

# SCATTERING FOR THE NONLINEAR SCHRÖDINGER EQUATION WITH A GENERAL ONE-DIMENSIONAL CONFINEMENT

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ABSTRACT. We consider the defocusing nonlinear Schrödinger equation in several space dimensions, in the presence of an external potential depending on only one space variable. This potential is bounded from below, and may grow arbitrarily fast at infinity. We prove existence and uniqueness in the associated Cauchy problem, in a suitable functional framework, as well as the existence of wave operators when the power of the nonlinearity is sufficiently large. Asymptotic completeness then stems from at least two approaches, which are briefly recalled.

## 1. INTRODUCTION

We consider the large time behavior for the nonlinear Schrödinger equation

$$(1.1) \quad i\partial_t u + \frac{1}{2}\Delta u = V(x)u + |u|^{2\sigma}u,$$

where  $u : (t, x, y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d-1} \rightarrow \mathbb{C}$ , with  $d \geq 2$ ,  $\Delta$  is the Laplacian in  $(x, y)$ , and  $0 < \sigma < \frac{2}{(d-2)_+}$  (where  $1/a_+$  stands for  $+\infty$  if  $a \leq 0$ , and for  $1/a$  if  $a > 0$ ): the nonlinearity is energy-subcritical in terms of the whole space dimension  $d$ . The external potential  $V$  depends only on  $x$ . More precisely, we suppose:

**Assumption 1.1.** *The potential  $V \in L^2_{\text{loc}}(\mathbb{R})$  is real-valued and bounded from below:*

$$\exists C_0, \quad V(x) + C_0 \geq 0, \quad \forall x \in \mathbb{R}.$$

It follows from [17, Theorem X.28] that

$$H = -\frac{1}{2}\Delta + V(x)$$

is essentially self-adjoint on  $C_0^\infty(\mathbb{R}^d)$ , with domain ([17, Theorem X.32])

$$D(H) = \{f \in L^2(\mathbb{R}^d), \quad -\frac{1}{2}\Delta f + Vf \in L^2(\mathbb{R}^d)\}.$$

The goal of this paper is to understand the large time dynamics in (1.1). This framework is to be compared with the analysis in [19], where there is no external potential ( $V = 0$ ), but where the  $x$  variable belongs to the torus  $\mathbb{T}$  (which is the only one-dimensional compact manifold without boundary). It is proven there that if a short range scattering theory is available for the nonlinearity  $|u|^{2\sigma}u$  in  $H^1(\mathbb{R}^{d-1})$ , that is if  $\frac{2}{d-1} < \sigma < \frac{2}{(d-2)_+}$ , then the solution of the Cauchy problem for  $(x, y) \in \mathbb{T} \times \mathbb{R}^{d-1}$  (is global and) is asymptotically linear as  $t \rightarrow \infty$ .

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This project was supported by the French ANR projects SchEq (ANR-12-JS01-0005-01) and BECASIM (ANR-12-MONU-0007-04).

In this paper, we prove the analogous result in the case of (1.1), as well as the existence of wave operators (Cauchy problem with behavior prescribed at infinite time). This extends some of the results from [1] where the special case of an harmonic potential  $V$  is considered. The properties related to the harmonic potentials are exploited to prove the existence of wave operators in the case of a multidimensional confinement ( $V(x) = |x|^2$ ,  $x \in \mathbb{R}^n$ ,  $n \geq 1$ ), a case that we do not consider in the present paper (see Remark 1.6): essentially, if the nonlinearity is short range on  $\mathbb{R}^{d-n}$ , then it remains short range on  $\mathbb{R}^d$  with  $n$  confined directions. Long range effects are described in [12], in the case  $n = d - 1$  and  $\sigma = 1$  (cubic nonlinearity, which is exactly the threshold to have long range scattering in one dimension). A technical difference with [19] is that for the Cauchy problem, we do not make use of inhomogeneous Strichartz for non-admissible pairs like established in [5, 7, 20], and for scattering theory, such estimates are not needed when  $d \leq 4$ .

We emphasize that here, the potential  $V$  can have essentially any behavior, provided that it remains bounded from below. It can be bounded (in which case the term ‘‘confinement’’ is inadequate), or grow arbitrarily fast as  $x \rightarrow \pm\infty$ . This is in sharp contrast with e.g. [14, 22, 23], where Strichartz estimates (with loss) are established in the presence of superquadratic potentials, or with [2], where a functional calculus adapted to confining potentials is developed: in all these cases, typically, an exponential growth of the potential is ruled out, since in this case, no pseudo-differential calculus is available.

Introduce the notation

$$M_x = -\frac{1}{2}\partial_x^2 + V(x) + C_0.$$

We define the spaces

$$B_x = \left\{ u \in L^2(\mathbb{R}), M_x^{1/2}u \in L^2(\mathbb{R}) \right\}, \quad \Sigma_y = \left\{ u \in H^1(\mathbb{R}^{d-1}), yu \in L^2(\mathbb{R}^{d-1}) \right\},$$

$$Z = L_y^2 B_x \cap L_x^2 H_y^1, \quad \tilde{Z} = L_y^2 B_x \cap L_x^2 \Sigma_y,$$

endowed with the norms

$$\|u\|_{B_x}^2 = \|u\|_{L_x^2(\mathbb{R})}^2 + \|M_x^{1/2}u\|_{L_x^2(\mathbb{R})}^2 = \|u\|_{L_x^2(\mathbb{R})}^2 + \langle M_x u, u \rangle,$$

$$\|u\|_{\Sigma_y}^2 = \|u\|_{L_y^2(\mathbb{R}^{d-1})}^2 + \|\nabla_y u\|_{L_y^2(\mathbb{R}^{d-1})}^2 + \|yu\|_{L_y^2(\mathbb{R}^{d-1})}^2,$$

and

$$\|u\|_Z^2 = \|u\|_{L_{x,y}^2(\mathbb{R}^d)}^2 + \|M_x^{1/2}u\|_{L_{x,y}^2(\mathbb{R}^d)}^2 + \|\nabla_y u\|_{L_{x,y}^2(\mathbb{R}^d)}^2, \quad \|u\|_{\tilde{Z}}^2 = \|u\|_Z^2 + \|yu\|_{L_{x,y}^2(\mathbb{R}^d)}^2.$$

The group  $e^{-itH}$  is unitary on  $Z$ , but not on  $\tilde{Z}$ , a property which is discussed in the proof of Lemma 2.6.

*Remark 1.2.* Note that  $B_x$  is the domain of the operator  $M_x^{1/2}$ , which is defined as a fractional power of the self-adjoint operator  $M_x$  acting on  $L^2(\mathbb{R})$ : for  $u \in B_x$ ,  $M_x^{1/2}u$  is defined by

$$M_x^{1/2}u = \int_0^\infty \lambda^{1/2} dE_\lambda(u),$$

where  $M_x = \int_0^\infty \lambda dE_\lambda$  is the spectral decomposition of  $M_x$ .

**Theorem 1.3** (Cauchy problem). *Let  $d \geq 2$ ,  $V$  satisfying Assumption 1.1 and  $0 < \sigma < \frac{2}{(d-2)_+}$ . Let  $t_0 \in \mathbb{R}$  and  $u_0 \in Z$ . There exists a unique solution  $u \in C(\mathbb{R}; Z)$  to (1.1) such*

that  $u|_{t=t_0} = u_0$ . The following two quantities are independent of time:

$$\text{Mass: } \|u(t)\|_{L^2_{xy}(\mathbb{R}^d)}^2,$$

$$\text{Energy: } \frac{1}{2} \|\nabla_{xy} u(t)\|_{L^2_{xy}(\mathbb{R}^d)}^2 + \frac{1}{\sigma+1} \|u(t)\|_{L^{2\sigma+2}_{xy}(\mathbb{R}^d)}^{2\sigma+2} + \int_{\mathbb{R}^d} V(x) |u(t, x, y)|^2 dx dy.$$

If in addition  $u_0 \in \tilde{Z}$ , then  $u \in C(\mathbb{R}; \tilde{Z})$ .

