Higher rank lattices are not coarse median

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Abstract. We show that symmetric spaces and thick affine buildings which are not of spherical type A_1^r have no coarse median in the sense of Bowditch. As a consequence, they are not quasi-isometric to a CAT(0) cube complex, answering a question of Haglund. Another consequence is that any lattice in a simple higher rank group over a local field is not coarse median.

Introduction

A metric space (X, d) is called metric median if for each $(x, y, z) \in X^3$, the three intervals $I(x, y)$, $I(y, z)$ and $I(x, z)$ intersect in a single point, where $I(x, y) = \{p \in X \mid d(x, p) + d(p, y) = d(x, y)\}$. This point is called the median of x, y and z. The rank of (X, d) is then defined as the maximal dimension r of an embedded cube $\{0, 1\}^r$. The relationship between groups and median metric spaces is rich and has been studied through many points of view: Haagerup property, property (T), action on a CAT(0) cube complex, action on a space with (measured) walls... (see [Che00], [CDH10], [CFI13], [CN05], [Bow13b], [Bow14]...).

Bowditch recently introduced the notion of a coarse median on a metric space (see [Bow13a]), in order to gather in the same setting CAT(0) cube complexes and Gromov hyperbolic spaces. A metric space is Gromov-hyperbolic if and only if every finite subset admits a good metric comparison with a tree (see for instance [GdlH90, Theorem 12, p. 33]). Bowditch's definition of a coarse median is having a good metric comparison of every finite subset with a metric median space, or equivalently with a CAT(0) cube complex according to Chepoi (see [Che00]).

Definition (Bowditch)**.** Let (X, d) be a metric space. A map $\mu : X^3 \to X$ is called a *coarse median* if there exist $k \in [0, +\infty)$ and $h : \mathbb{N} \to [0, +\infty)$ such that

- $\forall a, b, c, a', b', c' \in X, d(\mu(a, b, c), \mu(a', b', c')) \leqslant k(d(a, a') + d(b, b') + d(c, c')) + h(0)$.
- For each finite non-empty set $A \subset X$, with $|A| \leqslant p$, there exists a finite median algebra (Π, μ_Π) and maps $\pi : A \to \Pi$, $\lambda : \Pi \to X$ such that

$$\begin{aligned} \forall x, y, z \in X, d(\lambda\mu_\Pi(x, y, z), \mu(\lambda x, \lambda y, \lambda z)) &\leqslant h(p) \\ \text{and } \forall a \in A, d(a, \lambda(\pi(a))) &\leqslant h(p). \end{aligned}$$

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If Π can be chosen (independently of p) to have rank at most r, we say that μ has rank at most r.

A finitely generated group is said to be coarse median if some Cayley graph has a coarse median (not necessarily equivariant under the group action). Bowditch showed (see [Bow13a]) that if a group is coarse median, then its Dehn function is at most quadratic and it has property (RD). Furthermore, if a group has a coarse median of rank at most r, there is no quasi-isometric embedding of $\mathbb{R}^{r+1}$ into that group. Bowditch also showed that the existence of a coarse median is a quasi-isometry invariant, that a group is Gromov hyperbolic if and only if it is coarse median of rank 1, and that a group hyperbolic relative to coarse median groups is itself coarse median (see [Bow13c]). Furthermore, Bowditch showed that the mapping class group of a surface of genus g with p punctures is coarse median of rank $3g-3+p$, hereby recovering Behrstock and Minsky's result that the mapping class group has property (RD) (see [BM11]), and the rank theorem (see [Ham07] and [BM08]).

Since most known examples of coarse median groups have some nonnegative curvature features, Bowditch asked in [Bow13a] whether higher rank symmetric space, or even CAT(0) spaces, admit coarse medians. In this article, we give a negative answer to this question, by showing the following:

Theorem A. Let X be a symmetric space of non-compact type, or a thick affine building. There exists a coarse median on X if and only if the spherical type of X is A_1^r.

Note that the coarse median is not assumed to be equivariant by any group.

