk of law μ at time n .

The lamplighter interpretation is that at each step the lamplighter first switches the lamp at his position by a^{ε_1} and the lamp at distance k on the right by b^{ε_2} , secondly he moves by η and thirdly switches the lamp at his new position by $a^{\varepsilon'_1}$ and the lamp at distance k on the right by $b^{\varepsilon'_2}$. Note that the order of α and β in u_i has no influence unless $k = 0$ and $l > 2$.

3.1. Infinite dihedral lamp-group.

Lemma 3.1. *There exists positive constants $c_1, c_2 > 0$ depending only on k such that the switch-walk-switch random walk on $\Gamma(k, \infty, \infty) = \mathbb{Z} \wr D_\infty$ satisfies for all n :*

$$c_1 n^{\frac{3}{4}} \leq \mathbb{E}|Z_n^\infty| \leq c_2 n^{\frac{3}{4}}.$$

This lemma is a generalization of the estimation of the speed of switch-walk-switch random walk on $\mathbb{Z} \wr \mathbb{Z}$ due to Erschler [4]. Replacing the group \mathbb{Z} by D^∞ is harmless, but taking care of $k > 0$ requires to adapt her argument. The present lemma would follow directly from [4] if we knew that speed is invariant under change of generating set (which is an open question).

Proof. First observe that the Cayley graphs of \mathbb{Z} with respect to $\{-1, 1\}$ and that of D_∞ with respect to $\{a, b\}$ coincide, as well as the simple random walks on these graphs. We deduce that when ε_i are independent equidistributed in $\{0, 1\}$, we have $\mathbb{E}|a^{\varepsilon_1} b^{\varepsilon_2} \dots a^{\varepsilon_{n-1}} b^{\varepsilon_n}| \asymp \sqrt{n}$.

Now the lamp $f(x)$ of Z_n at x can be expressed as a product of a^{ε_j} and $b^{\varepsilon_{j'}}$, at times j when $i_1 + \dots + i_j = x$ (the lighter at 0 visits x) and j' when $i_1 + \dots + i_{j'} = x$ (the lighter at k visits x). This word can be rewritten by simplifying each sub- a -word and each sub- b -word (for instance $a^1 b^1 a^0 a^0$ should be rewritten as $a^1 b^0 a^0$ but not as a^1).

We denote $t_x^{(n)}$ the length of the word obtained, which we call the local time at x . A key observation is that the local times increase slower when k is bigger. This local time depends only on the random walk induced on \mathbb{Z} , as $t_x^{(n)}$ counts the number of alternations between times when the lighter at 0 visits x and times when the lighter at k visits x (i.e. when the lighter at 0 visits $x - k$).

With this definition of local times, the proof of [4] applies to the present setting as soon as we check that there exists $c > 0$ such that for any $\varepsilon > 0$ we have $\mathbb{P}(\sum_{x \in \mathbb{Z}} t_x^{(n)} \geq cn) \geq 1 - \varepsilon$, for n large. Indeed, this permits to get Erschler's Proposition 1 when ε is small enough to use Lemma 2. The present lemma is then nothing but Lemma 3 in [4] with $A = D_\infty$.

But when the trajectory induced on \mathbb{Z} is a path of length $2k$ from 0 to $-k$ and back to 0, we have $t_0^{(n)} \geq 2$, so the required sum increases by 2 with positive probability with every multiple of $2k$. This is sufficient. \square

Remark 3.2. Arguments of the same type show that the speed of the switch-walk-switch random walk in $\mathbb{Z} \wr D_l$ with finite l is asymptotically equivalent to \sqrt{n} up to multiplicative constants depending only on l and k .

3.2. Comparing finite dihedral lamp-groups.

Lemma 3.3. *There exists a universal constant $C > 0$ such that for any integers k, l, n , we have*

$$\mathbb{E}|Z_n^l| \leq \mathbb{E}|Z_n^{2l}| \leq C \mathbb{E}|Z_n^l|$$

with respect to the switch-walk-switch generating set.

