

Higher rank lattices are not coarse median

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ABSTRACT. We show that symmetric spaces and thick affine buildings which are not of spherical type A_1^r have no coarse median in the sense of Bowditch. As a consequence, they are not quasi-isometric to a CAT(0) cube complex, answering a question of Haglund. Another consequence is that any lattice in a simple higher rank group over a local field is not coarse median.

Introduction

A metric space (X, d) is called metric median if for each $(x, y, z) \in X^3$, the three intervals $I(x, y)$, $I(y, z)$ and $I(x, z)$ intersect in a single point, where $I(x, y) = \{p \in X \mid d(x, p) + d(p, y) = d(x, y)\}$. This point is called the median of x , y and z . The rank of (X, d) is then defined as the maximal dimension r of an embedded cube $\{0, 1\}^r$. The relationship between groups and median metric spaces is rich and has been studied through many points of view: Haagerup property, property (T), action on a CAT(0) cube complex, action on a space with (measured) walls... (see [Che00], [CDH10], [CFI13], [CN05], [Bow13b], [Bow14]...).

Bowditch recently introduced the notion of a coarse median on a metric space (see [Bow13a]), in order to gather in the same setting CAT(0) cube complexes and Gromov hyperbolic spaces. A metric space is Gromov-hyperbolic if and only if every finite subset admits a good metric comparison with a tree (see for instance [GdlH90, Theorem 12, p. 33]). Bowditch's definition of a coarse median is having a good metric comparison of every finite subset with a metric median space, or equivalently with a CAT(0) cube complex according to Chepoi (see [Che00]).

Definition (Bowditch). Let (X, d) be a metric space. A map $\mu : X^3 \rightarrow X$ is called a *coarse median* if there exist $k \in [0, +\infty)$ and $h : \mathbb{N} \rightarrow [0, +\infty)$ such that

- $\forall a, b, c, a', b', c' \in X, d(\mu(a, b, c), \mu(a', b', c')) \leq k(d(a, a') + d(b, b') + d(c, c')) + h(0)$.
- For each finite non-empty set $A \subset X$, with $|A| \leq p$, there exists a finite median algebra (Π, μ_Π) and maps $\pi : A \rightarrow \Pi$, $\lambda : \Pi \rightarrow X$ such that

$$\begin{aligned} \forall x, y, z \in X, d(\lambda\mu_\Pi(x, y, z), \mu(\lambda x, \lambda y, \lambda z)) &\leq h(p) \\ \text{and } \forall a \in A, d(a, \lambda(\pi(a))) &\leq h(p). \end{aligned}$$

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If Π can be chosen (independently of p) to have rank at most r , we say that μ has rank at most r .

A finitely generated group is said to be coarse median if some Cayley graph has a coarse median (not necessarily equivariant under the group action). Bowditch showed (see [Bow13a]) that if a group is coarse median, then its Dehn function is at most quadratic and it has property (RD). Furthermore, if a group has a coarse median of rank at most r , there is no quasi-isometric embedding of \mathbb{R}^{r+1} into that group. Bowditch also showed that the existence of a coarse median is a quasi-isometry invariant, that a group is Gromov hyperbolic if and only if it is coarse median of rank 1, and that a group hyperbolic relative to coarse median groups is itself coarse median (see [Bow13c]). Furthermore, Bowditch showed that the mapping class group of a surface of genus g with p punctures is coarse median of rank $3g - 3 + p$, hereby recovering Behrstock and Minsky's result that the mapping class group has property (RD) (see [BM11]), and the rank theorem (see [Ham07] and [BM08]).

Since most known examples of coarse median groups have some nonnegative curvature features, Bowditch asked in [Bow13a] whether higher rank symmetric space, or even CAT(0) spaces, admit coarse medians. In this article, we give a negative answer to this question, by showing the following:

Theorem A. Let X be a symmetric space of non-compact type, or a thick affine building. There exists a coarse median on X if and only if the spherical type of X is A_1^r .

Note that the coarse median is not assumed to be equivariant by any group.

Haglund asked if a higher rank symmetric space or affine building is quasi-isometric to a CAT(0) cube complex, and we give a negative answer:

Corollary B. Let X be a symmetric space of non-compact type, or a discrete, thick affine building. Then X is quasi-isometric to a CAT(0) cube complex if and only if the spherical type of X is A_1^r .

