TWO-PLAYER TOWER OF HANOI

JONATHAN CHAPPELON, URBAN LARSSON, AND AKIHIRO MATSUURA

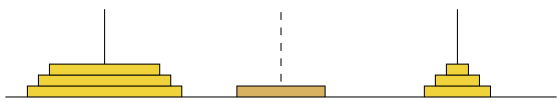
Abstract. The Tower of Hanoi game is a classical puzzle in recreational mathematics, which also has a strong record in pure mathematics. In a borderland between these two areas we find the characterization of the minimal number of moves, which is $2^n - 1$, to transfer a tower of n disks. But there are also other variations to the game, involving for example move edges weighted by real numbers. This gives rise to a similar type of problem, but where the final score seeks to be optimized. We study extensions of the one-player setting to two players, invoking classical winning conditions in combinatorial game theory such as the player who moves last wins, or the highest score wins. Here we solve both these winning conditions on three heaps.

1. Introduction/overview

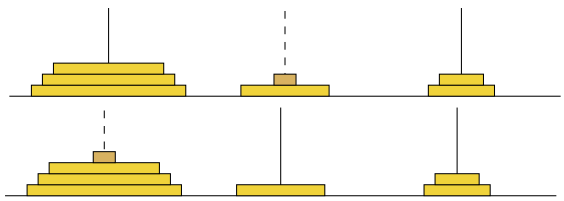
Tower of Hanoi is a classical puzzle, a one-player game [3, 12]. Here we let two players, Anh (first player) and Bao (second player), alternate turns and play a game on three or more pegs with various numbers of disks. We will begin by analyzing games under the following *impartial* rules [1, 2, 5].

Let $n \geqslant 1$ and $l \geqslant 3$ be positive integers. Two players alternate in transferring precisely one out of n disks (of different sizes) on l pegs. The starting position is as usual for the Tower of Hanoi; *the Tower* (i.e., all the disks) are placed on the starting peg in decreasing size, and at each stage of the game, a larger disk cannot be placed on top of a smaller. The current player cannot move the disk that the previous player just moved. The game ends when the tower has been transferred to some predetermined final peg. It is not allowed to transfer the tower to a non-final peg. We detail five ending conditions in Section 2.

Let us exemplify our game with a position on seven disks and three pegs, and where the disk at the dashed peg has just been moved by the previous player and hence cannot be moved by the current player; below it we find its two legal options.



Move options:



In a two-player game, a given winning condition usually provides the incitement to play. We can adapt the winning condition from the one-player game: the player who plays the last disk (on top o the rest of the tower) wins. And indeed, this corresponds to a classical convention for two-player games, that a player who cannot move loses; this is called *normal play*. If no player can force a win in this setting, then the game is declared drawn[1].

Date: March 7, 2015.

[1]Or for that matter, a game is declared drawn, in general, if both players chose non-victorious paths independent of optimality, but such plays will not be considered here, and is usually not considered in mathematical texts about combinatorial

One of our first observations, in Section 2, is that in spite of the loopy nature of our game(s), if we play on just three pegs, Anh will win the normal play (given perfect play). Bao's moves will be forced throughout the game and the proof is an adaptation of the well known one-player result. In fact, Anh's moves also have a restriction; she will always have to move the smallest disk. But, as we will see, it will not limit her capacities in the slightest. In that section, we also note that the game is drawn on four or more pegs.

As a side note, a complete theory of the disjunctive sum of loopy impartial games (on finite numbers of positions) has recently been developed [9, 18]. In a disjunctive sum of Tower of Hanoi games, Anh would not be able to control all moves, because the move sequence is not necessarily alternating in each game component. Therefore, in spite of the simple solutions of these normal play games, the disjunctive sum of games should offer new insights. We also note that it is non-trivial to count the minimal number of moves between two arbitrary Tower of Hanoi positions on three pegs [11], so we guess that a computation of generalized Sprague-Grundy values of the game will be difficult. In this paper, however, we build on another recent development of the original Tower of Hanoi.

In [15, 4, 16], some variations of Tower of Hanoi with some weighted cost are studied: In [15, 4], the authors consider the recurrence relations generalized from the one by the Frame-Stewart algorithm for the k-peg Tower of Hanoi problem, by giving arbitrary positive integers as coefficients of the recurrences and obtained the exact formula for them. In [16], the authors consider another generalization for the three-peg Tower of Hanoi problem, where each undirected edge between pegs has a positive weight and the problem is to transfer all the disks from one peg to another with the minimum sum of weights, instead of the minimum number of moves, and obtain an optimal algorithm for that problem.

A two-player interpretation of the weighted setting becomes the following *scoring game*, where the gain for a move from Peg i to Peg j (or reverse), equals a given real weight $w_{ij} = w_{ji}$. Play the above impartial game, but the player who obtains the largest score when the game ends wins; if the game terminates and none of the players can claim a victory then the game is a tie. We also adapt the convention of drawn games from normal play, so a game is declared a draw if no player can force a win, by terminating the game[2]. Disjunctive sum theory for scoring games has been studied e.g. [7, 17, 19, 14], and partial theories have been developed in different settings, but none yet for loopy scoring games. We will only consider the solution of a single game in this paper. Our main result is that Anh wins nearly all scoring two-player Tower of Hanoi games on three pegs. The reason is partly the same as for the normal play setting, but in this setting the optimal move sequence varies depending on the given weights, and the proof is non-trivial. The only case when Anh cannot force a win is when all weights are equal and non-positive; otherwise she can attain an arbitrary high score by adhering to certain intermediate repetitive patterns. Questions of termination are often very hard (e.g. Turing machines), but in our three-peg setting it will be easy to distinguish drawn from winning. The first player, namely Anh, controls all the moves under optimal play; if she cannot win, it will be easy to play drawn (see also Section 7.1). The question of minimizing the number of moves in the two-player setting is studied in Section 6.

2. Ending conditions for the two-player Tower of Hanoi

Consider the following variations of the two-player Tower of Hanoi; a winning condition is invoked when the tower has been transferred to

(EC1) a given peg, distinct from the starting peg;
(EC2) the starting peg, but the largest disk has to be moved at least once;
(EC3) the starting peg, but the smallest disk has to be moved at least once;
(EC4) any peg, but the largest disk has to be moved at least once; and
(EC5) any peg, but the smallest disk has to be moved at least once.

By convention, it is not meaningful to pile up all disks on a final peg. We disallow such moves in the two-player setting. Therefore (EC2) and (EC3) are not applicable when $n = 1$. This rule is a natural extension

games. In a disjunctive sum of games, however, a 'non-optimal' path in a distinct component could lead to a victory in the full game.

[2]But, for future reference, we note that there is also another choice for loopy scoring games, if one of the players repeatedly moves to more negative scores than the other player, then, even though the game might not terminate, one could define it as a loss for the 'more negative' player.

Figure 1. A graph representation of one-player TH for $n = 1$.