**Theorem 1.4** (Existence of wave operators). *Let  $d \geq 2$ , and  $V$  satisfying Assumption 1.1.*

1. *If  $u_- \in Z$  and  $\frac{2}{d-1} \leq \sigma < \frac{2}{(d-2)_+}$ , there exists  $u \in C(\mathbb{R}; Z)$  solution to (1.1) such that*

$$\|u(t) - e^{-itH} u_-\|_Z = \|e^{itH} u(t) - u_-\|_Z \xrightarrow{t \rightarrow -\infty} 0.$$

*This solution is such that*

$$u \in L^\infty(\mathbb{R}; Z) \cap L^p((-\infty, 0]; L_y^k L_x^2)$$

*for some pair  $(p, k)$  given in the proof, and it is unique in this class.*

2. *If  $u_- \in \tilde{Z}$  and  $\frac{2}{d} < \sigma < \frac{2}{(d-2)_+}$ , there exists a unique  $u \in C(\mathbb{R}; \tilde{Z})$  solution to (1.1) such that*

$$e^{itH} u \in L^\infty([-\infty, 0]; \tilde{Z}) \quad \text{and} \quad \|e^{itH} u(t) - u_-\|_{\tilde{Z}} \xrightarrow{t \rightarrow -\infty} 0.$$

In the second case, the lower bound  $\sigma > \frac{2}{d}$  is weaker than in the first case, so there is some gain in working in the smaller space  $\tilde{Z}$  rather than in  $Z$ . However, this lower bound is larger than in the corresponding result from [1] where only the case  $V(x) = x^2$  is considered. Indeed in [1], the general lower bound is  $\sigma > \frac{2d}{d+2} \frac{1}{d-1}$ , which is smaller than the present one as soon as  $d \geq 3$ . The main technical reason is that specific properties of the harmonic oscillator (typically, the fact that it generates a flow which is periodic in time) makes it possible to establish a larger set of Strichartz estimates than the one which we use in the present paper. In all cases, the expected borderline between short range and long range scattering is  $\sigma_c = \frac{1}{d-1}$  ( $d-1$  is the ‘‘scattering dimension’’), so our result is sharp in the case  $d = 2$ , and most likely only in this case.

**Theorem 1.5** (Asymptotic completeness). *Let  $d \geq 2$ ,  $V$  satisfying Assumption 1.1, and  $\frac{2}{d-1} < \sigma < \frac{2}{(d-2)_+}$ . For any  $u_0 \in Z$ , there exists a unique  $u_+ \in Z$  such that the solution to (1.1) with  $u|_{t=0} = u_0$  satisfies*

$$\|u(t) - e^{-itH} u_+\|_Z = \|e^{itH} u(t) - u_+\|_Z \xrightarrow{t \rightarrow +\infty} 0.$$

*Remark 1.6.* When a confinement is present (due either to a harmonic potential, or to a bounded geometry) in  $n$  directions, for a total space dimension  $d$ , it is expected that the ‘‘scattering dimension’’ is  $d - n$ . This was proven systematically in the case of a harmonic confinement in [1], complemented by [12]; see also [11, 18]. Therefore, to prove asymptotic completeness thanks to Morawetz estimates, it is natural to assume  $\sigma > \frac{2}{d-n}$  (essentially because it is not known how to take advantage of these estimates otherwise, except in the  $L^2$ -critical case, where many other tools are used). On the other hand, for the Cauchy problem to be locally well-posed at the  $H^1$ -level, it is necessary to assume  $\sigma \leq \frac{2}{d-2}$  if  $d \geq 3$ . For the above two conditions to be consistent in the energy-subcritical case  $\sigma < \frac{2}{d-2}$ , we readily see that the only possibility is  $n = 1$ , as in [19] and the present paper. To treat the case  $n = 2$ , the analysis of a doubly critical case would be required:  $L^2$ -critical in  $\mathbb{R}^{d-n}$  with  $\sigma = \frac{2}{d-n}$ , and energy-critical in  $\mathbb{R}^d$  with  $\sigma = \frac{2}{d-2}$ .

## 2. TECHNICAL PRELIMINARIES

## 2.1. Sobolev embeddings.

**Lemma 2.1.**  $B_x$  is continuously embedded into  $H_x^1(\mathbb{R})$ .

*Proof.* Since  $V$  is bounded from below, we have

$$\begin{aligned} \|u\|_{H_x^1(\mathbb{R})}^2 &\leq \|u\|_{L_x^2(\mathbb{R})}^2 + \|\partial_x u\|_{L_x^2(\mathbb{R})}^2 + 2 \int_{\mathbb{R}} (V(x) + C_0) |u(x)|^2 dx \\ &\leq \|u\|_{L_x^2(\mathbb{R})}^2 + 2 \langle M_x u, u \rangle \lesssim \|u\|_{B_x}^2, \end{aligned}$$

hence the result.  $\square$

Introduce, for  $\gamma, s \geq 0$ , the anisotropic Sobolev space

$$H_y^\gamma H_x^s = (1 - \Delta_y)^{-\gamma/2} (1 - \partial_x^2)^{-s/2} L_{x,y}^2,$$

endowed with the norm

$$\|u\|_{H_y^\gamma H_x^s}^2 = \int_{\mathbb{R} \times \mathbb{R}^{d-1}} \langle \xi \rangle^{2s} \langle \eta \rangle^{2\gamma} |\hat{u}(\xi, \eta)|^2 d\xi d\eta,$$

where  $\hat{u}$  denotes the Fourier transform of  $u$  in both  $x$  and  $y$  variables.  $\dot{H}_y^\gamma H_x^s$  denotes the corresponding homogeneous space, endowed with the norm

$$\|u\|_{\dot{H}_y^\gamma H_x^s}^2 = \int_{\mathbb{R} \times \mathbb{R}^{d-1}} \langle \xi \rangle^{2s} |\eta|^{2\gamma} |\hat{u}(\xi, \eta)|^2 d\xi d\eta.$$

**Lemma 2.2.** If  $\varepsilon \in (0, 1/2)$ ,  $s = \frac{1}{2} + \varepsilon$  and  $\gamma = \frac{1}{2} - \varepsilon$ , then

$$\|u\|_{\dot{H}_y^\gamma H_x^s} \leq \|u\|_{H_y^\gamma H_x^s} \lesssim \|u\|_Z, \quad \forall u \in Z.$$

*Proof.* From Young inequality and Lemma 2.1,

$$\begin{aligned} \|u\|_{H_y^\gamma H_x^s}^2 &= \int_{\mathbb{R} \times \mathbb{R}^{d-1}} \langle \xi \rangle^{2\gamma} \langle \eta \rangle^{2s} |\hat{u}(\xi, \eta)|^2 d\xi d\eta \\ &\lesssim \int_{\mathbb{R} \times \mathbb{R}^{d-1}} [(1 + \xi^2) + (1 + |\eta|^2)] |\hat{u}(\xi, \eta)|^2 d\xi d\eta \lesssim \|u\|_{L_y^2 H_x^1}^2 + \|u\|_{L_x^2 \dot{H}_y^1}^2, \end{aligned}$$

hence the result.  $\square$

## 2.2. Anisotropic Gagliardo-Nirenberg inequality.

**Proposition 2.3.** Let  $k, s, \gamma > 0$  such that

$$(2.1) \quad s > 1/2 \quad \text{and} \quad \frac{1}{2} > \frac{1}{k} > \frac{1}{2} - \frac{\gamma}{d-1} > 0.$$

Then  $H_y^\gamma H_x^s \subset L_y^k L_x^\infty$ , and there exists  $C > 0$  such that for every  $u \in H_y^\gamma H_x^s$ ,

$$\|u\|_{L_y^k L_x^\infty} \leq C \|u\|_{L_y^2 H_x^s}^{1-\delta} \|u\|_{\dot{H}_y^\gamma H_x^s}^\delta, \quad \text{where } \delta = \frac{d-1}{\gamma} \left( \frac{1}{2} - \frac{1}{k} \right).$$

*Proof.* We first use the Sobolev inequality in the  $x$  variable and Minkowski inequality (which is possible because  $k > 2$ ). We get

$$(2.2) \quad \|u\|_{L_y^k L_x^\infty} \lesssim \|u\|_{L_y^k H_x^s} = \|\langle \xi \rangle^s \mathcal{F}_x u(\xi, y)\|_{L_y^k L_\xi^2} \lesssim \|\langle \xi \rangle^s \mathcal{F}_x u(\xi, y)\|_{L_\xi^2 L_y^k},$$

where  $\mathcal{F}_x$  denotes the Fourier transform in the  $x$  variable. Similarly, we denote by  $\mathcal{F}_y$  the Fourier transform in  $y$  and  $\widehat{u}(\xi, \eta) = (\mathcal{F}_x \mathcal{F}_y u)(\xi, \eta)$ . Then for a fixed value of  $\xi \in \mathbb{R}$ , Hausdorff-Young inequality yields

$$(2.3) \quad \|\mathcal{F}_x u(\xi, y)\|_{L_y^k} \lesssim \|\widehat{u}(\xi, \eta)\|_{L_\eta^{k'}}.$$