Haglund asked if a higher rank symmetric space or affine building is quasi-isometric to a CAT(0) cube complex, and we give a negative answer:

Corollary B. Let X be a symmetric space of non-compact type, or a discrete, thick affine building. Then X is quasi-isometric to a CAT(0) cube complex if and only if the spherical type of X is A_1^r

Note that the CAT(0) cube complex is not assumed to be of finite dimension, and it could also be endowed with the L^p distance for any $p \in [1,\infty]$.

Also note that Corollary B still holds if we consider non-discrete thick affine buildings and non-discrete CAT(0) cube complexes.

Furthermore, for uniform lattices in semisimple Lie groups, Property (RD) implies the Baum-Connes conjecture without coefficient (see [Laf98]). Property (RD) has been proved notably for uniform lattices in $\mathrm{SL}(3,\mathbb{K})$, where $\mathbb{K}$ is a local field (see [RRS98], [Laf00] and [Cha03]). Valette conjectured that uniform lattices in semisimple Lie groups satisfy property (RD). Since being coarse median implies property (RD), one could ask if this could be a way to prove property (RD) for higher rank uniform lattices, and the answer is negative:

Corollary C. Let $\mathbb{K}$ be a local field, let G be the group of $\mathbb{K}$-points of a simple algebraic group without compact factors, and let Γ be a lattice in G. Then Γ is coarse median if and only if G has $\mathbb{K}$-rank 1.

Note that, due to property (T), higher rank lattices do not admit unbounded actions on median metric spaces (see [CDH10]). But in the coarse median setting this is not a consequence of property (T), since for instance every hyperbolic group with property (T) is coarse median.

Organization of the paper: We will prove the main part of Theorem A by assuming the existence of a coarse median on a symmetric space or affine building X. In Section 1, we will recall the general definitions of median algebras. In Section 2, we consider the asymptotic cone of X, and the ultralimit of the coarse median. In Section 3 we prove the existence of a convex cube. In Section 4, by considering the gate projection, we obtain a contradiction. In Section 5, we prove Corollaries B and C.

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1 Median algebras

Definition 1.1. Let X be a set. A symmetric map $\mu : X^3 \to X$ is called a *median* on X if:

$$\forall a, b, c, d, e \in X, \mu(a, b, \mu(c, d, e)) = \mu(\mu(a, b, c), \mu(a, b, d), e). \tag{1}$$

The pair (X, μ) is called a *median algebra.*

Remark.

- There is a unique median on the set $\{0, 1\}$.
- We can consider the product median on the n-cube $\{0, 1\}^n$.

Definition 1.2. Let (X, μ), (X', μ') be median algebras. A map $f : X \to X'$ is called *median* if $\forall x, y, z \in X, \mu'(f(x), f(y), f(z)) = f(\mu(x, y, z))$. If furthermore f is injective, it is called a *median embedding.*

Definition 1.3. Let (X, μ) be a median algebra. If every median embedding of an n-cube $\{0, 1\}^n \to X$ satisfies $n \leqslant r$, we say that X has *rank* at most r.

Definition 1.4. Let (X, μ) be a median algebra. If $a, b \in X$, the *interval* between a and b is $I(a, b) = \{c \in X \mid \mu(a, b, c) = c\}$. A subset $C \subset X$ is called *convex* if $\forall a, b \in C, I(a, b) \subset C$.

If (X, d) is a metric space, a weakening of the notion of metric median is the following.

Definition 1.5. Let (X, d) be a metric space. An abstract median μ on X is called

- continuous if $\mu : X^3 \to X$ is a continuous map,
- *Lipschitz* if there exists a constant $k \in [0, +\infty)$ such that μ is k-Lispchitz with respect to each variable, i.e.

$$\forall a, b, c, a', b', c' \in X, d(\mu(a, b, c), \mu(a', b', c')) \leqslant k(d(a, a') + d(b, b') + d(c, c')),$$

- *locally convex* if each point of X has a basis of neighbourhoods consisting of convex subsets of X.

Definition 1.6. Let (X, d) be a metric space, let μ be a continuous median on X, and let $C \subset X$ be a non-empty closed, locally compact convex subset of X. Then for each $x \in X$, there exists a unique $\pi_C(x) \in C$, called the *gate projection* of x onto C, such that $\forall y \in C, \pi_C(x) \in I(x, y)$. The map $\pi_C : X \to C$ is called the gate projection, it is a continuous map. If μ is k-Lipschitz, then π_C is a k-Lipschitz map.