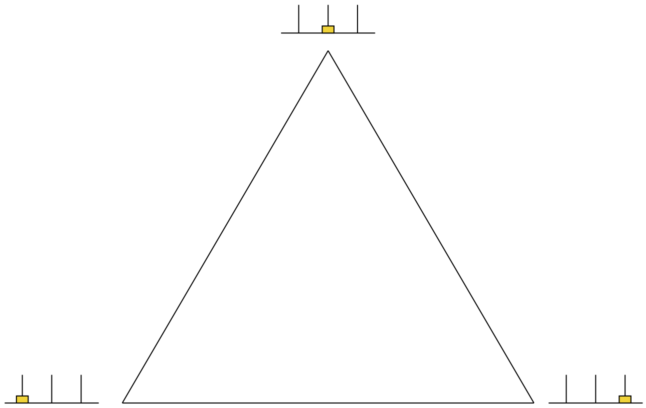
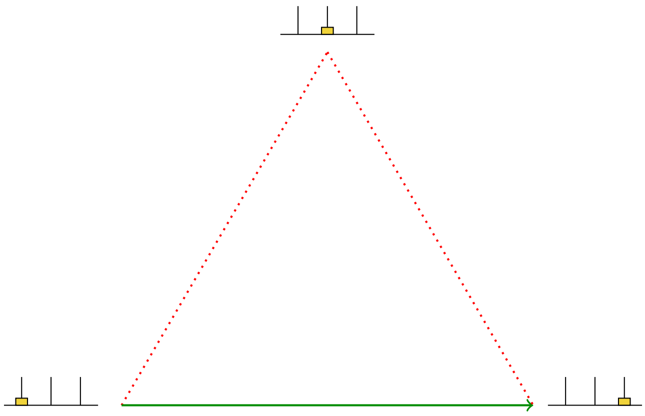


Figure 2. A graph representation of two-player TH for $n = 1$, (EC1).



of the one-player setting.[3] In Figure 1 to Figure 4, we illustrate the idea of going from the one-player setting to the two-player setting using the standard graph representation. Each edge will now be directed, and the direction depends on the previous move. We did not indicate the directed edges in Figure 3, because it depends on the initial choice, once the first move is made the minimal path is given. In Figure 5 we sketch how the Tower of Hanoi game converges to the Sierpinski gasket.

3. General play

In this section we regard 'an odd or even number of moves' in the one-player game in the same sense as for the two-player games; no disk will be moved twice unless some other disk has been moved between. Each odd numbered move thus moves the smallest disk, and in fact we could equivalently have chosen to let Anh lead the game. The initial position throughout the paper is all disks on Peg 1.

[3]Note that if we would have allowed the tower to be transferred to a non-final peg, then the game would terminate on that peg because the current player cannot move the disk that the previous player just moved, which is a contradiction in terms; so some special rule would have been required for that situation. We argue that our choice is a natural way forward, but perhaps there are other interesting ways as well.

Figure 3. A graph representation of one-player TH for $n = 2$.

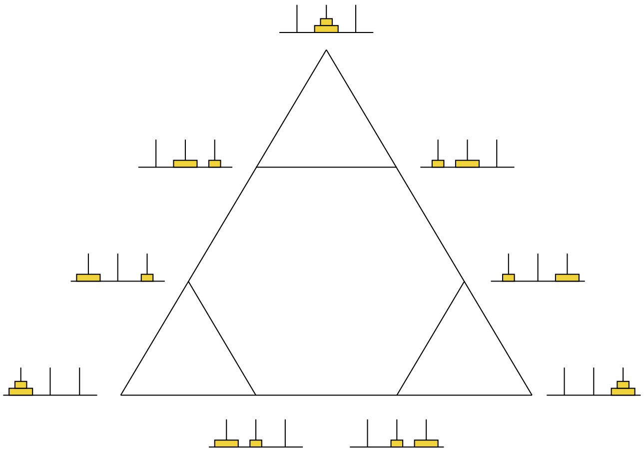
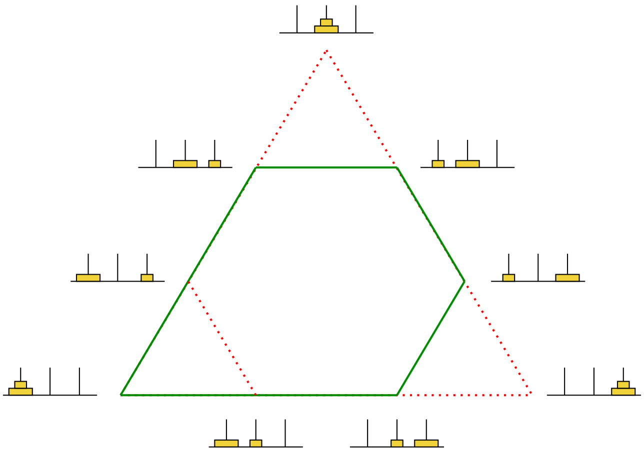


Figure 4. A graph of two-player TH for $n = 2$, (EC2); see also first row in Table 1.



Theorem 1. *For three pegs, $n \geqslant 2$ disks can be transferred from an initial position to any position using an odd number of moves. That is, in the two-player version, the first player can force play to any position in an odd number of moves.*

Proof. We use induction on n and the fact that the smallest disk is always moved on the odd numbered moves. For $n = 2$, Table 1 gives examples of sequences for transferring the two disks from Peg 1 to each of the nine possible positions, using in each case an odd number of moves. Suppose now that the result is true for $n \geqslant 2$ disks. We want to transfer $n + 1$ disks from an initial position, say from Peg 1, to a position P. We distinguish different cases depending on the position of the largest disk LD in P. If LD is on Peg 1 in P, we have just to transfer the n other disks from Peg 1 to the position P and we do it by using an odd number of moves by induction hypothesis. Otherwise, if LD is on, say Peg 2 in P, we transfer the n smallest disks from Peg 1 to Peg 3 using n_1 moves, the LD from Peg 1 to Peg 2, and finally the n smallest disks from Peg 3 to the position P using n_2 moves. Thus, we have transferred the $n + 1$ disks from Peg 1 to the position P using $n_1 + n_2 + 1$ moves and, since n_1 and n_2 are odd by induction hypothesis, this number of moves is odd. If LD is on peg 3, the argument is the same. This completes the proof. □

Figure 5. The case $n = 3$. The largest disk is moved along the middle edge as in the case for $n = 2$, but as n grows, the middle edge will take on a smaller and smaller proportion of the side of the triangle. Hence the graph representation of Tower of Hanoi will converge to the Sierpinski gasket, the minimal normal-play (EC1) path being along the base of the triangle.

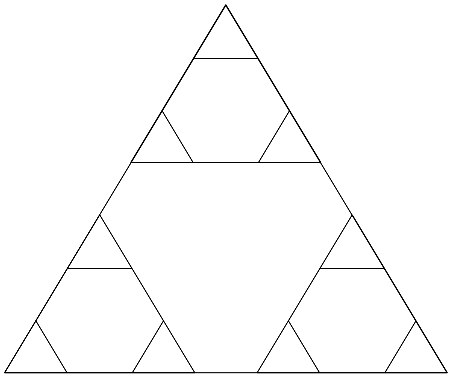
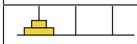
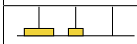
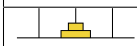
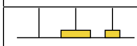
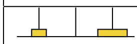
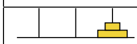


Table 1. Any Tower of Hanoi position on two disks and three pegs can be reached in an odd number of moves.

Position	Sequence of moves	Number of moves
	13 − 12 − 13 − 23 − 12 − 13 − 12	7
	12	1
	13	1
	13 − 12 − 13	3
	13 − 12 − 23	3
	12 − 13 − 12 − 23 − 13	5
	12 − 13 − 12	3
	13 − 12 − 13 − 23 − 12	5
	12 − 13 − 23	3

Corollary 2. *For three pegs, $n \geqslant 2$ disks can be transferred from an initial position to any intermediate position using an even number of moves.*

Proof. Let P be an intermediate position and suppose that the smallest disk is on Peg p. Since P is an intermediate position, we know that one of the two other pegs, Peg q or Peg r, contains at least one disk. Let QD be the smallest disk of the disks on Peg q and Peg r and suppose WLOG that QD is on Peg q. Let P' be the position where all the disks are positioned like in P, except QD which is on Peg r, instead of Peg q in P. Since all the disks can be transferred from Peg 1 to the position P' using an odd number moves by Theorem 1, we reach the position P by only adding one move of transferring QD from Peg r to Peg q. Indeed, the smallest disk was moved in the previous move to achieve the position P'. Thus, we have transferred all the disks from Peg p to the position P using an even number of moves. □

4. Normal play two-player games

In the normal play variation of the two-player Tower of Hanoi, Anh can avoid drawn simply by adhering to the well known *minimal algorithm* for the one-player Tower of Hanoi (Bao's moves will be forced all through the game), using precisely $2^n - 1$ moves. However, she can also choose freely among all odd-length move paths.

