Varying the direction of propagation in reaction-diffusion equations in periodic media

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Contents

Abstract

We consider a multidimensional reaction-diffusion equation of either ignition or monostable type, involving periodic heterogeneity, and analyze the dependence of the propagation phenomena on the direction. We prove that the (minimal) speed of the underlying pulsating fronts depends continuously on the direction of propagation, and so does its associated profile provided it is unique up to time shifts. We also prove that the spreading properties [24] are actually uniform with respect to the direction.

Key Words: periodic media, monostable nonlinearity, ignition nonlinearity, pulsating traveling front, spreading properties.

AMS Subject Classifications: 35K57, 35B10.

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1 Introduction

In this work, we focus on the heterogeneous reaction-diffusion equation

$$\partial_t u = \mathrm{div}(A(x)\nabla u) + q(x)\cdot\nabla u + f(x,u), \quad t\in\mathbb{R}, x\in\mathbb{R}^N. \tag{1}$$

Here $A=(A_{i,j})_{1\leq i,j\leq N}$ is a matrix field, and $q=(q_1,...,q_N)$ is a vector field, to be precised later. The nonlinearity f is of either the monostable type (not necessarily with the KPP assumption) or ignition type, which we will define below. We would like to understand, in the periodic framework, how the propagation phenomena depend on the direction.

On the one hand, we prove that the minimal (and, in the ignition case, unique) speed of the well known pulsating fronts depends continuously on the direction of propagation. On the other hand, we prove that the spreading properties are in some sense uniform with respect to the direction, thus improving the seminal result of Weinberger [24]. While in the ignition case, these properties will mostly follow from the well known uniqueness of the pulsating traveling wave, such uniqueness does not hold true in the monostable case where the set of admissible speeds is infinite. Our argument will be inspired by [10], [4], and will rely on an approximation of the monostable nonlinearity by some well-chosen ignition nonlinearities.

1.1 Main assumptions

Let L_1,...,L_N be given positive constants. A function $h:\mathbb{R}^N\to\mathbb{R}$ is said to be *periodic* if

$$h(x_1,...,x_k+L_k,...,x_N)=h(x_1,...,x_N),$$

for all $1\leq k\leq N$, all $(x_1,...,x_N)\in\mathbb{R}^N$. In such case, $\mathcal{C}=(0,L_1)\times\cdots\times(0,L_N)$ is called the cell of periodicity. Through this work, we put ourselves in the spatially periodic framework and assume that

$$\begin{aligned}&\text{for all } 0\leq i,j\leq N, \text{ the functions } A_{i,j}:\mathbb{R}^N\to\mathbb{R}\,,\ q_i:\mathbb{R}^N\to\mathbb{R} \quad \text{are periodic,}\\ &\text{for all } u\in\mathbb{R}_+, \text{ the function } f(\cdot,u):\mathbb{R}^N\to\mathbb{R} \text{ is periodic.}\end{aligned} \tag{2}$$

Moreover, we assume that $A=(A_{i,j})_{1\leq i,j\leq N}$ is a C^3 matrix field which satisfies

$$\begin{aligned}&A(x) \text{ is a symmetric matrix for any } x\in\mathbb{R}^N,\\ &\exists 0<a_1\leq a_2<\infty, \quad \forall(x,\xi)\in\mathbb{R}^N\times\mathbb{R}^N, \quad a_1|\xi|^2\leq\textstyle\sum_{i,j}A_{i,j}(x)\xi_i\xi_j\leq a_2|\xi|^2.\end{aligned} \tag{3}$$

Concerning the advection term, we assume that $q=(q_1,...,q_N)$ is a $C^{1,\delta}$ vector field, for some $\delta>0$, which satisfies

$$\mathrm{div}\, q=0 \text{ in } \mathbb{R}^N \quad \text{and} \quad \forall 0\leq i\leq N, \int_{\mathcal{C}} q_i=0. \tag{4}$$

The advection term in the equation is mostly motivated by combustion models where the dynamics of the medium also plays an essential role. In such a context, the fact that the flow q has zero divergence carries the physical meaning that the medium is incompressible.

Furthermore, we will assume that f satisfies either of the following two assumptions.

Assumption 1.1 (Monostable nonlinearity)**.** *The function $f:\mathbb{R}^N\times\mathbb{R}_+\to\mathbb{R}$ is of class $C^{1,\alpha}$ in (x,u) and C^2 in u, and nonnegative on $\mathbb{R}^N\times[0,1]$. Concerning the steady states of the periodic equation* (1)*, we assume that*

(i) the constants 0 and 1 are steady states (that is, $f(\cdot,0)\equiv f(\cdot,1)\equiv 0$ in $\mathbb{R}^N$);

(ii) $\forall u\in(0,1),\ \exists x\in\mathbb{R}^N,\quad f(x,u)>0.$

(iii) there exists some $\rho>0$ such that $f(x,u)$ is nonincreasing with respect to u in the set $\mathbb{R}^N\times(1-\rho,1]$.

Notice that, if $0 \leq p(x) \leq 1$ is a periodic stationary state, then $p \equiv 0$ or $p \equiv 1$. Indeed, since $f(x,p) \geq 0$, the strong maximum principle enforces p to be identically equal to its minimum, thus constant and, by (ii), the constant has to be 0 or 1.

Assumption 1.2 (Ignition nonlinearity)**.** *The function $f : \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}$ is locally Lipschitz-continuous on $\mathbb{R}^N \times \mathbb{R}_+$. Concerning the steady states of the periodic equation* (1)*, we assume that*

(i) there exists $0 < \theta < 1$ such that

$$\forall 0 \leq u \leq \theta, \quad \forall x \in \mathbb{R}^N, \qquad f(x,u) = 0,$$

as well as

$$\forall x \in \mathbb{R}^N, \qquad f(x,1) = 0.$$

(ii) $\forall u \in (\theta, 1), \ \exists x \in \mathbb{R}^N, \quad f(x,u) > 0.$

(iii) there exists some $\rho > 0$ such that $f(x,u)$ is nonincreasing with respect to u in the set $\mathbb{R}^N \times (1-\rho, 1]$.

Notice that, similarly as above, (ii) implies that if $\theta \leq p(x) \leq 1$ is a periodic stationary state then $p \equiv \theta$ or $p \equiv 1$.

1.2 Comments and related results

Under Assumption 1.1, Assumption 1.2, equation (1) is referred to as the monostable equation, the ignition equation respectively. Both sets of assumptions arise in various fields of physics and the life sciences, and especially in combustion and population dynamics models where propagation phenomena are involved. Indeed, a particular feature of these equations is the formation of traveling fronts, that is particular solutions describing the transition at a constant speed from one stationary solution to another. Such solutions have proved in numerous situations their utility in describing the spatio-temporal dynamics of a population, or the propagation of a flame modelled by a reaction-diffusion equation.

Equation (1) is a heterogeneous version of the well known reaction-diffusion equation

$$\partial_t u = \Delta u + f(u), \tag{5}$$

where typically f belongs to one of the three following classes: monostable, ignition and bistable. Homogeneous reaction-diffusion equations have been extensively studied in the literature (see [16], [2, 3], [11], [9], [23] among others) and are known to support the existence of monotone traveling fronts. In particular, for monostable nonlinearities, there exists a critical speed c^* such that all speeds $c \geq c^*$ are admissible, while in the bistable and ignition cases, the admissible speed $c = c^*$ is unique. Moreover, in both cases, the speed c^* corresponds to the so-called spreading speed of propagation of compactly supported initial data.

Among monostable nonlinearities, one can distinguish the ones satisfying the Fisher-KPP assumption, namely $u \mapsto \frac{f(u)}{u}$ is maximal at 0 (meaning that the growth per capita is maximal at small densities), the most famous example being introduced by Fisher [12] and Kolmogorov, Petrovsky and Piskunov [17] to model the spreading of advantageous genetic features in a population:

$$\partial_t u = \Delta u + u(1-u).$$

Let us notice that our work stands in the larger class of monostable nonlinearities.

Nevertheless, much attention was more recently devoted to the introduction of some heterogeneity, taking various forms such as advection, spatially dependent diffusion or reaction term. Taking such a matter into account is essential as far as models are concerned, the environment being rarely homogeneous and may depend in a non trivial way on the position in space (patches, periodic media, or more general heterogeneity...). We refer to the seminal book of Shigesada and

Kawasaki [22], and the enlightening introduction in [7] where the reader can find very precise and various references. As far as combustion models are concerned, one can consult [9], [25] and the references therein.

Traveling front solutions in heterogeneous versions of (5) with periodicity in space, in time, or more general media are studied in [24], [10], [15], [26], [4], [8], [19], [20] among others. For very general reaction-diffusion equations, we refer to [5] for a definition of generalized transition waves and their properties.

In this work, we restrict ourselves to the spatially periodic case, which provides insightful information on the role and influence of the heterogeneity on the propagation, as well as a slightly more common mathematical framework. In this periodic setting, let us mention the following keystone results for ignition and monostable nonlinearities. Weinberger [24] exhibited a direction dependent spreading speed for planar-shaped initial data and proved, in the monostable case, that this spreading speed is also the minimal speed of pulsating traveling waves moving in the same direction. His approach relies on a discrete formalism, in contrast with the construction of both monostable and ignition pulsating traveling waves by Berestycki and Hamel [4], via more flexible PDE technics. In this PDE framework, note also the work of Berestycki, Hamel and Roques [8] where KPP pulsating fronts are constructed without assuming the nonnegativity of the nonlinearity. Our main goal is to study how these results behave when we vary the direction of propagation.

Let us give another motivation for our analysis of the dependence of the propagation on the direction. Our primary interest was actually to study the *sharp interface limit* $\varepsilon \to 0$ of

$$\partial_t u^\varepsilon = \varepsilon \Delta u^\varepsilon + \frac{1}{\varepsilon} f\left(\frac{x}{\varepsilon}, u^\varepsilon\right), \tag{6}$$

arising from the hyperbolic space-time rescaling $u^\varepsilon(t,x) := u\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right)$ of (1), with $A \equiv Id$, $q \equiv 0$. The parameter $\varepsilon > 0$ measures the thickness of the diffuse interfacial layer. As this thickness tends to zero, (6) converges — in some sense— to a limit interface, whose motion is governed by the minimal speed (in each direction) of the underlying pulsating fronts. This dependence of the speed on the (moving) normal direction is in contrast with the homogeneous case and makes the analysis quite involved. In particular, it turns out that we need to improve (by studying the uniformity with respect to the direction) the known spreading properties [24], [4], for both ignition and monostable nonlinearities in periodic media. We refer to [1] for this singular limit analysis, using some of the results of the present work.

2 Main results

Before stating our main results in subsection 2.2, let us recall the classical results on both pulsating fronts and spreading properties in subsection 2.1.

2.1 Pulsating fronts and spreading properties: known results

The definition of the so-called pulsating traveling wave was introduced by Xin [25] in the framework of flame propagation. It is the natural extension, in the periodic framework, of classical traveling waves. Due to the interest of taking into account the role of the heterogeneity of the medium on the propagation of solutions, a lot of attention was later drawn on this subject. As far as monostable and ignition pulsating fronts are concerned, we refer to the seminal works of Weinberger [24], Berestycki and Hamel [4]. Let us also mention [8], [13], [14], [19] for related results.

For the sake of completeness, let us first recall the definition of a pulsating traveling wave for the equation (1), as stated in [4].

Definition 2.1 (Pulsating traveling wave). *A pulsating traveling wave solution, with speed $c > 0$ in the direction $n \in \mathbb{S}^{N-1}$, is an entire solution $u(t,x)$ — $t \in \mathbb{R}$, $x \in \mathbb{R}^N$ — of* (1) *satisfying*

$$\forall k \in \prod_{i=1}^{N} L_i\mathbb{Z}, \qquad u(t,x) = u\left(t + \frac{k \cdot n}{c}, x + k\right),$$

for any $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$, along with the asymptotics

$$u(-\infty, \cdot) = 0 < u(\cdot,\cdot) < u(+\infty,\cdot) = 1,$$

where the convergences in $\pm\infty$ are understood to hold locally uniformly in the space variable.