Replacing with other equivalent length functions just modifies the inequalities by multiplicative constants, for instance with respect to the usual generating sets

$\tau = (+1, \mathbf{1})$, $\alpha = (0, a\delta_0)$, $\beta = (0, b\delta_k)$ by a factor ≤ 5 . The first inequality is obvious since it holds pointwise.

In a dihedral group D_{2l} , let $Y_t = a^{\varepsilon_1} b^{\varepsilon_2} a^{\varepsilon_3} b^{\varepsilon_4} \dots a^{\varepsilon_t}$ with ε_i independent taking value 0 or 1 with probability $\frac{1}{2}$ (the last letter is b when t is even). Denote $p_t(x) = \mathbb{P}(d(Y_t, \{e, a\}) = x)$ the distribution of the distance between Y_t and the edge $\{e, a\}$ in the Cayley graph of D_{2l} .

Fact 3.4. *For each $t \in \mathbb{Z}^+$, the distribution $(p_t(0), \dots, p_t(l-1))$ is non-increasing.*

Proof. This is obvious for $t = 0$, and we have for t even $p_{t+1}(2x) = p_{t+1}(2x+1) = \frac{1}{2}(p_t(2x) + p_t(2x+1))$, and for t odd $p_{t+1}(2x+1) = p_{t+1}(2x+2) = \frac{1}{2}(p_t(2x+1) + p_t(2x+2))$ and $p_{t+1}(0) = p_t(0)$ and $p_{t+1}(l-1) = p_t(l-1)$, so the fact follows by induction. \square

As $d(Y_t, \{a, e\}) \leq d(Y_t, e) = |Y_t| \leq d(Y_t, \{a, e\}) + 1$, we deduce that (for $l \geq 4$)

$$(1) \quad \mathbb{P}\left(|Y_t| \in \left[\frac{3l}{2}, 2l\right]\right) \leq \mathbb{P}\left(|Y_t| \in \left[\frac{l}{2}, \frac{3l}{2}\right]\right)$$

In order to prove Lemma 3.3, we condition our samples by the random walk obtained by projection on the cyclic base-group. This fixes the local time function $t^{(n)} : \mathbb{Z} \rightarrow \mathbb{Z}^+$ introduced in the previous proof and the final position i_n . Lemma 3.3 reduces to finding a universal constant C such that for any $t^{(n)}$ and i_n we have

$$(2) \quad \mathbb{E}(|Z_n^{2l}| | t^{(n)}, i_n) \leq C \mathbb{E}(|Z_n^l| | t^{(n)}, i_n)$$

Under this conditioning, $Z_n^{2l} = (i_n, f_n^{2l})$ is described by the random products $f_n^{2l}(x) = a^{\varepsilon_1(x)} b^{\varepsilon_2(x)} \dots a^{\varepsilon_{t^{(n)}(x)}(x)}$ for $x \in \mathbb{Z}$ of fixed length $t^{(n)}(x)$, starting either by a or by b (and the latter is determined by the projection on the cyclic group), where all the $\varepsilon_i(x)$ for $i \in \mathbb{Z}^+$ and $x \in \mathbb{Z}$ are independent.

Then the word length $|Z_n^{2l}|$ is obtained as the solution of the following travelling salesman problem. The traveller starts at 0 and ends at i_n . Meanwhile, he must for each $x \in \mathbb{Z}$ visit the sites x and $x - k$ respectively $A(x)$ and $B(x)$ times, in order that $A(x)$ and $B(x)$ are respectively the number of a 's and b 's in a word representing $f_n^{2l}(x)$ and these visits to x and $x - k$ have to be made in the order of appearance of the letters a and b in this word. However there is no constraint in the order of visits corresponding to distinct integers x and x' .

The word length $|Z_n^l|$ is obtained by the same travelling salesman problem, where we replace $f_n^{2l}(x)$ by its projection $f_n^l(x)$ onto D_l . Inequality (2) is only true in average because of the existence of words where $f_n^{2l}(x) = (ab)^l$ which is very long in D_{2l} , but trivial in D_l . Since there is no dependance on the length n of the random product Z_n^{2l} , we drop the subscript to ease notations. Our first aim is to obtain a good approximation of the solution of this salesman problem.