Note that the CAT(0) cube complex is not assumed to be of finite dimension, and it could also be endowed with the L^p distance for any $p \in [1, \infty]$.

Also note that Corollary B still holds if we consider non-discrete thick affine buildings and non-discrete CAT(0) cube complexes.

Furthermore, for uniform lattices in semisimple Lie groups, Property (RD) implies the Baum-Connes conjecture without coefficient (see [Laf98]). Property (RD) has been proved notably for uniform lattices in $SL(3, \mathbb{K})$, where \mathbb{K} is a local field (see [RRS98], [Laf00] and [Cha03]). Valette conjectured that uniform lattices in semisimple Lie groups satisfy property (RD). Since being coarse median implies property (RD), one could ask if this could be a way to prove property (RD) for higher rank uniform lattices, and the answer is negative:

Corollary C. Let \mathbb{K} be a local field, let G be the group of \mathbb{K} -points of a simple algebraic group without compact factors, and let Γ be a lattice in G . Then Γ is coarse median if and only if G has \mathbb{K} -rank 1.

Note that, due to property (T), higher rank lattices do not admit unbounded actions on median metric spaces (see [CDH10]). But in the coarse median setting this is not a

consequence of property (T), since for instance every hyperbolic group with property (T) is coarse median.

Organization of the paper: We will prove the main part of Theorem A by assuming the existence of a coarse median on a symmetric space or affine building X . In Section 1, we will recall the general definitions of median algebras. In Section 2, we consider the asymptotic cone of X , and the ultralimit of the coarse median. In Section 3 we prove the existence of a convex cube. In Section 4, by considering the gate projection, we obtain a contradiction. In Section 5, we prove Corollaries B and C.

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1 Median algebras

Definition 1.1. Let X be a set. A symmetric map $\mu : X^3 \rightarrow X$ is called a *median* on X if:

$$\forall a, b, c, d, e \in X, \mu(a, b, \mu(c, d, e)) = \mu(\mu(a, b, c), \mu(a, b, d), e). \quad (1)$$

The pair (X, μ) is called a *median algebra*.

Remark.

- There is a unique median on the set $\{0, 1\}$.
- We can consider the product median on the n -cube $\{0, 1\}^n$.

Definition 1.2. Let $(X, \mu), (X', \mu')$ be median algebras. A map $f : X \rightarrow X'$ is called *median* if $\forall x, y, z \in X, \mu'(f(x), f(y), f(z)) = f(\mu(x, y, z))$. If furthermore f is injective, it is called a *median embedding*.

Definition 1.3. Let (X, μ) be a median algebra. If every median embedding of an n -cube $\{0, 1\}^n \rightarrow X$ satisfies $n \leq r$, we say that X has *rank* at most r .

Definition 1.4. Let (X, μ) be a median algebra. If $a, b \in X$, the *interval* between a and b is $I(a, b) = \{c \in X \mid \mu(a, b, c) = c\}$. A subset $C \subset X$ is called *convex* if $\forall a, b \in C, I(a, b) \subset C$.

If (X, d) is a metric space, a weakening of the notion of metric median is the following.

Definition 1.5. Let (X, d) be a metric space. An abstract median μ on X is called

- *continuous* if $\mu : X^3 \rightarrow X$ is a continuous map,
- *Lipschitz* if there exists a constant $k \in [0, +\infty)$ such that μ is k -Lipschitz with respect to each variable, i.e.

$$\forall a, b, c, a', b', c' \in X, d(\mu(a, b, c), \mu(a', b', c')) \leq k(d(a, a') + d(b, b') + d(c, c')),$$

- *locally convex* if each point of X has a basis of neighbourhoods consisting of convex subsets of X .

Definition 1.6. Let (X, d) be a metric space, let μ be a continuous median on X , and let $C \subset X$ be a non-empty closed, locally compact convex subset of X . Then for each $x \in X$, there exists a unique $\pi_C(x) \in C$, called the *gate projection* of x onto C , such that $\forall y \in C, \pi_C(x) \in I(x, y)$. The map $\pi_C : X \rightarrow C$ is called the gate projection, it is a continuous map. If μ is k -Lipschitz, then π_C is a k -Lipschitz map.