Omitting the dependence of the right hand side in  $\xi$ , let us denote by  $v(\eta) = \widehat{u}(\xi, \eta)$ . It follows from the triangle and Hölder inequality that for any  $R > 0$ ,

$$(2.4) \quad \begin{aligned} \|v\|_{L_\eta^{k'}} &\leq \|v\|_{L^{k'}(|\eta| < R)} + \|v\|_{L^{k'}(|\eta| > R)} \\ &\lesssim \|\mathbf{1}_{\{|\eta| < R\}}\|_{L^p(|\eta| < R)} \|v\|_{L_\eta^2} + \|\eta\|^{-\gamma} \|v\|_{L^p(|\eta| > R)} \|\eta\|^\gamma \|v\|_{L_\eta^2} \\ &\lesssim R^{(d-1)/p} \|v\|_{L_\eta^2} + R^{(d-1)/p-\gamma} \|\eta\|^\gamma \|v\|_{L_\eta^2}, \end{aligned}$$

where  $p$  is given by  $1/p = 1/2 - 1/k$ . Note that (2.1) implies that  $\gamma p > d - 1$ , and therefore  $|\eta|^{-\gamma} \in L^p(|\eta| > R)$ . Optimizing in  $R$  in the right hand side of (2.4), we get

$$(2.5) \quad \|v\|_{L_\eta^{k'}} \lesssim \|v\|_{L_\eta^2}^{1-\delta} \|\eta\|^\gamma \|v\|_{L_\eta^2}^\delta,$$

where  $\delta = \frac{d-1}{\gamma p} \in (0, 1)$ . Combining (2.2), (2.3) and (2.5), Hölder inequality yields

$$\begin{aligned} \|u\|_{L_y^k L_x^\infty} &\lesssim \left( \int \langle \xi \rangle^{2s(1-\delta)} \|\widehat{u}\|_{L_\eta^2}^{2(1-\delta)} \langle \xi \rangle^{2s\delta} \|\eta\|^\gamma \|\widehat{u}\|_{L_\eta^2}^{2\delta} d\xi \right)^{1/2} \\ &\lesssim \left( \int \langle \xi \rangle^{2s} \|\widehat{u}\|_{L_\eta^2}^2 d\xi \right)^{(1-\delta)/2} \left( \int \langle \xi \rangle^{2s} \|\eta\|^\gamma \|\widehat{u}\|_{L_\eta^2}^2 d\xi \right)^{\delta/2} \\ &= \|u\|_{L_y^2 H_x^s}^{1-\delta} \|u\|_{\dot{H}_y^\gamma H_x^s}^\delta. \end{aligned}$$

□

**Corollary 2.4.** *Let  $2 < k < \frac{2(d-1)}{(d-2)_+}$ . Then  $Z$  is continuously embedded in  $L_y^k L_x^\infty$ .*

*Proof.* Pick  $\varepsilon > 0$  small enough such that

$$\frac{1}{2} - \frac{1/2 - \varepsilon}{d-1} = \frac{d-2}{2(d-1)} + \frac{\varepsilon}{d-1} < \frac{1}{k}.$$

Then  $(s, \gamma) = (1/2 + \varepsilon, 1/2 - \varepsilon)$  satisfy the assumptions of Proposition 2.3 and Lemma 2.2. Thus, using also Lemma 2.1,

$$\|u\|_{L_y^k L_x^\infty} \lesssim \|u\|_{L_y^2 B_x}^{1-\delta} \|u\|_Z^\delta \lesssim \|u\|_Z.$$

□

**2.3. Strichartz estimates.** Following the idea from [18], with the generalization from [1] (noticing that the spectral decomposition from the proof in [18] is not needed), we have, since  $M_x$  commutes with  $H$ :

**Proposition 2.5.** *Let  $d \geq 2$ . We have*

$$\|e^{-itH} u_0\|_{L_t^q L_y^r L_x^2} + \left\| \int_0^t e^{-i(t-s)H} F(s) ds \right\|_{L_t^{q_1} L_y^{r_1} L_x^2} \lesssim \|u_0\|_{L_y^2 L_x^2} + \|F\|_{L_t^{q_2} L_y^{r_2} L_x^2},$$

*provided that the pairs are  $(d-1)$ -admissible, that is*

$$\frac{2}{q} + \frac{d-1}{r} = \frac{2}{q_1} + \frac{d-1}{r_1} = \frac{2}{q_2} + \frac{d-1}{r_2} = \frac{d-1}{2},$$

*with  $(q, r) \neq (2, \infty)$  if  $d = 3$ .*

**2.4. Vectorfields.** We introduce the notation

$$\begin{aligned} A_0(t) &= A_0 = \text{Id}, & A_1(t) &= A_1 = M_x^{1/2}, & A_2(t) &= A_2 = \nabla_y, \\ A_3(t) &= y + it\nabla_y = ite^{i|y|^2/(2t)}\nabla_y \left( \cdot e^{-i|y|^2/(2t)} \right) = e^{-itH} y e^{itH}. \end{aligned}$$

The operator  $A_3$  is the standard Galilean operator on  $\mathbb{R}^{d-1}$ , see e.g. [4], so the last identity stems from the fact that  $e^{-itM_x}$  commutes with both  $e^{i\frac{t}{2}\Delta_y}$  and  $y$ . We readily have:

**Lemma 2.6.** *The operators  $A_j$  satisfy the following properties:*

- *Commutation: for  $j \in \{0, \dots, 3\}$ ,  $[i\partial_t - H, A_j] = 0$ .*
- *Action on the nonlinearity: for all  $j \in \{0, \dots, 3\}$ ,*

$$\|A_j(|u|^{2\sigma}u)\|_{L_x^2} \lesssim \|u\|_{L_x^\infty}^{2\sigma} \|A_j u\|_{L_x^2}.$$

- *Equivalence of norms: for all  $u \in C_0^\infty(\mathbb{R}^d)$ , we have, uniformly in  $t \in \mathbb{R}$ ,*

$$(2.6) \quad \|e^{itH}u\|_Z = \|u\|_Z \approx \sum_{j=0}^2 \|A_j u\|_{L_{xy}^2}, \quad \|e^{itH}u\|_{\bar{Z}} \approx \sum_{j=0}^3 \|A_j(t)u\|_{L_{xy}^2}.$$

- *Gagliardo-Nirenberg inequalities: for all  $g \in \Sigma_y$ ,  $2 \leq p < \frac{2}{(d-3)_+}$ ,*

$$\begin{aligned} \|g\|_{L_y^p} &\leq C \|g\|_{L^2}^{1-\delta} \|A_2 g\|_{L_y^2}^\delta, \\ \|g\|_{L_y^p} &\leq \frac{C}{|t|^\delta} \|g\|_{L^2}^{1-\delta} \|A_3(t)g\|_{L_y^2}^\delta, \quad t \neq 0, \end{aligned}$$

where  $C$  is independent of  $t$ , and  $\delta = (d-1)\left(\frac{1}{2} - \frac{1}{p}\right)$ .

*Proof.* The commutation property is straightforward. For the action on the nonlinearity, it is trivial in the case of  $A_0$  and  $A_2$ . For  $A_3$ , it stems classically from the fact that  $A_3$  is the gradient in  $y$  conjugated by an exponential of modulus one and that the nonlinearity we consider is gauge invariant. Concerning  $A_1$ , we compute

$$\begin{aligned} \|M_x^{1/2}(|u|^{2\sigma}u)\|_{L_x^2}^2 &= \langle M_x(|u|^{2\sigma}u), |u|^{2\sigma}u \rangle \\ &= \frac{1}{2} \|\partial_x(|u|^{2\sigma}u)\|_{L_x^2}^2 + \int_{-\infty}^{+\infty} (V(x) + C_0) |u|^{4\sigma+2} dx \\ &\leq (2\sigma+1)^2 \|u\|_{L_x^\infty}^{4\sigma} \left( \frac{1}{2} \|\partial_x u\|_{L_x^2}^2 + \int_{-\infty}^{+\infty} (V(x) + C_0) |u|^2 dx \right) \\ &= (2\sigma+1)^2 \|u\|_{L_x^\infty}^{4\sigma} \|M_x^{1/2}u\|_{L_x^2}^2. \end{aligned}$$

Recall that  $A_0, A_1$  and  $A_2$  commute with  $e^{itH}$ , which is unitary on  $L^2(\mathbb{R}^d)$ , hence the first equivalence of norms. The identity  $A_3(t) = e^{-itH} y e^{itH}$  yields the second equivalence of norms, uniformly in time: note that  $\|e^{itH}u\|_{\bar{Z}}$  is equivalent to  $\|u\|_{\bar{Z}}$  only locally in time, due to the factor  $t$  in the identity  $A_3(t) = y + it\nabla_y$ .