Now we recall the definition of walls in a median algebra.

Definition 1.7. Let (X, μ) be a median algebra. A *wall* in X is a pair $W = \{H^+(W), H^-(W)\}$, where $H^+(W)$ and $H^-(W)$ are non-empty convex disjoints subsets of X whose union is equal to X.

Lemma 1.8. *[Bow13a, Lemma 6.1] Let (X, μ) be a median algebra, and let A, B be disjoint convex subsets of M. There exists a wall $W = \{H^+(W), H^-(W)\}$ in X separating A from B, i.e. such that $A \subset H^{\pm}(W)$ and $B \subset H^{\mp}(W)$.*

Lemma 1.9. *[Bow13a, Lemma 7.3] Let (X, d) be a metric space, and let μ be a continuous locally convex median on X. Let a, b be distinct points of X. There exists a wall $W = \{H^+(W), H^-(W)\}$ in X strongly separating a from b, i.e. such that $a \in X \backslash \overline{H^-(W)}$ and $b \in X \backslash \overline{H^+(W)}$.*

Lemma 1.10. *[Bow13a, Lemma 7.5] Let X be a metric space, and let μ be a continuous locally convex median on X. For each wall $W = \{H^+(W), H^-(W)\}$ in X, the subset $L(W) = \overline{H^+(W)} \cap \overline{H^-(W)}$ is a convex median subalgebra of X, of rank at most $r - 1$ if the rank of μ is r.*

2 Ultralimits of spaces and coarse medians

In [KL97], Kleiner and Leeb developed a geometric definition of spherical and affine buildings, and in particular they studied their asymptotic cones.

Theorem 2.1. *[KL97, Theorem 1.2.1] Let X be a symmetric space of non-compact type or a thick affine building. Then any asymptotic cone of X is a thick affine building of the same spherical type as X.*

They also proved that any tangent cone of an affine building is an affine building:

Theorem 2.2. *[KL97, Theorem 5.1.1] Let (X, d) be an affine building, let ω be a non-principal ultrafilter on $\mathbb{N}$, let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X and let $(\lambda_n)_{n \in \mathbb{N}}$ be a sequence in $(0, +\infty)$ such that $\lim_{\omega} \lambda_n = +\infty$. Let $(X_\infty, d_\infty, x_\infty)$ be the ω-ultralimit of $(X_n, \lambda_n d, x_n)$. Then (X_∞, d_∞) is an affine building. Furthermore, if X is thick, then X_∞ is thick. The affine Weyl group of X_∞ acts transitively on each apartment of X_∞.*

One motivation for Bowditch's definition of coarse median is that it behaves well when one considers asymptotic cones.

Theorem 2.3. *[Bow13a, Theorem 2.3] Let (X, d) be a metric space, and let μ be a (k, h)-coarse median on X. Then on any asymptotic cone (X_∞, d_∞) of (X, d), μ defines a k-Lipschitz, locally convex median μ_∞.*

So Theorem A is a consequence of the following.

Theorem 2.4. *Let X be a thick affine building of spherical type different from A_1^r. There is no locally convex Lipschitz median on X.*

Proof. [Theorem 2.4 implies Theorem A] Assume first that the spherical type of the symmetric space or affine building X is A_1^r. If X is a symmetric space, it is a product of rank 1 symmetric spaces, which are Gromov hyperbolic, so X has a coarse median. If X is an affine building, if we endow it with the L^1 metric it becomes a metric median space. In particular, this median is a coarse median with respect to any usual metric on X, which is equivalent to the L^1 metric.

Conversely, if the spherical type of the symmetric space or affine building X is not A_1^r, by Theorem 2.1 any asymptotic cone X_∞ is a thick affine building. If there existed a coarse median μ on X, it would give rise by Theorem 2.3 to a locally convex Lipschitz median on X_∞, which contradicts Theorem 2.4 since the spherical type of X_∞ is not A_1^r. Hence there is no coarse median on X. □

3 Existence of a convex cube

In this Section, we will show the following result.

