The fundamental group of a Hopf linear category

Dedicated to Eduardo N. Marcos for his 60th birthday

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Abstract

We define the fundamental group of a Hopf algebra over a field. For this we consider gradings of Hopf algebras and Galois coverings. The latter are given by linear categories with new additional structure which we call Hopf linear categories over a finite group. We compute some examples.

2010 MSC: 16T05 16W50 14H30 18D10 18D20

1 Introduction

The main purposes of this paper are to initiate the theory of the fundamental group of a Hopf algebra over a field, relative to a finite group, and to compute it in some examples. In this context there is no analogous of homotopy theory of loops available, so the situation differs from the algebraic topology setting. Instead we use the approach developed in recent years for considering an intrinsic fundamental group "à la Grothendieck" of an associative algebra over a field or more generally of a small category over a field, see [7, 5, 6]. As quoted in [7], we follow methods closely related to the way in which the fundamental group is considered in algebraic geometry, after A. Grothendieck and C. Chevalley, see for instance [11].

Gradings by different groups of a given Hopf algebra constitute the main tool. The grading groups are a sort of approximation to the fundamental group that we define in this paper. More precisely, in order to retain only useful information, we consider connected gradings, namely gradings where the degrees of the homogeneous components generate the grading group.

Note that to have a grading of a Hopf algebra H as defined by S. Montgomery in [13], the group which grades is required to be abelian in order to insure that the same group grades $H \otimes H$ so that the comultiplication $\Delta : H \longrightarrow H \otimes H$ is homogeneous. This differs from the associative algebra context, where the groups grading an algebra are arbitrary. In case the abelian group Γ grading a Hopf algebra is finite, a Galois covering exists using a smash product category with Galois group Γ – see also E. Beneish and W. Chin [2]. The linear category that we obtain has an additional structure that we consider in the first section. More precisely, we define a Hopf k-category over a finite group G to be a small k-category whose objects are the elements of G, equipped with a comultiplication, a counit and an

*This work has been supported by the projects UBACYT 20020130100533BA, PIP-CONICET 112–201101–00617, PICT 2011–1510 and MATHAMSUD-GR2HOPF. The second author is a research member of CONICET (Argentina).

antipode. Note that this structure extend the one considered by A. Virelizier in [15]. In particular a Hopf k-category over a trivial group is a Hopf k-algebra. The smash product of a Hopf k-category $\mathcal{H}$ over a finite group G with respect to a finite abelian group Γ is a Hopf k-category over $G \times \Gamma$. If a universal connected grading of a Hopf k-category exists, or equivalently a universal Galois covering exists, the involved group would be the fundamental group. We exhibit in Proposition 4 an example where this occurs.

The fundamental group of $\mathcal{H}$ is obtained by considering all the connected gradings by finite abelian groups – which provide Galois coverings – and morphisms between them. An element of the fundamental group is a coherent family of elements in the groups which are grading $\mathcal{H}$ in a connected way. Consequently this group is abelian. This property can be compared with the fact that the usual fundamental group of an H-space is abelian. Recall that an H-space is a topological space with additional structure, namely an associative continuous product with a neutral element, see for instance [12]. Note that a Hopf k-category is a k-category with additional structure as well, namely a comultiplication, a counit and an antipode.

We observe two facts. Firstly the fundamental group of the trivial Hopf k-category over any group is trivial. Secondly this group is invariant with respect to isomorphisms of Hopf k-categories.

We compute the fundamental group of Taft Hopf k-categories over finite cyclic groups. Finally, we use the classification of gradings of diagonal algebras obtained by S. Dăscălescu in [10] in order to prove the following. If G is a finite abelian group and k is a field such that the number of $|G|$-th roots of unity is $|G|$, then the Hopf algebra kG has trivial fundamental group. Note that the preceding hypotheses imply that kG is semisimple.

2 Hopf linear categories

Let k be a field. A *k-category* $\mathcal{B}$ is a small category with set of objects $\mathcal{B}_0$ which is enriched over k-vector spaces. In other words, for $x, y \in \mathcal{B}_0$ the set of morphisms ${}_y\mathcal{B}_x$ from x to y is a k-vector space and composition of morphisms is k-bilinear. The endomorphism spaces are k-algebras and each morphism space ${}_y\mathcal{B}_x$ is a ${}_y\mathcal{B}_y - {}_x\mathcal{B}_x$-bimodule.