Corollary 3. *For three pegs and $n \geqslant 1$ disks, the two-player Tower of Hanoi game terminates and the first player wins. This is true for any ending condition and also from any non-final position, provided that the previous player did not move the smallest disk.*

Proof. From Theorem 1, the number of moves for the one-player Tower of Hanoi can be odd. We use any such sequence in the two-player game. In the one-player game, every odd move transfers the smallest of the n disks, which here means that, in each move, Anh will move the smallest of the n disks on top of an empty peg or a larger disk. This forces Bao to move a larger disk at each stage of the game. Precisely one such move is possible since Anh's move of the smallest disk occupies one out of precisely three pegs. Since the number of moves used here is odd, Anh wins. □

Games on four pegs are mostly loopy.

Theorem 4. *The two-player Tower of Hanoi on four or more pegs is a draw game if the number of disks is three or more, given any ending condition.*

Proof. Suppose that there are three or more disks. Then Bao's moves are never forced; he never has to place the second smallest disk on top of the third smallest (analogously for Anh). □

For completeness, let us also give the rest of the four-peg observations. For (EC1-3), if there are two disks, Bao never has to move the largest disk to a final peg and hence the game is drawn. For (EC1), if there is only one disk, then Anh wins in the first move. For (EC2,3), if there is only one disk, the special rule is invoked and Bao wins in his first move. For (EC4,5), if there are two disks, Bao has to move the largest disk to a final peg and hence loses. If there is only one disk, then Anh wins in the first move.

5. Scoring play: two-player games with weights

As stated in the introduction, for the scoring variation of the normal play setting, we provide real weights to the *move edges*, in the three-peg case, w_{12}, w_{13} and w_{23} respectively. As usual, the two players alternate in moving, and a player gets $w_{ij} = w_{ji}$ points for a move along edge $\{i, j\}$. The player who has most points when the game ends wins. We begin by giving the solution of the game with less than three disks.

We will use $A_{ij}(n)$ and $B_{ij}(n)$, for the total points for Anh and Bao respectively, of the two-player Tower of Hanoi game, for transferring n disks from Peg i to Peg j by a given algorithm, for example the minimal algorithm, and we let the *total score* be $\Delta_{ij}(n) = A_{ij}(n) - B_{ij}(n)$, or just $\Delta(n)$. Hence, if $\Delta(n) > 0$ then Anh wins, if $\Delta(n) = 0$, then the game is tie, and otherwise Bao wins.

Theorem 5. *Consider the two-player Tower of Hanoi game on two disks, three pegs and three weights of real numbers w_{12}, w_{13}, and w_{23}. In case of (EC4) or (EC5), the first player wins if either of the following inequalities holds:*

$$w_{12} + w_{23} - w_{13} > 0 \tag{1}$$

$$3w_{13} - w_{12} - w_{23} > 0 \tag{2}$$

$$w_{13} + w_{23} - w_{12} > 0 \tag{3}$$

$$3w_{12} - w_{13} - w_{23} > 0 \tag{4}$$

$$w_{12} + w_{13} - w_{23} > 0 \tag{5}$$

In case of (EC2) or (EC3), she wins if (5) holds. In case of (EC1), she wins if (1) or (2) holds. Otherwise the game is a draw.

If the game is played on only one disk, then (EC2,3) are not applicable. For (EC1) the first player wins if $w_{13} > 0$; loses if $w_{13} < 0$; and the game is a tie otherwise. The second player wins (EC4,5) if $w_{12} < 0$ and $w_{13} < 0$. The game is a tie if at least one of these weights is 0 and the others are non-positive. Otherwise the first player wins.

Proof. Notice that in this game, Anh will only move the small disk and Bao will only move the large one. There are only six possibilities. If Anh is not able to force a win, then she will resort to drawn, using either of the two possibilities:

$$12 - (13 - 12 - 23)^{\infty}$$

or

$$13 - (12 - 13 - 23)^{\infty},$$

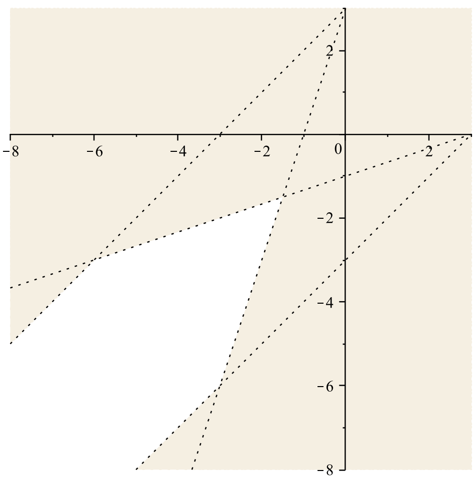
where ij denotes the current player's move between Peg i to Peg j, and where $(\cdot)^{\infty}$ denotes an infinite repetition of a given move sequence. But, in case it is to her advantage, she can interrupt either of these two sequences of moves, and use either of the following six sequences:

$$12 - (13 - 12 - 23)^{2k} - 13 - 23$$
$$13 - (12 - 13 - 23)^{2k+1} - 13$$
$$12 - (13 - 12 - 23)^{2k+1} - 12$$
$$13 - (12 - 13 - 23)^{2k} - 12 - 23$$
$$12 - (13 - 12 - 23)^{2k+1} - 13 - 12 - 13$$
$$13 - (12 - 13 - 23)^{2k+1} - 12 - 13 - 12$$

for k a non-negative integer. In the first two cases she will end the game on Peg 3 for (EC1,4,5), corresponding to the cases (1) and (2) respectively. For the two middle move sequences, she will end on Peg 2 for (EC4,5), corresponding to the cases (3) and (4) respectively. To terminate the game on Peg 1, only valid for (EC2-5), she uses one of the two last move sequences. They are symmetric and both result in the inequality (5). In either case, the largest disk has been moved.

In each of the above cases we evaluate the value of $A(2)-B(2)$ and then the triangular inequalities appear, and it is clear that the difference of total points is independent of the choice of k in each case. □

FIGURE 6. Consider (EC1) for $n = 2$. Here $w_{23} = -3$ and the other weights represent the x- and y-axes. The game is drawn in the white area. Compare this picture with the result for $n \geqslant 3$ in Theorem 6, where the class of drawn games would have been represented by a single white dot at $(-3, -3)$.



It turns out that the case $n \geqslant 3$ has fewer drawn games, allowing a simpler description; it relies on the general ideas in Section 2. By relabeling the pegs, it suffices to analyze the case of transferring the disks from Peg 1 to Peg 3 and the case from Peg 1 to Peg 1 .

Theorem 6. *Given $n \geqslant 3$ disks, three pegs and three weights of real numbers w_{13}, w_{12}, and w_{23}. Then, for the two-player Tower of Hanoi game, the first player wins any game, except in the case $w_{12} = w_{13} = w_{23} \leqslant 0$ for which the game is a draw.*

Proof. If $w_{12} = w_{13} = w_{23} = \alpha$, it is easy to see that, with any strategy, if the game is not drawn (because of infinite play), then the total score is $\Delta(n) = \alpha$ for any $n \geqslant 1$ because the total number of moves is always odd if the game finish. The result follows in this case.