Proposition 3.1. *Let X be a connected metric space, with a Lipschitz locally convex median of rank r. There exists a median, biLipschitz embedding of the r-cube $[0,1]^r$ into X with convex image.*

Fix X a geodesic metric space, and $\mu : X^3 \to X$ a Lipschitz median.

Definition 3.2. A continuous path $p : I \to X$, where $I \subset \mathbb{R}$ is an interval, is said to be *monotone* is for each $t_1 < t_2 < t_3$ in I, we have $\mu(p(t_1), p(t_2), p(t_3)) = p(t_2)$.

To prove Proposition 3.1, we will need the following two Lemmas.

Lemma 3.3. *Let X be a connected metric space, with a continuous locally convex median μ, and let $f : \{0,1\}^r \to X$ be a median embedding of the r-cube, and let W be a wall in X strongly separating $f(0,\dots,0)$ and $f(1,0,\dots,0)$. There exists a median embedding $g : \{0,1\}^r \to X$ such that $\forall t \in \{0\} \times \{0,1\}^{r-1}, g(t) = f(t)$ and $\forall t \in \{1\} \times \{0,1\}^{r-1}, g(t) \in L(W)$.*

Proof. Note that if we knew that $L(W)$ was locally compact, projecting the half-cube $\{1\} \times \{0,1\}^{r-1}$ using the gate projection onto $L(W)$ would immediately give the result.

Since intervals are connected, we can consider $a \in I(f(0,\dots,0), f(1,0,\dots,0)) \cap L(W)$. Define

$$\begin{array}{rcl} g : \{0,1\}^r & \to & X \\ t \in \{0\} \times \{0,1\}^{r-1} & \mapsto & f(t) \\ t \in \{1\} \times \{0,1\}^{r-1} & \mapsto & \mu(f(0,t_2,\dots,t_r), a, f(t)). \end{array}$$

Since $L(W)$ is convex, we deduce that $\forall t \in \{1\} \times \{0,1\}^{r-1}, g(t) \in L(W)$.

Using repeatedly Property (1), we prove that g is a median map. As a consequence, if for some $t, t' \in \{0,1\}^r$ we have $g(t) = g(t')$, then

$$\begin{array}{rcl} f(0,t_2,\dots,t_r) & = & \mu(g(0,t_2,\dots,t_r), g(t), g(0,t'_2,\dots,t'_r)) \\ & = & \mu(g(0,t_2,\dots,t_r), g(t'), g(0,t'_2,\dots,t'_r)) \\ & = & f(0,t'_2,\dots,t'_r), \end{array}$$

hence $(0, t_2, \dots, t_r) = (0, t'_2, \dots, t'_r)$, so $t = t'$: g is injective. $\square$

Lemma 3.4. *Let (X, μ) be a median algebra. Assume there exists a median embedding of the r-cube $f : [0,1]^r \to X$, such that the image by f of any edge of $[0,1]^r$ is convex. Then the image of f is convex in X.*

Proof. For each $k \in [\![1, r]\!]$, let $e_k = (0, \dots, 0, 1, 0, \dots, 0)$ (the 1 is in the k^{th} position). Let $x \in I(f(0), f(e_1 + \dots + e_r))$. For each $k \in [\![1, r]\!]$, since the image by f of the edge $[0, e_k]$ is convex, we deduce that $I(f(0), f(e_k)) = f([0, e_k])$, so there exists $t_k \in [0,1]$ such that $\mu(f(0, \dots, 0), x, f(e_k)) = f(t_k e_k)$. We will show by induction on $k \in [\![0, r]\!]$ that $f(t_1 e_1 + \dots + t_k e_k) = \mu(f(0), x, f(e_1 + \dots + e_k))$.