Now we recall the definition of walls in a median algebra.

Definition 1.7. Let (X, μ) be a median algebra. A *wall* in X is a pair $W = \{H^+(W), H^-(W)\}$, where $H^+(W)$ and $H^-(W)$ are non-empty convex disjoint subsets of X whose union is equal to X .

Lemma 1.8. [Bow13a, Lemma 6.1] Let (X, μ) be a median algebra, and let A, B be disjoint convex subsets of M . There exists a wall $W = \{H^+(W), H^-(W)\}$ in X separating A from B , i.e. such that $A \subset H^\pm(W)$ and $B \subset H^\mp(W)$.

Lemma 1.9. [Bow13a, Lemma 7.3] Let (X, d) be a metric space, and let μ be a continuous locally convex median on X . Let a, b be distinct points of X . There exists a wall $W = \{H^+(W), H^-(W)\}$ in X strongly separating a from b , i.e. such that $a \in X \setminus \overline{H^-(W)}$ and $b \in X \setminus \overline{H^+(W)}$.

Lemma 1.10. [Bow13a, Lemma 7.5] Let X be a metric space, and let μ be a continuous locally convex median on X . For each wall $W = \{H^+(W), H^-(W)\}$ in X , the subset $L(W) = \overline{H^+(W)} \cap \overline{H^-(W)}$ is a convex median subalgebra of X , of rank at most $r - 1$ if the rank of μ is r .

2 Ultralimits of spaces and coarse medians

In [KL97], Kleiner and Leeb developed a geometric definition of spherical and affine buildings, and in particular they studied their asymptotic cones.

Theorem 2.1. [KL97, Theorem 1.2.1] Let X be a symmetric space of non-compact type or a thick affine building. Then any asymptotic cone of X is a thick affine building of the same spherical type as X .

They also proved that any tangent cone of an affine building is an affine building:

Theorem 2.2. [KL97, Theorem 5.1.1] Let (X, d) be an affine building, let ω be a non-principal ultrafilter on \mathbb{N} , let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X and let $(\lambda_n)_{n \in \mathbb{N}}$ be a sequence in $(0, +\infty)$ such that $\lim_{\omega} \lambda_n = +\infty$. Let $(X_\infty, d_\infty, x_\infty)$ be the ω -ultralimit of $(X_n, \lambda_n d, x_n)$. Then (X_∞, d_∞) is an affine building. Furthermore, if X is thick, then X_∞ is thick. The affine Weyl group of X_∞ acts transitively on each apartment of X_∞ .

One motivation for Bowditch's definition of coarse median is that it behaves well when one considers asymptotic cones.

Theorem 2.3. [Bow13a, Theorem 2.3] Let (X, d) be a metric space, and let μ be a (k, h) -coarse median on X . Then on any asymptotic cone (X_∞, d_∞) of (X, d) , μ defines a k -Lipschitz, locally convex median μ_∞ .

So Theorem A is a consequence of the following.

Theorem 2.4. *Let X be a thick affine building of spherical type different from A_1^r . There is no locally convex Lipschitz median on X .*

Proof. [Theorem 2.4 implies Theorem A] Assume first that the spherical type of the symmetric space or affine building X is A_1^r . If X is a symmetric space, it is a product of rank 1 symmetric spaces, which are Gromov hyperbolic, so X has a coarse median. If X is an affine building, if we endow it with the L^1 metric it becomes a metric median space. In particular, this median is a coarse median with respect to any usual metric on X , which is equivalent to the L^1 metric.

Conversely, if the spherical type of the symmetric space or affine building X is not A_1^r , by Theorem 2.1 any asymptotic cone X_∞ is a thick affine building. If there existed a coarse median μ on X , it would give rise by Theorem 2.3 to a locally convex Lipschitz median on X_∞ , which contradicts Theorem 2.4 since the spherical type of X_∞ is not A_1^r . Hence there is no coarse median on X . \square

3 Existence of a convex cube

In this Section, we will show the following result.