Let $\{i, j, k\} = \{1, 2, 3\}$ and suppose that w_{ij} is the smallest integer among w_{ij}, w_{ik} and w_{jk}. Then, we have $w_{ik} + w_{jk} > 2w_{ij}$. Let P be an intermediate position where the two smallest disks are on Peg k and let the smallest disk among the disks on Peg i and Peg j be on Peg i. We know, from Corollary 2, that the $n \geqslant 3$ disks can be transferred from the initial position on Peg 1 to the intermediate position P using an even sequence of moves s_1 and, from Theorem 1, that all the disks can be transferred from the final position on Peg f to the intermediate position P using an odd sequence of moves s_2. Then, it is clear that the reverse sequence of s_2 is an odd sequence ${s_2}^{-1}$ of moves transferring all the disks from the intermediate position P to the final position. Thus, the sequence of moves $s_1{s_2}^{-1}$ is a finite strategy for transferring $n \geqslant 3$ disks from Peg 1 to Peg f. Let p be the total score $\Delta(n)$ associated with this strategy. If $p > 0$, Anh wins following this strategy and the result is then proved. Suppose now that $p \leqslant 0$. We consider a new strategy where Anh forces Bao to play as follows:

1) Anh starts the game and forces Bao to play the even number of moves of s_1, where the n disks are transferred from the initial position on Peg 1 to the intermediate position P. After that, since s_1 is even, it is Anh's turn.
2) Anh continues by forcing Bao to play an even number of moves, whose sequence is denoted as s_3 and for which the total score $\Delta(n)$ is incremented by a positive value and where the n disks are always at the position P after s_3. Note that after this, it is always Anh's turn since s_3 is even. This step is repeated until the total score $\Delta(n)$ is sufficiently large.
3) Anh finishes the game by forcing Bao to play the odd number of moves of ${s_2}^{-1}$, where the n disks are transferred from the intermediate position P to the final position on Peg f.

It remains to detail the sequence s_3 and to verify that it effectively increments the total score $\Delta(n)$ by a positive value. In fact, the sequence s_3 consists of the 7 moves that transfer the two smallest disks from Peg k to Peg k, as already seen in the proof of Theorem 1, and the move of the smallest disk of Peg i and Peg j from Peg i to Peg j, and we repeat these 8 moves twice so that we return to the intermediate position P. There are two possible sequences for s_3, that are

$$(ik - jk - ik - ij - jk - ik - jk - ij)^2 \text{ or } (jk - ik - jk - ij - ik - jk - ik - ij)^2 .$$

In both cases, the total score $\Delta(n)$ is incremented by $2\,(w_{ik} + w_{jk} - 2w_{ij})$, which is strictly positive by hypothesis. So we have proved that Anh wins the game by using the following strategy $s_1{s_3}^{\lambda}{s_2}^{-1}$, where

$$\lambda = \left\lfloor \frac{-p}{2\,(w_{ik} + w_{jk} - 2w_{ij})} \right\rfloor + 1.$$

This completes the proof. □

We can adapt this proof to scoring play from an arbitrary position of disks for the three-peg case. But we have to remember that in the two-player setting a position carries also a memory of the last move.

Corollary 7. *Consider an arbitrary Tower of Hanoi position on $n \geqslant 3$ disks, three pegs and three weights of real numbers w_{13}, w_{12}, and w_{23}. Then the first player wins, unless $w_{12} = w_{13} = w_{23} \leqslant 0$, and provided the previous player did not move the smallest disk.*

Proof. Apply Theorem 1 and the proof of Theorem 6 (omitting s_1). □

6. The minimal number of moves for winning

6.1. **The minimal number of moves for winning normal play.** Traditionally, in the one-player setting, the interest has often been focused on the minimal number of moves for transferring the tower. In this section we analyze our variations of the two-player game in this sense. It is not a big surprise that the minimal number of moves required for Anh to win normal play is the same as the number of moves in the one-player minimal algorithm, but let us sum up the state of the art before we move on to the more challenging analysis of minimum number of moves for winning scoring play.

Theorem 8. *The minimum number of moves for transferring $n \geqslant 1$ disks from one peg to another peg is $2^n - 1$. The minimum number of moves for transferring $n \geqslant 2$ disks from one peg to the same peg is $2^{n+1} - 1$, if the largest disk has to be moved; and it is seven if only the smallest disk has to be moved.*

Proof. If we want to transfer $n \geqslant 1$ disks from Peg 1 to Peg 3, it is well known that the minimum number of moves needed is exactly $2^n - 1$. Recall here in few words how to obtain this result. We prove it by induction on n. For $n = 1$, the result is clear. Now, suppose that the result is true for transferring $n - 1$ disks from one peg to another peg. If we want to transfer the nth disk from Peg 1 to Peg 3, the $n - 1$ smallest disks must be on Peg 2. So, we transfer the $n - 1$ smallest disks from Peg 1 to Peg 2 using $2^{n-1} - 1$ moves by induction hypothesis, the largest disk from Peg 1 to Peg 3 and finally the $n - 1$ smallest disks from Peg 2 to Peg 3 using also $2^{n-1} - 1$ moves by induction hypothesis. This is the reason why the minimal number of moves needed for transferring n disks from Peg 1 to Peg 3 is $2^n - 1$.

Now, we want to transfer $n \geqslant 2$ disks from Peg 1 to Peg 1 with the condition that all disks have been moved at least twice (the largest disk has to be moved). We proceed by induction on $n \geqslant 2$. For $n = 2$, it is easy to check that the two minimal sequences of moves are of length 7, that are

$$12 - 13 - 12 - 23 - 13 - 12 - 13 \quad \text{and} \quad 13 - 12 - 13 - 23 - 12 - 13 - 12.$$

Suppose that the result is true for transferring $n - 1$ disks from one peg to the same peg. First, if we want to transfer the nth disk from Peg 1 to another peg, Peg i with $\{i, j\} = \{2, 3\}$, the $(n - 1)$th smallest disks have to be transferred from Peg 1 to Peg j using at least $2^{n-1} - 1$ moves. Then, we distinguish two cases.

Case 1. If we want to transfer the largest disk from Peg i to Peg 1, the $n - 1$ smallest disks have to be transferred from Peg j to Peg j using at least $2^n - 1$ moves by induction hypothesis. Finally, we transfer the $n - 1$ smallest disks from Peg j to Peg 1 with at least $2^{n-1} - 1$ moves. So, the number of moves is at least $2^{n+1} - 1$ if we follow this strategy.

Case 2. If we want to transfer the largest disk from Peg i to Peg j, the $n - 1$ smallest disks have to be transferred from Peg j to Peg 1 using at least $2^{n-1} - 1$ moves. After that, for transferring the largest disk from Peg j to Peg 1, the $n - 1$ smallest disks have to be transferred from Peg 1 to Peg i using at least $2^{n-1} - 1$ moves. Finally, we transfer the $n - 1$ smallest disks from Peg i to Peg 1 using at least $2^{n-1} - 1$ moves. Thus, the number of moves is at least $2^{n+1} - 1$ if we follow this strategy.

In all cases, we have proved that the minimal number of moves for transferring n disks from Peg 1 to Peg 1 is exactly $2^{n+1} - 1$, following the two possible strategies that have been represented in Figure 7.

If we only require that the smallest disk, instead of the largest disk, has to be moved, the minimal number of moves for transferring $n \geqslant 2$ disks from one peg to the same peg is seven. The result is obtained when we only move the two smallest disks and let the $n - 2$ largest disks unmoved on the starting peg. □

Now, we estimate the minimal number of moves for winning the normal play two-player Tower of Hanoi.

Theorem 9. *Let $\mathrm{M}_l(n)$ denote the minimal number of moves needed for winning a normal play game on $l \geqslant 3$ pegs and $n \geqslant 1$ disks. Then,*

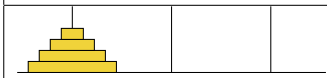
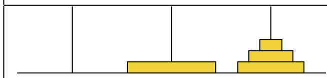
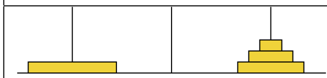
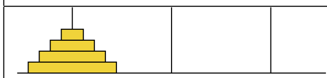
$$\mathrm{M}_3(n) = \begin{cases} 2^n - 1, & \text{for } n \geqslant 1, \text{ for (EC1,4)}, \\ \begin{cases} 2^{n+1} - 1, & \text{for } n \geqslant 2, \text{ for (EC2)}, \\ 7, & \text{for } n \geqslant 2, \text{ for (EC3)}, \end{cases} \\ \begin{cases} 2^n - 1, & \text{for } n \leqslant 2, \text{ for (EC5)}, \\ 7, & \text{for } n \geqslant 3, \text{ for (EC5)}. \end{cases} \end{cases}$$

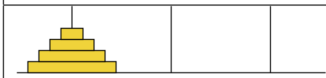
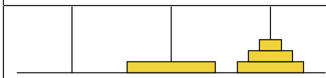
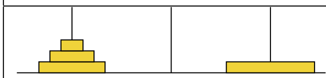
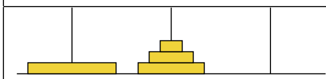
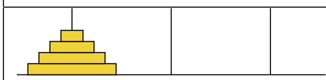
$$\mathrm{M}_l(n) = \begin{cases} 1, & \text{for } n = 1,\ l \geqslant 4 \text{ and for (EC1,4,5)}, \\ 3, & \text{for } n = 2,\ l \geqslant 4 \text{ and for (EC4,5)}, \\ \infty, & \text{otherwise for } l \geqslant 4. \end{cases}$$

Proof. Apply the results in this section to the two-player setting.