Finally, the Gagliardo-Nirenberg inequalities stated in the lemma are the classical ones, using once more the factorization of  $A_3$ .  $\square$

## 3. CAUCHY PROBLEM

In this section, we prove Theorem 1.3. The existence part relies on a standard fixed point argument, adapted to the present framework. Since the problem is invariant by translation in time, we may assume  $t_0 = 0$ . Duhamel's formula reads

$$u(t) = e^{-itH}u_0 - i \int_0^t e^{-i(t-s)H} (|u|^{2\sigma}u)(s)ds =: \Phi(u)(t).$$

This Cauchy problem will be solved thanks to a fixed point argument in a ball of the Banach space

$$Z_T = \{u \in L^\infty([0, T]; Z), \quad A_j u \in L^q([0, T]; L_y^r L_x^2), \forall j \in \{0, 1, 2\}\},$$

where  $(q, r)$  is a  $(d-1)$ -admissible pair that will be fixed later. The space  $Z_T$  is naturally equipped with the norm

$$\|u\|_{Z_T} = \sum_{j=0}^2 \left( \|A_j u\|_{L_T^\infty L_{xy}^2} + \|A_j u\|_{L_T^q L_y^r L_x^2} \right).$$

Denote  $L_T^q X = L^q([0, T]; X)$ . Proposition 2.5 and the first point of Lemma 2.6 imply, for  $j \in \{0, 1, 2\}$ :

$$\|A_j \Phi(u)\|_{L_T^\infty L_{xy}^2} + \|A_j \Phi(u)\|_{L_T^q L_y^r L_x^2} \lesssim \|A_j u_0\|_{L_{xy}^2} + \|A_j (|u|^{2\sigma}u)\|_{L_T^{q'} L_y^{r'} L_x^2}.$$

The second point of Lemma 2.6 and Hölder inequality yield

$$\|A_j (|u|^{2\sigma}u)\|_{L_T^{q'} L_y^{r'} L_x^2} \lesssim \|u\|_{L_T^\theta L_y^k L_x^\infty}^{2\sigma} \|A_j u\|_{L_T^q L_y^r L_x^2},$$

where  $\theta$  and  $k$  are given by

$$(3.1) \quad \frac{1}{q'} = \frac{2\sigma}{\theta} + \frac{1}{q}, \quad \frac{1}{r'} = \frac{2\sigma}{k} + \frac{1}{r}.$$

We infer

$$(3.2) \quad \|\Phi(u)\|_{Z_T} \lesssim \|u_0\|_Z + \|u\|_{L_T^\theta L_y^k L_x^\infty}^{2\sigma} \|u\|_{Z_T}.$$

Let us now explain how the parameters  $q, r, \theta, k$  are chosen.

**Case  $d = 2$ .** We choose  $r \in (2, \infty)$  if  $\sigma \geq 1$ ,  $2 < r < \frac{2}{1-\sigma}$  if  $0 < \sigma < 1$ , and  $(q, r)$  the corresponding 1-admissible pair. Then, (3.1) defines a number  $k$  that belongs to  $(2, \infty)$ .

**Case  $d = 3$ .**  $(q, r)$  is a 2-admissible pair with  $r \in (2, \infty)$  such that

$$\frac{1}{4} < \frac{1}{2\sigma} \left(1 - \frac{2}{r}\right) =: \frac{1}{k} < \frac{1}{2}.$$

Note that this is made possible thanks to the assumption  $\sigma < 2$ .

**Case  $d \geq 4$ .** As  $(q, r)$  describes the set of all  $(d-1)$ -admissible pairs,  $r$  varies between the two extremal values 2 and  $\frac{2(d-1)}{d-3}$ , and therefore  $\frac{1}{2\sigma}(1 - \frac{2}{r})$  varies between 0 and  $\frac{1}{\sigma(d-1)}$ , where the latter number is larger than  $\frac{d-2}{2(d-1)}$  thanks to the assumption  $\sigma < 2/(d-2)$ .

Thus, one can choose  $2 < r < \frac{2(d-1)}{d-3}$  such that if  $k$  is defined by (3.1),

$$\frac{d-2}{2(d-1)} < \frac{1}{k} < \frac{1}{2}.$$

For these choices of the parameters, Corollary 2.4 and Hölder inequality in time imply

$$(3.3) \quad \|u\|_{L_T^\theta L_y^k L_x^\infty} \lesssim \|u\|_{L_T^\theta Z} \lesssim T^{1/\theta} \|u\|_{Z_T}.$$

Note that we have chosen admissible pairs such that  $q > 2$ . Thus, since  $\theta$  is defined by (3.1),  $1/\theta > 0$ . From the combination of (3.2) and (3.3), we deduce that if  $u$  belongs to the ball  $B(R, Z_T)$  of  $Z_T$  with radius  $R > 0$  centered at the origin, we have

$$(3.4) \quad \|\Phi(u)\|_{Z_T} \leq C_1 \|u_0\|_Z + CT^{2\sigma/\theta} R^{2\sigma+1}.$$

Choosing  $R = 2C_1 \|u_0\|_Z$  and  $T = T(\|u_0\|_Z) > 0$  sufficiently small,  $B(R, Z_T)$  is stable by  $\Phi$ . Then, we note that  $B(R, Z_T)$  endowed with the norm

$$\|u\|_{B(R, Z_T)} = \|u\|_{L_T^\infty L_{xy}^2} + \|u\|_{L_T^q L_y^r L_x^2}$$

is a complete metric space (Kato's method, see e.g. [4]). For  $u_2, u_1 \in B(R, Z_T)$ , the same estimates as above yield

$$\begin{aligned} \|\Phi(u_2) - \Phi(u_1)\|_{L_T^\infty L_{xy}^2} + \|\Phi(u_2) - \Phi(u_1)\|_{L_T^q L_y^r L_x^2} \\ \lesssim \left( \|u_2\|_{L_T^\theta L_y^k L_x^\infty}^{2\sigma} + \|u_1\|_{L_T^\theta L_y^k L_x^\infty}^{2\sigma} \right) \|u_2 - u_1\|_{L_T^q L_y^r L_x^2} \\ \lesssim T^{2\sigma/\theta} \left( \|u_2\|_{Z_T}^{2\sigma} + \|u_1\|_{Z_T}^{2\sigma} \right) \|u_2 - u_1\|_{L_T^q L_y^r L_x^2} \\ \lesssim T^{2\sigma/\theta} R^{2\sigma} \|u_2 - u_1\|_{L_T^q L_y^r L_x^2}. \end{aligned}$$

Therefore,  $\Phi$  is a contraction on  $B(R, Z_T)$  endowed with the above norm, provided that  $T = T(\|u_0\|_Z)$  is sufficiently small, hence the existence of a local solution in  $Z$ .

The conservation of mass and energy follows from standard arguments (see e.g. [4]). Under Assumption 1.1, this implies an a priori bound for  $\|u(t)\|_Z$ , and so the solution  $u$  is global in time,  $u \in L^\infty(\mathbb{R}; Z)$ .

Unconditional uniqueness as stated in Theorem 1.3 follows from the same approach as in [19]. If  $u_1, u_2 \in C([0, T]; Z)$  are two solutions of (1.1) with the same initial datum, then

$$u_2(t) - u_1(t) = -i \int_0^t e^{-i(t-s)H} (|u_2|^{2\sigma} u_2 - |u_1|^{2\sigma} u_1)(s) ds.$$

Resuming the same estimates as above, we now have, for  $0 < \tau \leq T$ :

$$\begin{aligned} \|u_2 - u_1\|_{L_T^q L_y^r L_x^2} &\lesssim \left( \|u_2\|_{L_T^\theta L_y^k L_x^\infty}^{2\sigma} + \|u_1\|_{L_T^\theta L_y^k L_x^\infty}^{2\sigma} \right) \|u_2 - u_1\|_{L_T^q L_y^r L_x^2} \\ &\lesssim \tau^{2\sigma/\theta} \left( \|u_2\|_{Z_T}^{2\sigma} + \|u_1\|_{Z_T}^{2\sigma} \right) \|u_2 - u_1\|_{L_T^q L_y^r L_x^2}, \end{aligned}$$

and uniqueness follows by taking  $\tau > 0$  sufficiently small.