Proposition 3.1. *Let X be a connected metric space, with a Lipschitz locally convex median of rank r . There exists a median, biLipschitz embedding of the r -cube $[0, 1]^r$ into X with convex image.*

Fix X a geodesic metric space, and $\mu : X^3 \rightarrow X$ a Lipschitz median.

Definition 3.2. A continuous path $p : I \rightarrow X$, where $I \subset \mathbb{R}$ is an interval, is said to be *monotone* if for each $t_1 < t_2 < t_3$ in I , we have $\mu(p(t_1), p(t_2), p(t_3)) = p(t_2)$.

To prove Proposition 3.1, we will need the following two Lemmas.

Lemma 3.3. *Let X be a connected metric space, with a continuous locally convex median μ , and let $f : \{0, 1\}^r \rightarrow X$ be a median embedding of the r -cube, and let W be a wall in X strongly separating $f(0, \dots, 0)$ and $f(1, 0, \dots, 0)$. There exists a median embedding $g : \{0, 1\}^r \rightarrow X$ such that $\forall t \in \{0\} \times \{0, 1\}^{r-1}, g(t) = f(t)$ and $\forall t \in \{1\} \times \{0, 1\}^{r-1}, g(t) \in L(W)$.*

Proof. Note that if we knew that $L(W)$ was locally compact, projecting the half-cube $\{1\} \times \{0, 1\}^{r-1}$ using the gate projection onto $L(W)$ would immediately give the result.

Since intervals are connected, we can consider $a \in I(f(0, \dots, 0), f(1, 0, \dots, 0)) \cap L(W)$. Define

$$\begin{aligned} g : \{0, 1\}^r &\rightarrow X \\ t \in \{0\} \times \{0, 1\}^{r-1} &\mapsto f(t) \\ t \in \{1\} \times \{0, 1\}^{r-1} &\mapsto \mu(f(0, t_2, \dots, t_r), a, f(t)). \end{aligned}$$

Since $L(W)$ is convex, we deduce that $\forall t \in \{1\} \times \{0, 1\}^{r-1}, g(t) \in L(W)$.

Using repeatedly Property (1), we prove that g is a median map. As a consequence, if for some $t, t' \in \{0, 1\}^r$ we have $g(t) = g(t')$, then

$$\begin{aligned} f(0, t_2, \dots, t_r) &= \mu(g(0, t_2, \dots, t_r), g(t), g(0, t'_2, \dots, t'_r)) \\ &= \mu(g(0, t_2, \dots, t_r), g(t'), g(0, t'_2, \dots, t'_r)) \\ &= f(0, t'_2, \dots, t'_r), \end{aligned}$$

hence $(0, t_2, \dots, t_r) = (0, t'_2, \dots, t'_r)$, so $t = t'$: g is injective. \square

Lemma 3.4. *Let (X, μ) be a median algebra. Assume there exists a median embedding of the r -cube $f : [0, 1]^r \rightarrow X$, such that the image by f of any edge of $[0, 1]^r$ is convex. Then the image of f is convex in X .*

Proof. For each $k \in \llbracket 1, r \rrbracket$, let $e_k = (0, \dots, 0, 1, 0, \dots, 0)$ (the 1 is in the k^{th} position). Let $x \in I(f(0), f(e_1 + \dots + e_r))$. For each $k \in \llbracket 1, r \rrbracket$, since the image by f of the edge $[0, e_k]$ is convex, we deduce that $I(f(0), f(e_k)) = f([0, e_k])$, so there exists $t_k \in [0, 1]$ such that $\mu(f(0), \dots, 0, x, f(e_k)) = f(t_k e_k)$. We will show by induction on $k \in \llbracket 0, r \rrbracket$ that $f(t_1 e_1 + \dots + t_k e_k) = \mu(f(0), x, f(e_1 + \dots + e_k))$.