One can easily check that, for any $c > 0$ and $n \in \mathbb{S}^{N-1}$, $u(t,x)$ is a pulsating traveling wave with speed c in the direction n if and only if it can be written in the form

$$u(t,x) = U(x \cdot n - ct, x),$$

where $U(z,x)$ — $z \in \mathbb{R}$, $x \in \mathbb{R}^N$— satisfies

$$\text{for all } z \in \mathbb{R},\ U(z,\cdot) : \mathbb{R}^N \to \mathbb{R} \text{ is periodic,}$$

$$U(-\infty,\cdot) = 1 < U(\cdot,\cdot) < U(+\infty,\cdot) = 0 \quad \text{uniformly w.r.t. the space variable,}$$

along with the following equation

$$\begin{aligned} &\mathrm{div}_x(A\nabla_x U) + (n \cdot An)\,\partial_{zz}U + \mathrm{div}_x(An\,\partial_z U) + \partial_z(n \cdot A\nabla_x U) \\ &\quad + q \cdot \nabla_x U + (q \cdot n)\,\partial_z U + c\partial_z U + f(x,U) = 0, \quad \text{on } \mathbb{R} \times \mathbb{R}^N. \end{aligned} \tag{7}$$

We can now recall the results of [4] (see also [24] for the monostable case), on the existence of pulsating traveling waves for the spatially periodic monostable and ignition equations. Precisely, the following holds.

Theorem 2.2 (Monostable and ignition pulsating fronts, [4],[24]).

- *Assume that f is of the spatially periodic monostable type, i.e. f satisfies* (2) *and Assumption 1.1. Then for any $n \in \mathbb{S}^{N-1}$, there exists $c^*(n) > 0$ such that pulsating traveling waves with speed c in the direction n exist if and only if $c \geq c^*(n)$.*
- *Assume that f is of the spatially periodic ignition type, i.e. f satisfies* (2) *and Assumption 1.2. Then for any $n \in \mathbb{S}^{N-1}$, there exists a unique (up to time shift) pulsating traveling wave, whose speed we denote by $c^*(n) > 0$.*

Furthermore, in both cases, any pulsating traveling wave is increasing in time.

The introduction of these pulsating traveling waves was motivated by their expected role in describing the large time behavior of solutions of (1) for a large class of initial data. In this context, let us state the result of [24] for planar-shaped initial data.

Theorem 2.3 (Spreading properties, [24]). *Assume that f is of the spatially periodic monostable or ignition type, i.e. f satisfies* (2) *and either of the two Assumptions 1.1 and 1.2. Let u_0 be a nonnegative and bounded initial datum such that $\|u_0\|_\infty < 1$ and*

$$\exists C > 0, \qquad x \cdot n \geq C \ \Rightarrow\ u_0(x) = 0,$$

$$\liminf_{x \cdot n \to -\infty} u_0(x) > 0 \ \ (\textit{monostable case}), \qquad \liminf_{x \cdot n \to -\infty} u_0(x) > \theta \ \ (\textit{ignition case})$$

for some $n \in \mathbb{S}^{N-1}$.

Then the solution u of (1)*, with initial datum u_0, spreads with speed $c^*(n)$ in the n-direction in the sense that*

$$\forall c < c^*(n), \qquad \lim_{t \to +\infty} \sup_{x \cdot n \leq ct} |1 - u(t,x)| = 0, \tag{8}$$

$$\forall c > c^*(n), \qquad \lim_{t \to +\infty} \sup_{x \cdot n \geq ct} u(t,x) = 0. \tag{9}$$

Remark 2.4 (Link between spreading speed and wave speed)**.** *In [24], Weinberger was actually concerned with a more general discrete formalism where pulsating waves are not always known to exist. Therefore, the fact that the spreading speed and the minimal wave speed are one and the same was only explicitly stated in the monostable case.*

However, under the ignition Assumption 1.2 and benefiting from the results in [4], it is clear by a simple comparison argument that the solution associated with any such initial datum spreads at most with the wave speed $c^*(n)$, namely (9) *holds true. Furthermore, one may check, using for instance $U^*(x \cdot n - (c^*(n) - \alpha)t, x) - \delta$ as a subsolution of* (1)*, where U^* is the pulsating wave with speed $c^*(n)$ and $\alpha > 0$, $\delta > 0$ are small enough, that* (8) *also holds true, at least for some large enough initial data. Thus, the spreading speed exhibited by Weinberger must be $c^*(n)$, as one would expect.*

We will use a very similar argument in Section 7, which is why we omit the details. Moreover, it is a simplification of a classical argument, which originates from [11] in the homogeneous framework, and usually aims at proving the stronger property that the profile of such a solution $u(t, x)$ of the Cauchy problem converges to that of the ignition pulsating wave. We refer for instance to the work of Zlatoš [27], which dealt with a fairly general multidimensional heterogeneous (not necessarily periodic) framework, and covers the above result under the additional assumption that $f(x, u)$ is bounded from below by a standard homogeneous ignition nonlinearity.

Various features of pulsating fronts and many generalizations of spreading properties have been studied recently. Nevertheless, as far as we know, nothing is known on the dependence of these results on the direction of propagation. Our results stand in this new framework and are stated in the next subsection.

2.2 Pulsating fronts and spreading properties: varying the direction

As recalled above, the periodic ignition equation admits a unique pulsating traveling wave in any direction $n \in \mathbb{S}^{N-1}$, while the periodic monostable equation (1) admits pulsating traveling waves in any direction $n \in \mathbb{S}^{N-1}$, for any speed larger than some critical $c^*(n) > 0$. The latter is a consequence of the former, as was proved in [4] by approximating the monostable equation with an ignition type equation. With some modifications of their argument, we will prove the following continuity property.

Theorem 2.5 (Continuity of minimal speeds)**.** *Assume that f is of the spatially periodic monostable or ignition type, i.e. f satisfies* (2) *and either of the two Assumptions 1.1 or 1.2.*

Then the mapping $n \in \mathbb{S}^{N-1} \mapsto c^*(n)$ is continuous.

In the Fisher-KPP case the continuity of the velocity map $n \mapsto c^*(n)$, even if not explicitly stated, seems to follow from the characterization of $c^*(n)$ (see [24], [4]). However, for other types of nonlinearities (and in particular, in the more general monostable case), such a property seems to be far from obvious.

For the sake of completeness, let us state the continuity of the profile of the ignition wave, which will be proved simultaneously.

Theorem 2.6 (Continuity of ignition waves)**.** *If f satisfies* (2) *and the ignition Assumption 1.2, then the mapping*

$$n \in \mathbb{S}^{N-1} \mapsto U^*(z, x; n)$$

is continuous with respect to the uniform topology, where

$$u^*(t, x; n) = U^*(x \cdot n - c^*(n)t, x; n)$$

is the unique pulsating traveling wave in the n direction, normalized by $\min_{x \in \mathbb{R}^N} U^*(0, x; n) = \frac{1+\theta}{2}$.

In Section 3, we deal with the ignition case, proving both the continuity of the speed (Theorem 2.5) and that of the profile (Theorem 2.6). To do so we take advantage of the uniqueness of the pulsating wave in each direction.

Then, in Section 4, we approach our original monostable equation by some ignition type problems, and prove that the associated ignition speeds converge to $c^*(n)$ not only pointwise (as in [4]), but even uniformly with respect to $n \in \mathbb{S}^{N-1}$. The continuity of the minimal speed (Theorem 2.5) then immediately follows. Unfortunately, the lack of a rigorous uniqueness result of the monostable pulsating wave with minimal speed (at least up to our knowledge) prevents us from stating continuity of its profile with respect to the speed of propagation. We refer to [14] for uniqueness results in the Fisher-KPP case and discussion on the general monostable framework.

We also stated above the well known fact that for any planar-like initial data in some direction n, the associated solution of (1) spreads in the n direction with speed $c^*(n)$. Our main result consists in improving (compare Theorem 2.7 with Theorem 2.3) this property by adding some uniformity with respect to $n \in \mathbb{S}^{N-1}$, as follows.

Theorem 2.7 (Uniform spreading)**.** *Assume that f is of the spatially periodic monostable or ignition type, i.e. f satisfies (2) and either Assumption 1.1 or Assumption 1.2. Let a family of nonnegative initial data $(u_{0,n})_{n\in\mathbb{S}^{N-1}}$ be such that*

$$\exists C > 0, \qquad \forall n \in \mathbb{S}^{N-1}, \qquad x \cdot n \geq C \ \Rightarrow \ u_{0,n}(x) = 0, \tag{10}$$

$$\left.\begin{array}{ll} \exists \mu > \theta & (\textit{ignition case}) \\ \exists \mu > 0 & (\textit{monostable case}) \end{array}\right\}, \qquad \exists K > 0, \qquad \inf_{n\in\mathbb{S}^{N-1},\ x\cdot n \leq -K} u_{0,n}(x) \geq \mu, \tag{11}$$

$$\sup_{n\in\mathbb{S}^{N-1}} \sup_{x\in\mathbb{R}^N} u_{0,n}(x) < 1. \tag{12}$$

We denote by $(u_n)_{n\in\mathbb{S}^{N-1}}$ the associated family of solutions of (1).

Then, for any $\alpha > 0$ and $\delta > 0$, there exists $\tau > 0$ such that for all $t \geq \tau$,

$$\sup_{n\in\mathbb{S}^{N-1}} \sup_{x\cdot n \leq (c^*(n)-\alpha)t} |1 - u_n(t,x)| \leq \delta, \tag{13}$$

$$\sup_{n\in\mathbb{S}^{N-1}} \sup_{x\cdot n \geq (c^*(n)+\alpha)t} u_n(t,x) \leq \delta. \tag{14}$$

The difficult part is again to deal with the monostable case. The proof of the lower spreading property (13) will again rely on an ignition approximation of the monostable equation, whose traveling waves will serve as nontrivial subsolutions of (1). This is performed in Section 5. Then, Section 6 is devoted to the proof of the upper spreading property: we prove (14) in subsection 6.1 and, for sake of completeness, relax assumption (12) in subsection 6.2.

Last, in Section 7, we prove Theorem 2.7 in the ignition case.

3 Continuity of ignition waves

Let us here consider a periodic nonlinearity f of the ignition type, namely satisfying Assumption 1.2. As announced, we will prove simultaneously the continuity of both mappings $n \mapsto c^*(n)$ and $n \mapsto U^*(z, x; n)$, where we recall that $c^*(n)$ and $U^*(x \cdot n - c^*(n)t, x; n)$ denote respectively the unique admissible speed and the unique pulsating wave in the direction n, normalized by

$$\min_{x\in\mathbb{R}^N} U^*(0, x; n) = \frac{1+\theta}{2}. \tag{15}$$

Proofs of Theorem 2.5 (ignition case) and Theorem 2.6. We first claim (we postpone the proof to the end of this section) that

$$\kappa := \inf_{n\in\mathbb{S}^{N-1}} c^*(n) > 0. \tag{16}$$

Let us now prove that that $c^*(n)$ is also bounded from above, using

$$(t, x) \mapsto v(t, x) := \min\{1, \theta + Ce^{-\lambda(x\cdot n - 2a_1\lambda t)}\}$$

as a supersolution of (1). Here C and λ are positive constants to be chosen later, and a_1 comes from hypothesis (3). Indeed, when $v < 1$, it satisfies

$$\begin{aligned} & \partial_t v - \operatorname{div}(A(x)\nabla v) - q(x)\cdot\nabla v - f(x,v) \\ = \ & \left[2a_1\lambda^2 - (n\cdot An)\lambda^2 + \lambda \operatorname{div}(An) + \lambda q\cdot n\right] \times Ce^{-\lambda(x\cdot n - 2a_1\lambda t)} - f(x,v) \\ \geq \ & \left[a_1\lambda^2 - \lambda\,|\operatorname{div}(An)| - \lambda|q\cdot n| - M\right] \times Ce^{-\lambda(x\cdot n - 2a_1\lambda t)} > 0, \end{aligned} \tag{17}$$

where

$$M := \sup_{x\in\mathbb{R}^N, u\in[0,1]} \frac{f(x,u)}{|u-\theta|} < +\infty \tag{18}$$

comes from the Lipschitz continuity of f, and the last inequality holds provided that λ is large enough, independently of $n \in \mathbb{S}^{N-1}$. As 1 is a solution of (1), it is then clear that v is a generalized supersolution of (1). Then, choosing $C > 0$ so that $v(t = 0, x)$ lies above the traveling wave $u^*(t = 0, x; n) = U^*(x \cdot n, x; n)$ at time 0, we can apply the comparison principle and obtain that $c^*(n) \leq 2a_1\lambda$. Putting this fact together with (16), we conclude that

$$0 < \kappa := \inf_{n\in\mathbb{S}^{N-1}} c^*(n) \leq \sup_{n\in\mathbb{S}^{N-1}} c^*(n) =: K < +\infty. \tag{19}$$