For $k = 0$ this is immediate using Property (1), so assume that for some $k < r$ we have $f(t_1 e_1 + \dots + t_k e_k) = \mu(f(0), x, f(e_1 + \dots + e_k))$. Then

$$\begin{aligned} f(t_1 e_1 + \dots + t_{k+1} e_{k+1}) &= \mu(f(t_1 e_1 + \dots + t_k e_k), f(t_{k+1} e_{k+1}), f(e_1 + \dots + e_{k+1})) \\ &= \mu(\mu(f(0), x, f(e_1 + \dots + e_k)), \mu(f(0), x, f(e_{k+1})), f(e_1 + \dots + e_{k+1})) \\ &= \mu(f(0), x, \mu(f(e_1 + \dots + e_k), f(e_{k+1}), f(e_1 + \dots + e_{k+1})) \\ &= \mu(f(0), x, f(e_1 + \dots + e_{k+1})). \end{aligned}$$

As a consequence, for $k = r$ we deduce that

$$f(t_1 e_1 + \dots + t_r e_r) = \mu(f(0), x, f(e_1 + \dots + e_r)) = x,$$

as $x \in I(f(0), f(e_1 + \dots + e_r))$. So we have proved that the image of f is equal to the interval $I(f(0), f(e_1 + \dots + e_r))$, which is convex. $\square$

We can now prove Proposition 3.1.

Proof. [Proof of Proposition 3.1] Since the rank of the median μ is r, consider a median embedding $f : \{0,1\}^r \to X$. Applying $2r$ times Lemma 3.3, up to replacing f by another median embedding of $\{0,1\}^r$ into X, we can assume that for each $i \in [\![1, r]\!]$ and $\varepsilon \in \{0, 1\}$, the image under f of the codimension 1 face $\{0,1\}^{i-1} \times \{\varepsilon\} \times \{0,1\}^{r-1-i}$ is included in a closed convex subspace $L(W_{i,\varepsilon})$ of X, where $W_{i,\varepsilon}$ is a wall of X. According to Lemma 1.10, each $L(W_{i,\varepsilon})$ has rank at most $r-1$, and since it contains the image by f of the $(r-1)$-cube $\{0,1\}^{i-1} \times \{\varepsilon\} \times \{0,1\}^{r-1-i}$, we deduce that each $L(W_{i,\varepsilon})$ has rank $r-1$.

For $i, j \in [\![1, r]\!]$ distinct and $\varepsilon, \varepsilon' \in \{0,1\}$, since $L(W_{i,\varepsilon}) \cap L(W_{j,\varepsilon'}) = L(L(W_{i,\varepsilon}) \cap W_{j,\varepsilon'})$, where $L(W_{i,\varepsilon}) \cap W_{j,\varepsilon'}$ is a wall in the rank $r-1$ median algebra $L(W_{i,\varepsilon})$, we deduce by Lemma 1.10 that $L(W_{i,\varepsilon}) \cap L(W_{j,\varepsilon'})$ has rank $r-2$.

By induction, we prove that for each $p \in [\![1, r]\!]$, for each distinct $i_1, \dots, i_p \in [\![1, r]\!]$ and each $\varepsilon_1, \dots, \varepsilon_p \in \{0,1\}$, the intersection $\bigcap_{1 \leqslant k \leqslant p} L(W_{i_k, \varepsilon_k})$ has rank $r - p$.

For each $k \in [\![1, r]\!]$, let $e_k = (0, \dots, 0, 1, 0, \dots, 0)$ (the 1 is in the k^{th} position). Hence for each $k \in [\![1, r]\!]$, the points $f(0)$ and $f(e_k)$ are contained in a convex rank 1 closed subspace. In particular, there exists an injective, monotone, biLipschitz path p_k from $f(0)$ to $f(e_k)$, with convex image.

We will show by induction on $k \in [\![0, r]\!]$ that we can extend f to a biLipschitz median embedding from $[0,1]^k \times \{0,1\}^{r-k}$ into X. The case $k = 0$ is already true.

Assume we have extended f to a biLipschitz median embedding $f : [0,1]^k \times \{0,1\}^{r-k} \to X$ for some $k < r$. Define

$$\begin{array}{rcl} f : [0,1]^k \times [0,1] \times \{0,1\}^{r-k-1} & \to & X \\ (t,u,v) \in [0,1]^k \times [0,1] \times \{0,1\}^{r-k-1} & \mapsto & \mu(f(t,0,v), p_{k+1}(u), f(1,\dots,1)). \end{array}$$

See Figure 1. Since p_{k+1} and μ are biLipschitz, we deduce that f is biLipschitz on $[0,1]^k \times [0,1] \times \{0,1\}^{r-k-1}$.