For each, $Z = (i, f) \in \mathbb{Z} \wr D_l$, we call range of Z both the set $\{x | f(x) / \langle b \rangle \neq e\} \cup \{x - k | f(x) / \langle a \rangle \neq e\} \cup \{0, i\}$ and the difference between its maximum and minimum (the context will make clear which one we refer to). It always satisfies

$|Z| \geq \text{Range}(Z)$. In fact, the range of Z is the set of integers visited by the projection on \mathbb{Z} of a minimal word representing Z .

We denote $L = L_Z$ the function on \mathbb{Z} taking non-negative value $L(x)$ such that $b^{\varepsilon_1(x)}(ab)^{L(x)}a^{\varepsilon_2(x)}$ with $\varepsilon_i \in \{0, 1\}$ is a minimal word representing $f(x)$. The length of Z is approximated in terms of the function L and the range as follows.

Lemma 3.5. *For any $l, k \geq 0$ and $Z = (i, f)$, we have*

$$\sum_{x \in \mathbb{Z}} 2 \max(L(y) \mathbb{1}_{[y-k, y]}(x)) \leq |Z| \leq \sum_{x \in \mathbb{Z}} 2 \max(L(y) \mathbb{1}_{[y-k, y]}(x)) + 5 \text{Range}(Z),$$

with respect to the switch-walk-switch generating set.

Proof. For each y in \mathbb{Z} , the solution of the travelling salesman problem has to visit y and $y-k$ at least $L(y)$ times and alternatingly. If x belongs to $[y-k, y)$ the salesman has to cross at least $2L(y)$ times the edge right of x . This gives the first inequality. (The choice of ε_i guarantees that any other word representing $f(x)$ contains $(ab)^{L(x)}$ as subword.)

To show the second inequality, it is sufficient to describe a path in \mathbb{Z} that permits to represent Z . The first part of the path consists in moving from 0 to the maximum of the range, crossing on the way all sites $x-k$ where $\varepsilon_1(x) \neq 0$ (and turning on the lamps there appropriately). This has length less than twice the range. The last part of the path consists in moving from the minimum of the range to i , crossing on the way all sites x where $\varepsilon_2(x) \neq 0$ (and turning on the lamps), which also has length less than twice the range.

The central part of the path consists in moving from the maximum to the minimum of the range, switching on the way each lamp at x on $(ab)^{L(x)}$, this can be done in less than $\sum_{x \in \mathbb{Z}} 2 \max(L(y) \mathbb{1}_{[y-k, y]}(x)) + \text{Range}(Z)$ steps. Let us prove it by induction on $\sum_{x \in \mathbb{Z}} L(x)$. If the sum is zero, the path just goes from the maximum to the minimum. In general, the salesman should first travel from the maximum of the range to the maximum x_1 of the support of L , then to the maximum x'_1 of the points x such that $L(y) = 0$ for all $y \in [x, x+k)$, then back to x_1 , switching on the a -lamps between x_1 and x'_1+k and the b -lamps between x'_1 and x_1-k (the interval $[x'_1+k, x_1]$ is the rightmost k -connected component of the support of L). The rest of the path is described by induction applied between x_1 and the minimum of the range and to L' satisfying $L'(x) = L(x) - 1$ whenever x is in the rightmost k -connected component of the support of L . \square

Proof of Lemma 3.3. It is sufficient to prove inequality (2). First observe that if $\forall x \in \mathbb{Z}, t(x) \leq \frac{l}{2}$, then $|Z^{2l}| = |Z^l|$. Indeed, this assumption implies that $|f^{2l}(x)| \leq \frac{l}{2}$, so its image in the quotient $D_{2l} \rightarrow D_l$ also has $|f^l(x)| \leq \frac{l}{2}$. If a path γ in \mathbb{Z} is the projection of a switch-walk-switch word representing Z^l , it permits to write $f^l(x)$ at each x . Thus the same path permits to write one of its two preimages $f^{2l}(x)$ or $f^{2l}(x)(ab)^l$. Now as the second preimage has length $\geq \frac{3l}{2}$ it contains as a subword any reduced word of length $\leq \frac{l}{2}$, in particular $f^{2l}(x)$. Thus keeping the same projection γ on \mathbb{Z} and only modifying the lamp-factors, we can find a word of length $|\gamma|$ representing Z^{2l} .