We now let some sequence of directions $n_k \to n \in \mathbb{S}^{N-1}$. As we have just shown, the sequence $c^*(n_k)$ is bounded and, up to extraction of a subsequence, $c^*(n_k) \to c > 0$. We also choose the shifts z_k so that, for all k, $\max_{x\in\mathbb{R}^N} U^*(z_k, x; n_k) = \theta$. In particular, recalling that U^* is monotonically decreasing with respect to its first variable, we have for all k that

$$\forall z \geq z_k, \forall x \in \mathbb{R}^N, \quad 0 < U^*(z, x; n_k) \leq \theta.$$

Then

$$u_k(t,x) := U^*(z_k + x\cdot n_k, x + c^*(n_k)tn_k; n_k)$$

satisfies

$$\partial_t u_k = \operatorname{div}(A(x)\nabla u_k) + q(x)\cdot\nabla u_k + c^*(n_k)\nabla u_k\cdot n_k, \tag{20}$$

for all $t \in \mathbb{R}$ and all x in the half-space $x \cdot n_k \geq 0$ (recall that U^* solves (7) and that, in the ignition case, $f(x, u) = 0$ if $0 \leq u \leq \theta$).

Let us now find a supersolution of (20) of the exponential type, namely

$$\overline{u}_k(t,x) := \phi_k(t,x) \times e^{-\lambda_0\, x\cdot n_k}, \tag{21}$$

where ϕ_k will be a well-chosen positive and bounded function.

For any $n \in \mathbb{S}^{N-1}$, one may define (see Proposition 5.7 in [4]) the principal eigenvalue problem

$$\begin{cases} -L_{n,\lambda}\phi_{n,\lambda} = \mu(n,\lambda)\phi_{n,\lambda} \ \text{ in } \mathbb{R}^N, \\ \phi_{n,\lambda} > 0 \text{ is periodic,} \end{cases}$$

where

$$L_{n,\lambda}\phi := \operatorname{div}(A\nabla\phi) + \lambda^2(n\cdot An)\phi - \lambda(\operatorname{div}(An\phi) + n\cdot A\nabla\phi) + q\cdot\nabla\phi - \lambda(q\cdot n + \kappa)\phi,$$

with $\kappa > 0$ given by (16). In the sequel, the eigenfunction $\phi_{n,\lambda}$ is normalized so that

$$\min_{x\in\mathcal{C}} \phi_{n,\lambda}(x) = \theta.$$

As stated in Proposition 5.7 of [4], the function $\lambda \mapsto \mu(n, \lambda)$ is concave and satisfies, for any n, that $\mu(n, 0) = 0$ (any positive constant is clearly a principal eigenfunction of $-L_{n,0}$), and $\partial_\lambda\mu(n, 0) = \kappa > 0$.

It follows that one can find some small $\lambda_0 > 0$ such that, for any $n \in \mathbb{S}^{N-1}$,

$$\mu(n, \lambda_0) > 0.$$

Indeed, proceed by contradiction and assume that for any $j \in \mathbb{N}^*$, there exists n_j such that $\mu(n_j, 1/j) \leq 0$. Then, by $\mu(n_j, 0) = 0$ and by concavity, one has that $\mu(n_j, \lambda) \leq 0$ for all $\lambda > \frac{1}{j}$. By uniqueness of the principal normalized eigenfunction, it is straightforward to check that $\mu(n, \lambda)$ depends continuously on both n and λ, as well as $\phi_{n,\lambda}$ with respect to the uniform topology. Thus, one can pass to the limit and conclude that $\mu(n_\infty, \lambda) \leq 0$ for some $n_\infty = \lim n_j$ (up to extraction of a subsequence) and all $\lambda \geq 0$. This contradicts the fact that $\partial_\lambda \mu(n_\infty, 0) = \kappa > 0$.

Notice that, by continuity of the eigenfunction with respect to n and λ in the uniform topology, it is clear that for any bounded set Λ,

$$\max_{n \in \mathbb{S}^{N-1}} \max_{\lambda \in \Lambda} \max_{x \in \mathcal{C}} \phi_{n,\lambda}(x) < +\infty. \tag{22}$$

Choosing λ_0 as above and

$$\phi_k(t,x) := \phi_{n_k,\lambda_0}(x + c^*(n_k) t n_k),$$

in (21), one gets that

$$\begin{aligned} & \partial_t \overline{u}_k - \mathrm{div}\,(A(x)\nabla \overline{u}_k) - q(x) \cdot \nabla \overline{u}_k - c^*(n_k) \nabla \overline{u}_k \cdot n_k \\ = \; & [c^*(n_k) n_k \cdot \nabla \phi_{n_k,\lambda_0} - L_{n_k,\lambda_0} \phi_k + \lambda_0(-\kappa + c^*(n_k))\phi_k - c^*(n_k) n_k \cdot \nabla \phi_{n_k,\lambda_0}] \times e^{-\lambda_0 (x \cdot n_k)} \\ = \; & [\mu(n_k, \lambda_0) + \lambda_0 (c^*(n_k) - \kappa)]\, \overline{u}_k > 0. \end{aligned}$$

In other words, as announced, $\overline{u}_k$ is a supersolution of (20).

Let us now prove that

$$\forall t \in \mathbb{R},\ \forall x \cdot n_k \geq 0, \quad u_k(t,x) \leq \overline{u}_k(t,x). \tag{23}$$

Proceed by contradiction and define a sequence of points $(t_j, x_j)_{j \in \mathbb{N}}$ such that

$$u_k(t_j, x_j) - \overline{u}_k(t_j, x_j) \to \sup_{t \in \mathbb{R}, x \cdot n_k \geq 0} (u_k(t,x) - \overline{u}_k(t,x)) > 0.$$

Now write $x_j = (x_j \cdot n_k) n_k + y_j$ for any $j \geq 0$. Note that, since $u_k(t,x)$ and $\overline{u}_k(t,x)$ both tend to 0 as $x \cdot n_k \to +\infty$ uniformly with respect to t, then $(x_j \cdot n_k)_{j \in \mathbb{N}}$ must be bounded. Thus, up to extraction of a subsequence, we can assume that $x_j \cdot n_k \to a \geq 0$ as $j \to \infty$. Moreover, since y_j is orthogonal to n_k, since ϕ_{n_k,λ_0} is periodic and since U^* is periodic with respect to its second variable, we can assume without loss of generality that $y_j + c^*(n_k) t_j n_k \in \mathcal{C}$ the cell of periodicity. As y_j is orthogonal to n_k for all $j \in \mathbb{N}$, we can extract a subsequence such that both $y_j \to y_\infty \in \mathbb{R}^N$ and $t_j \to t_\infty \in \mathbb{R}$.

Finally, $u_k - \overline{u}_k$ reaches its positive maximum, over $t \in \mathbb{R}$ and $x \cdot n_k \geq 0$, at $(t = t_\infty, x = a n_k + y_\infty)$. Moreover, as

$$\forall x \cdot n_k = 0, \quad u_k(0,x) \leq \theta \leq \overline{u}_k(0,x),$$

the maximum is reached at an interior point, which contradicts the parabolic maximum principle. Thus, (23) is proved.

Now, by standard parabolic estimates and up to extraction of a subsequence, we can assume that, as $k \to \infty$, the sequence $u^*\left(t - \frac{z_k}{c^*(n_k)}, x; n_k\right) = U^*(x \cdot n_k - c^*(n_k) t + z_k, x; n_k)$ converges locally uniformly, along with its derivatives, to a solution $u_\infty(t,x)$ of (1). Moreover, u_∞ satisfies

$$\forall l \in \Pi_{i=1}^N L_i \mathbb{Z}, \quad u_\infty(t,x) = u_\infty\left(t + \frac{l \cdot n}{c}, x + l\right).$$

In a similar way than the discussion after Definition 2.1 of pulsating waves, this means that $u_\infty(t,x) = U_\infty(x\cdot n - ct, x)$ where $U_\infty(z,x)$ is periodic with respect to its second variable and satisfies

$$\begin{aligned}\mathrm{div}_x(A\nabla_x U) + (n\cdot An)\,\partial_{zz}U + \mathrm{div}_x(An\partial_z U) + \partial_z(n\cdot A\nabla_x U)\\ + q\cdot\nabla_x U + (q\cdot n)\,\partial_z U + c\partial_z U + f(x,U) = 0 \quad \text{ on } \mathbb{R}\times\mathbb{R}^N.\end{aligned}$$

It is then straightforward to retrieve that the sequence $U^*(z+z_k, x; n_k)$ also converges, along with its derivatives, to this function $U_\infty(z,x)$. In particular, U_∞ is nonincreasing with respect to its first variable, and satisfies the inequalities

$$0 \le U_\infty(z,x) \le 1.$$

Furthermore, noticing that $u^*\left(t - \frac{z_k}{c^*(n_k)}, x; n_k\right) = u_k(t, x - c^*(n_k)tn_k)$, it follows from passing to the limit in (23), and thanks to (22), that

$$u_\infty(t,x) \le Ae^{-\lambda_0(x\cdot n - ct)},$$

for some $A>0$ and all $x\cdot n \ge ct$.

Thus, $U_\infty(x\cdot n - ct, x) \le Ae^{-\lambda_0(x\cdot n-ct)}$, for all $t\in\mathbb{R}$ and $x\cdot n\ge ct$. This means that $U_\infty(z,x)$ converges exponentially to 0 as $z\to+\infty$, uniformly with respect to its second variable:

$$\forall z\ge 0,\ \forall x\in\mathbb{R}^N, \quad U_\infty(z,x)\le Ae^{-\lambda_0 z}. \tag{24}$$

By monotonicity with respect to its first variable, $U_\infty(z,x)$ converges as $z\to-\infty$ to some periodic function $p(x)$. Or, equivalently, $u_\infty(t,x)$ converges as $t\to+\infty$ to the same function $p(x)$. By standard parabolic estimates, we get that $p(x)$ is a periodic and stationary solution of (1). Let us show that $p\equiv 1$. From our choice of the shifts z_k and up to extraction of another subsequence, there exists some x_∞ such that $U_\infty(0,x_\infty)=\theta$, hence $\max p\ge\theta$. Assume first that $\max p = \theta$. Then $u_\infty(t,x)\le\theta$ for all $t\in\mathbb{R}$ and $x\in\mathbb{R}^N$ and, by the strong maximum principle, $u_\infty\equiv\theta$. This contradicts the inequality (24) above. Therefore, $\max p>\theta$. Using again the strong maximum principle and the fact that $f(x,u)=0$ for all $u\le\theta$ and $x\in\mathbb{R}^N$, we reach another contradiction if $\min p\le\theta$. Therefore $\min p>\theta$ and, thanks to part (*ii*) of our ignition Assumption 1.2, $p\equiv 1$ the unique periodic stationary solution of (1) above θ.

From the above analysis, we conclude that $U_\infty(\cdot,\cdot) = U^*(\cdot+Z,\cdot;n)$ the unique pulsating traveling wave in the n direction with speed $c=c^*(n)$, where Z is the unique shift such that $\max_{x\in\mathbb{R}^N} U^*(Z,x;n)=\theta$. This in fact proves, by uniqueness of the limit, that the whole sequence $c^*(n_k)$ converges to $c^*(n)$, and that the whole sequence $U^*(\cdot+z_k,\cdot;n_k)$ converges locally uniformly to $U^*(\cdot+Z,\cdot;n)$. This in particular shows the continuity of the map $n\mapsto c^*(n)$, that is Theorem 2.5 in the ignition case.