□

Figure 7. Two possible strategies for transferring n disks from one peg to the same peg.

Position	Number of Moves
	0
	2^{n-1}
	2^n
	$2^{n-1}-1$
Total	$2^{n+1}-1$

Position	Number of Moves
	0
	2^{n-1}
	2^{n-1}
	2^{n-1}
	$2^{n-1}-1$
Total	$2^{n+1}-1$

6.2. **The minimal number of moves for winning scoring play.** When the players move blindly (ignoring winning) and just follows the classical minimal algorithm, we obtain the total scores according to the following two lemmas.

Lemma 10. *Given $n \geqslant 1$ disks, three pegs and three weights of real numbers w_{12}, w_{13}, and w_{23}. Then, for the two-players' weighted Tower of Hanoi game of transferring n disks from Peg 1 to Peg 3 by the minimal algorithm, the total score is*

- $\Delta_{13}(n) = w_{13}$ *if n is odd;*
- $\Delta_{13}(n) = w_{12} + w_{23} - w_{13}$ *if n is even.*

Proof. The statement is that Anh gets w_{13} (or $w_{12} + w_{23} - w_{13}$) more points than Bao when n is odd (or n is even, resp.). We prove it by induction on n.

When $n = 1$, Anh gets $\Delta_{13}(n) = w_{13}$ points by moving one disk from Peg 1 to Peg 3. When $n = 2$, by using the usual minimal algorithm, the difference of points is $\Delta_{13}(n) = w_{12} + w_{23} - w_{13}$.

Now suppose that the statement is true for $n-1$. Anh takes the strategy based on the minimal algorithm.

When n is odd, then the movement of the smaller $n-1$ disks from Peg 1 to Peg 2 gives Anh $\Delta_{12}(n-1) = w_{13} + w_{23} - w_{12}$ more points since $n-1$ is even and by the assumption of induction. Then, Bao gets w_{13} points by moving the largest disk from Peg 1 to Peg 3. Finally, the movement of the smaller $n-1$ disks from Peg 2 to Peg 3 gives Anh $\Delta_{23}(n-1) = w_{12} + w_{13} - w_{23}$ more points. So, the difference of the total points is $\Delta_{13}(n) = (w_{13} + w_{23} - w_{12}) - w_{13} + (w_{12} + w_{13} - w_{23}) = w_{13}$, when n is odd.

When n is even, using the same argument, Anh gets w_{12} and w_{23} more points in the phases of moving the smaller $n-1$ disks from Peg 1 to Peg 2 and Peg 2 to Peg 3, respectively by the assumption of induction. So, the difference of the total points is $\Delta_{13}(n) = w_{12} + w_{23} - w_{13}$.

□

Lemma 11. *Given $n \geqslant 2$ disks, three pegs and three weights of real numbers w_{12}, w_{13}, and w_{23}. Then, for the two-players' weighted Tower of Hanoi game of transferring n disks from Peg 1 to Peg 1 by the minimal algorithm, if we suppose that the largest disk be moved, the total score is*

- $\Delta_{11}(n) = 3w_{23} - w_{12} - w_{13}$ *if n is odd;*
- $\Delta_{11}(n) = w_{12} + w_{13} - w_{23}$ *if n is even.*

Proof. By induction on $n \geqslant 2$. For $n = 2$, we have already seen in the proof of Theorem 1 that the two minimal sequences of moves for transferring two disks from Peg 1 to Peg 1 are

$$12 - 13 - 12 - 23 - 13 - 12 - 13 \quad \text{and} \quad 13 - 12 - 13 - 23 - 12 - 13 - 12.$$

In all cases, we obtain that $\Delta_{11}(2) = w_{12} + w_{13} - w_{23}$. Suppose now that the result is true for $n-1 \geqslant 2$. There are two minimal algorithms for transferring n disks from Peg 1 to Peg 1. We recall them below. The first minimal algorithm is:

i) the $n-1$ smallest disks are transferred from Peg 1 to Peg 3 using the $2^{n-1}-1$ moves of the minimal algorithm,
ii) the largest disk is transferred from Peg 1 to Peg 2,
iii) the $n-1$ smallest disks are transferred from Peg 3 to Peg 1 using the $2^{n-1}-1$ moves of the minimal algorithm,
iv) the largest disk is transferred from Peg 2 to Peg 3,
v) the $n-1$ smallest disks are transferred from Peg 1 to Peg 2 using the $2^{n-1}-1$ moves of the minimal algorithm,
vi) the largest disk is transferred from Peg 3 to Peg 1,
vii) the $n-1$ smallest disks are transferred from Peg 2 to Peg 1 using the $2^{n-1}-1$ moves of the minimal algorithm.

Thus, we have

$$\begin{aligned}\Delta_{11}(n) &= \Delta_{13}(n-1) - w_{12} + \Delta_{31}(n-1) - w_{23} + \Delta_{12}(n-1) - w_{13} + \Delta_{21}(n-1) \\ &= 2\left(\Delta_{13}(n-1) + \Delta_{12}(n-1)\right) - \left(w_{12} + w_{13} + w_{23}\right).\end{aligned}$$

Then, from Lemma 10, if n is even, we obtain that

$$\Delta_{11}(n) = 2\left(w_{13} + w_{12}\right) - \left(w_{12} + w_{13} + w_{23}\right) = w_{12} + w_{13} - w_{23}$$

and, if n is odd, we have

$$\Delta_{11}(n) = 2\left(\left(w_{12} + w_{23} - w_{13}\right) + \left(w_{13} + w_{23} - w_{12}\right)\right) - \left(w_{12} + w_{13} + w_{23}\right) = 3w_{23} - w_{12} - w_{13}.$$

Finally, we consider the second minimal algorithm, that is:

i) the $n-1$ smallest disks are transferred from Peg 1 to Peg 3 using the $2^{n-1}-1$ moves of the minimal algorithm,
ii) the largest disk is transferred from Peg 1 to Peg 2,
iii) the $n-1$ smallest disks are transferred from Peg 3 to Peg 3 using the $2^{n}-1$ moves of the minimal algorithm,
iv) the largest disk is transferred from Peg 2 to Peg 1,
v) the $n-1$ smallest disks are transferred from Peg 3 to Peg 1 using the $2^{n-1}-1$ moves of the minimal algorithm.

Thus, we have

$$\Delta_{11}(n) = \Delta_{13}(n-1) - w_{12} + \Delta_{33}(n-1) - w_{12} + \Delta_{31}(n-1) = 2\Delta_{13}(n-1) + \Delta_{33}(n-1) - 2w_{12}.$$

Then, from Lemma 10 for $\Delta_{13}(n-1)$ and by induction hypothesis for $\Delta_{33}(n-1)$, if n is even, we have

$$\Delta_{11}(n) = 2w_{13} + \left(3w_{12} - w_{13} - w_{23}\right) - 2w_{12} = w_{12} + w_{13} - w_{23}$$

and, if n is odd, we have

$$\Delta_{11}(n) = 2\left(w_{12} + w_{23} - w_{13}\right) + \left(w_{13} + w_{23} - w_{12}\right) - 2w_{12} = 3w_{23} - w_{12} - w_{13}.$$

This completes the proof. □

We estimate the minimal number of moves for winning scoring Tower of Hanoi under the five ending conditions (EC1-5). We recall that the starting peg is Peg 1 and for (EC1) we suppose the tower is transferred to Peg 3.