For the general case, let X denote the set of points x where $t(x) \geq \frac{l}{2}$. To each sample we associate $A = \{x \in \mathbb{Z} \mid |f^{2l}(x)| \geq \frac{l}{2}\}$ and we condition by subsets A of X to get

$$\mathbb{E}(|Z^{2l}| \mid t, i) = \sum_{A \subset X} \mathbb{E}(|Z^{2l}| \mid t, i, A) \mathbb{P}(\{x \in \mathbb{Z} \mid |f^{2l}(x)| \geq \frac{l}{2}\} = A)$$

From now on we fix t, i, A and the value of f^{2l} outside A . We define $Z_{\text{cut}}^{2l} = (i, f_{\text{cut}}^{2l})$ by $f_{\text{cut}}^{2l}(x) = f^{2l}(x)$ for $x \notin A$, and $f_{\text{cut}}^{2l} = (ab)^{\frac{l}{4}}$ of length $\frac{l}{2}$ for $x \in A$. We have $|Z^{2l}| \leq 9|Z_{\text{cut}}^{2l}|$. Indeed, if we have an appropriate path for Z_{cut}^{2l} , up to going through back and forth four times, we can write any representative word at x instead of $(ab)^{\frac{l}{4}}$, and a ninth travel ends at i . Moreover, our first observation shows that $|Z_{\text{cut}}^{2l}| = |Z_{\text{cut}}^l|$.

Now consider the random partition of A into $A^+ \cup A^-$, where A^+ is the set of points x in A where $|f^{2l}(x)| \in [\frac{l}{2}, \frac{3l}{2}]$. By (1), we have for each x in A that $\mathbb{P}(x \in A^- \mid t, i, A) \leq \frac{1}{2}$. We define the discretized element Z_{rand}^l by $f_{\text{rand}}^l(x) = f^l(x)$ for $x \notin A$, $f_{\text{rand}}^l(x) = (ab)^{\frac{l}{4}}$ for $x \in A^+$, and $f_{\text{rand}}^l(x) = e$ for $x \in A^-$. We have $|Z_{\text{rand}}^l| \leq 3|Z^l|$. Indeed, $f^l(x)$ is a subword of $f_{\text{rand}}^l(x)$ except when $f^l(x) = (ba)^{\frac{l}{4}}$, so switching the lamps of an appropriate path for Z^l is also appropriate except when $f^l(x) = (ba)^{\frac{l}{4}}$, but this word can be written at x on the way back. A third travel ends at i .

To get the lemma, there remains to show that

$$(3) \quad |Z_{\text{cut}}^l| \leq 22\mathbb{E}(|Z_{\text{rand}}^l| \mid t, i, A, f|_{\mathbb{Z} \setminus A})$$

By Lemma 3.5, we have

$$\mathbb{E}(|Z_{\text{rand}}^l| \mid t, i, A) \geq \mathbb{E} \sum_{x \in \mathbb{Z}} 2 \max_{y \in (x, x+k]} (L_{\text{rand}}(y)) \geq \sum_{x \in \mathbb{Z}} 2 \max_{y \in (x, x+k]} \mathbb{E} L_{\text{rand}}(y).$$

Now for $y \notin A$, $L_{\text{rand}}(y) = L(y) = L_{\text{cut}}(y)$ and for $y \in A$, $L_{\text{rand}}(y) = \frac{l}{4}$ with probability $\mathbb{P}(f^{2l}(y) \in [\frac{l}{2}, \frac{3l}{2}]) \geq \frac{1}{2}$ and $L_{\text{rand}}(y) = 0$ with probability $\mathbb{P}(f^{2l}(y) \in (\frac{3l}{2}, 2l]) \leq \frac{1}{2}$ and $L_{\text{cut}}(y) = \frac{l}{4}$, so $\mathbb{E} L_{\text{rand}}(y) \geq \frac{1}{2} L_{\text{cut}}(y)$ for any y in \mathbb{Z} . We deduce $\mathbb{E}(|Z_{\text{rand}}^l| \mid t, i, A) \geq \sum_{x \in \mathbb{Z}} \max(L_{\text{cut}}(y) \mathbf{1}_{[y-k, y)}(x))$.