Let us now conclude the proof of Theorem 2.6. Let us first prove that the sequence of shifts z_k is bounded. The normalization (15) implies that $U^*(0,y_k;n_k)=\frac{1+\theta}{2}$, for some $y_k\in\mathcal{C}$ that (up to some subsequence) converges to some $y\in\mathcal{C}$. Since $U^*(z_k,y_k;n_k)\to U^*(Z,y;n)\le\theta$ and $U^*(0,y;n)=\frac{1+\theta}{2}$, the monotonicity of traveling waves enforces $z_k\ge 0$ for k large enough. Now proceed by contradiction and assume that (up to some subsequence) $z_k\to+\infty$. Then, for all $-z_k\le z\le 0$,

$$U^*(z+z_k,y_k;n_k)\le U^*(0,y_k;n_k)=\frac{1+\theta}{2}.$$

Passing to the limit as $k\to+\infty$, we get that

$$U_\infty(z,y)\le\frac{1+\theta}{2},$$

for all $z\le 0$. This contradicts the fact that U_∞ is a pulsating traveling wave and converges to 1 as $z\to-\infty$.

From the boundedness of the sequence z_k, we can now rewrite the convergence as follows: the sequence $U^*(\cdot,\cdot;n_k)$ converges locally uniformly to $U^*(\cdot,\cdot;n)$. It now remains to prove that this convergence is in fact uniform with respect to both variables. Note first that uniformity with respect to the second variable immediately follows from the periodicity. Furthermore, for a given $\delta > 0$, let $K > 0$ be such that, for any $x \in \mathbb{R}^N$,

$$0 \le U^*(z, x; n) \le \frac{\delta}{2} \quad \text{and} \quad 1 - \frac{\delta}{2} \le U^*(-z, x; n) \le 1, \text{ for all } z \ge K. \tag{25}$$

From the locally uniform convergence with respect to the first variable, we have, for any k large enough,

$$\|U^*(\cdot, \cdot; n_k) - U^*(\cdot, \cdot; n)\|_{L^\infty([-K,K]\times\mathbb{R})} \le \frac{\delta}{2}.$$

In particular, $U^*(K, x; n_k) \le \delta$ and $1 - \delta \le U^*(-K, x; n_k)$, so that, by monotonicity with respect to the first variable, for any $x \in \mathbb{R}^N$ and k large enough,

$$0 \le U^*(z, x; n_k) \le \delta \quad \text{and} \quad 1 - \delta \le U^*(-z, x; n_k) \le 1, \text{ for all } z \ge K. \tag{26}$$

Combining (25) and (26), we get

$$\|U^*(\cdot, \cdot; n_k) - U^*(\cdot, \cdot; n)\|_{L^\infty((-\infty,-K)\cup(K,\infty)\times\mathbb{R})} \le \delta,$$

for any k large enough. As a result the convergence of $U^*(\cdot, \cdot; n_k)$ to $U^*(\cdot, \cdot; n)$ is uniform in $\mathbb{R} \times \mathbb{R}^N$. This ends the proof of the continuity of ignition waves, that is Theorem 2.6. □

Proof of claim (16). Proceed by contradiction and assume that there exists a sequence $n_k \in \mathbb{S}^{N-1}$ such that $c^*(n_k) \to 0$.

Now for any k, recall that the pulsating wave is normalized by

$$\min_{x\in\mathbb{R}^N} U^*(0, x; n_k) = \frac{1+\theta}{2}. \tag{27}$$

Up to extraction of a subsequence, we can assume as above that $n_k \to n$ and

$$u^*(t, x; n_k) \to u_\infty(t, x),$$

where the convergence is understood to hold locally uniformly, and $u_\infty(t, x)$ is a solution of (20). By the strong maximum principle, it is clear that $0 < u_\infty < 1$. We also know, by the monotonicity of $U^*(\cdot, \cdot; n_k)$ with respect to its first variable, by (27) and by passing to the limit, that

$$u_\infty(t, x) \ge \frac{1+\theta}{2}, \quad \forall x \cdot n \le 0.$$

Applying Weinberger's result (see Theorem 2.3 as well as Remark 2.4), we get that the solution spreads at least at speed $c^*(n)$. In particular, as $t \to +\infty$, $u_\infty(t, x)$ converges locally uniformly to 1.

On the other hand, we fix $x \in \mathbb{R}^N$ and $s \ge 0$, then we let some vector $l \in \Pi_{i=1}^N L_i\mathbb{Z}$ be such that $l \cdot n > 0$. In particular, for any large k, one also has that $l \cdot n_k \ge \frac{l \cdot n}{2} > 0$. Then, for all large k, using the fact that $c^*(n_k) \to 0$ and the monotonicity of $u^*(\cdot, \cdot; n_k)$ with respect to its first variable, we have that

$$u^*(s, x; n_k) \le u^*\left(\frac{l \cdot n_k}{c^*(n_k)}, x; n_k\right) = u^*(0, x - l; n_k).$$

By passing to the limit as $k \to +\infty$, we obtain that

$$u_\infty(s, x) \le u_\infty(0, x - l) < 1,$$

for all $x \in \mathbb{R}^N$ and $s \ge 0$. This contradicts the locally uniform convergence of $u_\infty(t, x)$ to 1 as $t \to +\infty$. The claim is proved. □

4 Continuity of the monostable minimal speed

Let us here consider a periodic nonlinearity f of the monostable type, namely satisfying Assumption 1.1. We will prove the continuity of the mapping $n \mapsto c^*(n)$, that is Theorem 2.5. To do so, we introduce a family $f_\varepsilon(x,u)$, for small $\varepsilon > 0$, of ignition nonlinearities which serve as approximations from below of the monostable nonlinearity $f(x,u)$. Our aim is to prove that, by passing to the limit as $\varepsilon \to 0$, we indeed retrieve the dynamics of the monostable equation. This will be enough to prove Theorem 2.5.

The family $(f_\varepsilon)_\varepsilon$, for small enough $\varepsilon > 0$, is chosen as follows:

$$\forall x \in \mathbb{R}^N, \quad \begin{cases} \forall u \in [-\varepsilon, 0], & f_\varepsilon(x,u) = 0 \\ \forall u \in [0, 1-\varepsilon], & f_\varepsilon(x,u) = f(x,u) \\ \forall u \in \left[1-\varepsilon, 1-\frac{\varepsilon}{2}\right], & f_\varepsilon(x,u) = f\left(x, 1-\varepsilon + 2(u-(1-\varepsilon))\right). \end{cases}$$

Notice that $\|f_\varepsilon - f\|_{L^\infty(-\varepsilon,1)} \to 0$ as $\varepsilon \to 0$, and that, thanks to Assumption 1.1 (iii), f_ε lies below f and $0 < \varepsilon < \varepsilon'$ implies $f_\varepsilon \geq f_{\varepsilon'}$. Also, the equation

$$\partial_t u = \mathrm{div}(A(x)\nabla u) + q(x)\cdot \nabla u + f_\varepsilon(x,u), \tag{28}$$

where u is to take values between $-\varepsilon$ and $1-\frac{\varepsilon}{2}$, is of the ignition type in the sense of Assumption 1.2 (where 0, θ, 1 are replaced by $-\varepsilon$, 0 and $1-\frac{\varepsilon}{2}$ respectively). In particular, for each $n \in \mathbb{S}^{N-1}$, there exists a unique ignition pulsating traveling wave

$$u_\varepsilon^*(t,x;n) = U_\varepsilon^*(x\cdot n - c_\varepsilon^*(n)t, x; n)$$

of (28) in the n direction with speed $c_\varepsilon^*(n) > 0$, normalized by

$$\min_{x\in\mathbb{R}^N} U_\varepsilon^*(0,x;n) = \frac{1}{2}.$$

Furthermore, we have already proved in the previous section that the mappings $n \mapsto c_\varepsilon^*(n)$ and $n \mapsto U_\varepsilon^*(\cdot,\cdot;n)$ are continuous (with respect to the uniform topology).

Theorem 4.1 (Convergence of speeds)**.** *Assume that f is of the spatially periodic monostable type, i.e. f satisfies* (2) *and Assumption 1.1. Let $f_\varepsilon(x,u)$ be defined as above.*

Then, as $\varepsilon \to 0$, $c_\varepsilon^*(n) \nearrow c^*(n)$ uniformly with respect to $n \in \mathbb{S}^{N-1}$.

As mentioned before, pointwise convergence was shown in [4], where the goal was to prove existence of monostable traveling waves for the range of speeds $[c^*(n), +\infty)$. Here we prove that the convergence is actually uniform, which together with the continuity of speeds in the ignition case, immediately insures the continuity of $n \mapsto c^*(n)$, that is Theorem 2.5 in the monostable case.

Proof. First note that, for any fixed $n \in \mathbb{S}^{N-1}$ and $\varepsilon > 0$, $c_\varepsilon^*(n) \leq c^*(n)$. Indeed, recalling that $U_\varepsilon^*(z,x;n)$ connects $1-\frac{\varepsilon}{2}$ to $-\varepsilon$, one can find some shift $Z \in \mathbb{R}$ such that $U_\varepsilon^*(z+Z,x;n) \leq U^*(z,x;n)$, where U^* denotes a monostable pulsating traveling wave — connecting 1 to 0— with the minimal speed $c^*(n)$. By a comparison argument, it follows that $c_\varepsilon^*(n) \leq c^*(n)$. It is also very similar to check that, for any $n \in \mathbb{S}^{N-1}$, $0 < \varepsilon < \varepsilon'$ implies $c_\varepsilon^*(n) \geq c_{\varepsilon'}^*(n)$.

Let us now consider some sequences $\varepsilon_k \to 0$ and $n_k \to n$. Consider the estimate (19) where κ and K should *a priori* depend on ε. First, it is clear from the above that $\kappa(\varepsilon) := \inf_n c_\varepsilon^*(n)$ is nonincreasing with respect to ε. Also, since

$$\sup_{0<\varepsilon\leq\varepsilon_0} M_\varepsilon := \sup_{0<\varepsilon\leq\varepsilon_0} \; \sup_{x\in\mathbb{R}^N, u\in[-\varepsilon,1-\frac{\varepsilon}{2}]} \frac{f_\varepsilon(x,u)}{|u|} < +\infty$$

(compare with (18)), arguing as we did to derive (19), we see that $K(\varepsilon) := \sup_n c_\varepsilon^*(n)$ is uniformly bounded from above. As a result, we have

$$0 < \kappa := \inf_{0<\varepsilon\leq\varepsilon_0} \; \inf_{n\in\mathbb{S}^{N-1}} c^*(n) \leq \sup_{0<\varepsilon\leq\varepsilon_0} \; \sup_{n\in\mathbb{S}^{N-1}} c^*(n) =: K < +\infty. \tag{29}$$

Hence, we can assume, up to extraction of a subsequence, that $c^*_{\varepsilon_k}(n_k) \to c_\infty > 0$ as $k \to \infty$. In order to prove Theorem 4.1, we have to prove that $c_\infty = c^*(n)$.

We begin by showing that $U^*_{\varepsilon_k}(z, x; n_k)$ converges as $k \to \infty$ to a monostable pulsating traveling wave of (1), up to extraction of a subsequence. Indeed, proceeding as before, one can use standard parabolic estimates to extract a converging subsequence of pulsating ignition traveling waves, such that

$$U^*_{\varepsilon_k}(z, x; n_k) \to U_\infty(z, x),$$

as $k \to +\infty$ locally uniformly with respect to $(z, x) \in \mathbb{R} \times \mathbb{R}^N$. Furthermore, $0 \leq U_\infty(z, x) \leq 1$ solves (7) with $c = c_\infty$, is nonincreasing with respect to z, periodic with respect to x, and satisfies $\min_{x \in \mathbb{R}^N} U_\infty(0, x) = \frac{1}{2}$. In particular, U_∞ converges as $z \to \pm\infty$ to two periodic stationary solutions of (1), which under the monostable Assumption 1.1 can only be 0 and 1. We can conclude that U_∞ is a monostable pulsating traveling wave with speed c_∞, hence $c_\infty \geq c^*(n)$.