Definition 2.1 *Let Λ be a k-algebra and let E be a finite set of orthogonal idempotents of Λ which is complete, i.e. $\sum_{e\in E} e = 1$. The* Peirce category *of Λ with respect to E is the k-category $\mathcal{B}_{\Lambda,E}$ whose set of objects is E and ${}_y(\mathcal{B}_{\Lambda,E})_x = y\Lambda x$ for all $x, y \in E$. Composition is inferred from the product in Λ.*

The reverse construction for a k-category $\mathcal{B}$ with a finite number of objects is the *sum-algebra* $\oplus\mathcal{B}$ which is the direct sum of all vector spaces of morphisms, equipped with the product inferred from the matrix product combined with composition in $\mathcal{B}$.

Definition 2.2 *Let G be a finite group. A* Hopf k-category *over G is a k-category $\mathcal{H}$ whose objects are the elements of G, provided with the following additional data for each pair of objects $x, y \in G$.*

1. *A k-linear comultiplication map*

$$ {}_y\Delta_x : {}_y\mathcal{H}_x \longrightarrow \bigoplus_{\substack{x''x'=x\\ y''y'=y}} {}_{y''}\mathcal{H}_{x''} \otimes {}_{y'}\mathcal{H}_{x'}. $$

In the following we make use of the following convention: if two morphisms g and f cannot be composed because the target object $\tau(f)$ of f is different from the source object $\sigma(g)$ of g, then $gf = 0$.
The comultiplication verifies that whenever $g \in {}_z\mathcal{H}_y$ and $f \in {}_y\mathcal{H}_x$ are composable,

$$ {}_z\Delta_x(gf) = {}_z\Delta_y(g)\, {}_y\Delta_x(f). $$

Moreover

$$ {}_x\Delta_x({}_x1_x) = \sum_{x''x'=x} {}_{x''}1_{x''} \otimes {}_{x'}1_{x'}. $$

2. *A counit*

$$ {}_y\epsilon_x : {}_y\mathcal{H}_x \to k, $$

which is a k-functor from $\mathcal{H}$ to the single object category with endomorphism algebra reduced to k.

3. *An antipode S which is a contravariant functor of $\mathcal{H}$ such that $S(x) = x^{-1}$.*

The previous maps verify the conditions bellow, analogous to those defining a Hopf k-algebra. We state them using using Sweedler's notation

$$ \Delta\left({}_yf_x\right) = \sum_{\substack{x''x'=x\\ y''y'=y}} {}_{y''}f''_{x''} \otimes {}_{y'}f'_{x'}. $$

1. *Coassociativity: given objects x and y, both maps from ${}_y\mathcal{H}_x$ to*

$$ \bigoplus_{\substack{x'''x''x'=x\\ y'''y''y'=y}} {}_{y'''}\mathcal{H}_{x'''} \otimes {}_{y''}\mathcal{H}_{x''} \otimes {}_{y'}\mathcal{H}_{x'} $$

induced by the comultiplication coincide.

2. *Counit: if x and y are objects such that at least one of them is different from 1_G, then ${}_y\epsilon_x = 0$, and given ${}_yf_x \in {}_y\mathcal{H}_x$,*

$$ {}_1\epsilon_1({}_1f''{}_1)\ \ {}_yf'{}_x = {}_yf_x \text{ and } {}_yf''{}_x\ \ {}_1\epsilon_1({}_1f'{}_1) = {}_yf_x. $$

3. *Antipode: for ${}_yf_x \in {}_y\mathcal{H}_x$,*

$$ \sum_{\substack{x''x'=x\\ y''y'=y}} S({}_{y''}f''_{x''})\ {}_{y'}f'_{x'} = {}_y\epsilon_x({}_yf_x) \text{ and } \sum_{\substack{x''x'=x\\ y''y'=y}} {}_{y''}f''_{x''}\ S({}_{y'}f'_{x'}) = {}_y\epsilon_x({}_yf_x). $$

Alternatively, this situation may be described in terms of a new structure defined as follows. Given a small k-category $\mathcal{B}$, the category $\mathcal{PB}$ of *parts* of $\mathcal{B}$ has as set of objects all finite subsets of $\mathcal{B}_0$. For E and F finite subsets of $\mathcal{B}_0$, we define

$$ {}_F(\mathcal{PB})_E = \bigoplus_{\substack{x\in E\\ y\in F}} {}_y\mathcal{B}_x. $$

Composition of morphisms in $\mathcal{PB}$ is given by the matrix product combined with composition in $\mathcal{B}$.