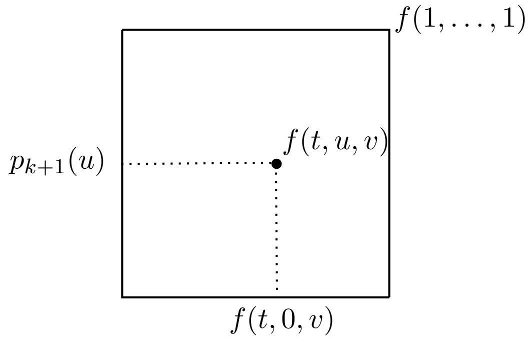

Figure 1: Extending f

So f is extended to a biLipschitz map $f : [0,1]^r \to X$. If $t \in [0,1]^r$ and $k \in [\![1,r]\!]$, notice that in $[0,1]^r$ the median of $(t, 0, e_k)$ is equal to $t_k e_k$. So $\mu(f(t), f(0), f(e_k)) = f(t_k e_k) = p_k(t_k)$. Since each path $p_1, \dots, p_r$ is injective, we deduce that f itself is injective.

By using several times Property (1), we prove that f preserves the medians. Hence f is a median embedding, and by Lemma 3.4 the image of f is convex. □

Let us recall Bowditch's definition of the separation dimension of a space, which is a good notion of dimension when working with medians on a metric space.

Definition 3.5 (Bowditch)**.** If X is a Hausdorff topological space, define the *separation dimension* of X inductively as follows:

- If $X = \emptyset$, then the separation dimension of X is -1.
- X has separation dimension at most $n \in \mathbb{N}$ if for any distinct $x, y \in X$, there exist closed subsets A, B of X such that $x \notin B$, $y \notin A$, $X = A \cup B$ and $A \cap B$ has dimension at most $n - 1$.

Remark. If X is a Hausdorff metric space, then the inductive dimension of X is at most equal to the separation dimension. Conversely, we have the following.

Lemma 3.6. *[HW41, Section III.6] If X is a locally compact Hausdorff metric space, then the inductive dimension of X equals the separation dimension.*

The following Lemma is immediate.

Lemma 3.7. *Let X, Y be Hausdorff topological spaces, and let $f : X \to Y$ be a continuous injective map. Then the separation dimension of X is at most equal to the separation dimension of Y.*

We deduce the following.

Corollary 3.8. *Let X be a connected metric space, with a Lipschitz locally convex median of rank r. Then the separation dimension of X equals r.*

Proof. According to [Bow13a, Theorem 2.2], the separation dimension of X is bounded above by r. According to 3.1, there exists an embedding of $[0, 1]^r$ into X, so according to Lemma 3.7 the separation dimension of X is precisely equal to r. □

Finally, for affine buildings, we have the following.

Corollary 3.9. *Let X be an affine building of rank r. Then any locally convex Lipschitz median on X has rank r.*

Proof. According to [Kra11, Theorem B], X has separation dimension equal to r. According to Corollary 3.8, any locally convex Lipschitz median on X has rank r. □

4 Proof of Theorem A

According to Section 2, we only need to prove Theorem 2.4. Consider a thick affine building X of spherical type different from A_1^r. By contradiction, we will assume that there exists a k-Lipschitz, locally convex median μ on X.