To complete the proof of Theorem 1.3, we just have to check that the extra regularity  $u_0 \in \tilde{Z}$  is propagated by the flow. To do so, it suffices to replace the space  $Z_T$  with

$$\tilde{Z}_T = \{u \in L^\infty((0, T), Z), \quad A_j(t)u \in L^q((0, T); L_y^r L_x^2), \forall j \in \{0, 1, 2, 3\}\},$$

that is, to add the field  $A_3$ . The second point of Lemma 2.6, and the above computations then yield

$$\begin{aligned} \|A_3\Phi(u)\|_{L_T^\infty L_{xy}^2} + \|A_3\Phi(u)\|_{L_T^q L_y^r L_x^2} &\lesssim \|yu_0\|_{L_{xy}^2} + \|u\|_{L_T^\theta L_y^k L_x^\infty}^{2\sigma} \|A_3u\|_{L_T^q L_y^r L_x^2} \\ &\lesssim \|yu_0\|_{L_{xy}^2} + T^{2\sigma/\theta} \|u\|_{Z_T}^{2\sigma} \|A_3u\|_{L_T^q L_y^r L_x^2}. \end{aligned}$$

The above fixed point argument can then be resumed: we construct a local solution in  $\tilde{Z}$ ,  $u \in C([-T, T]; \tilde{Z}) \cap L^\infty(\mathbb{R}; Z)$ . The latest property and the previous estimate show that  $A_3u \in C(\mathbb{R}; L_{xy}^2)$  is global in time.



## 4. EXISTENCE OF WAVE OPERATORS

To prove the existence of wave operators, we construct a fixed point for the related Duhamel's formula,

$$(4.1) \quad u(t) = e^{-itH}u_- - i \int_{-\infty}^t e^{-i(t-s)H} (|u|^{2\sigma}u)(s)ds =: \Phi_-(u)(t),$$

on some time interval  $(-\infty, -T]$  for  $T$  possibly very large but finite. According to the regularity assumption on  $u_-$ , we construct a solution in  $Z$  or in  $\tilde{Z}$ . This solution is actually global in time from either case of Theorem 1.3. We therefore focus on the construction of a fixed point for  $\Phi_-$ , as well as on uniqueness. In a similar fashion as in Section 3, we denote  $L_T^a X = L^a((-\infty, -T]; X)$ .

**4.1. Wave operators in  $Z$ .** Resume the  $(d-1)$ -admissible pair  $(q, r)$  used in Section 3, and  $(\theta, k)$  given by (3.1). For  $(q_1, r_1)$  a  $(d-1)$ -admissible pair, and  $j \in \{0, 1, 2\}$ , Strichartz estimates and Hölder inequality yield:

$$\begin{aligned} \|A_j \Phi_-(u)\|_{L_T^{q_1} L_y^{r_1} L_x^2} &\lesssim \|A_j u_-\|_{L_{xy}^2} + \|A_j (|u|^{2\sigma}u)\|_{L_T^{q'} L_y^{r'} L_x^2} \\ &\lesssim \|A_j u_-\|_{L_{xy}^2} + \|u\|_{L_T^\theta L_y^k L_x^\infty}^{2\sigma} \|A_j u\|_{L_T^q L_y^r L_x^2}. \end{aligned}$$

By construction,

$$2 \leq k < \frac{2(d-1)}{(d-2)_+} < \frac{2(d-1)}{(d-3)_+},$$

so we can find  $p$  such that  $(p, k)$  is  $(d-1)$ -admissible. Putting the definition of admissible pairs and (3.1) together, we get

$$1 - \frac{2\sigma}{\theta} = \frac{2}{q} = (d-1) \left( \frac{1}{2} - \frac{1}{r} \right) = \frac{(d-1)\sigma}{k} = \sigma \left( \frac{d-1}{2} - \frac{2}{p} \right).$$

By assumption,  $\sigma \geq \frac{2}{d-1}$ , so  $p \leq \theta$ , and there exists  $\beta \in (0, 1]$  such that

$$\|u\|_{L_T^p L_y^k L_x^\infty} \leq \|u\|_{L_T^p L_y^k L_x^\infty}^\beta \|u\|_{L_T^\infty L_y^k L_x^\infty}^{1-\beta}.$$

Corollary 2.4 implies

$$\|A_j \Phi_-(u)\|_{L_T^{q_1} L_y^{r_1} L_x^2} \lesssim \|A_j u_-\|_{L_{xy}^2} + \|u\|_{L_T^p L_y^k L_x^\infty}^{2\sigma\beta} \|u\|_{L_T^\infty Z}^{2\sigma(1-\beta)} \|A_j u\|_{L_T^q L_y^r L_x^2}.$$

Now the one-dimensional Gagliardo-Nirenberg inequality

$$\|f\|_{L_x^\infty} \leq \sqrt{2} \|f\|_{L_x^2}^{1/2} \|\partial_x f\|_{L_x^2}^{1/2}$$

and according to the proof of Lemma 2.1, we have

$$(4.2) \quad \begin{aligned} \|A_j \Phi_-(u)\|_{L_T^{q_1} L_y^{r_1} L_x^2} &\leq C \|A_j u_-\|_{L_{xy}^2} \\ &+ C \|u\|_{L_T^p L_y^k L_x^2}^{\sigma\beta} \|A_1 u\|_{L_T^p L_y^k L_x^2}^{\sigma\beta} \|u\|_{L_T^\infty Z}^{2\sigma(1-\beta)} \|A_j u\|_{L_T^q L_y^r L_x^2}. \end{aligned}$$

for  $C$  sufficiently large. We can now define

$$\begin{aligned} B_T &:= \left\{ u \in C([-\infty, -T]; Z), \right. \\ &\quad \|A_j u\|_{L_T^q L_y^r L_x^2} + \|A_j u\|_{L_T^\infty L_{xy}^2} \leq 4C \|A_j u_-\|_{L_{xy}^2}, \quad j \in \{0, 1, 2\}, \\ &\quad \left. \|A_j u\|_{L_T^p L_y^k L_x^2} \leq 2 \|A_j e^{-itH} u_-\|_{L_T^p L_y^k L_x^2}, \quad j \in \{0, 1\} \right\}. \end{aligned}$$

From Strichartz estimates, we know that for  $j \in \{0, 1\}$ ,

$$A_j e^{-itH} u_- \in L^p(\mathbb{R}; L_y^k L_x^2), \quad \text{so} \quad \|A_j e^{-itH} u_-\|_{L_T^p L_T^k L_x^2} \rightarrow 0 \quad \text{as } T \rightarrow +\infty.$$

Since  $\beta > 0$ , we infer that  $\Phi_-$  maps  $B_T$  to itself, for  $T$  sufficiently large, by (4.2), and since the same estimates yield, for  $j \in \{0, 1\}$ ,

$$\begin{aligned} \|A_j \Phi_-(u)\|_{L_T^p L_y^k L_x^2} &\leq \|A_j e^{-itH} u_-\|_{L_T^p L_y^k L_x^2} \\ &\quad + C \|u\|_{L_T^p L_y^k L_x^2}^{\sigma\beta} \|A_1 u\|_{L_T^p L_y^k L_x^2}^{\sigma\beta} \|u\|_{L_T^\infty Z}^{2\sigma(1-\beta)} \|A_j u\|_{L_T^q L_y^r L_x^2}. \end{aligned}$$

We have also, for  $u_2, u_1 \in B_T$ , and typically  $(q_1, r_1) \in \{(q, r), (\infty, 2)\}$ :

$$\begin{aligned} \|\Phi_-(u_2) - \Phi_-(u_1)\|_{L_T^{q_1} L_y^{r_1} L_x^2} &\lesssim \max_{j=1,2} \|u_j\|_{L_T^\theta L_y^k L_x^\infty}^{2\sigma} \|u_2 - u_1\|_{L_T^q L_y^r L_x^2} \\ &\lesssim \|e^{-itH} u_-\|_{L_T^p L_y^k L_x^2}^{\sigma\beta} \|A_1 e^{-itH} u_-\|_{L_T^p L_y^k L_x^2}^{\sigma\beta} \|u_-\|_{L_T^\infty Z}^{2\sigma(1-\beta)} \|u_2 - u_1\|_{L_T^q L_y^r L_x^2}. \end{aligned}$$

Up to choosing  $T$  larger,  $\Phi_-$  is a contraction on  $B_T$ , so  $\Phi_-$  has a unique fixed point in  $B_T$ , which solves (4.1). Uniqueness as stated in Theorem 1.4 is an easy consequence of the above estimates.