We consider a functor of k-categories

$$\Delta : \mathcal{H} \to \mathcal{P}(\mathcal{H} \otimes \mathcal{H})$$

defined on objects by $\Delta(x) = \{(x'', x') \mid x''x' = x\}$ and through the family $\{{}_y\Delta_x\}_{x,y\in\mathcal{B}_0}$. The above definition of a Hopf category can be translated into this setting.

Example 2.3 *The trivial Hopf k-category $\mathcal{T}_G$ over a finite group G has G as set of objects, the zero morphism space between different objects while ${}_x(\mathcal{T}_G)_x = k$ for each object x. We set*

$${}_x\Delta_x({}_x1_x) = \sum_{x''x'=x} {}_{x''}1_{x''} \otimes {}_{x'}1_{x'},$$

while if $x \neq 1$ we put $\epsilon({}_x1_x) = 0$, and $\epsilon({}_11_1) = 1$. The antipode is given by $S({}_x1_x) = {}_{x^{-1}}1_{x^{-1}}$.

Recall that a free k-category is a k-category $\mathcal{L}$ given by a set of objects $\mathcal{L}_0$ and a family of vector spaces $\{{}_yV_x\}_{x,y\in\mathcal{L}_0}$. The morphisms in $\mathcal{L}$ are generated by this family, *i.e.* they are the concatenated tensor products of these vector spaces.

Example 2.4 *[3] Let $C_n = \langle t | t^n = 1\rangle$ be a cyclic group of order $n \geq 2$ and let $\mathcal{C}$ be the free k-category which has C_n as set of objects and such that $\{a_u : u \to tu\}_{u\in C_n}$ generates the morphisms. Fixing $q \in k^*$, the category $\mathcal{C}$ is in fact a Hopf k-category, with comultiplication induced by*

$${}_{t^{i+1}}\Delta_{t^i}(a_{t^i}) = \sum_{t^jt^k=t^i} t^j \otimes a_{t^k} + q^k a_{t^j} \otimes t^k.$$

See Lemma 3.1 of [3].

In case q is a n-th primitive root of unity, the two-sided ideal I of $\mathcal{C}$ generated by all compositions of length n of the a_i's is a Hopf ideal, that is $\mathcal{C}_q^n = \mathcal{C}/I$ inherits a Hopf k-category structure. Note that the sum-algebra $\oplus\mathcal{C}_q^n$ is a Hopf algebra isomorphic to the n-th-Taft algebra. We call $\mathcal{C}_q^n$ the n-th-*Taft category*.

More generally, the examples with underlying free k-category are given by direct applications of [9] as follows.

Let G be a finite group, let k^G be the Hopf algebra of functions from G to k and let B be a k^G-Hopf bimodule. Since k^G is a semisimple algebra, B has a unique decomposition as direct sum of its isotypic k^G-bimodule components $\delta_yB\delta_x$, where δ_x denotes the Dirac mass on x, namely $\delta_x(y)$ is zero for y different from x and $\delta_x(x) = 1$.

Let $\mathcal{H}_B$ be the free k-linear category with set of objects G and whose family of vector spaces is $\{\delta_yB\delta_x\}_{x,y\in G}$. The comultiplication on each generator is the sum of the left and right comodule structure morphisms, extended in the unique possible way to a global comultiplication. The antipode and the counit exist, see [9].