We now prove that $c_\infty = c^*(n)$. Notice that f_ε lies below f but, since the direction varies, we cannot use a simple comparison argument to conclude that $c_\infty \leq c^*(n)$. Instead, we will use a sliding method as in [4]. To do so, we shall need the following lemma.

Lemma 4.2 (Some uniform estimates)**.** *There exists $C > 0$ such that, for any small $\varepsilon > 0$ and $n \in \mathbb{S}^{N-1}$, the ignition pulsating traveling wave $U^*_\varepsilon(z, x; n)$ satisfies*

$$|\partial_{zz} U^*_\varepsilon(\cdot, \cdot; n)| \leq -C \partial_z U^*_\varepsilon(\cdot, \cdot, n), \quad |\nabla_x \partial_z U^*_\varepsilon(\cdot, \cdot; n)| \leq -C \partial_z U^*_\varepsilon(\cdot, \cdot, n).$$

Proof. Let us define $u^*_\varepsilon(t, x) := U^*_\varepsilon(x \cdot n - c^*_\varepsilon(n)t, x; n)$. Then $v(t, x) := \partial_t u^*_\varepsilon(t, x) > 0$ satisfies

$$\partial_t v = \mathrm{div}(A(x)\nabla v) + q(x) \cdot \nabla v + v\, \partial_u f_\varepsilon(x, u^*_\varepsilon), \qquad \text{a.e. in } \mathbb{R} \times \mathbb{R}^N.$$

From our definition of the ignition approximation $f_\varepsilon(x, u)$, we see that $\|\partial_u f_\varepsilon\|_{L^\infty(\mathbb{R}^N \times (-\varepsilon, 1 - \frac{\varepsilon}{2}))}$ is uniformly bounded, independently on small $\varepsilon > 0$ and $n \in \mathbb{S}^{N-1}$. Therefore, from the interior parabolic L^p-estimates (see [21, Theorem 48.1] for instance) and Sobolev embedding theorem, one gets

$$\forall (t_0, x_0) \in \mathbb{R} \times \mathbb{R}^N, \ |\partial_t v(t_0, x_0)| + |\nabla_x v(t_0, x_0)| \leq C_1 \max_{t_0 - 1 \leq t \leq t_0, |x - x_0| \leq 1} v(t, x), \tag{30}$$

for some $C_1 > 0$ which is independent on t_0, x_0, small $\varepsilon > 0$ and $n \in \mathbb{S}^{N-1}$.

Furthermore, for any $n \in \mathbb{S}^{N-1}$, choose N integers $k_i(n) \in \{-1, 0, 1\}$ such that

$$k(n)L \cdot n = \max_{k_1, \ldots, k_N \in \{-1, 0, 1\}} (k_1 L_1, \ldots k_N L_N) \cdot n,$$

where $k(n)L := (k_1(n)L_1, \ldots k_N(n)L_N)$. Then

$$0 < \inf_{n \in \mathbb{S}^{N-1}} k(n)L \cdot n \leq \sup_{n \in \mathbb{S}^{N-1}} k(n)L \cdot n < +\infty,$$

and hence, thanks to (29),

$$0 < \inf_{0 < \varepsilon \leq \varepsilon_0, n \in \mathbb{S}^{N-1}} \frac{k(n)L \cdot n}{c^*_\varepsilon(n)} \leq \sup_{0 < \varepsilon \leq \varepsilon_0, n \in \mathbb{S}^{N-1}} \frac{k(n)L \cdot n}{c^*_\varepsilon(n)} < +\infty.$$

By the parabolic Harnack inequality for strong solutions (see [18, Chapter VII] for instance), we get

$$\forall (t_0, x_0) \in \mathbb{R} \times \mathbb{R}^N, \ \max_{t_0 - 1 \leq t \leq t_0, |x - x_0| \leq 1} v(t, x) \leq C_2 v\left(t_0 + \frac{k(n)L \cdot n}{c^*_\varepsilon(n)}, x_0 + k(n)L\right), \tag{31}$$

for some $C_2 > 0$ which is also independent on t_0, x_0, small $\varepsilon > 0$ and $n \in \mathbb{S}^{N-1}$.

Combining (30), (31) and the space-time periodicity of the traveling wave, we get

$$\forall (t_0, x_0) \in \mathbb{R} \times \mathbb{R}^N, \ |\partial_t v(t_0, x_0)| + |\nabla_x v(t_0, x_0)| \leq C_3 v(t_0, x_0),$$

with $C_3 = C_1 C_2$. Now recall that $U_\varepsilon^*(z,x;n) = u_\varepsilon^*\left(\frac{x\cdot n - z}{c_\varepsilon^*(n)}, x\right)$. Thus

$$|\partial_{zz} U_\varepsilon^*| = \frac{1}{c_\varepsilon^*(n)^2} |\partial_t v| \leq \frac{C_3}{c_\varepsilon^*(n)^2} v = -\frac{C_3}{c_\varepsilon^*(n)} \partial_z U_\varepsilon^*,$$

$$|\nabla_x \partial_z U_\varepsilon^*| \leq \left| \frac{-1}{c_\varepsilon^*(n)^2} \partial_t v\, n - \frac{1}{c_\varepsilon^*(n)} \nabla_x v \right| \leq \left(\frac{1}{c_\varepsilon^*(n)^2} + \frac{1}{c_\varepsilon^*(n)} \right) C_3 v = -\left(\frac{1}{c_\varepsilon^*(n)} + 1 \right) C_3 \partial_z U_\varepsilon^*.$$

Since $\kappa = \inf_{0<\varepsilon\leq\varepsilon_0} \inf_{n\in\mathbb{S}^{N-1}} c_\varepsilon^*(n) > 0$, this proves the lemma. □

Let us now go back to the proof of Theorem 4.1. Proceed by contradiction and assume that $c_\infty \geq c^*(n) + \delta$ for some $\delta > 0$. We plug $U_{\varepsilon_k}^*(\cdot,\cdot;n_k)$ into equation (7) satisfied by $U^*(\cdot,\cdot;n)$ and, thanks to the above lemma, get

$$\begin{aligned}
& \mathrm{div}_x(A\nabla_x U_{\varepsilon_k}^*) + (n\cdot An)\,\partial_{zz} U_{\varepsilon_k}^* + \mathrm{div}_x(An\,\partial_z U_{\varepsilon_k}^*) + \partial_z(n\cdot A\nabla_x U_{\varepsilon_k}^*) \\
& \quad + q\cdot\nabla_x U_{\varepsilon_k}^* + (q\cdot n)\,\partial_z U_{\varepsilon_k}^* + c^*(n)\partial_z U_{\varepsilon_k}^* + f(x, U_{\varepsilon_k}^*) \\
= \; & (n\cdot An - n_k\cdot An_k)\partial_{zz} U_{\varepsilon_k}^* + \mathrm{div}_x((An - An_k)\times\partial_z U_{\varepsilon_k}^*) \\
& \quad + \partial_z((n-n_k)\cdot A\nabla_x U_{\varepsilon_k}^*) + (q\cdot(n-n_k))\partial_z U_{\varepsilon_k}^* \\
& \quad + (c^*(n) - c_{\varepsilon_k}^*(n_k))\partial_z U_{\varepsilon_k}^* + f(x, U_{\varepsilon_k}^*) - f_{\varepsilon_k}(x, U_{\varepsilon_k}^*) \\
\geq \; & \left[4a_2 C|n-n_k| + \mathrm{div}_x A(n-n_k) + |q||n-n_k| - \frac{\delta}{2} \right] \partial_z U_{\varepsilon_k}^* \\
\geq \; & -\frac{\delta}{3}\partial_z U_{\varepsilon_k}^* > 0,
\end{aligned} \tag{32}$$

provided k is large enough, and where $a_2 > 0$ comes from (3). We now use the sliding method. From the asymptotics

$$U_{\varepsilon_k}^*(+\infty,\cdot;n_k) = -\varepsilon_k < 0 = U^*(+\infty,\cdot;n),$$

$$U_{\varepsilon_k}^*(-\infty,\cdot;n_k) = 1 - \frac{\varepsilon_k}{2} < 1 = U^*(+\infty,\cdot;n),$$

one can define

$$\tau_0 := \inf\{\tau : U_{\varepsilon_k}^*(z+\tau, x; n_k) < U^*(z,x;n), \forall z\in\mathbb{R}, \forall x\in\mathbb{R}^N\} \in \mathbb{R}.$$

Then, using again the asymptotics as well as the periodicity with respect to x of any pulsating wave, there is some first touching point $(z_0, x_0) \in \mathbb{R}\times\mathbb{R}^N$ such that

$$U_{\varepsilon_k}^*(z_0+\tau_0, x_0; n_k) = U^*(z_0, x_0; n), \quad \text{and} \quad U_{\varepsilon_k}^*(\cdot+\tau_0,\cdot;n_k) \leq U^*(\cdot,\cdot;n).$$

Substracting the equation (7) satisfied by $U^*(z,x;n)$ to the inequality (32) satisfied by $U_{\varepsilon_k}^*(z+\tau_0, x; n_k)$ above, and estimating it at point (z_0, x_0), we get that

$$0 \geq -\frac{\delta}{3}\partial_z U_{\varepsilon_k}^*(z_0+\tau_0, x_0; n_k) > 0,$$

a contradiction. Hence, $c_\infty = c^*(n)$, and the convergence of $c_\varepsilon^*(n)$ to $c^*(n)$ is uniform. □

Remark 4.3 (On the convergence of profiles)**.** *The argument above also shows that the ignition traveling waves converge locally uniformly, up to a subsequence, to a traveling wave with minimal speed of the monostable equation. Proceeding as in Section 3 and thanks to the monotonicity of traveling waves, one can check that this convergence is actually uniform in time and space. In particular, they do not flatten as the parameter $\varepsilon \to 0$. However, as the uniqueness of the monostable traveling wave with minimal speed is not known [14], we cannot conclude on the convergence of the whole sequence.*

5 The uniform lower spreading

In this section and the next, we will prove Theorem 2.7 under the monostable assumption. The easier ignition case will be dealt with in the last section.

We begin here with the uniform lower spreading property (13) of Theorem 2.7. The argument again relies on the approximation from below by an ignition type problem, and follow the footsteps of the proof of Theorem 4.1.

Proof of (13). Recall that $f_\varepsilon(x,u)$ is an ignition type nonlinearity which approximates $f(x,u)$ from below as $\varepsilon \to 0$. We still denote $u_\varepsilon^*(t,x) = U_\varepsilon^*(x \cdot n - c_\varepsilon^*(n)t, x; n)$ the unique ignition pulsating traveling wave of (28) in the direction n, normalized by $\min_{x\in\mathbb{R}^N} U_\varepsilon^*(0,x;n) = \frac{1}{2}$.

As $f_\varepsilon \le f$, it is clear that u_ε^* is a subsolution of (1), whose speed is arbitrary close to $c^*(n)$ as $\varepsilon \to 0$ thanks to Theorem 4.1. This leads back to Weinberger's result [24], namely the fact that for any planar-like initial datum in the n direction, the solution of (1) spreads with speed "at least" $c^*(n)$ in the n direction.

Let us now make this spreading property uniform with respect to the family of solutions $(u_n)_{n\in\mathbb{S}^{N-1}}$, as stated in Theorem 2.7. In the following μ and K are as in assumption (11) (monostable case). Let $\alpha > 0$ and $\delta > 0$ be given. In view of assumption (12) and the comparison principle we have $u_n(t,x) \le 1$. Hence to prove (13), we need to find $\tau > 0$ so that

$$\inf_{n\in\mathbb{S}^{N-1}} \inf_{x\cdot n \le (c^*(n)-\alpha)t} u_n(t,x) \ge 1-\delta, \tag{33}$$

holds for all $t \ge \tau$.