**4.2. Wave operators in  $\tilde{Z}$ .** In the case  $u_- \in \tilde{Z}$ , we consider the whole set of vector fields,  $(A_j)_{0 \leq j \leq 3}$ . For  $(q, r)$  a  $(d-1)$ -admissible pair to be chosen later, we define

$$\tilde{Z}_T = \{u \in C((-\infty, -T]; \tilde{Z}), \quad A_j(t)u \in L_T^q L_y^r L_x^2 \cap L_T^\infty L_{xy}^2, \forall j \in \{0, 1, 2, 3\}\}.$$

We have, for all  $(d-1)$ -admissible pairs  $(q_1, r_1)$ , and all  $j \in \{0, 1, 2, 3\}$ ,

$$(4.3) \quad \|A_j \Phi_-(u)\|_{L_T^{q_1} L_y^{r_1} L_x^2} \lesssim \|u_-\|_{\tilde{Z}} + \|u\|_{L_T^\theta L_y^k L_x^\infty}^{2\sigma} \|A_j u\|_{L_T^q L_y^r L_x^2},$$

where  $\theta$  and  $k$  are again given by (3.1). If

$$(4.4) \quad H^{1/2-}(\mathbb{R}_y^{d-1}) \hookrightarrow L^k(\mathbb{R}_y^{d-1}), \quad \text{that is,} \quad 2 \leq k < \frac{2(d-1)}{d-2},$$

we can find  $s$  and  $\gamma$  satisfying (2.1) and  $s + \gamma = 1$ . To obtain explicit time decay, apply Proposition 2.3 to  $v = e^{-i|y|^2/(2t)} u$ . This yields

$$\|u\|_{L_y^k L_x^\infty} = \|v\|_{L_y^k L_x^\infty} \lesssim \|v\|_{L_y^k H_x^s} \lesssim \|v\|_{L_y^2 H_x^s}^{1-\delta} \|v\|_{\dot{H}_y^\gamma H_x^s}^\delta,$$

where  $\delta$  is defined by

$$\delta\gamma = (d-1) \left( \frac{1}{2} - \frac{1}{k} \right).$$

Then, since  $\gamma + s = 1$ , it follows from the Young inequality as in Lemma 2.2 that

$$(4.5) \quad \begin{aligned} \|v\|_{\dot{H}_y^\gamma H_x^s} &= |t|^{-\gamma} \left( \int |t\eta|^{2\gamma} (1 + \xi^2)^s |\widehat{v}(\xi, \eta)|^2 d\xi d\eta \right)^{1/2} \\ &\lesssim |t|^{-\gamma} \left( \int (|t\eta|^2 + (1 + \xi^2)) |\widehat{v}(\xi, \eta)|^2 d\xi d\eta \right)^{1/2} \\ &\lesssim |t|^{-\gamma} \left( \|A_3(t)u\|_{L_x^2 L_y^2} + \|u\|_{L_y^2 H_x^1} \right), \end{aligned}$$

where in the last line, we have used Plancherel formula and

$$A_3(t)u = ite^{i|y|^2/(2t)} \nabla_y e^{-i|y|^2/(2t)} u = ite^{i|y|^2/(2t)} \nabla_y v.$$

Then, we deduce from (4.5) and Lemma 2.2 that for any  $u \in \tilde{Z}_T$  and  $t \leq -T$ , we have

$$(4.6) \quad \|u(t)\|_{L_y^k L_x^\infty} \lesssim \frac{1}{|t|^{(d-1)(\frac{1}{2}-\frac{1}{k})}} \sum_{j=0}^3 \|A_j u\|_{L_T^\infty L_{xy}^2} \lesssim \frac{1}{|t|^{(d-1)(\frac{1}{2}-\frac{1}{k})}} \|u\|_{\tilde{Z}_T}.$$

Then, provided  $t \mapsto |t|^{-(d-1)(1/2-1/k)}$  belongs to  $L^\theta(-\infty, -1)$ , (4.3) and (4.6) imply that for every  $u \in \tilde{Z}_T$ ,

$$(4.7) \quad \|A_j \Phi_-(u)\|_{\tilde{Z}_T} \lesssim \|u_-\|_{\tilde{Z}} + T^{2\sigma(\frac{1}{\theta}-(d-1)(\frac{1}{2}-\frac{1}{k}))} \|u\|_{\tilde{Z}_T}^{2\sigma+1}.$$

Let us now explain how the parameters  $\theta, k, q, r$  are chosen. Since  $\sigma > 1/(d-1)$ , one can choose  $q > 2$  large enough such that

$$(4.8) \quad (d-1)\sigma > \frac{2}{q} + 1.$$

Then,  $r$  is chosen such that  $(q, r)$  is a  $(d-1)$ -admissible pair, in such a way that (4.8) becomes

$$(d-1) \left( \sigma + \frac{1}{r} - \frac{1}{2} \right) > 1,$$

which is equivalent to

$$(d-1) \left( \sigma - \frac{2\sigma}{k} \right) = (d-1) \left( \sigma - 1 + \frac{2}{r} \right) > 1 - (d-1) \left( \frac{1}{2} - \frac{1}{r} \right) = 1 - \frac{2}{q} = \frac{2\sigma}{\theta},$$

where  $\theta$  and  $k$  are defined by (3.1). This is precisely the condition  $\theta(d-1)(\frac{1}{2}-\frac{1}{k}) > 1$  which ensures that the right hand side of (4.6) belongs to  $L^\theta$ . In terms of  $k$ , (4.8) is equivalent to

$$\frac{1}{k} < 1 - \frac{1}{(d-1)\sigma}.$$

This condition is consistent with (4.4) if and only if

$$\frac{d-2}{2(d-1)} < 1 - \frac{1}{(d-1)\sigma},$$

which is equivalent to  $\sigma > \frac{2}{d}$ .

The rest of the proof is similar to the proof of local well-posedness of the Cauchy problem: we take  $R$  and  $T$  sufficiently large so that the ball of radius  $R$  in  $\tilde{Z}_T$  is stable under the action of  $\Phi_-$ , and so that  $\Phi_-$  is a contraction on this ball, equipped with the distance  $\|u\|_{L_T^\infty L_{xy}^2} + \|u\|_{L_T^q L_y^r L_x^2}$ , in view of the previous estimates and

$$\|\Phi_-(u_2) - \Phi_-(u_1)\|_{L_T^{q_1} L_y^{r_1} L_x^2} \lesssim \max_{j=1,2} \|u_j\|_{L_T^\theta L_y^k L_x^\infty}^{2\sigma} \|u_2 - u_1\|_{L_T^q L_y^r L_x^2}.$$

In view of (2.6), the solution that we have constructed satisfies

$$e^{itH} u \in L^\infty((-\infty, -T]; \tilde{Z}).$$

Uniqueness in this class follows from (2.6) and the same approach as for the Cauchy problem. If  $u_1$  and  $u_2$  are two solutions of (1.1) satisfying

$$e^{itH} u_j \in L^\infty((-\infty, -T]; \tilde{Z}), \quad \|e^{itH} u_j(t) - u_-\|_{\tilde{Z}} \xrightarrow[t \rightarrow -\infty]{} 0, \quad j = 1, 2,$$

then for  $\tau > T$ ,

$$\|u_2 - u_1\|_{L_\tau^q L_y^r L_x^2} \lesssim \max_{j=1,2} \|u_j\|_{L_\tau^\theta L_y^k L_x^\infty}^{2\sigma} \|u_2 - u_1\|_{L_\tau^q L_y^r L_x^2},$$

and (4.6) implies

$$\|u_2 - u_1\|_{L_t^q L_y^r L_x^2} \lesssim \tau^{2\sigma(\frac{1}{\theta} - (d-1)(\frac{1}{2} - \frac{1}{k}))} \|u_2 - u_1\|_{L_t^q L_y^r L_x^2}.$$

Choosing  $\tau$  sufficiently large, we have  $u_2 = u_1$  for  $t \leq -\tau$ , and Theorem 1.3 yields  $u_2 \equiv u_1$ .