Theorem 2.5 *Let G be a finite group and let k be a field. There is a one-to-one correspondence between Hopf k-categories over G and the embeddings of k^G in Hopf k-algebras.*

Proof. Let $\mathcal{H}$ be a Hopf k-category over G and let $\oplus\mathcal{H}$ be the corresponding sum-algebra. The data of $\mathcal{H}$ provides a Hopf algebra structure on $\oplus\mathcal{H}$. The inclusion of k^G in $\oplus\mathcal{H}$ is obtained by sending each δ_x to the element of the direct sum of all morphisms which is zero everywhere except in ${}_x\mathcal{H}_x$, where its value is ${}_x1_x$.

Reciprocally, consider an embedding of Hopf k-algebras $k^G \subset H$.The set $E = \{\delta_x\}_{x\in G}$ is a complete finite system of orthogonal idempotents of H.

Let $\mathcal{B}_{H,E}$ be the corresponding Peirce category, whose objects are in one-to-one correspondence with elements of G. The comultiplication Δ of H restricted to $\delta_y H \delta_x = {}_y\left(\mathcal{B}_{H,E}\right)_x$ provides ${}_y\Delta_x$. The family $({}_y\Delta_x)_{x,y\in G}$ verifies the requested properties. Similarly, the counit $\epsilon : H \to k$ restricts to ${}_y\epsilon_x : \delta_y H\delta_x \to k$ and the antipode S provides ${}_yS_x : \delta_y H\delta_x \to \delta_{x^{-1}} H \delta_{y^{-1}}$. ◇

Remark 2.6 *Let G be a finite group. The identity embedding of k^G into itself corresponds to the trivial Hopf k-category $\mathcal{T}_G$ over G, see Example 2.3.*

3 Gradings and the smash product

We first recall the definition of a connected grading of a k-category $\mathcal{B}$ and the corresponding smash product, see for instance [4].

Definition 3.1 A *grading* X of $\mathcal{B}$ by a group Γ_X is a direct sum decomposition of each morphism space

$$ {}_y\mathcal{B}_x = \bigoplus_{s\in\Gamma_X} X^s{}_y\mathcal{B}_x $$

compatible with composition: for $x, y, z \in \mathcal{B}_0$ and $s, t \in \Gamma_X$,

$$ \left(X^t{}_z\mathcal{B}_y\right)\left(X^s{}_y\mathcal{B}_x\right) \subset X^{ts}{}_z\mathcal{B}_x. $$

In case $X^s{}_y\mathcal{B}_x$ is non-zero, this vector space is called the *homogeneous component of degree s from x to y.*

A *virtual morphism* is a pair (f, ϵ) where f is a morphism and $\epsilon = 1$ or -1. Source and target objects of virtual morphisms remain unchanged if $\epsilon = 1$ and are reversed if $\epsilon = -1$. A *walk* from x to y is a sequence $w = (f_n, \epsilon_n)\dots(f_1, \epsilon_1)$ where the morphisms f_i are non-zero, $\epsilon_i = \pm 1$ and $\tau(f_i, \epsilon_i) = \sigma(f_{i+1}, \epsilon_{i+1})$ for $i = 1, \dots, n-1$, with $\sigma(f_1, \epsilon_1) = x$ and $\tau(f_n, \epsilon_n) = y$.

A *homogeneous walk* w from x to y in $\mathcal{B}$ is a sequence as above, where each f_i is homogeneous in the graded category $\mathcal{B}$, of X-degree denoted $\text{deg}_X f_i$. The X-degree of w is defined as follows:

$$ \text{deg}_X w = (\text{deg} f_n)^{\epsilon_n} \dots (\text{deg} f_1)^{\epsilon_1}. $$

The set of homogeneous walks from x to y is denoted ${}_yHW_X(\mathcal{B})_x$. In this way we obtain a groupoid with set of objects $\mathcal{B}_0$, namely a category $HW_X(\mathcal{B})$ where all morphisms are invertible.

We assume that $\mathcal{B}$ is connected, that is from any object x in $\mathcal{B}$ we can reach any other object y by a non-zero walk. The grading X of $\mathcal{B}$ is *connected* if given any two objects x, y in $\mathcal{B}$, the degree map

$$ \text{deg}_X : {}_yHW_X(\mathcal{B})_x \to \Gamma_X $$

is surjective.

The following result is straightforward.