Proposition 4.1. *There exists $x \in X$ such that in a tangent cone $(X_\infty, d_\infty, x_\infty, \mu_\infty)$ of (X, d, x, μ) at x, the ultralimit F_∞ of some apartment F containing x is convex and median-isomorphic to $(\mathbb{R}^r, L^1)$ by an affine isomorphism.*

Proof. According to Corollary 3.9, the median μ has rank r. According to Proposition 3.1, there exists a biLipschitz, median embedding f of $[0, 1]^r$ into X with convex image. According to [KL97, Corollary 6.2.3], the image of f intersects finitely many apartments of X. Consider a non-empty open subset U of $[0, 1]^r$ such that $f(U)$ lies in one apartment F of X. The map $f|_U : U \to F$ is biLipschitz, hence it is differentiable almost everywhere: pick a point $t \in U$ where f is differentiable. Since f is biLipschitz, the differential of f at t is invertible. Then in any tangent cone of (X, d, x, μ) at $x = f(t)$, the ultralimit of F is convex and median-isomorphic to $(\mathbb{R}^r, L^1)$, by an affine isomorphism. □

According to Proposition 4.1, up to considering a tangent cone of X and using Theorem 2.2, we can assume that there exists a convex apartment F of X with a median, affine isomorphism with $(\mathbb{R}^r, L^1)$. Since F is convex, closed and locally compact, we can consider $\pi_F : X \to F$ the gate projection onto F.

Lemma 4.2. *For each $x \in X \backslash F$, and for each apartment F' of X containing x such that $F \cap F'$ is a half-apartment, we have $\pi_F(x) \in F \cap F'$.*

Proof. By contradiction, assume that there exists such an $x \in X \backslash F$ and an apartment F' containing x such that $F \cap F'$ is a half-apartment, and such that $\pi_F(x) \notin F \cap F'$. Fix a Lipschitz embedding ι of the $(r-1)$-ball $\mathbb{B}^{r-1}$ into $F \cap F'$. Extend ι to a Lipschitz embedding of the half r-ball $\mathbb{B}^{r,+}$ into $F' \backslash \mathring{F}$, where $\mathbb{B}^{r-1}$ is the equatorial sphere of $\mathbb{B}^r$. Extend ι to a Lipschitz map $\iota : \mathbb{B}^r \to F \cup F'$ by setting, for $z \in \mathbb{B}^{r,-}$, $\iota(z) = \pi_F(\iota(-z))$. Since $\pi_F(x) \in F \backslash F'$ and ι is Lipschitz, we deduce that $(\iota(\mathbb{B}^r) \backslash \iota(\partial \mathbb{B}^r)) \cap F$ has non-empty interior.

For each $z \in (\partial \mathbb{B}^r)^+ = \mathbb{S}^{r-1,+}$, we have $\iota(-z) = \pi_F(\iota(z))$. Consider the following map:

$$\begin{array}{rcl} \widetilde{\iota'} : \mathbb{S}^{r-1,+} \times [0,1] & \to & X \\ (z,t) & \mapsto & \mu(\iota(z), \pi_F(\iota(z)), [(1-t)\iota(z) + t\pi_F(\iota(z))]), \end{array}$$

where $[(1-t)\iota(z)+t\pi_F(\iota(z))]$ is the unique point on the CAT(0) geodesic segment between $\iota(z)$ and $\pi_F(\iota(z))$ at distance $td(\iota(z), \pi_F(\iota(z)))$ from $\iota(z)$ (see Figure 2).