On the other hand, the range of Z_{rand}^l is equal to the range of Z_{cut}^l with probability $\geq \frac{1}{4}$ (when both the minimal and maximal lamps are on), so $\mathbb{E}(|Z_{\text{rand}}^l| \mid t, i, A) \geq \mathbb{E}(\text{Range}(Z_{\text{rand}}^l) \mid t, i, A) \geq \frac{1}{4} \text{Range}(Z_{\text{cut}}^l)$. The two previous inequality and Lemma 3.5 imply (3).

Piling up the inequalities between Z^{2l} , Z_{cut}^{2l} , Z_{cut}^l , Z_{rand}^l and Z^l , we obtain Lemma 3.3 with $C = 594$. \square

4. PROOF OF THEOREM 1.1

4.1. A baby case. The following proposition is the core of Theorem 1.1. It is not used directly in the proof, but serves as a guideline.

Proposition 4.1. *Let C be the universal constant of Lemma 3.3, and let $\lambda \in (\frac{1}{2}, \frac{3}{4})$. For any $n_0 \in \mathbb{Z}_{\geq 0}$, there exists k, l integers such that the speed of the switch-walk-switch random walk on $\Gamma(k, l, \infty)$ satisfies*

- (1) $\forall n \leq n_0, \mathbb{E}|Z_n^{\Gamma(k, l, \infty)}| = \mathbb{E}|Z_n^{\Gamma(0, 2, \infty)}|,$
- (2) $\forall n \geq 0, \mathbb{E}|Z_n^{\Gamma(k, l, \infty)}| \leq n^\lambda,$
- (3) $\exists N \geq n_0, \mathbb{E}|Z_n^{\Gamma(k, l, \infty)}| \geq \frac{N^\lambda}{C},$
- (4) $\exists C_1 > 0, \mathbb{E}|Z_n^{\Gamma(k, l, \infty)}| \leq C_1 \sqrt{n}$ for n large enough.

Proof. By Lemma 2.4, choose k large enough so that the balls of radius n_0 in $\Gamma(k, l, \infty)$ and $\Gamma(0, 2, \infty)$ coincide. This gives (1). Remark 3.2 ensures (4). Now let l be minimal such that there exists N with $\mathbb{E}|Z_N^{\mathbb{Z}_N^{D_{2l}}}| > N^\lambda$ (this exists by Lemma 3.1). Minimality gives (2) and we obtain (3) by Lemma 3.3. \square

4.2. The case $\lambda \in [\frac{1}{2}, \frac{3}{4}]$. Let $\Delta = \Delta(k, l, m)$ be the infinite diagonal product of the groups $\Gamma(k_s, l_s, m_s)$ as defined in section 2.3. This group is 3-solvable. We assume that the sequence $k = (k_s)$ tends to infinity, and that all values of k, l, m are finite.

For each $s \in \mathbb{Z}_{\geq 1}$, denote Λ_s the diagonal product from $k = 1$ to $s - 1$ of these groups, which is a finite group, and M_s the diagonal product from s to infinity. The group Δ is the diagonal product $\Lambda_s \times M_s$.

By Lemma 2.2, the group M_s admits $\Gamma(0, 2, \infty)$ as quotient, and if $\frac{k_s-1}{2} > R$, the balls of radius R in $\Lambda_s \times M_s$ and in $\Delta_s = \Lambda_s \times \Gamma(0, 2, \infty)$ coincide.