Theorem 12. *Let* $\mathrm{M}_3(n)$ *denote the minimal number of moves needed for a winning game on three pegs and* $n \geqslant 1$ *disks. Then, for* $n \leqslant 2$*,*

- *for (EC1):*

$$\mathrm{M}_3(1) = \begin{cases} 1 & \text{if } w_{13} \neq 0, \\ \infty & \text{otherwise,} \end{cases}$$

$$3 \leqslant \mathrm{M}_3(2) \leqslant \begin{cases} 3 & \textit{if } w_{12}+w_{23} > w_{13}, \\ 5 & \textit{if } 3w_{13} > w_{12}+w_{23}, \\ \infty & \textit{otherwise}, \end{cases}$$

- *for (EC2,3):*

$$\mathrm{M}_3(1) = \infty,$$

$$7 \leqslant \mathrm{M}_3(2) \leqslant \begin{cases} 7 & \textit{if } w_{12}+w_{13} > w_{23}, \\ \infty & \textit{otherwise}, \end{cases}$$

- *for (EC4,5):*

$$\mathrm{M}_3(1) = \begin{cases} 1 & \textit{if } \max\{w_{12}, w_{13}\} \neq 0, \\ \infty & \textit{otherwise}, \end{cases}$$

$$3 \leqslant \mathrm{M}_3(2) \leqslant \begin{cases} 3 & \textit{if } w_{12}+w_{23} > w_{13} \textit{ or } w_{13}+w_{23} > w_{12}, \\ 5 & \textit{if } 3w_{13} > w_{12}+w_{23} \textit{ or } 3w_{12} > w_{13}+w_{23}, \\ 7 & \textit{if } w_{12}+w_{13} > w_{23}, \\ \infty & \textit{otherwise}. \end{cases}$$

For $n \geqslant 3$, if $w_{12} = w_{13} = w_{23} = \alpha$, we have

$$\mathrm{M}_3(n) = \begin{cases} 2^n - 1 & \textit{if } \alpha > 0, \\ \infty & \textit{if } \alpha \leqslant 0. \end{cases}$$

Otherwise, suppose that

$$\beta_1 = \begin{cases} w_{13} & \textit{if } n \textit{ odd}, \\ w_{12}+w_{23}-w_{13} & \textit{if } n \textit{ even}, \end{cases}$$

$$\beta_2 = \begin{cases} 3w_{23}-w_{12}-w_{13} & \textit{if } n \textit{ odd}, \\ w_{12}+w_{13}-w_{23} & \textit{if } n \textit{ even}, \end{cases}$$

$$\beta_3 = \begin{cases} \max\{w_{13}, w_{12}\} & \textit{if } n \textit{ odd}, \\ \max\{w_{12}+w_{23}-w_{13}, w_{13}+w_{23}-w_{12}\} & \textit{if } n \textit{ even}, \end{cases}$$

$$\textit{and } \gamma = \max\left\{w_{ij}+w_{ik}-2w_{jk} \mid \{i,j,k\} = \{1,2,3\}\right\}.$$

Then,

- *for (EC1):*

$$\mathrm{M}_3(n) = 2^n - 1 \textit{ if } \beta_1 > 0,$$

otherwise,

$$2^n \leqslant \mathrm{M}_3(n) \leqslant 2^n + 15 + 16\left\lfloor \frac{-\beta_1}{2\gamma} \right\rfloor,$$

except for $n = 3$ with (i) $\gamma = w_{13}+w_{23}-2w_{12}$ and (ii) $\gamma = w_{12}+w_{13}-2w_{23}$, where

$$8 \leqslant \mathrm{M}_3(n) \leqslant \min\left\{27 + 16\left\lfloor \frac{2(w_{12}+w_{23})-3w_{13}}{2\gamma} \right\rfloor, 29 + 16\left\lfloor \frac{-w_{13}}{2\gamma} \right\rfloor\right\},$$

- *for (EC2):*

$$\mathrm{M}_3(n) = 2^{n+1} - 1 \textit{ if } \beta_2 > 0,$$

$$2^{n+1} \leqslant \mathrm{M}_3(n) \leqslant 2^{n+1} + 15 + 16\left\lfloor \frac{-\beta_2}{2\gamma} \right\rfloor \textit{ otherwise},$$

- *for (EC3):*

$$\mathrm{M}_3(n) = \begin{cases} 7 & \textit{if } w_{12}+w_{13} > w_{23}, \\ 15 & \textit{if } w_{12}+w_{13} \leqslant w_{23} \textit{ and } \beta_2 > 0, \end{cases}$$

$$16 \leqslant \mathrm{M}_3(n) \leqslant 31 + 16\left\lfloor \frac{-\beta_2}{2\gamma} \right\rfloor \textit{ otherwise},$$

- *for (EC4):*

$$\mathrm{M}_3(n) = 2^n - 1 \text{ if } \beta_3 > 0,$$

otherwise,

$$2^n \leqslant \mathrm{M}_3(n) \leqslant \min\left\{2^n + 15 + 16\left\lfloor\frac{-\beta_3}{2\gamma}\right\rfloor, 2^{n+1} + 15 + 16\left\lfloor\frac{-\beta_2}{2\gamma}\right\rfloor\right\},$$

except for (i) $n = 3$ *and* $\gamma = w_{13} + w_{23} - 2w_{12}$*, where*

$$8 \leqslant \mathrm{M}_3(3) \leqslant \min\left\{23 + 16\left\lfloor\frac{-w_{12}}{2\gamma}\right\rfloor, 27 + 16\left\lfloor\frac{2(w_{12}+w_{23}) - 3w_{13}}{2\gamma}\right\rfloor, 29 + 16\left\lfloor\frac{-w_{13}}{2\gamma}\right\rfloor, 31 + 16\left\lfloor\frac{-\beta_2}{2\gamma}\right\rfloor\right\},$$

for (ii) $n = 3$ *and* $\gamma = w_{12} + w_{23} - 2w_{13}$*, where*

$$8 \leqslant \mathrm{M}_3(3) \leqslant \min\left\{23 + 16\left\lfloor\frac{-w_{13}}{2\gamma}\right\rfloor, 27 + 16\left\lfloor\frac{2(w_{13}+w_{23}) - 3w_{12}}{2\gamma}\right\rfloor, 29 + 16\left\lfloor\frac{-w_{12}}{2\gamma}\right\rfloor, 31 + 16\left\lfloor\frac{-\beta_2}{2\gamma}\right\rfloor\right\},$$

and for (iii) $n = 3$ *and* $\gamma = w_{12} + w_{13} - 2w_{23}$*, where*

$$8 \leqslant \mathrm{M}_3(3) \leqslant \min\left\{27 + 16\left\lfloor\frac{2(w_{12}+w_{23}) - 3w_{13}}{2\gamma}\right\rfloor, 29 + 16\left\lfloor\frac{-w_{13}}{2\gamma}\right\rfloor, 27 + 16\left\lfloor\frac{2(w_{13}+w_{23}) - 3w_{12}}{2\gamma}\right\rfloor, 29 + 16\left\lfloor\frac{-w_{12}}{2\gamma}\right\rfloor, 31 + 16\left\lfloor\frac{-\beta_2}{2\gamma}\right\rfloor\right\},$$

- *for (EC5):*

$$7 \leqslant \mathrm{M}_3(n) \leqslant \begin{cases} 7 & \text{if } w_{12} + w_{13} > w_{23} \text{ or } (n = 3 \text{ and } \beta_3 > 0), \\ 15 & \text{if } (w_{12} + w_{13} \leqslant w_{23} \text{ and } \beta_2 > 0) \text{ or } (n = 4 \text{ and } \beta_3 > 0), \\ 2^n - 1 & \text{if } \beta_3 > 0, \end{cases}$$

otherwise,

$$8 \leqslant \mathrm{M}_3(n) \leqslant \min\left\{31 + 16\left\lfloor\frac{-\beta_2}{2\gamma}\right\rfloor, 2^n + 15 + 16\left\lfloor\frac{-\beta_3}{2\gamma}\right\rfloor\right\},$$

except for the cases (i), (ii), and (iii) that are the same with (EC4).