For $k = 0$ this is immediate using Property (1), so assume that for some $k < r$ we have $f(t_1 e_1 + \dots + t_k e_k) = \mu(f(0), x, f(e_1 + \dots + e_k))$. Then

$$\begin{aligned} f(t_1 e_1 + \dots + t_{k+1} e_{k+1}) &= \mu(f(t_1 e_1 + \dots + t_k e_k), f(t_{k+1} e_{k+1}), f(e_1 + \dots + e_{k+1})) \\ &= \mu(\mu(f(0), x, f(e_1 + \dots + e_k)), \mu(f(0), x, f(e_{k+1})), f(e_1 + \dots + e_{k+1})) \\ &= \mu(f(0), x, \mu(f(e_1 + \dots + e_k), f(e_{k+1}), f(e_1 + \dots + e_{k+1}))) \\ &= \mu(f(0), x, f(e_1 + \dots + e_{k+1})). \end{aligned}$$

As a consequence, for $k = r$ we deduce that

$$f(t_1 e_1 + \dots + t_r e_r) = \mu(f(0), x, f(e_1 + \dots + e_r)) = x,$$

as $x \in I(f(0), f(e_1 + \dots + e_r))$. So we have proved that the image of f is equal to the interval $I(f(0), f(e_1 + \dots + e_r))$, which is convex. \square

We can now prove Proposition 3.1.

Proof. [Proof of Proposition 3.1] Since the rank of the median μ is r , consider a median embedding $f : \{0, 1\}^r \rightarrow X$. Applying $2r$ times Lemma 3.3, up to replacing f by another median embedding of $\{0, 1\}^r$ into X , we can assume that for each $i \in \llbracket 1, r \rrbracket$ and $\varepsilon \in \{0, 1\}$, the image under f of the codimension 1 face $\{0, 1\}^{i-1} \times \{\varepsilon\} \times \{0, 1\}^{r-1-i}$ is included in a closed convex subspace $L(W_{i,\varepsilon})$ of X , where $W_{i,\varepsilon}$ is a wall of X . According to Lemma 1.10, each $L(W_{i,\varepsilon})$ has rank at most $r-1$, and since it contains the image by f of the $(r-1)$ -cube $\{0, 1\}^{i-1} \times \{\varepsilon\} \times \{0, 1\}^{r-1-i}$, we deduce that each $L(W_{i,\varepsilon})$ has rank $r-1$.

For $i, j \in \llbracket 1, r \rrbracket$ distinct and $\varepsilon, \varepsilon' \in \{0, 1\}$, since $L(W_{i,\varepsilon}) \cap L(W_{j,\varepsilon'}) = L(L(W_{i,\varepsilon}) \cap W_{j,\varepsilon'})$, where $L(W_{i,\varepsilon}) \cap W_{j,\varepsilon'}$ is a wall in the rank $r-1$ median algebra $L(W_{i,\varepsilon})$, we deduce by Lemma 1.10 that $L(W_{i,\varepsilon}) \cap L(W_{j,\varepsilon'})$ has rank $r-2$.

By induction, we prove that for each $p \in \llbracket 1, r \rrbracket$, for each distinct $i_1, \dots, i_p \in \llbracket 1, r \rrbracket$ and each $\varepsilon_1, \dots, \varepsilon_p \in \{0, 1\}$, the intersection $\bigcap_{1 \leq k \leq p} L(W_{i_k, \varepsilon_k})$ has rank $r-p$.

For each $k \in \llbracket 1, r \rrbracket$, let $e_k = (0, \dots, 0, 1, 0, \dots, 0)$ (the 1 is in the k^{th} position). Hence for each $k \in \llbracket 1, r \rrbracket$, the points $f(0)$ and $f(e_k)$ are contained in a convex rank 1 closed subspace. In particular, there exists an injective, monotone, biLipschitz path p_k from $f(0)$ to $f(e_k)$, with convex image.

We will show by induction on $k \in \llbracket 0, r \rrbracket$ that we can extend f to a biLipschitz median embedding from $[0, 1]^k \times \{0, 1\}^{r-k}$ into X . The case $k = 0$ is already true.

Assume we have extended f to a biLipschitz median embedding $f : [0, 1]^k \times \{0, 1\}^{r-k} \rightarrow X$ for some $k < r$. Define

$$\begin{aligned} f : [0, 1]^k \times [0, 1] \times \{0, 1\}^{r-k-1} &\rightarrow X \\ (t, u, v) \in [0, 1]^k \times [0, 1] \times \{0, 1\}^{r-k-1} &\mapsto \mu(f(t, 0, v), p_{k+1}(u), f(1, \dots, 1)). \end{aligned}$$

See Figure 1. Since p_{k+1} and μ are biLipschitz, we deduce that f is biLipschitz on $[0, 1]^k \times [0, 1] \times \{0, 1\}^{r-k-1}$.