## 5. ASYMPTOTIC COMPLETENESS

In this section, we prove Theorem 1.5. Three approaches are available to prove asymptotic completeness for nonlinear Schrödinger equations (without potential). The initial approach ([8]) consists in working with a  $\Sigma$  regularity. This makes it possible to use the operator  $x + it\nabla$ , whose main properties are essentially those stated in Lemma 2.6, and to which an important evolution law (the pseudo-conformal conservation law) is associated. This law provides important a priori estimates, from which asymptotic completeness follows very easily in the case  $\sigma \geq 2/d$ , and less easily for some range of  $\sigma$  below  $2/d$ ; see e.g. [4]. Unfortunately, this conservation law seems to be bound to isotropic frameworks: an analogous identity is available in the presence on an isotropic quadratic potential ([3]), but in our present framework, anisotropy seems to rule out a similar algebraic miracle.

The second historical approach relaxes the localization assumption on the data, and allows to work in  $H^1(\mathbb{R}^d)$ , provided that  $\sigma > 2/d$ . It is based on Morawetz inequalities: asymptotic completeness is then established in [13, 9] for the case  $d \geq 3$ , and in [15] for the low dimension cases  $d = 1, 2$ , by introducing more intricate Morawetz estimates.

The most recent approach to prove asymptotic completeness in  $H^1$  relies on the introduction of interaction Morawetz estimates in [6], an approach which has been revisited since, in particular in [16] and [10]. In the anisotropic case, interaction Morawetz have been used in [1] and [19] with two different angles: in both cases, it starts with the choice of an anisotropic weight in the virial computation from [10, 16], but the interpretations of this computation are then different. We start by presenting a unified statement of this approach in the next paragraph.

**5.1. Morawetz estimates.** For  $(x, y) \in \mathbb{R}^d$  and  $\mu > 0$ , we denote by  $Q(x, y, \mu)$  a dilation of the unit cube centered in  $(x, y)$ ,

$$Q(x, y, \mu) = (x, y) + [-\mu, \mu]^d.$$

**Proposition 5.1.** *Let  $u \in C(\mathbb{R}; Z)$  be as in Theorem 1.3. For every  $\mu > 0$ , there exists  $C_\mu > 0$  such that*

$$\begin{aligned} \left\| |\nabla_y|^{\frac{4-d}{2}} R \right\|_{L_{ty}^2(\mathbb{R} \times \mathbb{R}^{d-1})}^2 + \int_{\mathbb{R}} \left( \sup_{(x_0, y_0) \in \mathbb{R}^d} \iint_{Q(x_0, y_0, \mu)} |u(t, x, y)|^2 dx dy \right)^{\sigma+2} dt \\ \leq C_\mu \sup_{t \in \mathbb{R}} \|u(t)\|_{H_{xy}^1}^4 \lesssim \|u_0\|_Z^4, \end{aligned}$$

where

$$R(t, y) = \int_{-\infty}^{+\infty} |u(t, x, y)|^2 dx$$

is the marginal of the mass density.

*Proof.* We resume the computations from [1, Section 5], and simply recall the main steps.

To shorten the notations, we set  $z = (x, y)$ . Following [10], we write that if  $u$  is a solution to (1.1), then we have

$$(5.1) \quad \begin{cases} \partial_t \rho + \operatorname{div} J = 0 \\ \partial_t J + \operatorname{div} (\operatorname{Re}(\nabla \bar{u} \otimes \nabla u)) + \frac{\sigma}{\sigma+1} \nabla \rho^{\sigma+1} + \rho \nabla V = \frac{1}{4} \nabla \Delta \rho, \end{cases}$$

where  $\rho(t, z) := |u(t, z)|^2$  and  $J(t, z) := \operatorname{Im}(\bar{u} \nabla u)(t, z)$ . Let us define the virial potential

$$I(t) := \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \rho(t, z) a(z - z') \rho(t, z') dz dz' = \frac{1}{2} \langle \rho, a * \rho \rangle,$$

where  $a$  is a sufficiently smooth even weight function which will be eventually a function of  $y$  only. Here  $\langle \cdot, \cdot \rangle$  denotes the scalar product in  $L^2(\mathbb{R}^d)$ . By using (5.1), we see that the time derivative of  $I(t)$  reads

$$(5.2) \quad \frac{d}{dt} I(t) = -\langle \rho, \nabla a * J \rangle = \iint \rho(t, z') \nabla a(z - z') \cdot J(t, z) dz' dz =: M(t),$$

where  $M(t)$  is the Morawetz action. By using again the balance laws (5.1) we have

$$(5.3) \quad \begin{aligned} \frac{d}{dt} M(t) &= -\langle J, \nabla^2 a * J \rangle + \langle \rho, \nabla^2 a * \operatorname{Re}(\nabla \bar{u} \otimes \nabla u) \rangle + \frac{\sigma}{\sigma+1} \langle \rho, \Delta a * \rho^{\sigma+1} \rangle \\ &\quad - \langle \rho, \nabla a * (\rho \nabla V) \rangle - \frac{1}{4} \langle \rho, \Delta a * \Delta \rho \rangle \\ &= -\langle \operatorname{Im}(\bar{u} \nabla u), \nabla^2 a * \operatorname{Im}(\bar{u} \nabla u) \rangle + \langle \rho, \nabla^2 a * (\nabla \bar{u} \otimes \nabla u) \rangle \\ &\quad + \frac{\sigma}{\sigma+1} \langle \rho, \Delta a * \rho^{\sigma+1} \rangle - \langle \rho, \nabla a * (\rho \nabla V) \rangle - \frac{1}{4} \langle \rho, \Delta a * \Delta \rho \rangle, \end{aligned}$$

where in the second term we dropped the real part because of the symmetry of  $\nabla^2 a$  (here, the notation  $\nabla^2 a * \operatorname{Re}(\nabla \bar{u} \otimes \nabla u)$  stands for  $\sum_{j,k} \partial_{jk}^2 a * \operatorname{Re}(\partial_k \bar{u} \partial_j u)$ ). Leaving out the details presented in [1] and [19], the computation shows that if  $\nabla^2 a$  is non-negative and if  $a$  depends on  $y$  only (so we have  $\nabla a(z_1) \cdot \nabla V(z_2) = 0$  for all  $z_1, z_2 \in \mathbb{R}^d$ ), then we have:

$$(5.4) \quad \frac{d}{dt} M(t) \geq \frac{1}{2} \langle \nabla_y \rho, \Delta_y a * \nabla_y \rho \rangle + \frac{\sigma}{\sigma+1} \langle \rho, \Delta_y a * \rho^{\sigma+1} \rangle.$$

Now we consider two choices for the weight  $a$ . First, for  $a(y) = |y|$ , we have indeed  $\nabla^2 a \geq 0$  as a symmetric matrix, and for  $d \geq 3$ ,  $\Delta_y a(y) = \frac{d-2}{|y|}$ : it is, up to a multiplicative constant, the integral kernel of the operator  $(-\Delta_y)^{-\frac{d-2}{2}}$ , that is,

$$\left( (-\Delta_y)^{-\frac{d-2}{2}} f \right) (y) = \int_{\mathbb{R}^{d-1}} \frac{c}{|y - y'|} f(y') dy'.$$

Thus, by recalling  $z = (x, y)$ , we obtain

$$\begin{aligned} &\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{|y - y'|} \nabla_y \rho(t, z') \cdot \nabla_y \rho(t, z) dz' dz \\ &= \iiint_{\mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d-1}} \nabla_y \rho(t, x, y) \cdot \nabla_y (-\Delta_y)^{-\frac{d-2}{2}} \rho(t, x', y) dx dx' dy. \end{aligned}$$

Hence, if we define the marginal of the mass density

$$R(t, y) := \int_{\mathbb{R}} \rho(t, x, y) dx,$$

the last integral also reads

$$\int_{\mathbb{R}^{d-1}} \left| |\nabla_y|^{\frac{4-d}{2}} R(t, y) \right|^2 dy.$$

We now plug this expression into (5.4) and we integrate in time. Furthermore, the second term in the right hand side in (5.4) is positive. We then infer

$$(5.5) \quad \int_{-T}^T \int_{\mathbb{R}^{d-1}} \left| |\nabla_y|^{\frac{4-d}{2}} R(t, y) \right|^2 dy dt \leq C \sup_{t \in [-T, T]} |M(t)|.$$

Furthermore, with our choice of the weight  $a$ , we have

$$|M(t)| = \left| \iint \rho(t, z') \frac{y - y'}{|y - y'|} \cdot \text{Im}(\bar{u} \nabla_y u)(t, z) dz' dz \right| \leq \|u_0\|_{L^2(\mathbb{R}^d)}^3 \|\nabla_y u(t)\|_{L^2(\mathbb{R}^d)},$$

hence the first part of Proposition 5.1 in the case  $d \geq 3$ . In the case  $d = 2$ , the choice  $a(y) = |y|$  leads to  $a''(y) = 2\delta_0$ , and the conclusion remains the same.