In view of Theorem 4.1, we can fix $\varepsilon > 0$ small enough so that, for all $n \in \mathbb{S}^{N-1}$,

$$c_\varepsilon^*(n) \ge c^*(n) - \frac{\alpha}{2}. \tag{34}$$

We then claim that one can find some $t_\varepsilon > 0$ such that

$$u_n(t_\varepsilon, x) \ge 1 - \frac{\varepsilon}{2}, \tag{35}$$

for all $n \in \mathbb{S}^{N-1}$ and all x such that $x \cdot n \le -K$. We insist on the fact that t_ε does not depend on $n \in \mathbb{S}^{N-1}$. To prove (35), let us define

$$\mathcal{S} = \{x \in \mathbb{R}^N : \ x \cdot n \le c^*(n) \text{ for all } n \in \mathbb{S}^{N-1}\}.$$

We know from Theorem 2.5 that the mapping $n \mapsto c^*(n)$ is positive and continuous, hence $\mathcal{S}$ has nonempty interior. It is then known (see Theorem 2.3 in [24], as well as Remark 2.4 above) that for compactly supported initial data "with large enough support", the associated solution of (1) converges locally uniformly to 1 as $t \to +\infty$ (in fact, even uniformly on the expanding sets $t\mathcal{S}'$ for any subset $\mathcal{S}'$ of the interior of $\mathcal{S}$; also, under the additional assumption that 0 is linearly unstable with respect to the periodic problem, this is even true for any non trivial and compactly supported initial datum, regardless of its size [7], [6]). More precisely, let u_R be the solution of (1) associated with the initial datum $u_{0,R}(x) = \mu \times \chi_{B_R}(x)$, where R is a large but fixed positive constant (depending on μ) which we can assume to be larger than $2\sqrt{N}\max_i L_i$. Here B_R denotes the ball of radius R centered at the origin. Then u_R converges locally uniformly to 1 as $t \to +\infty$. In particular,

$$u_R(t_\varepsilon, x) \ge 1 - \frac{\varepsilon}{2},$$

for some $t_\varepsilon > 0$ and all $x \in B_{2R}$. Besides, for $x_0 \in \Pi_{i=1}^N L_i\mathbb{Z}$ such that $x_0 \cdot n \le -K - R$, we have — thanks to (11)— that $u_n(0, x + x_0) \ge u_R(0,x)$. Then, by the comparison principle,

$$\forall x \in B_{2R}, \quad u_n(t_\varepsilon, x + x_0) \ge u_R(t_\varepsilon, x) \ge 1 - \frac{\varepsilon}{2}.$$

Since $R > 2\sqrt{N}\max_i L_i$, for all $x \cdot n \leq -K$, there exists $x_0 \in \Pi_{i=1}^N L_i\mathbb{Z}$ such that $x_0 \cdot n \leq -K - R$ and $x \in B_{2R}(x_0)$. Thus, we obtain $u_n(t_\varepsilon, x) \geq 1 - \frac{\varepsilon}{2}$, for all $n \in \mathbb{S}^{N-1}$ and $x \cdot n \leq -K$, that is claim (35).

Now, recall that $U_\varepsilon^*(\cdot, \cdot; n)$ is the pulsating traveling wave of equation (28) in the direction n, connecting $1 - \frac{\varepsilon}{2}$ to $-\varepsilon$. Hence, it follows from (35) that, for any $n \in \mathbb{S}^{N-1}$, one can find some shift Z_n such that

$$u_n(t_\varepsilon, x) \geq U_\varepsilon^*(x \cdot n - c_\varepsilon^*(n)t_\varepsilon + Z_n, x; n). \tag{36}$$

Actually, it suffices to select

$$Z_n := \min\{z \in \mathbb{R} : \min_{x \in \mathcal{C}} U_\varepsilon^*(-K - c_\varepsilon^*(n)t_\varepsilon + z, x; n) \leq 0\} \in (0, \infty).$$

Moreover, from the uniform continuity of ignition traveling waves with respect to the direction, namely Theorem 2.6, it is straightforward that the family $(U_\varepsilon^*(z, x; n))_{n \in \mathbb{S}^{N-1}}$ converges to $-\varepsilon$ as $z \to +\infty$ uniformly with respect to $n \in \mathbb{S}^{N-1}$. Therefore, we can also define the bounded real number $Z := \sup_{n \in \mathbb{S}^{N-1}} Z_n \in (0, \infty)$, so that (36) is improved to

$$\forall n \in \mathbb{S}^{N-1}, \quad u_n(t_\varepsilon, x) \geq U_\varepsilon^*(x \cdot n - c_\varepsilon^*(n)t_\varepsilon + Z, x; n).$$

Then we can apply the parabolic comparison principle to get

$$\forall t \geq t_\varepsilon, \forall x \in \mathbb{R}^N, \forall n \in \mathbb{S}^{N-1}, \quad u_n(t, x) \geq U_\varepsilon^*(x \cdot n - c_\varepsilon^*(n)t + Z, x; n). \tag{37}$$

Therefore it follows from (34), (37) and the monotonicity of the front that

$$u_n(t, x) \geq U_\varepsilon^*\left(-\frac{\alpha}{2}t + Z, x; n\right), \tag{38}$$

for all $n \in \mathbb{S}^{N-1}$, all $t \geq t_\varepsilon$ and all x such that $x \cdot n \leq (c^*(n) - \alpha)t$. Using again the uniform continuity of ignition traveling waves with respect to the direction, namely Theorem 2.6, one can find some shift $Z' > 0$ such that, for all $n \in \mathbb{S}^{N-1}$,

$$z \leq -Z' \;\Rightarrow\; U_\varepsilon^*(z, x; n) \geq 1 - \varepsilon. \tag{39}$$

Up to decreasing ε, we can assume that $\varepsilon < \delta$ without loss of generality. Now choose $\tau \geq t_\varepsilon$ such that $-\frac{\alpha}{2}\tau + Z \leq Z'$. Then, we get from (38) and (39) that

$$u_n(t, x) \geq 1 - \delta,$$

for all $n \in \mathbb{S}^{N-1}$, $t \geq \tau$ and x such that $x \cdot n \leq (c^*(n) - \alpha)t$. We have thus proved (33), and hence (13). □

6 The uniform upper spreading

We conclude here the proof of Theorem 2.7 (monostable case), by proving the uniform upper spreading (14) in subsection 6.1. Then in subsection 6.2 we again prove (14) — together with the uniform lower spreading property (13)— when assumption (12) is relaxed.

6.1 Proof of (14)

We begin by proving some kind of uniform steepness of the monostable minimal waves, which in turn will easily imply (14).

Proposition 6.1 (Steepness of critical waves). *Assume that f is of the spatially periodic monostable type, i.e. f satisfies* (2) *and Assumption 1.1.*

Let $u^*(t, x; n) = U^*(x \cdot n - c^*(n)t, x; n)$ be a family of increasing in time pulsating traveling waves of (1)*, with minimal speed $c^*(n)$ in each direction $n \in \mathbb{S}^{N-1}$, normalized by $U^*(0, 0; n) = \frac{1}{2}$.*

Then, the asymptotics $U^*(-\infty, x; n) = 1$, $U^*(\infty, x; n) = 0$ are uniform with respect to $n \in \mathbb{S}^{N-1}$. Moreover, for any $K > 0$, we have $\inf_{n \in \mathbb{S}^{N-1}} \inf_{|z| \leq K} \inf_{x \in \mathbb{R}^N} -\partial_z U^*(z, x; n) > 0$ and $\inf_{n \in \mathbb{S}^{N-1}} \inf_{|z| \leq K} \inf_{x \in \mathbb{R}^N} U^*(z, x; n) > 0$.

Remark 6.2 (Lack of uniqueness)**.** *Such a family of traveling waves is always known to exist. However, the uniqueness of the traveling wave with minimal speed in each direction is not known. We shall prove that any sequence of increasing in time traveling waves with minimal speed in the directions $n_k \to n$ converges, up to extraction of a subsequence, to an increasing in time traveling wave with minimal speed in the direction n, as we did in the ignition case. The proposition then easily follows, but the lack of uniqueness is the reason we state this result in a slightly different way.*

Proof. Proceeding as explained in the above remark, choose some sequence $n_k \to n \in \mathbb{S}^{N-1}$. As before, one can extract a subsequence such that $u^*(\cdot, \cdot; n_k)$ converges locally uniformly to a solution u_∞ of (1). By the continuity of the speeds $c^*(n)$ with respect to n, as proved in Theorem 2.5, the function u_∞ also satisfies

$$\forall l \in \prod_{i=1}^{N} L_i \mathbb{Z}, \quad u_\infty(t,x) = u_\infty\left(t + \frac{l \cdot n}{c^*(n)}, x + l\right).$$

Moreover, it is nondecreasing in time, hence increasing in time by applying the strong maximum principle to $\partial_t u_\infty$. In particular, it converges to two spatially periodic stationary solutions as $t \to \pm\infty$ which, as before and thanks to the monostable assumption, must be 0 and 1. As announced, u_∞ is an increasing in time traveling wave with minimal speed in the direction n. Reasoning by contradiction, it is now straightforward to prove Proposition 6.1. □

Proof of (14). First, from Proposition 6.1 above, and hypotheses (10)—(12), one can find some shift $K_1 > 0$ large enough so that, for any $n \in \mathbb{S}^{N-1}$, $u_{0,n}(x) \leq U^*(x \cdot n - K_1, x; n)$. Thus, by the comparison principle,

$$u_n(t,x) \leq U^*(x \cdot n - c^*(n)t - K_1, x; n).$$

For any $\alpha > 0$ and $\delta > 0$, let τ be such that $U^*(\alpha\tau - K_1, x; n) \leq \delta$, for all $n \in \mathbb{S}^{N-1}$ and $x \in \mathbb{R}^N$, which is again made possible by Proposition 6.1. Then (14) immediately follows. □

6.2 Relaxing assumption (12)

We here consider the case when the family $(u_{0,n})_{n \in \mathbb{S}^{N-1}}$ does not necessarily satisfy (12), but is only uniformly bounded: there is $M > 0$ such that

$$\forall x \in \mathbb{R}^N, \forall n \in \mathbb{S}^{N-1}, \quad u_{0,n}(x) \leq M. \tag{40}$$

We prove that, in such a situation, the uniform lower and upper spreading properties (13) and (14) remain true if we make the following additional assumptions on the behavior of f, and in particular on its behavior above the stationary state p.

Assumption 6.3 (Additional assumptions)**.** *(i) There is $\phi(t,x)$ a solution of* (1) *such that $\phi(0,\cdot) \geq M$, and $\phi(t,x)$ converges uniformly to* 1 *as $t \to +\infty$.*

(ii) The steady state 0 of (1) *is linearly unstable with respect to periodic perturbations.*

(iii) There exists some $\rho > 0$ such that $f(x,u)$ is nonincreasing with respect to u in the set $\mathbb{R}^N \times (1-\rho, 1+\rho)$.

The first part of this assumption holds true, for instance, if $f(x,s) < 0$ for all $x \in \mathbb{R}^N$ and $s > 1$. As we will see below, the second part can be expressed in terms of some principal eigenvalue problem, and holds true as soon as $\partial_u f(x,0)$ is positive on a non empty set. The last part is a natural extension of *(iii)* of Assumption 1.1.

Combining (33), whose proof does not require assumption (12), and a comparison of the solutions $(u_n)_{n \in \mathbb{S}^{N-1}}$ with ϕ given by the above assumption, it is clear that the lower spreading property (13) still holds true. In the sequel, we prove the upper spreading property (14). We start with the following proposition, whose proof is identical to that of Proposition 6.1 and does not require Assumption 6.3.

Proposition 6.4 (Steepness of noncritical waves)**.** *Assume that f is of the spatially periodic monostable type, i.e. f satisfies* (2) *and Assumption 1.1.*

For any $\alpha > 0$, let $u_\alpha(t,x;n) = U_\alpha(x \cdot n - (c^*(n)+\alpha)t, x; n)$ be a family of increasing in time pulsating traveling waves of (1)*, in direction n, with speed $c^*(n)+\alpha$, normalized by $U_\alpha(0,0;n) = \frac{1}{2}$.*

Then, the asymptotics $U_\alpha(-\infty,x;n) = 1$, $U_\alpha(\infty,x;n) = 0$ are uniform with respect to $n \in \mathbb{S}^{N-1}$. Moreover, for any $K > 0$, we have $\inf_{n\in\mathbb{S}^{N-1}} \inf_{|z|\leq K} \inf_{x\in\mathbb{R}^N} -\partial_z U_\alpha(z,x;n) > 0$ and $\inf_{n\in\mathbb{S}^{N-1}} \inf_{|z|\leq K} \inf_{x\in\mathbb{R}^N} U_\alpha(z,x;n) > 0$

We now turn to the proof of the upper spreading property (14), which relies on the construction of a suitable family of supersolutions that were already used in [14] (following an idea of [11]).