Lemma 3.2 *Let $\mathcal{B}$ be a single object k-category graded by a group Γ, equivalently let B be a graded k-algebra. The grading is connected if and only if the degrees of the homogeneous components of B generate Γ.*

Gradings and Galois coverings are related as follows. A *smash product category* $\mathcal{B}\#X$ is inferred from a grading X of $\mathcal{B}$, see [4, 5]:

$$(\mathcal{B}\#X)_0 = \mathcal{B}_0 \times \Gamma_X \text{ and } {}_{(y,t)}(\mathcal{B}\#X)_{(x,s)} = X^{t^{-1}s}{}_y\mathcal{B}_x.$$

One can easily check that there is a well-defined composition inherited from composition in $\mathcal{B}$, which makes $\mathcal{B}\#X$ a k-category. Moreover, if $\mathcal{B}$ and X are connected, the category $\mathcal{B}\#X$ is connected and the natural functor $F_X : \mathcal{B}\#X \to \mathcal{B}$ is a Galois covering of $\mathcal{B}$. Up to isomorphism, this construction provides all Galois coverings of $\mathcal{B}$.

In case there exists a *universal* grading, its group is by definition the fundamental group of $\mathcal{B}$. This occurs for some families of categories [6] but it is not always the case. In order to deal with the general situation, in [6, 7] a fundamental group *à la Grothendieck* is defined as follows.

A *morphism $\mu : X \to Y$ of connected gradings* is a group morphism $\mu : \Gamma_X \to \Gamma_Y$ such that there exists a homogeneous autofunctor $J : \mathcal{B} \to \mathcal{B}$ which is the identity on objects and such that given $b_0 \in \mathcal{B}_0$, the diagram

$$\begin{array}{ccc} {}_{b_0}HW_X(\mathcal{B})_{b_0} & \xrightarrow{HW(J)} & {}_{b_0}HW_Y(\mathcal{B})_{b_0} \\ \Big\downarrow{\scriptstyle \deg_X} & & \Big\downarrow{\scriptstyle \deg_Y} \\ \Gamma(X) & \xrightarrow[\mu]{} & \Gamma(Y) \end{array}$$

commutes. Note that in general J is not unique.

Definition 3.3 *The* fundamental group $\Pi_1(\mathcal{B}, b_0)$ *is the set of families (γ_X), where $\gamma_X \in \Gamma_X$ and X varies amongst the connected gradings of $\mathcal{B}$, such that for any morphism of connected gradings $\mu : X \to Y$ the equality $\mu(\gamma_X) = \gamma_Y$ holds. If such is the case, we say that (γ_X) is a coherent family. The product of coherent families is the pointwise product.*

Next we recall the definition of a graded Hopf algebra, see S. Montgomery [13, 10.5]. Note that M. Aguiar and S. Mahajan in [1] consider graded Hopf algebras as a setting where their results can be extended. Graded Hopf algebras are also used by F. Patras in [14].

Definition 3.4 *Let H be a Hopf k-algebra. A* grading *X of H by an abelian group Γ_X is a grading of the underlying associative algebra, such that comultiplication, counit and antipode are homogeneous.*

If $k^G \subset H$ is an embedding of Hopf algebras, a grading as above which verifies in addition that k^G is of trivial degree is called a k^G-grading.

Remark 3.5 *As quoted in the introduction, the fact that Γ_X is abelian insures that $H \otimes H$ is a graded algebra.*

Definition 3.6 *Let $\mathcal{H}$ be a Hopf k-category over G. A* Hopf grading *X of $\mathcal{H}$ by an abelian group Γ_X is a structure obtained by transporting a k^G- grading of $\oplus\mathcal{H}$ using Theorem 2.5.*

Example 3.7 *For $n \geq 2$, let q be a n-th primitive root of unity and let $\mathcal{C}_q^n$ be the n-th-Taft k-category. Let X be a grading of the underlying k-category. Since the spaces of morphisms are at most one dimensional, they are homogeneous. Let w be the homogeneous walk $w = (a_{t^{n-1}}, 1), \dots, (a_t, 1), (a_1, 1)$.*

The image of the homomorphism $\deg_X : {}_1HW(\mathcal{C}_q^n)_1 \to \Gamma_X$ is cyclic, generated by $\deg_X w = (\deg_X a_{t^{n-1}}) \dots (\deg_X a_t)(\deg_X a_1)$. Consequently, if X is connected then Γ_X is cyclic.