Let $\lambda \in [\frac{1}{2}, \frac{3}{4}]$ et $\varepsilon(n) \rightarrow \infty$. To prove Theorem 1.1, it is sufficient to construct by induction on s integers $k_1 < \dots < k_{s-1}$, l_1, \dots, l_{s-1} and m_1, \dots, m_{s-1} such that the switch-walk-switch random walk on $\Delta_s = \Lambda_s \times \Gamma(0, 2, \infty)$ satisfies:

- (4) $\forall n \geq 1, n^{\frac{1}{2}} \leq \mathbb{E}|Z_n^{\Delta_s}| \leq n^\lambda$ and
 $\exists n_1 < N_1 < \dots < n_{s-1} < N_{s-1}, \mathbb{E}|Z_{n_i}^{\Delta_s}| \leq n_i^{\frac{1}{2}} \varepsilon(n_i)$ and $\mathbb{E}|Z_{N_i}^{\Delta_s}| \geq \frac{N_i^\lambda}{\varepsilon(N_i)},$

and that $\frac{k_s-1}{2} > N_{s-1}$, so the balls of radius N_{s-1} in Γ_s and in Γ coincide. (To get the Theorem with $\lambda = \frac{3}{4}$, the function n^λ should be replaced by $\frac{n^{\frac{3}{4}}}{\log n}$, which does not alter the proof.)

There is nothing to prove for $s = 0$. Assume this is true for s , and construct k_s, l_s, m_s . First observe that as Λ_s is a finite group, Lemma 2.1 gives two constants C_1, C_2 such that $\mathbb{E}|Z_n^{\Gamma(0, 2, \infty)}| \leq \mathbb{E}|Z_n^{\Delta_s}| \leq C_1 \mathbb{E}|Z_n^{\Gamma(0, 2, \infty)}| + C_2$. By Remark 3.2, we can choose n_s such that

- (5) $\forall n \geq n_s, \mathbb{E}|Z_n^{\Delta_s}| \leq n^{\frac{1}{2}} \varepsilon(n)$ and $\frac{n^\lambda - C_2}{CC_1} \geq \frac{n^\lambda}{\varepsilon(n)},$

where C is the universal constant of Lemma 3.3.

Now we choose k_s such that $\frac{k_s-1}{2} > n_s$, which guarantees by Lemma 2.4 that the balls of radius n_s in $\Gamma(0, 2, \infty)$ and in M_s coincide (and in particular also their speed functions for $n \leq n_s$), and such that $\frac{k_s-1}{2}$ is bigger than the radius R of Lemma 2.1 with $F = \Lambda_s$ and $\Gamma = \Gamma(0, 2, \infty)$ so that the constants C_1, C_2 are valid for $\Lambda_s \times M_s$ instead of Δ_s .

For l even, we consider the family of auxiliary groups $G_l = \Lambda_s \times \Gamma(k_s, l, \infty)$. As l is even, these groups all admit $\Gamma(0, 2, \infty)$ as quotient, so Lemma 2.1 applies to both G_l and G_{2l} . Moreover, Lemma 3.3 relates the speed in $\Gamma(k_s, l, \infty)$ and $\Gamma(k_s, 2l, \infty)$. Combining these results, we get

$$(6) \quad \frac{\mathbb{E}|Z_n^{G_l}| - C_2}{C_1} \leq \mathbb{E}|Z_n^{G_{2l}}| \leq C_1 C \mathbb{E}|Z_n^{G_l}| + C_2.$$

Now let l_s be the minimal l such that there exists $N = N_s$ with $\mathbb{E}|Z_N^{G_{2l}}| > N^\lambda$. This l_s exists by Lemma 3.1. By (6) and the second condition (5), we have

$$(7) \quad \forall n \geq 1, \mathbb{E}|Z_n^{\Lambda_s \times \Gamma(k_s, l_s, \infty)}| \leq n^\lambda \text{ and } \mathbb{E}|Z_{N_s}^{\Lambda_s \times \Gamma(k_s, l_s, \infty)}| \geq \frac{N_s^\lambda}{\varepsilon(N_s)}$$

Now we choose $m_s > 2N_s$ in order that the balls of radius N_s in $\Lambda_s \times \Gamma(k_s, l_s, \infty)$, in $\Lambda_s \times \Gamma(k_s, l_s, m_s) \times \Gamma(0, 2, \infty)$ and in $\Lambda_s \times \Gamma(k_s, l_s, m_s) \times M_{s+1} = \Delta$ all coincide as soon as $\frac{k_s+1-1}{2} > N_s$.