Proof of (14). Let $\alpha > 0$ and $\delta > 0$ be given. We need to find $\tau > 0$ so that estimate (14) holds for all $t \geq \tau$.

First, we need to introduce some notations, and some well known results (see [25], [4], [14] among others). We begin with the principal eigenvalue problem

$$\begin{cases} -L_{0,n,\lambda}\phi_{n,\lambda} = \mu_0(n,\lambda)\phi_{n,\lambda} \ \text{ in } \mathbb{R}^N, \\ \phi_{n,\lambda} \text{ is periodic, } \phi_{n,\lambda} > 0, \ \|\phi_{n,\lambda}\|_\infty = 1, \end{cases} \tag{41}$$

where

$$L_{0,n,\lambda}\phi = \operatorname{div}(A\nabla\phi) + \lambda^2(n\cdot An)\phi - \lambda(\operatorname{div}(An\phi) + n\cdot A\nabla\phi) + q\cdot\nabla\phi - \lambda(q\cdot n)\phi + \partial_u f(x,0)\phi.$$

This arises, similarly as in Section 3, when looking for moving exponential solutions of the type $e^{-\lambda(x\cdot n - ct)}\phi_{n,\lambda}(x)$ of the linearized problem around 0. Such solutions exist if and only if

$$c \geq c^*_{lin}(n) := \min_{\lambda>0} \frac{-\mu_0(n,\lambda)}{\lambda},$$

which is well-defined thanks to the linear instability of 0, which reads as $\mu_0 = \mu_0(n,0) < 0$. Moreover, it is known that $c^*(n) \geq c^*_{lin}(n)$ [13]. We introduce $\lambda(n)$ the smallest positive solution of $-\mu_0(n,\lambda) = \left(c^*(n) + \frac{\alpha}{4}\right)\lambda$. It is standard that $\mu_0(n,\lambda)$ is continuous with respect to n and, as it is known to be concave, $\lambda(n)$ is also continuous with respect to n. In particular

$$0 < \min_{n\in\mathbb{S}^{N-1}} \lambda(n) \leq \max_{n\in\mathbb{S}^{N-1}} \lambda(n) < +\infty.$$

Let some smooth and nonincreasing $\chi(z)$ be such that

$$\chi(z) = \begin{cases} 1 \text{ if } z < -1, \\ 0 \text{ if } z > 1, \end{cases}$$

and define, for $s \geq 0$ (a shift to be fixed later),

$$\Phi(t,x) = \Phi_s(t,x;n) := \chi(\xi_s) + (1-\chi(\xi_s))\phi_{n,\lambda(n)}(x)e^{-\lambda(n)\xi_s},$$

where

$$\xi_s = \xi_s(t,x;n) = x\cdot n - \left(c^*(n) + \frac{\alpha}{2}\right)t - s.$$

Note that Φ is nonnegative and, along with its derivatives, is bounded uniformly with respect to n and s.

Let us now define various positive constants. Choose $0 < \eta < \delta$ small enough so that

$$\forall x \in \mathbb{R}^N, \ \forall 0 \leq u \leq \eta, \quad |\partial_u f(x,u) - \partial_u f(x,0)| \leq \frac{\alpha}{4} \min_{n\in\mathbb{S}^{N-1}} \lambda(n), \tag{42}$$

$$\forall x \in \mathbb{R}^N, \ \forall 1-\eta \leq u \leq 1+\eta, \quad \partial_u f(x,u) \leq 0. \tag{43}$$

Now, by Proposition 6.4, there is $K > 1$ large enough such that, for all $n \in \mathbb{S}^{N-1}$, $x \in \mathbb{R}^N$,

$$\xi > K \Rightarrow 0 \leq U_{\frac{\alpha}{4}}(\xi, x; n) \leq \frac{\eta}{2}, \quad \xi < -K \Rightarrow 1 - \frac{\eta}{2} \leq U_{\frac{\alpha}{4}}(\xi, x; n) \leq 1. \tag{44}$$

Then, by Proposition 6.4 again, we have

$$\gamma := \inf_{n \in \mathbb{S}^{N-1}} \inf_{|z| \leq K, x \in \mathbb{R}^N} -\partial_z U_{\frac{\alpha}{4}}(z, x; n) > 0. \tag{45}$$

Last, we define

$$\epsilon_1 := \frac{\eta}{2\|\Phi\|_\infty}, \ \epsilon_2 := \frac{\alpha\gamma}{4(\|\partial_t \Phi\|_\infty + \|\mathrm{div}(A\nabla\Phi)\|_\infty + \|q \cdot \nabla\Phi\|_\infty + \|\Phi\|_\infty \|\partial_u f\|_{L^\infty(\mathbb{R}^N \times (0, 1+\frac{\eta}{2}))})}$$

and

$$\epsilon := \min(\epsilon_1, \epsilon_2) > 0. \tag{46}$$

Now, we are going to show that

$$v(t,x) = v_s(t,x;n) := U_{\frac{\alpha}{4}}\left(x \cdot n - \left(c^*(n) + \frac{\alpha}{2}\right)t - s, x; n\right) + \epsilon\Phi(t,x) = U_{\frac{\alpha}{4}}(\xi_s, x; n) + \epsilon\Phi(t,x)$$

is a supersolution of the monostable equation (1). Straightforward computations and the mean value Theorem yield

$$\begin{aligned} \mathcal{L}[v](t,x) &:= \partial_t v(t,x) - \mathrm{div}(A(x)\nabla v(t,x)) - q(x) \cdot \nabla v(t,x) - f(x, v(t,x)) \\ &= \epsilon\left[\partial_t \Phi(t,x) - \mathrm{div}(A(x)\nabla\Phi(t,x)) - q(x)\cdot\nabla\Phi(t,x) - \Phi(t,x)\partial_u f(x, \theta(t,x))\right] \\ &\quad -\frac{\alpha}{4}\partial_z U_{\frac{\alpha}{4}}(\xi_s, x; n), \end{aligned}$$

for some

$$U_{\frac{\alpha}{4}}(\xi_s, x; n) \leq \theta(t,x) \leq U_{\frac{\alpha}{4}}(\xi_s, x; n) + \epsilon\Phi(t,x).$$

We distinguish three regions, depending on the values of ξ_s.

First, if $|\xi_s| \leq K$, the nonnegativity of $\mathcal{L}[v](t,x)$ is obtained thanks to $-\frac{\alpha}{4}\partial_z U_{\frac{\alpha}{4}}(\xi_s, x; n) \geq \frac{\alpha}{4}\gamma$ by (45) and the definition of ϵ in (46).

Next, if $\xi_s > K$, then $\Phi(t,x)$ reduces to $\phi_{n,\lambda(n)}(x)e^{-\lambda(n)\xi_s}$ and, dropping $-\frac{\alpha}{4}\partial_z U_{\frac{\alpha}{4}}(\xi_s, x; n)$ which is positive, we arrive at

$$\begin{aligned} \frac{1}{\epsilon}\mathcal{L}[v](t,x) &\geq \left[\lambda(n)\left(c^*(n) + \frac{\alpha}{2}\right) + \mu_0(n, \lambda(n)) + \partial_u f(x,0) - \partial_u f(x, \theta(t,x))\right]\phi_{n,\lambda(n)}(x)e^{-\lambda(n)\xi_s} \\ &\geq \left(\frac{\alpha}{4}\lambda(n) + \partial_u f(x,0) - \partial_u f(x,\theta(t,x))\right)\phi_{n,\lambda(n)}(x)e^{-\lambda(n)\xi_s}. \end{aligned}$$

But, when $\xi_s > K$, (44) and $\epsilon \leq \epsilon_1$ imply $0 \leq \theta(t,x) \leq \eta$, and the nonnegativity of $\mathcal{L}[v](t,x)$ is obtained thanks to (42).

Last, we consider the case where $\xi_s < -K$, so that $\Phi(t,x)$ reduces to 1. Hence

$$\frac{1}{\epsilon}\mathcal{L}[v](t,x) \geq -\partial_u f(x, \theta(t,x)).$$

But, when $\xi_s < -K$, (44) and $\epsilon \leq \epsilon_1$ imply $1 - \eta \leq \theta(t,x) \leq 1 + \eta$, and the nonnegativity of $\mathcal{L}[v](t,x)$ is obtained thanks to (43). Hence, $v_s(t,x;n)$ is a supersolution of (1).

Thanks to (40), we get by the comparison principle that, for all $n \in \mathbb{S}^{N-1}$, all $t \geq 0$, all $x \in \mathbb{R}^N$, $u_n(t,x) \leq \phi(t,x)$, where ϕ is given by Assumption 6.3. Now choose $T > 0$ such that $\phi(T,x) \leq 1 + \frac{\epsilon}{2}$, and get that

$$\forall n \in \mathbb{S}^{N-1}, \ \forall x \in \mathbb{R}^N, \quad u_n(T,x) \leq 1 + \frac{\epsilon}{2}. \tag{47}$$

Using the comparison principle and a computation identical to that of (17), we get that, for any large $\lambda > 0$, there is $C > 0$ — independent on n thanks to (10) and (40)— such that

$$\forall n \in \mathbb{S}^{N-1},\ \forall t \geq 0,\ \forall x \in \mathbb{R}^N, \quad u_n(t,x) \leq Ce^{-\lambda(x\cdot n - 2a_1\lambda t)}.$$

In particular $u_n(T,\cdot)$ decays faster than any exponential as $x \cdot n \to +\infty$, namely

$$\forall \lambda > 0, \quad u_n(T,x)e^{\lambda x\cdot n} \to 0 \ \text{ as } x\cdot n \to +\infty, \ \text{ uniformly w.r.t. } n \in \mathbb{S}^{N-1}. \tag{48}$$

Observe that, for all $s \geq 0$,

$$\forall n \in \mathbb{S}^{N-1},\ \forall x\cdot n \geq \left(c^*(n) + \frac{\alpha}{2}\right)T+s+1,\ v_s(T,x;n) \geq \epsilon\phi_{n,\lambda(n)}(x)e^{-\lambda(n)x\cdot n} \geq \epsilon\gamma e^{-\lambda_{max}x\cdot n}, \tag{49}$$

where $\gamma := \min_{n\in\mathbb{S}^{N-1}} \min_{x\in\mathbb{R}^N} \phi_{n,\lambda(n)}(x) > 0$ and $\lambda_{max} := \max_{n\in\mathbb{S}^{N-1}} \lambda(n) < \infty$ (recall that $n \mapsto \lambda(n)$ is continuous and so is $(n,\lambda) \mapsto \phi_{n,\lambda}$). Now, select $A > 1$ large enough so that, for all $s \geq 0$,

$$\forall n \in \mathbb{S}^{N-1},\ \forall x \cdot n \leq \left(c^*(n) + \frac{\alpha}{2}\right)T + s - A,\ v_s(T,x;n) \geq 1 + \frac{\epsilon}{2}, \tag{50}$$

which is possible thanks to Proposition 6.4, and more precisely the uniform with respect to n asymptotics of $U_{\frac{\alpha}{4}}(z,x;n)$ as $z \to -\infty$. Proposition 6.4 also enables to define

$$\kappa := \inf_{n\in\mathbb{S}^{N-1}} \inf_{-A\leq z\leq 1} \inf_{x\in\mathbb{R}^N} U_{\frac{\alpha}{4}}(z,x;n) > 0,$$

so that, for all $s \geq 0$,

$$\forall n \in \mathbb{S}^{N-1},\ \forall \left(c^*(n) + \frac{\alpha}{2}\right)T + s - A \leq x\cdot n \leq \left(c^*(n) + \frac{\alpha}{2}\right)T + s + 1,\ v_s(T,x;n) \geq \kappa. \tag{51}$$

In view of (48), we can now select a large enough shift $s_0 > A$ so that

$$\forall n \in \mathbb{S}^{N-1},\ \forall x \cdot n \geq \left(c^*(n) + \frac{\alpha}{2}\right)T + s_0 - A,\ u_n(T,x) \leq \min\{\epsilon\gamma, \kappa\}e^{-\lambda_{max}x\cdot n}.$$

Combining this with (47), (49), (50), (51), we have that, for all $n \in \mathbb{S}^{N-1}$ and $x \in \mathbb{R}^N$,

$$u_n(T,x) \leq v_{s_0}(T,x;n).$$

Then, by the comparison principle, for all $t \geq T$, $x \in \mathbb{R}^N$, $n \in \mathbb{S}^{N-1}$,

$$0 \leq u_n(t,x) \leq U_{\frac{\alpha}{4}}\left(x\cdot n - \left(c^*(n) + \frac{\alpha}{2}\right)t - s_0, x; n\right) + \epsilon\Phi(t,x).$$

Hence, when $x \cdot n \geq (c^*(n) + \alpha)t$, we have, since $\epsilon\|\Phi\|_\infty \leq \frac{\eta}{2} \leq \frac{\delta}{2}$, that

$$0 \leq u_n(t,x) \leq U_{\frac{\alpha}{4}}\left(\frac{\alpha}{2}t - s_0, x; n\right) + \frac{\delta}{2} \leq \delta,$$

as soon as $t \geq \tau$, where $\tau > 0$ is large enough (again independently on n by Proposition 6.4). This proves (14). □

7 The uniform spreading: the ignition case

For the sake of completeness, we give here the main steps to prove Theorem 2.7 in the (simpler) ignition case. We will see that it follows from the continuity of ignition waves, Theorem 2.6, together with the standard idea explained in Remark 2.4. We will briefly sketch at the end of this section how the hypothesis (12) can again be relaxed.