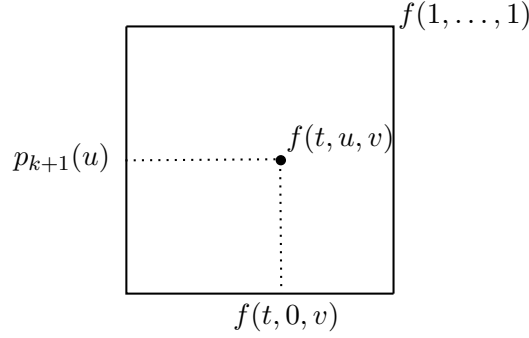


Figure 1: Extending f

So f is extended to a biLipschitz map $f : [0, 1]^r \rightarrow X$. If $t \in [0, 1]^r$ and $k \in \llbracket 1, r \rrbracket$, notice that in $[0, 1]^r$ the median of $(t, 0, e_k)$ is equal to $t_k e_k$. So $\mu(f(t), f(0), f(e_k)) = f(t_k e_k) = p_k(t_k)$. Since each path p_1, \dots, p_r is injective, we deduce that f itself is injective.

By using several times Property (1), we prove that f preserves the medians. Hence f is a median embedding, and by Lemma 3.4 the image of f is convex. \square

Let us recall Bowditch's definition of the separation dimension of a space, which is a good notion of dimension when working with medians on a metric space.

Definition 3.5 (Bowditch). If X is a Hausdorff topological space, define the *separation dimension* of X inductively as follows:

- If $X = \emptyset$, then the separation dimension of X is -1 .
- X has separation dimension at most $n \in \mathbb{N}$ if for any distinct $x, y \in X$, there exist closed subsets A, B of X such that $x \notin B$, $y \notin A$, $X = A \cup B$ and $A \cap B$ has dimension at most $n - 1$.

Remark. If X is a Hausdorff metric space, then the inductive dimension of X is at most equal to the separation dimension. Conversely, we have the following.

Lemma 3.6. [HW41, Section III.6] *If X is a locally compact Hausdorff metric space, then the inductive dimension of X equals the separation dimension.*

The following Lemma is immediate.

Lemma 3.7. *Let X, Y be Hausdorff topological spaces, and let $f : X \rightarrow Y$ be a continuous injective map. Then the separation dimension of X is at most equal to the separation dimension of Y .*

We deduce the following.

Corollary 3.8. *Let X be a connected metric space, with a Lipschitz locally convex median of rank r . Then the separation dimension of X equals r .*

Proof. According to [Bow13a, Theorem 2.2], the separation dimension of X is bounded above by r . According to 3.1, there exists an embedding of $[0, 1]^r$ into X , so according to Lemma 3.7 the separation dimension of X is precisely equal to r . \square

Finally, for affine buildings, we have the following.

Corollary 3.9. *Let X be an affine building of rank r . Then any locally convex Lipschitz median on X has rank r .*

Proof. According to [Kra11, Theorem B], X has separation dimension equal to r . According to Corollary 3.8, any locally convex Lipschitz median on X has rank r . \square

4 Proof of Theorem A

According to Section 2, we only need to prove Theorem 2.4. Consider a thick affine building X of spherical type different from A_1^r . By contradiction, we will assume that there exists a k -Lipschitz, locally convex median μ on X .