Remark 4.2. The existence of l_s is due to an intermediate value argument. In this view-point, Lemma 3.3 is a statement of continuity in l of the speed function of $\Gamma(k, l, \infty)$, uniform in k .

Remark 4.3. By choosing n_s even larger at step (5) in the previous proof, we can get arbitrarily large intervals where $\mathbb{E}|Z_n^\Delta|$ is close to \sqrt{n} . In contrast, we have no control on the width of intervals where $\mathbb{E}|Z_n^\Delta|$ is close to n^λ . Note that if we try to reduce n_s , for instance to obtain a bigger lower exponent, we lose control over the upper exponent because the second condition (5) and thus (7) may fail.

4.3. The general case. To obtain the case $i = 2$ (i.e. $\lambda \in [\frac{3}{4}, \frac{7}{8}]$), the idea is essentially to replace $\Delta(k, l, m)$ by $\mathbb{Z} \wr \Delta(k, l, m)$ and use Erschler [4] that $\mathbb{E}|Z_n^{\mathbb{Z} \wr \Delta}|$ is comparable with $\sqrt{n \mathbb{E}|Z_n^\Delta|}$. However, this is known only under the hypothesis that $\mathbb{E}|Z_n^\Delta|$ is concave, which is not clear in our case. In practice, we get Theorem 1.1 by running the machinery of the previous proof, "interpolating" between $\mathbb{Z} \wr (\mathbb{Z} \wr D_2)$ and $\mathbb{Z} \wr (\mathbb{Z} \wr D_\infty)$.

For $i \geq 1$, let $\Gamma_{i+1}(k, l, m) = \mathbb{Z}/m\mathbb{Z} \wr \Gamma_i(k, l, m)$, where $\Gamma_1(k, l, m) = \Gamma_1(k, l, m)$ with switch-walk-switch generating set. As marked generating set for Γ_{i+1} , we use the switch-walk-switch generating set where switch generators are the marked generators of Γ_i at position 0.

Lemma 4.4. *With the above notations, for any integers $i \geq 1$, $l \geq 0$ and $k \geq 0$*

- (1) *if $2k < m$, the balls of radius $\frac{k-1}{2}$ in $\Gamma_i(k, l, m)$ and $\Gamma_i(0, 2, \infty)$ coincide.*
- (2) *the speed in $\Gamma_i(k, \infty, \infty)$ is comparable with $n^{1-\frac{1}{2^i+1}}$.*
- (3) *the groups $\Gamma_i(k, 2, \infty)$ and $\Gamma_i(0, 2, \infty)$ are isomorphic as marked groups, and their speed is comparable with $n^{1-\frac{1}{2^i}}$.*
- (4) *there is a universal constant C_i such that*

$$\frac{1}{C_i} \mathbb{E}|Z_n^{\Gamma_i(k, l, \infty)}| \leq \mathbb{E}|Z_n^{\Gamma_i(k, 2l, \infty)}| \leq C_i \mathbb{E}|Z_n^{\Gamma_i(k, l, \infty)}|.$$

Proof. This follows from Lemma 2.2 and 3.1, Remarks 2.3 and 3.2, Lemma 3.3 and Erschler [4]. \square

The group with speed oscillating between $n^{1-\frac{1}{2^i}}$ and n^λ will be a diagonal product $\Delta_i = \Delta_i(k, l, m)$ of $\Gamma_i(k_s, l_s, m_s)$ with appropriate sequences $k = (k_s)$ tending to ∞ , $l = (l_s)$ and $m = (m_s)$ with $m_s > 2k_s$. The proof of section 4.2 applies, replacing the references to previous sections by Lemma 4.4.

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