Lemma 3.8 *Let X be a connected Hopf grading of the n-th-Taft k-category $\mathcal{C}_q^n$ over C_n. The morphisms $\{a_u : u \to tu\}_{u \in C_n}$ of the free category have the same degree and the grading group Γ_X is cyclic of order coprime with n.*

Proof. Since comultiplication is homogeneous, we infer that all the free vector spaces of generators of the category have same degree that we call γ; as a consequence, $\deg_X w = \gamma^n$. The precedent discussion shows that γ^n is a generator of Γ_X, which is thus finite of order m, coprime with n. $\diamond$

Theorem 3.9 *Let $\mathcal{H}$ be a Hopf k-category over a finite group G. Let X be a Hopf grading of $\mathcal{H}$ by an abelian group Γ_X. If Γ_X is finite, then the smash product category $\mathcal{H}\#X$ is a Hopf k-category over the group $G \times \Gamma_X$.*

Remark 3.10 *We require that the grading group is finite in order to insure that the group $G \times \Gamma_X$ of the Hopf k-category obtained via the smash product is finite.*

Proof. For simplicity we consider a single object category $\mathcal{H}$, hence the set of objects of the smash product is Γ_X.

Each homogeneous element f of $\mathcal{H}$ gives rise to a family of morphisms in $\mathcal{H}\#X$

$$\left\{ {}_t f_{t\deg_X(f)} \right\}_{t \in \Gamma_X}.$$

Letting f vary we obtain all the morphisms of $\mathcal{H}\#X$. On the other hand, the coproduct of $\mathcal{H}$ provides

$$\Delta(f) = \sum f'' \otimes f',$$

where f'' and f' are homogeneous and verify $\deg_X(f) = \deg_X(f'')\deg_X(f')$.

In order to define the coproduct of a morphism, we consider the family

$$\left\{ {}_v f''_{v\deg_X(f'')} \otimes {}_u f'_{u\deg_X(f')} \right\}$$

with $u, v \in \Gamma_X$. Our purpose is to set a Hopf category structure on $\mathcal{H}\#X$, consequently we retain the members of this family verifying

$$vu = t \text{ and } v\deg_X(f'')u\deg_X(f') = t\deg_X(f).$$

Since Γ_X is abelian, the second requirement is a consequence of the first one. The coproduct of the smash product is thus defined as follows

$$\Delta\left({}_t f_{t\deg_X(f)}\right) = \sum_{t,x} {}_{tx^{-1}} f''_{tx^{-1}\deg_X(f'')} \otimes {}_x f'_{x\deg_X(f')}.$$

In order to prove that Δ is coassociative, consider the following computations:

$$(1\otimes\Delta)\Delta\left({}_t f_{t\deg_X(f)}\right) = \sum_{t,x,y} {}_{tx^{-1}}f'''_{tx^{-1}\deg_X(f''')} \otimes {}_{xy^{-1}}f''_{xy^{-1}\deg_X(f'')} \otimes {}_y f'_{y\deg_X(f')},$$

$$\Delta(\Delta\otimes 1)\left({}_t f_{t\deg_X(f)}\right) = \sum_{t,u,v} {}_{tu^{-1}v^{-1}}f'''_{tu^{-1}v^{-1}\deg_X(f''')} \otimes {}_v f''_{v\deg_X(f'')} \otimes {}_u f'_{u\deg_X(f')}.$$

Replacing v by xy^{-1} and y by u both expressions coincide. The verification for the antipode and the counit are straightforward. $\diamond$

Recall that the category of left modules over a Hopf algebra H admits a monoidal structure given by the tensor product over the base field k and the comultiplication Δ on H. Considering, as usual, linear functors from a Hopf k-category $\mathcal{H}$ to k-vector as left $\oplus\mathcal{H}$-modules, provides an immediate extension of this construction and a monoidal structure on such functors which are simply called $\mathcal{H}$-modules.

Note that the following well known result holds also under the assumption that the group is abelian.