Proof of Theorem 2.7 in the ignition case. First, the proof of the uniform upper spreading (14) is the same as that of subsection 6.1 in the monostable case, using Theorem 2.6 (continuity of ignition waves) instead of Proposition 6.1 (steepness of critical waves).

Let us now prove the uniform lower spreading (13). Let $\alpha > 0$ and $\delta > 0$ be given. We may reduce δ without loss of generality, and assume that $\delta < \rho$ where ρ is given by part (iii) of Assumption 1.2. Using the same arguments as in the proof of (35), we get the existence of some time $t_\delta > 0$ such that

$$u_n(t_\delta, x) \geq 1 - \frac{\delta}{2},$$

for all $n \in \mathbb{S}^{N-1}$ and all x such that $x \cdot n \leq -K$. Now let, as usual, $U^*(x \cdot n - c^*(n)t, x; n)$ be the unique ignition pulsating wave in the direction n, normalized by $\min_{x \in \mathbb{R}^N} U^*(0, x; n) = \frac{1+\theta}{2}$. Thanks to the inequality above and the continuity of the mapping $n \mapsto U^*(\cdot, \cdot; n)$ with respect to the uniform topology, it is clear that there exists some shift $Z > 0$ such that, for all $n \in \mathbb{S}^{N-1}$,

$$U^*(x \cdot n + Z, x; n) - \frac{\delta}{2} \leq u_n(t_\delta, x), \quad \forall x \in \mathbb{R}^N.$$

We then check that $\underline{u}(t,x) := U^*(x \cdot n + Z - (c^*(n) - \frac{\alpha}{2})t, x; n) - \frac{\delta}{2}$ is a subsolution of (1). Indeed,

$$\partial_t \underline{u} - \mathrm{div}\,(A(x)\nabla \underline{u}) - q(x) \cdot \nabla \underline{u} - f(x, \underline{u}) = \frac{\alpha}{2}\partial_z U^* + f(x, U^*) - f(x, \underline{u}).$$

Assume first that $\underline{u} \leq \theta - \frac{\delta}{2}$. Then $f(x, \underline{u}) = f(x, U^*(x \cdot n + Z - (c^*(n) - \frac{\alpha}{2})t, x; n) = 0$, which together with the monotonicity of U^* with respect to its first variable, gives the wanted inequality. Assume then that $\underline{u} \geq 1 - \rho$. Then, by the monotonicity of f with respect to u in the range $[1-\rho, 1]$, we again obtain the wanted inequality.

It remains to prove that $\underline{u}$ is a subsolution when $\theta - \frac{\delta}{2} \leq \underline{u} \leq 1 - \rho$ or, equivalently, when $\theta \leq U^*(x \cdot n - (c^*(n) - \frac{\alpha}{2})t + Z, x; n) \leq 1 - \rho + \frac{\delta}{2}$. Recall first that $1 - \rho + \frac{\delta}{2} < 1 - \frac{\rho}{2}$. Using again the continuity of the ignition wave with respect to the direction, we have that there exists some $R > 0$ such that, for all $n \in \mathbb{S}^{N-1}$,

$$z \geq R \;\Rightarrow\; U^*(z + Z, x; n) < \theta, \quad z \leq -R \;\Rightarrow\; U^*(z + Z, x; n) > 1 - \frac{\rho}{2},$$

and, furthermore,

$$\max_{n \in \mathbb{S}^{N-1}} \max_{|z| \leq R} \max_{x \in \mathbb{R}^N} \partial_z U^*(z + Z, x; n) < 0.$$

Up to reducing δ again, we may assume that

$$\max_{n \in \mathbb{S}^{N-1}} \max_{|z| \leq R} \max_{x \in \mathbb{R}^N} \partial_z U^*(z + Z, x; n) < -\frac{M\delta}{\alpha},$$

where M is a Lipschitz constant of f. Therefore, when $\theta - \frac{\delta}{2} \leq \underline{u} \leq 1 - \rho$, then $|x \cdot n - (c^*(n) - \frac{\alpha}{2})t| \leq R$ and

$$\begin{aligned} & \partial_t \underline{u} - \mathrm{div}\,(A(x)\nabla \underline{u}) - q(x) \cdot \nabla \underline{u} - f(x, \underline{u}) \\ = \; & \frac{\alpha}{2}\partial_z U^* + f(x, U^*) - f(x, \underline{u}) \\ \leq \; & \frac{\alpha}{2}\partial_z U^* + M\frac{\delta}{2} \leq 0, \end{aligned}$$

that is the wanted inequality.

We can therefore apply the comparison principle and conclude that

$$U^*\left(x \cdot n + Z - (c^*(n) - \frac{\alpha}{2})t, x; n\right) - \frac{\delta}{2} \leq u_n(t_\delta + t, x),$$

for all $x \in \mathbb{R}^N$ and $t \geq 0$.

Noting that there exists some other shift $Z' > 0$ such that, for all $n \in \mathbb{S}^{N-1}$,

$$z \leq -Z' \Rightarrow U^*(z, x; n) - \frac{\delta}{2} \geq 1 - \delta,$$

we get the uniform lower spreading (13) as in the end of Section 5. □

Relaxing hypothesis (12). In order to relax (12), assume now that f satisfies Assumption 1.2 and parts (i) and (iii) of Assumption 6.3. As above, one can then show that $U^*(x \cdot n - (c^*(n) + \frac{\alpha}{2})t, x; n) + \frac{\delta}{2}$ is a supersolution of (1). Then, as in Section 6.2, one can find some time T and some shift s_0 such that, for all n, the solution $u_n(T, x)$ lies below $U^*(x \cdot n + s_0 - (c^*(n) + \frac{\alpha}{2})T, x; n) + \frac{\delta}{2}$ in the whole space. It is then straightforward to obtain the wanted uniform upper spreading (14). □

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References

[1] M. Alfaro and T. Giletti, *Asymptotic analysis of a monostable equation in periodic media*, in preparation.

[2] D. G. Aronson and H. F. Weinberger, *Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation*, Partial differential equations and related topics (Program, Tulane Univ., New Orleans, La., 1974), 5–49. Lecture Notes in Math., Vol. 446, Springer, Berlin, 1975.

[3] D. G. Aronson and H. F. Weinberger, *Multidimensional nonlinear diffusion arising in population genetics*, Adv. in Math. **30** (1978), no. 1, 33–76.

[4] H. Berestycki and F. Hamel, *Front propagation in periodic excitable media*, Comm. Pure Appl. Math. **55** (2002), no. 8, 949–1032.

[5] H. Berestycki and F. Hamel, *Generalized transition waves and their properties*, Comm. Pure Appl. Math. **65**, (2012), no. 5, 592–648.

[6] H. Berestycki, F. Hamel and G. Nadin, *Asymptotic spreading in heterogeneous diffusive excitable media*, J. Funct. Anal. **255** (2008), 2146–2189.

[7] H. Berestycki, F. Hamel and L. Roques, *Analysis of the periodically fragmented environment model. I. Species persistence*, J. Math. Biol. **51** (2005), 5–113.

[8] H. Berestycki, F. Hamel and L. Roques, *Analysis of the periodically fragmented environment model. II. Biological invasions and pulsating traveling fronts*, J. Math. Pures Appl. **84** (2005), 1101–1146.

[9] H. Berestycki, B. Nicolaenko and B. Scheurer, *Traveling wave solutions to combustion models and their singular limits*, SIAM J. Math. Anal. **16** (1985), no. 6, 1207–1242.

[10] H. Berestycki and L. Nirenberg, *Travelling fronts in cylinders*, Ann. Inst. H. Poincaré Anal. Non Linéaire **9** (1992), no. 5, 497–572.

[11] P. C. Fife and J. B. McLeod, *The approach of solutions of nonlinear diffusion equations to travelling front solutions*, Arch. Rational Mech. Anal. **65** (1977), 335–361.

[12] R. A. Fisher, *The wave of advance of advantageous genes*, Ann. of Eugenics **7** (1937), 355–369.

[13] F. Hamel, *Qualitative properties of monostable pulsating fronts: exponential decay and monotonicity*, J. Math. Pures Appl. **89** (2008), 355–399.

[14] F. Hamel and L. Roques, *Uniqueness and stability properties of monostable pulsating fronts*, J. Eur. Math. Soc. **13** (2011), 345–390.

[15] W. Hudson and B. Zinner, *Existence of traveling waves for reaction diffusion equations of Fisher type in periodic media*, Boundary value problems for functional-differential equations, 187–199, World Sci. Publ., River Edge, NJ, 1995.

[16] Ja. I. Kanel, *Stabilization of solutions of the Cauchy problem for equations encountered in combustion theory*, (Russian) Mat. Sb. **59** (1962), 245–288.

[17] A. N. Kolmogorov, I. G. Petrovsky and N. S. Piskunov, *Etude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique*, Bulletin Université d'Etat Moscou, Bjul. Moskowskogo Gos. Univ., 1937, 1–26.

[18] G. M. Lieberman, *Second Order Parabolic Differential Equations*, World Scientific Publishing Co. Inc., River Edge, NJ, 1996.

[19] G. Nadin, *Traveling fronts in space-time periodic media*, J. Math. Pures Appl. (9) **92** (2009), no. 3, 232–262.

[20] J. Nolen and L. Ryzhik, *Traveling waves in a one-dimensional heterogeneous medium*, Ann. Inst. H. Poincaré Anal. Non Linéaire **26** (2009), no. 3, 1021–1047.

[21] P. Quittner and P. Souplet, *Superlinear Parabolic Problems*, Birkhäuser Verlag, Basel, 2007.

[22] N. Shigesada and K. Kawasaki, *Biological Invasion: Theory and Practise*, Oxford University Press, 1997.

[23] A. Volpert, V. Volpert, V. Volpert, *Travelling Wave Solutions of Parabolic Systems*, Translations of Mathematical Monographs, vol. 140, AMS Providence, RI, 1994.

[24] H. Weinberger, *On spreading speed and travelling waves for growth and migration*, J. Math. Biol. **45** (2002), 511–548.

[25] J. Xin, *Existence of planar flame fronts in convective-diffusive periodic media*, Arch. Ration. Mech. Anal. **121** (1992), 205–233.

[26] J. Xin, *Front propagation in heterogeneous media*, SIAM Rev. **42** (2000), no. 2, 161–230.

[27] A. Zlatoš, *Generalized traveling waves in disordered media: Existence, uniqueness, and stability*, Arch. Ration. Mech. Anal. 208 (2013), 447-480.