Proposition 4.1. *There exists $x \in X$ such that in a tangent cone $(X_\infty, d_\infty, x_\infty, \mu_\infty)$ of (X, d, x, μ) at x , the ultralimit F_∞ of some apartment F containing x is convex and median-isomorphic to (\mathbb{R}^r, L^1) by an affine isomorphism.*

Proof. According to Corollary 3.9, the median μ has rank r . According to Proposition 3.1, there exists a biLipschitz, median embedding f of $[0, 1]^r$ into X with convex image. According to [KL97, Corollary 6.2.3], the image of f intersects finitely many apartments of X . Consider a non-empty open subset U of $[0, 1]^r$ such that $f(U)$ lies in one apartment F of X . The map $f|_U : U \rightarrow F$ is biLipschitz, hence it is differentiable almost everywhere: pick a point $t \in U$ where f is differentiable. Since f is biLipschitz, the differential of f at t is invertible. Then in any tangent cone of (X, d, x, μ) at $x = f(t)$, the ultralimit of F is convex and median-isomorphic to (\mathbb{R}^r, L^1) , by an affine isomorphism. \square

According to Proposition 4.1, up to considering a tangent cone of X and using Theorem 2.2, we can assume that there exists a convex apartment F of X with a median, affine isomorphism with (\mathbb{R}^r, L^1) . Since F is convex, closed and locally compact, we can consider $\pi_F : X \rightarrow F$ the gate projection onto F .

Lemma 4.2. *For each $x \in X \setminus F$, and for each apartment F' of X containing x such that $F \cap F'$ is a half-apartment, we have $\pi_F(x) \in F \cap F'$.*

Proof. By contradiction, assume that there exists such an $x \in X \setminus F$ and an apartment F' containing x such that $F \cap F'$ is a half-apartment, and such that $\pi_F(x) \notin F \cap F'$. Fix a Lipschitz embedding ι of the $(r - 1)$ -ball \mathbb{B}^{r-1} into $F \cap F'$. Extend ι to a Lipschitz embedding of the half r -ball $\mathbb{B}^{r,+}$ into $F' \setminus \overset{\circ}{F}$, where \mathbb{B}^{r-1} is the equatorial sphere of \mathbb{B}^r . Extend ι to a Lipschitz map $\iota : \mathbb{B}^r \rightarrow F \cup F'$ by setting, for $z \in \mathbb{B}^{r,-}$, $\iota(z) = \pi_F(\iota(-z))$. Since $\pi_F(x) \in F \setminus F'$ and ι is Lipschitz, we deduce that $(\iota(\mathbb{B}^r) \setminus \iota(\partial \mathbb{B}^r)) \cap F$ has non-empty interior.

For each $z \in (\partial\mathbb{B}^r)^+ = \mathbb{S}^{r-1,+}$, we have $\iota(-z) = \pi_F(\iota(z))$. Consider the following map:

$$\begin{aligned} \tilde{\iota} : \mathbb{S}^{r-1,+} \times [0, 1] &\rightarrow X \\ (z, t) &\mapsto \mu(\iota(z), \pi_F(\iota(z)), [(1-t)\iota(z) + t\pi_F(\iota(z))]), \end{aligned}$$

where $[(1-t)\iota(z) + t\pi_F(\iota(z))]$ is the unique point on the CAT(0) geodesic segment between $\iota(z)$ and $\pi_F(\iota(z))$ at distance $td(\iota(z), \pi_F(\iota(z)))$ from $\iota(z)$ (see Figure 2).