Proposition 3.11 *Let H be a Hopf algebra graded by a group Γ_X. If Γ_X is abelian, then the category of graded H-modules is monoidal.*

Proof. Let M and N be graded H-modules with homogeneous components $\{X^sM\}_{s\in\Gamma_X}$ and $\{X^sN\}_{s\in\Gamma_X}$. The tensor product $M\otimes_k N$ decomposes as direct sum of vector spaces $\{X^s(M\otimes N)\}_{s\in\Gamma_X}$ where

$$X^s(M\otimes N) = \bigoplus_{s''s'=s} X^{s''}M\otimes X^{s'}N.$$

In case Γ_X is abelian, this decomposition provides a structure of graded H-module on $M\otimes N$ as follows. Given a homogeneous element $h\in X^\gamma H$, we know that

$$\Delta(h)\in\bigoplus_{\gamma''\gamma'=\gamma} X^{\gamma''}H\otimes X^{\gamma'}H.$$

Let $m\otimes n$ be an element of $X^{s''}M\otimes X^{s'}N$ with $s''s'=s$. Acting by h gives

$$h(m\otimes n)\in\bigoplus_{\gamma''\gamma'=\gamma}(X^{\gamma''}H)(X^{s'}N)\otimes(X^{\gamma'}H)(X^{s''}M)\subset\bigoplus_{\gamma''\gamma'=\gamma}X^{\gamma''s''}M\otimes X^{\gamma's'}N.$$

The total degree of each term of this last direct sum is $\gamma''s''\gamma's'$. Since Γ_X is abelian, this degree is γs as required. $\diamond$

Using the above results the proof of the following result is clear.

Theorem 3.12 *Let $\mathcal{H}$ be a Hopf k-category over G with a grading X whose structure group Γ_X is finite and abelian. There is an isomorphism of monoidal categories between Γ_X-graded modules over $\mathcal{H}$, and modules over the smash product $\mathcal{H}\#X$.*

4 Fundamental group of a Hopf linear category

In this section we will follow the lines of [7] as described in the previous section, in order to define the fundamental group of a Hopf k-category $\mathcal{H}$ over a finite group G. Let X and Y be Hopf gradings of $\mathcal{H}$. A morphism $\mu : X \to Y$ is a morphism of groups $\mu : \Gamma_X \to \Gamma_Y$ such that there exists at least one homogeneous autofunctor $J : \mathcal{H} \to \mathcal{H}$ of Hopf k-categories such that the following diagram commutes for a given object $x_0 \in \mathcal{H}_0$:

$$\begin{array}{ccc} {}_{x_0}HW_X(\mathcal{B})_{x_0} & \xrightarrow{HW(J)} & {}_{x_0}HW_Y(\mathcal{B})_{x_0} \\ \big\downarrow{\scriptstyle \deg_X} & & \big\downarrow{\scriptstyle \deg_Y} \\ \Gamma_X & \xrightarrow[\mu]{} & \Gamma_Y \end{array} .$$

As mentioned in the Introduction, following methods closely related to the way in which the fundamental group is considered in algebraic geometry, we define the fundamental group of a Hopf k-category $\mathcal{H}$ as follows.

Definition 4.1 *Let $\mathcal{H}$ be a Hopf k-category over a finite group G and let x_0 be a fixed object in $\mathcal{H}_0$. An element of $\Pi_1^{\mathsf{H}}(\mathcal{H}, x_0)$ is a family (γ_X) where X varies amongst the connected Hopf gradings of $\mathcal{H}$ with abelian finite grading group, $\gamma_X \in \Gamma_X$, and the family verifies $\mu(\gamma_X) = \gamma_Y$ for $\mu : X \to Y$ a morphism of Hopf gradings. The product of $\Pi_1^{\mathsf{H}}(\mathcal{H}, x_0)$ is the pointwise product of the families.*

Remark 4.2 *The fundamental group of a Hopf k-category over a finite group G is abelian since all the Γ_X's are abelian.*

Lemma 4.3 *The fundamental group of the trivial Hopf k-category over a finite group is zero.*

The proof of the following result is straightforward.