Proof. We first prove the results for $n = 1$ and 2. For (EC1) and $n = 1$, the number of moves for a winning game is 1 if $w_{13} \neq 0$ because what Anh can do in the first turn is to only move the disk from Peg 1 to Peg 3 (note that if $w_{13} < 0$, Bao wins). For $n = 2$ and if $w_{12} + w_{23} > w_{13}$, the minimal algorithm for transferring the two disks from Peg 1 to Peg 3 is used for Anh to win. For $n = 2$ and if $3w_{13} > w_{12} + w_{23}$, the move sequence $13 - 12 - 13 - 23 - 13$ with $\Delta(2) = 3w_{13} - w_{12} - w_{23}$ is used. For (EC2,3), in the case of $n = 1$, the game can not be started because the disk is not allowed to be moved from Peg 1. For $n = 2$ and if $w_{12} + w_{13} > w_{23}$, one of the sequences with seven moves used in the proof of Theorem 8 is adopted, resulting in $\Delta(2) = w_{12} + w_{13} - w_{23}$. For (EC4,5), the result for $n = 1$ is obtained by combining the conditions for moving the disk to Peg 2 and to Peg 3. For $n = 2$, the result is obtained by considering all the cases of Anh completing the tower on Peg 1, 2, and 3 and by employing the move sequences in (EC1-3).

For $n \geqslant 3$, if $w_{12} = w_{13} = w_{23} = \alpha \leqslant 0$, Anh can not get a positive score so she uses the strategy of escaping the game to end and achives a draw. If $\alpha > 0$, Anh wins using the minimal algorithm with difference of score $\Delta(n) = \alpha > 0$. Otherwise, that is, when $n \geqslant 3$ and not all w_{12}, w_{13}, and w_{23} are equal, Anh uses the common strategy for all ending conditions (EC1-5) of using the minimal algorithm for transferring the tower from Peg 1 to a final peg if that algorithm can be regarded as the concatenation of the move sequences s_1 and s_2^{-1} used in the proof of Theorem 6. For all (EC1-5) and $n \geqslant 4$, and for most of the cases with $n = 3$, this strategy works. Otherwise, that is, in some exceptional cases with $n = 3$, we construct new move sequences. We state the detail for each of the ending conditions below.

For (EC1) and $n \geqslant 3$, except for the two cases ($n = 3$ and $\gamma = w_{13} + w_{23} - 2w_{12}$) and ($n = 3$ and $\gamma = w_{12} + w_{13} - 2w_{23}$), the minimal algorithm for transferring the tower from Peg 1 to Peg 3 can be used as concatenation of s_1 and s_2^{-1} because the minimal algorithm reaches the intermediate position satisfying the condition for P in Theorem 6 with an even number of moves. Then it is further divided into two subcases depending on whether $\Delta(n)$ is positive. When $\Delta(n) > 0$, the minimal algorithm results in Anh's win with $\mathrm{M}_3(n) = 2^n - 1$. Otherwise, the move sequence s_3 in the proof of Theorem 6 is additionally used λ times in the notation of Theorem 6 for Anh to win, where by Lemma 10,

$$\lambda = \left\lfloor \frac{-\Delta(n)}{2\left(w_{ik} + w_{jk} - 2w_{ij}\right)} \right\rfloor + 1 = \left\lfloor \frac{-\beta_1}{2\gamma} \right\rfloor + 1.$$

Therefore, the minimal number of moves is bounded as

$$\mathrm{M}_3(n) \leqslant 2^n - 1 + 16\left(\left\lfloor \frac{-\beta_1}{2\gamma} \right\rfloor + 1\right) = 2^n + 15 + 16\left\lfloor \frac{-\beta_1}{2\gamma} \right\rfloor.$$

For $n = 3$ and $\gamma = w_{13} + w_{23} - 2w_{12}$, that is, if w_{12} is the smallest weight, the minimal algorithm can not reach the position satisfying the condition for P, having the two smallest disks on Peg 3 and the largest disk on either of the remaining pegs. So the following two move sequences are considered instead as candidates for $s_1 s_2^{-1}$. The sequence with a smaller number of moves is then actually used (we write the sequences in the format (s_1)-(s_2^{-1}).

$$(12 - 13 - 23 - 12) - (23 - 13 - 12 - 23 - 12 - 13 - 23) \quad \text{(11 moves)}$$
$$(13 - 12 - 13 - 23 - 13 - 12) - (23 - 13 - 12 - 23 - 12 - 13 - 23) \quad \text{(13 moves)}$$

The total scores for these sequences are $\Delta(n) = 2(w_{12}+w_{23})-3w_{13}$ and $\Delta(n) = w_{13}$, respectively. Therefore, $\mathrm{M}_3(3)$ is bounded as stated. Next, for $n = 3$ and $\gamma = w_{12} + w_{13} - 2w_{23}$, similarly to the exceptional case just mentioned, the minimal algorithm can not be used as $s_1 s_2^{-1}$. So, the following two sequences are considered as candidates for s_1 and s_2^{-1} and the one with a smaller number of moves is actually used.

$$(12 - 13 - 23 - 12 - 23 - 13 - 12 - 23) - (12 - 13 - 23) \quad \text{(11 moves)}$$
$$(12 - 13 - 23 - 12 - 13 - 23 - 13 - 12 - 13 - 23) - (12 - 13 - 23) \quad \text{(13 moves)}$$

For the first and the second sequences, the total scores are $\Delta(n) = 2(w_{12} + w_{23}) - 3w_{13}$ and $\Delta(n) = w_{13}$, respectively. Therefore, $\mathrm{M}_3(3)$ is bounded as stated.

For (EC2) and $n \geqslant 3$, the minimal algorithm for transferring the Tower from Peg 1 to itself, where the largest disk has to be moved, can be used as concatenation of s_1 and s_2^{-1}. More precisely, if $\gamma = w_{12}+w_{23}-2w_{13}$, the first minimal algorithm in Lemma 2 in which the role of Peg 2 and Peg 3 is exchanged is used. Then after the procedure ii) of the algorithm with an even number of moves, it reaches the intermediate position P, so this algorithm can be $s_1 s_2^{-1}$. Next, if $\gamma = w_{13} + w_{23} - 2w_{12}$, the first minimal algorithm in Lemma 2 can be $s_1 s_2^{-1}$. Finally, if $\gamma = w_{12} + w_{13} - 2w_{23}$, after the procedure iv) of the minimal algorithm with an even number of moves, it reaches the intermediate position P. So the minimal algorithm can be $s_1 s_2^{-1}$. Similarly to (EC1), it is further divided into two subcases depending on whether $\Delta(n)$ is positive. When $\Delta(n) > 0$, Anh wins with the minimal algorithm with $\mathrm{M}_3(n) = 2^{n+1} - 1$. Otherwise, by Lemma 2 the minimal number of moves is bounded as

$$\mathrm{M}_3(n) \leqslant 2^{n+1} - 1 + 16\left(\left\lfloor \frac{-\beta_2}{2\gamma} \right\rfloor + 1\right) = 2^{n+1} + 15 + 16\left\lfloor \frac{-\beta_2}{2\gamma} \right\rfloor.$$

For (EC3), recall that the Tower is moved to itself, but only the smallest disk has to be moved at least once. First, we examine the case when only the two smallest disks are to be moved during the game. Then Anh can win with the 7 moves in Table 1 if $\Delta(n) = w_{12} + w_{13} - w_{23} > 0$. Next, when more than two disks are moved during the game, then Anh uses the minimal algorithm of transferring the smallest three disks from Peg 1 to itself with 15 moves. Then, as shown in (EC2) with $n = 3$, if $\beta_2 > 0$ $M(n) = 2^{3+1} - 1 = 15$ for all $n \geqslant 3$. Otherwise,

$$\mathrm{M}_3(n) \leqslant 2^{3+1} - 1 + 16\left(\left\lfloor \frac{-\beta_2}{2\gamma} \right\rfloor + 1\right) = 31 + 16\left\lfloor \frac{-\beta_2}{2\gamma} \right\rfloor.$$