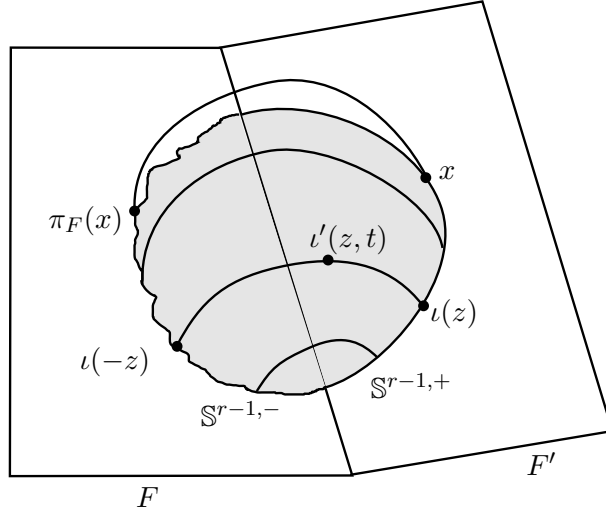


Figure 2: The sphere \mathbb{S}^r in X

The map $\tilde{\iota}$ is Lipschitz, and it is such that $\forall z \in \partial\mathbb{S}^{r-1,+} = \partial\mathbb{B}^{r-1}$, as $\iota(z) \in F \cap F'$ we have $\pi_F(\iota(z)) = \iota(z)$, so $\forall t, t' \in [0, 1]$, $\tilde{\iota}(z, t) = \tilde{\iota}(z, t')$. Consider the quotient of $\mathbb{S}^{r-1,+} \times [0, 1]$ by the equivalence relation $\forall z \in \partial\mathbb{S}^{r-1,+}, \forall t, t' \in [0, 1], (z, t) \sim (z, t')$: it is a topological ball \mathbb{B}^r . So $\tilde{\iota}$ induces a Lipschitz map $\iota' : \mathbb{B}^r \rightarrow X$ such that $\iota|_{\mathbb{S}^{r-1}} = \iota'|_{\mathbb{S}^{r-1}}$. This defines a Lipschitz map $\alpha : \mathbb{S}^r \rightarrow X$.

In $\alpha(\mathbb{S}^r)$, if we collapse the complement of a small open ball in $F \setminus F'$, we obtain a topological sphere \mathbb{S}^r . As a consequence, $H_r(\alpha(\mathbb{S}^r)) \neq 0$. According to [Kra11, Theorem B], X has topological dimension r , and since $\alpha(\mathbb{S}^r)$ is a compact subspace of X , we deduce that $H_r(\alpha(\mathbb{S}^r)) \rightarrow H_r(X)$ is an injection (see for instance [HW41, Theorem VIII.3']). Since X is contractible, this is a contradiction. \square

We can now conclude the proof of Theorem 2.4. Since X is thick and is not of spherical type A_1^r , there exists a Weyl wall W in F , and two singular geodesics γ, γ' in X , each intersecting W , such that γ and γ' intersect in $X \setminus F$. Let $x = \gamma \cap \gamma' \in X \setminus F$. Since γ is singular, the intersection of all apartments F' containing γ such that $F' \cap F$ is a half-apartment is equal to γ . According to Lemma 4.2, we deduce that $\pi_F(x) \in \gamma \cap F$. Similarly, $\pi_F(x) \in \gamma' \cap F$. This contradicts the assumption that γ and γ' intersect in $X \setminus F$.

This concludes the proof of Theorem 2.4 and of Theorem A.

5 Proof of Corollaries B and C

Proof. [Proof of Corollary B] In one direction, assume that the spherical type of X is A_1^r . Discrete affine buildings of spherical type A_1^r , endowed with the L^1 metric, are actual

CAT(0) cube complexes. If X is a symmetric space, it is isometric to a product of rank 1 symmetric spaces. According to [HW12, Theorem 1.8], every word-hyperbolic group is quasi-isometric to CAT(0) cube complex. So each rank 1 symmetric space is quasi-isometric to a CAT(0) cube complex, hence X itself is quasi-isometric to a CAT(0) cube complex.

Conversely, assume that the symmetric space or affine building X is quasi-isometric to a CAT(0) cube complex (Y, d_p) , possibly of infinite dimension, endowed with the L^p distance for some $p \in [1, \infty]$. Since (Y, d_p) is quasi-isometric to the metric space X which has finite dimension, we deduce that (Y, d_p) is quasi-isometric to (Y, d_1) . Since (Y, d_1) is a metric median space, we deduce that there exists a coarse median on X . According to Theorem A, we deduce that the spherical type of X is A_1^r . \square

Proof. [Proof of Corollary C]

If G has \mathbb{K} -rank 1, then the lattice Γ is (relatively) hyperbolic hence coarse median by [Bow13c].

Conversely, assume that Γ is coarse median. Since non-uniform lattices do not have property (RD), Γ is cocompact in G . So Γ , endowed with a word metric, is quasi-isometric to G , endowed with a left G -invariant metric. Let X be the symmetric space of non-compact type of G (if $\mathbb{K} = \mathbb{R}$ or \mathbb{C}) or the Bruhat-Tits Euclidean building of G (if \mathbb{K} is non-archimedean). Then G is quasi-isometric to X , and so X has a coarse median. According to Theorem A, X has spherical type A_1^r , so G has relative type A_1^r . Since G is simple, $r = 1$, and G has rank 1. \square

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