Now, as in [19], consider the weight  $a(y) = \langle y \rangle$ : we still have  $\nabla^2 a \geq 0$ . Resume (5.3): the computations from [19, 16] yield a rearrangement of the terms so that instead of (5.4), we now have

$$\frac{d}{dt} M(t) \geq \frac{\sigma}{\sigma + 1} \langle \rho, \Delta_y a * \rho^{\sigma+1} \rangle.$$

The right hand side is equal to

$$\frac{\sigma}{\sigma + 1} \iint \iint |u(t, x_1, y_1)|^2 \Delta a(y_1 - y_2) |u(t, x_2, y_2)|^{2\sigma+2} dx_1 dy_1 dx_2 dy_2.$$

Following [19], we note that

$$\inf_{Q(0,0,2\mu)} \Delta_y \langle \langle y \rangle \rangle > 0,$$

so the above term is bounded from below by constant times

$$\sup_{(x_0, y_0) \in \mathbb{R}^d} \iint_{Q(x_0, y_0, \mu)} \iint_{Q(x_0, y_0, \mu)} |u(t, x_1, y_1)|^2 |u(t, x_2, y_2)|^{2\sigma+2} dx_1 dy_1 dx_2 dy_2.$$

Hölder inequality yields

$$\iint_{Q(x_0, y_0, \mu)} |u(t, x_2, y_2)|^{2\sigma+2} dx_2 dy_2 \gtrsim \left( \iint_{Q(x_0, y_0, \mu)} |u(t, x_2, y_2)|^2 dx_2 dy_2 \right)^{\sigma+1}.$$

Finally, with this second choice for  $a$ , we still have

$$|M(t)| \leq \|u_0\|_{L^2_{xy}}^3 \|\nabla_y u(t)\|_{L^2_{xy}},$$

hence the result by integrating in time.  $\square$

**5.2. End of the argument.** To prove Theorem 1.5 in the case  $d \leq 4$ , one can resume the approach followed in [1, Section 6] which is readily adapted to our framework, the only difference being that the function space and the related set of vectorfields are not the same here.

However, as pointed out in [19], the fact that negative order derivatives are involved in the first term in Proposition 5.1 makes it delicate to use this term when  $d \geq 5$ , and requires fine harmonic analysis estimates in the case  $V = 0$ ; it is not clear whether or not these tools can be adapted to the present setting. This is why the second term in Proposition 5.1, which corresponds to the one considered in [19], is more efficient then, and allows to prove Theorem 1.5 for all  $d \geq 2$ .

The first step stems from [21]: Theorem 1.3 and Proposition 5.1 imply that

$$\|u(t)\|_{L^r_{xy}} \xrightarrow{t \rightarrow +\infty} 0, \quad \forall 2 < r < \frac{2d}{(d-2)_+}.$$

The end of the proof is presented in [19], and is readily adapted to our framework: it consists in choosing suitable Lebesgue exponents and applying inhomogeneous Strichartz estimates for non-admissible pairs, which follow in our case from [1, 7]. Since the proof is then absolutely the same as in [19], we choose not to reproduce it here.

## REFERENCES

- [1] P. ANTONELLI, R. CARLES, AND J. DRUMOND SILVA, *Scattering for nonlinear Schrödinger equation under partial harmonic confinement*, Comm. Math. Phys., 334 (2015), pp. 367–396.
- [2] N. BEN ABDALLAH, F. CASTELLA, AND F. MÉHATS, *Time averaging for the strongly confined nonlinear Schrödinger equation, using almost periodicity*, J. Differential Equations, 245 (2008), pp. 154–200.
- [3] R. CARLES, *Nonlinear Schrödinger equations with repulsive harmonic potential and applications*, SIAM J. Math. Anal., 35 (2003), pp. 823–843.
- [4] T. CAZENAVE, *Semilinear Schrödinger equations*, vol. 10 of Courant Lecture Notes in Mathematics, New York University Courant Institute of Mathematical Sciences, New York, 2003.
- [5] T. CAZENAVE AND F. WEISSLER, *Rapidly decaying solutions of the nonlinear Schrödinger equation*, Comm. Math. Phys., 147 (1992), pp. 75–100.
- [6] J. COLLIANDER, M. KEEL, G. STAFFILANI, H. TAKAOKA, AND T. TAO, *Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in  $\mathbb{R}^3$* , Ann. of Math. (2), 167 (2008), pp. 767–865.
- [7] D. FOSCHI, *Inhomogeneous Strichartz estimates*, J. Hyperbolic Differ. Equ., 2 (2005), pp. 1–24.
- [8] J. GINIBRE AND G. VELO, *On a class of nonlinear Schrödinger equations. II Scattering theory, general case*, J. Funct. Anal., 32 (1979), pp. 33–71.
- [9] ———, *Scattering theory in the energy space for a class of nonlinear Schrödinger equations*, J. Math. Pures Appl. (9), 64 (1985), pp. 363–401.
- [10] ———, *Quadratic Morawetz inequalities and asymptotic completeness in the energy space for nonlinear Schrödinger and Hartree equations*, Quart. Appl. Math., 68 (2010), pp. 113–134.
- [11] Z. HANI, B. PAUSADER, N. TZVETKOV, AND N. VISCIGLIA, *Modified scattering for the cubic Schrödinger equation on product spaces and applications*. preprint, archived at <http://arxiv.org/abs/1311.2275>.
- [12] Z. HANI AND L. THOMANN, *Asymptotic behavior of the nonlinear Schrödinger equation with harmonic trapping*. preprint. Archived at <http://arxiv.org/abs/1408.6213>.
- [13] J. E. LIN AND W. A. STRAUSS, *Decay and scattering of solutions of a nonlinear Schrödinger equation*, J. Funct. Anal., 30 (1978), pp. 245–263.
- [14] H. MIZUTANI, *Strichartz estimates for Schrödinger equations with variable coefficients and unbounded potentials II. Superquadratic potentials*, Commun. Pure Appl. Anal., 13 (2014), pp. 2177–2210.
- [15] K. NAKANISHI, *Energy scattering for nonlinear Klein-Gordon and Schrödinger equations in spatial dimensions 1 and 2*, J. Funct. Anal., 169 (1999), pp. 201–225.
- [16] F. PLANCHON AND L. VEGA, *Bilinear virial identities and applications*, Ann. Sci. Éc. Norm. Supér. (4), 42 (2009), pp. 261–290.
- [17] M. REED AND B. SIMON, *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*, Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1975.
- [18] N. TZVETKOV AND N. VISCIGLIA, *Small data scattering for the nonlinear Schrödinger equation on product spaces*, Comm. Partial Differential Equations, 37 (2012), pp. 125–135.
- [19] ———, *Well-posedness and scattering for NLS on  $\mathbb{R}^d \times \mathbb{T}$  in the energy space*. preprint. Archived at <http://arxiv.org/abs/1409.3938>, 2014.
- [20] M. C. VILELA, *Inhomogeneous Strichartz estimates for the Schrödinger equation*, Trans. Amer. Math. Soc., 359 (2007), pp. 2123–2136 (electronic).
- [21] N. VISCIGLIA, *On the decay of solutions to a class of defocusing NLS*, Math. Res. Lett., 16 (2009), pp. 919–926.
- [22] K. YAJIMA AND G. ZHANG, *Smoothing property for Schrödinger equations with potential superquadratic at infinity*, Comm. Math. Phys., 221 (2001), pp. 573–590.

- [23] ———, *Local smoothing property and Strichartz inequality for Schrödinger equations with potentials superquadratic at infinity*, J. Differential Equations, 202 (2004), pp. 81–110.

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