Proposition 4.4 *Let $\mathcal{H}$ and $\mathcal{H}'$ be isomorphic Hopf k-categories over the same group G. There is an isomorphism*

$$\Pi_1^{\mathsf{H}}(\mathcal{H}) \cong \Pi_1^{\mathsf{H}}(\mathcal{H}').$$

Note that two equivalent Hopf k-categories over the same group G are in fact isomorphic.

We compute now the fundamental group of the Taft k-category $\mathcal{C}_q^n$. We have already proved that only cyclic groups C_m of order m coprime to n can provide connected Hopf gradings of this category. Moreover, C_m must be generated by an element γ^n, with $\gamma \in C_m$. The next result follows immediately.

Proposition 4.5 *The fundamental group of the n-th Taft category $\mathcal{C}_q^n$ is the subgroup of $\prod_{(n,m)=1} \mathbb{Z}/m\mathbb{Z}$ consisting of families (x_m) verifying $\mu(x_m) = x_{m'}$, where $\mu : \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/m'\mathbb{Z}$ is the canonical projection in case m' divides m.*

Theorem 4.6 *Let G be a finite abelian group and let k be a field which contains $|G|$ distinct $|G|$-th roots of unity. Let kG be the group algebra considered as a Hopf k-category over the trivial group, namely a category with a single object which has kG as endomorphisms. The fundamental group $\Pi_1^{\mathsf{H}}(kG)$ is zero.*

Observe that as a consequence of the hypothesis, the characteristic of k is either zero or does not divide the order of G; thus kG is semisimple.

Proof. Recall that k^G is the dual Hopf algebra of kG. As an associative algebra, k^G is a diagonal algebra, which means that k^G is a product of copies of k, more precisely

$$k^G = \times_{x \in G} k\delta_x.$$

Under the assumption we have made for k, recall that kG and k^G are isomorphic Hopf k-algebras when G is abelian.

The classification of connected gradings of diagonal algebras has been performed by S. Dăscălescu in [10]. Besides the grading by the trivial group, they are in one to one correspondence with bases B of k^G which are multiplicative subgroups of the group of invertible elements of k^G. The corresponding grading has structure group B, given $b \in B$ the homogeneous component of degree b is kb.

We assert that none of these gradings is a Hopf grading. Indeed, for any Hopf grading the counit $\epsilon : k^G \longrightarrow k$ is homogeneous, namely $\epsilon(1) = 1$ and if f is homogeneous of non trivial degree, then $\epsilon(f) = 0$. However $\epsilon(b) = 1$ for all $b \in B$ since b is invertible. Hence the only connected Hopf grading of kG is the grading by the trivial group. $\diamond$

Remark 4.7 *Let kC_p be the group algebra of the cyclic group of order p and let k be a field of characteristic p. In [5] the classification of the gradings of this algebra is obtained and two families arise. For the first one an easy inspection shows that none of its members is a Hopf grading. Concerning the other family, some of its members could be Hopf gradings but this appears to be a difficult point to decide that we have been unable to perform in general. The fundamental group can be non trivial as the following result shows.*

Proposition 4.8 *Let k be a field of characteristic 3. The fundamental group of kC_3 as a Hopf algebra is the cyclic group of order 2.*

Proof. The Hopf algebras kC_3 and $k[x]/\left(x^3\right)$ are isomorphic, where

$$\Delta(x) = x \otimes x + x \otimes 1 + 1 \otimes x, \quad S(x) = x^2 - x \text{ and } \epsilon(x) = 0.$$

The classification of its gradings, as quoted in the previous remark, is given in Proposition 5.4 of [5]. Since the gradings of the first family are not Hopf gradings, we focus on a grading where the class of a polynomial $f = ax^2 + bx$ of valuation 1 is homogeneous of a certain degree t. We record that $a \neq 0$ since if $f = x$, the antipode is not homogeneous. So we suppose $f = x^2 + bx$. The antipode is homogeneous if and only if $b = 1$ in k. In order to insure that Δ is homogeneous an easy computation shows that the order of t has to be 2. Consequently, the only non-trivial connected Hopf grading of kC_3 is given by a cyclic of order 2, hence $\Pi_1^{\mathsf{H}}(kC_3) = C_2$. $\diamond$

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