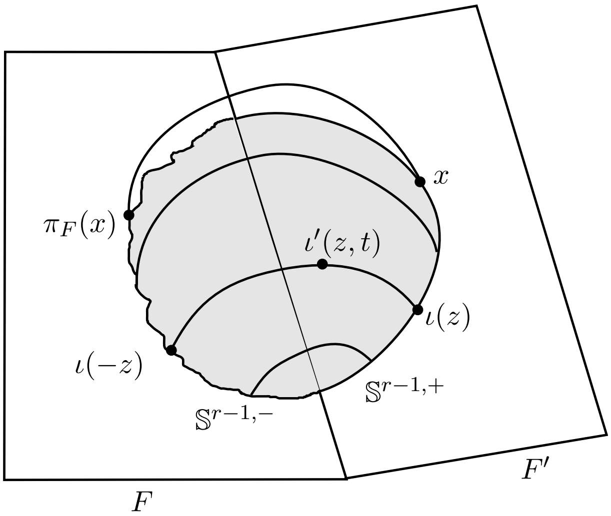

Figure 2: The sphere $\mathbb{S}^r$ in X

The map $\widetilde{\iota'}$ is Lipschitz, and it is such that $\forall z \in \partial \mathbb{S}^{r-1,+} = \partial \mathbb{B}^{r-1}$, as $\iota(z) \in F \cap F'$ we have $\pi_F(\iota(z)) = \iota(z)$, so $\forall t, t' \in [0,1], \widetilde{\iota'}(z,t) = \widetilde{\iota'}(z,t')$. Consider the quotient of $\mathbb{S}^{r-1,+} \times [0,1]$ by the equivalence relation $\forall z \in \partial \mathbb{S}^{r-1,+}, \forall t, t' \in [0,1], (z,t) \sim (z,t')$: it is a topological ball $\mathbb{B}^r$. So $\widetilde{\iota'}$ induces a Lipschitz map $\iota' : \mathbb{B}^r \to X$ such that $\iota|_{\mathbb{S}^{r-1}} = \iota'|_{\mathbb{S}^{r-1}}$. This defines a Lipschitz map $\alpha : \mathbb{S}^r \to X$.

In $\alpha(\mathbb{S}^r)$, if we collapse the complement of a small open ball in $F \backslash F'$, we obtain a topological sphere $\mathbb{S}^r$. As a consequence, $H_r(\alpha(\mathbb{S}^r)) \neq 0$. According to [Kra11, Theorem B], X has topological dimension r, and since $\alpha(\mathbb{S}^r)$ is a compact subspace of X, we deduce that $H_r(\alpha(\mathbb{S}^r)) \to H_r(X)$ is an injection (see for instance [HW41, Theorem VIII.3']). Since X is contractible, this is a contradiction. □

We can now conclude the proof of Theorem 2.4. Since X is thick and is not of spherical type A_1^r, there exists a Weyl wall W in F, and two singular geodesics γ, γ' in X, each interesecting W, such that γ and γ' intersect in $X \backslash F$. Let $x = \gamma \cap \gamma' \in X \backslash F$. Since γ is singular, the intersection of all apartments F' containg γ such that $F' \cap F$ is a half-apartment is equal to γ. According to Lemma 4.2, we deduce that $\pi_F(x) \in \gamma \cap F$. Similarly, $\pi_F(x) \in \gamma' \cap F$. This contradicts the assumption that γ and γ' intersect in $X \backslash F$.

This concludes the proof of Theorem 2.4 and of Theorem A.

5 Proof of Corollaries B and C

Proof. [Proof of Corollary B] In one direction, assume that the spherical type of X is A_1^r. Discrete affine buildings of spherical type A_1^r, endowed with the L^1 metric, are actual

CAT(0) cube complexes. If X is a symmetric space, it is isometric to a product of rank 1 symmetric spaces. According to [HW12, Theorem 1.8], every word-hyperbolic group is quasi-isometric to CAT(0) cube complex. So each rank 1 symmetric space is quasi-isometric to a CAT(0) cube complex, hence X itself is quasi-isometric to a CAT(0) cube complex.

Conversely, assume that the symmetric space or affine building X is quasi-isometric to a CAT(0) cube complex (Y, d_p), possibly of infinite dimension, endowed with the L^p distance for some $p \in [1, \infty]$. Since (Y, d_p) is quasi-isometric to the metric space X which has finite dimension, we deduce that (Y, d_p) is quasi-isometric to (Y, d_1). Since (Y, d_1) is a metric median space, we deduce that there exists a coarse median on X. According to Theorem A, we deduce that the spherical type of X is A_1^r. □

Proof. [Proof of Corollary C]
If G has $\mathbb{K}$-rank 1, then the lattice Γ is (relatively) hyperbolic hence coarse median by [Bow13c].

Conversely, assume that Γ is coarse median. Since non-uniform lattices do not have property (RD), Γ is cocompact in G. So Γ, endowed with a word metric, is quasi-isometric to G, endowed with a left G-invariant metric. Let X be the symmetric space of non-compact type of G (if $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) or the Bruhat-Tits Euclidean building of G (if $\mathbb{K}$ is non-archimedean). Then G is quasi-isometric to X, and so X has a coarse median. According to Theorem A, X has spherical type A_1^r, so G has relative type A_1^r. Since G is simple, $r = 1$, and G has rank 1. □

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