For (EC4), in which one can win by completing the Tower at Peg 1, 2, or 3, the algorithms and results for (EC1) with Peg 2 or Peg 3 as the final peg and (EC2) are employed. First, recall that

$$\beta_3 = \begin{cases} \max\{w_{13}, w_{12}\} & \text{if } n \text{ odd,} \\ \max\{w_{12}+w_{23}-w_{13}, w_{13}+w_{23}-w_{12}\} & \text{if } n \text{ even.} \end{cases}$$

When $\beta_3 > 0$, Anh can win by using the minimal algorithm to reach either of Peg 2 or Peg 3 with 2^n-1 moves. Otherwise, if pairs of n and γ are not the exceptional ones in (EC1) and (EC2), the minimal algorithms for reaching Peg 1, 2, or 3 with the repetitive part s_3^λ is used. Then, the minimal number of moves is bounded as

$$\mathrm{M}_3(n) \leqslant \min\left\{2^n + 15 + 16\left\lfloor\frac{-\beta_3}{2\gamma}\right\rfloor, 2^{n+1} + 15 + 16\left\lfloor\frac{-\beta_2}{2\gamma}\right\rfloor\right\}$$

as stated. The remaining cases are for (EC1) with Peg 2 or Peg 3 as the final peg with $n = 3$ and with either of the following: (i) $\gamma = w_{13} + w_{23} - w_{12}$, (ii) $\gamma = w_{12} + w_{23} - w_{13}$, and (iii) $\gamma = w_{12} + w_{13} - w_{23}$. When $\gamma = w_{13} + w_{23} - w_{12}$, we evaluate the numbers of moves for each case of reaching Peg 3, Peg 2, and Peg 1. When Peg 3 is the final peg, this is exactly one of the exceptional cases in (EC1), so the number of moves should be

$$\min\left\{27 + 16\left\lfloor\frac{2(w_{12}+w_{23}) - 3w_{13}}{2\gamma}\right\rfloor, 29 + 16\left\lfloor\frac{-w_{13}}{2\gamma}\right\rfloor\right\}.$$

When Peg 2 is the final peg, this case of $\gamma = w_{13} + w_{23} - w_{12}$ is not at all exceptional; thus, the minimal algorithm for transferring the three disks from Peg 1 to Peg 2 is used as the sequence $s_1 s_2^{-1}$. So, the number of moves is

$$2^3 - 1 + 16\left(\left\lfloor\frac{-w_{12}}{2\gamma}\right\rfloor + 1\right) = 23 + 16\left\lfloor\frac{-w_{12}}{2\gamma}\right\rfloor.$$

When Peg 1 is the final peg, there is no exception, so the minimal algorithm is used for $s_1 s_2^{-1}$ and the number of moves is

$$2^{3+1} - 1 + 16\left(\left\lfloor\frac{-\beta_2}{2\gamma}\right\rfloor + 1\right) = 31 + 16\left\lfloor\frac{-\beta_2}{2\gamma}\right\rfloor.$$

In all, the minimum of these numbers of moves is taken as the upper bound of $M_3(3)$ as stated for (i) of (EC4). For (ii), the argument is exactly the same with (i) by exchanging the role of Peg 2 and Peg 3. For (iii), since $\gamma = w_{12} + w_{13} - w_{23}$ has to be treated as an exceptional case for both Peg 2 and Peg 3 are final pegs, so the bound for $M_3(3)$ is obtained as stated.

Finally for (EC5), the algorithms and results used for (EC1) and (EC3) are employed for obtaining the bounds for $M_3(n)$. Since the argument is almost the same with (EC4), we omit the detail.

□

7. Discussion

We have chosen a number of ending conditions for two player variations of the classical Tower of Hanoi. There are probably many other ways to proceed. But one of the most intriguing questions is if there is some way to release Bao from his confinement of forced move play, which gives the two player game still the flavor of a one player game. Indeed each move by Bao is automatic without thinking and his input to the game has diminished to a purely mechanical matter. Still Anh's winning move paths in the scoring variation are non-trivial; in particular when it comes to minimizing the number of moves. Theorem 12 obviously leads to the big open question what the minimal numbers of moves may be in the open cases.

7.1. **When is Bao at advantage?** The analysis so far has been disadvantageous for Bao, apart from very special circumstances where Anh was forced to revert to infinite play. Let us discuss minor alternations of the game rules, which gives Bao a greater impact on the game. We settle with the standard impartial setting and leave possible further scoring analysis for a future study.

A standard variation for impartial games is to play misère rather than normal play, that is, the player who terminates the game loses. We need to impose a rule that makes such rules meaningful. A standard technique in combinatorial games is to use 'compulsory' moves. Here: a player must put a disk on a one size larger disk if such a move is available. This rule makes misère meaningful, because a final move is forced if the game goes this far. However there is no guarantee that the game will reach the second last position. In fact, since Anh still controls the game by always moving the smaller disk she will have to avoid that the game continues until the second last position. Now, the question arises whether she can, at least force drawn. Of course now Anh only has a choice every second move, and the analysis is straight forward. But we leave this to the reader, since we are looking for some rules where Bao actually wins.

Another variation which gives Bao more impact on the game, is to allow more than one move on a given disk in a direct sequence of moves. If the previous player moved disk d, then the current player can move it as well, unless the position would become exactly the same as another position in the current 'round'. Here 'round' stands for a circuit where only the given disk has been moved. Once another disk has been moved the current circuit is broken, and a new 'round' starts. For the three peg case, this gives an immediate advantage for Bao, but he will only be able to use it to play a draw game in the general case.

It turns out, that combining the two modifications in this section, but playing normal (EC5), Bao can force a win. Let us explain in the three-peg case and with two disks. Anh plays the smallest disk to Peg 3, say. Bao moves it to Peg 2, which is legal in this round (note that he is not forced to put it on top of the larger disk on Peg 1, since this move is not legal). Now, Anh has to play the larger disk to Peg 3 and hence Bao wins by putting the smaller disk on top of the larger. Note that the forced moves were not important in the case for two disks. Now, we let them play the three-disk case, still on three pegs. The move sequence will, for example, be *12* − 23 − *12* − 23 − *13* − 12 − *23* − 12 where Anh's moves are all forced (except the first one), and Bao wins. Bao cannot immediately adapt this strategy for the cases $n > 3$, because if $n = 4$, then Anh would get a parity advantage. We leave it as an open question to resolve this game in general.

7.2. **How about several players?** Yet another variation is to play our standard variation with several players. In the case for (EC1), three players, three pegs and two disks, then the first player cannot win under optimal play, but she can decide which one of the other players that will win. If this game is played with three disks, then it is drawn. This can be seen this way. No player will be forced to put the second smallest disk on a final peg and thereby giving the player just after the winning move. This follows because the player who would have moved the supposedly second last move of the smallest disk, can choose to put it on either the second or third smaller disk, either of which prevents a bad forced move of the second smallest disk. Open problems: classify the variations of normal and scoring play where several player Tower of Hanoi terminates.

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Institut de Mathématiques et de Modélisation de Montpellier, Université de Montpellier, Campus Triolet - CC051, place Eugène Bataillon, 34095 Montpellier Cedex 05, France
E-mail address: jonathan.chappelon@um2.fr

Corresponding author: Urban Larsson, Department of Mathematics and Statistics, Dalhousie University, 6316 Coburg Road, PO Box 15000, Halifax, Nova Scotia, Canada B3H 4R2, supported by the Killam Trust
E-mail address: urban031@gmail.com

Division of Information System Design, School of Sience and Engineering, Tokyo Denki University, Ishizaka, Hatoyama-cho, Hiki, Saitama, 350-0394, Japan
E-mail address: matsu@rd.dendai.ac.jp