

SYSTOLIC GEOMETRY AND SIMPLICIAL COMPLEXITY FOR GROUPS

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ABSTRACT. This paper introduces a new combinatorial invariant for finitely presentable groups called *simplicial complexity*. This complexity is strongly related to another invariant arising from systolic geometry and called *systolic area*. Beside its own interest, the simplicial complexity allows us to obtain new finitude results for systolic area and to precise its behaviour in terms of the 1-torsion of the group.

1. INTRODUCTION

We focus on groups which can be presented as the fundamental group of a finite 2-dimensional simplicial complex. Thus, throughout this article, by *group* we mean *finitely presentable group*, and by *simplicial complex* we mean *finite simplicial complex*.

First recall the definition of systolic area for 2-dimensional simplicial complexes and groups. Let X be a simplicial complex of dimension 2 and suppose that its fundamental group is not trivial. Given a piecewise smooth Riemannian metric h on X the systole denoted by $\text{sys}(X, h)$ is defined as the shortest length of a non-contractible closed curve in X . We call *systolic area* the number

$$\sigma(X) := \inf_h \frac{\text{area}(X, h)}{\text{sys}(X, h)^2}$$

where the infimum is taken over all piecewise smooth Riemannian metrics on X . Following [Gro96, p.337] the *systolic area* of a group G is the number

$$\sigma(G) := \inf_X \sigma(X)$$

where the infimum is taken over all 2-dimensional simplicial complexes X with fundamental group G .

It is straightforward that systolic area for free groups is zero. For a non-free group G , we have the following universal lowerbound:

$$\sigma(G) \geq \frac{\pi}{16}.$$

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This bound was proved by Rudyak & Sabourau in [RS08] where the authors also pointed the following two fundamental questions: given a non-free group G , is it true that

- (1) $\sigma(G) \geq 2/\pi$?
- (2) $\sigma(G * \mathbb{Z}) = \sigma(G)$?

The first question is motivated by the fact that systolic area for surfaces is known to be minimal for $\mathbb{R}P^2$ whose corresponding fundamental group is \mathbb{Z}_2 and that its value is precisely $2/\pi$ (see [Gro83, 5.2.B] and [Pu52]).

The second question is related to finitude problems for systolic area. It is easy to check that systolic area is subadditive for free products, and that in particular

$$(1.1) \quad \sigma(G * F_n) \leq \sigma(G)$$

where F_n denotes the free group of rank n . In [Gro96, p.337] Gromov raised the following question: given a positive number T how large is the set of isomorphism classes of groups with systolic area at most T ? Because of inequality (1.1) we may hope a finitude result only if we consider groups without free factor \mathbb{Z} (or at least with an uniformly bounded number of such factors). In [RS08] the authors proved such a finitude result and give a (non-sharp) upperbound for the cardinality. We give the exact statement of their result in the next paragraph, but first observe that an affirmative answer to the second question above would thus imply that the set of values of the systolic area function lying in a compact interval is always finite.

Let now precise the notion of groups without free factor \mathbb{Z} . According to [Kur60, §35]—see also [Mas67] for a topological version— for any group G there exist an unique integer n and an unique subgroup H (up to conjugation) such that G decomposes into a free product

$$(1.2) \quad G = H * F_n$$

where H can not be decomposed in its turn like in (1.2) with a positive n . We call this number n the *free index* of G . A group without free factor \mathbb{Z} is thus a group of zero free index.

Denote by $\mathcal{G}_\sigma(T)$ the set of isomorphism classes of groups G with free index zero such that $\sigma(G) \leq T$. According to [RS08] it is a finite set whose cardinality satisfies the upper bound

$$(1.3) \quad |\mathcal{G}_\sigma(T)| \leq A^{T^3}$$

for some explicit constant $A > 1$ and the lower bound

$$(1.4) \quad 2^T \leq |\mathcal{G}_\sigma(T)|$$

for T large enough. In this article we propose an alternative proof of the finitude of $\mathcal{G}_\sigma(T)$ which leads to an improvement of the upper bound (1.3). We also give a lower bound for the cardinality of the subset $\mathcal{A}_\sigma(T) \subset \mathcal{G}_\sigma(T)$ consisting of isomorphism classes of *finite abelian* groups.

Theorem 1.1. For $T \geq 2$

$$\left[2^{\frac{\pi}{1+2\sqrt{3}}T} \right] \leq |\mathcal{A}_\sigma(T)| \leq |\mathcal{G}_\sigma(T)| \leq B^{T^{1+\frac{B'}{\sqrt{\log T}}}},$$

where B and B' are explicit constants, and $[x]$ denotes the integral part of a number x . In particular for any positive ε

$$|\mathcal{G}_\sigma(T)| \leq B^{T^{1+\varepsilon}}$$

provided T is large enough.

Observe that $\pi/(1+2\sqrt{3}) \simeq 0.7$ so that our lower bound does not improve inequality (1.4). But it shows that an exponential asymptotic growth is already realized on the class of finite abelian groups.

The main tool to improve the upperbound of Rudyak & Sabourau is a new combinatorial invariant for finitely presentable groups introduced in section 2. Namely, if G is such a group, this new invariant called *simplicial complexity* and denoted by $\kappa(G)$ is the minimal number of 2-simplices of a 2-dimensional simplicial complex with fundamental group G , see Definition 2.1. For groups with zero free index the two invariants $\sigma(G)$ and $\kappa(G)$ are closely related. The central result of this article is the following comparison theorem.

Theorem 1.2. Let G be a group with zero free index. Then

$$2\pi\sigma(G) \leq \kappa(G) \leq C\sigma(G)^{1+\frac{C'}{\sqrt{\log_2(\sigma(G))}}}$$

for some explicit constants C and C' .

This theorem is the link between the different results in this paper and will be proven in section 3. It shows that for large values the systolic area σ is a quasi-linear function of the simplicial complexity κ . More precisely, for any positive $\varepsilon > 0$

$$2\pi\sigma(G) \leq \kappa(G) \leq (\sigma(G))^{1+\varepsilon}$$

provided $\kappa(G)$ is large enough. Observe that there is no hope for a linear upperbound as for surface groups it is known that systolic area is strictly sublinear (see [BS94, BPS12]) while the simplicial complexity is linear in terms of the genus (see Example 2 in section 2.1).

Using Theorem 1.2 the problem of estimating $|\mathcal{G}_\sigma(T)|$ transforms into a purely combinatorial problem. For a positive integer T we denote by $\mathcal{G}_\kappa(T)$ the set of isomorphism classes of groups G with zero free index such that $\kappa(G) \leq T$. We also denote by $\mathcal{A}_\kappa(T)$ the subset corresponding to finite abelian groups.

Theorem 1.3. For any $T \geq 2$

$$\left[2^{\frac{T-3}{14}} \right] \leq |\mathcal{A}_\kappa(T)| \leq |\mathcal{G}_\kappa(T)| \leq 2^{6T \log_2 T}.$$

The lower bound is proved by estimating the simplicial complexity of \mathbb{Z}_m . For the upper bound we code simplicial complexes with a minimal number of 2-simplices by some special

colored graphs and estimate their number, see section 4.

Here is another application of simplicial complexity. In [BPS12] is proved that for any group G

$$\sigma(G) \geq C \frac{b_1(G) + 1}{(\ln(b_1(G) + 2))^2}$$

for some universal constant C where $b_1(G)$ denotes the first real Betti number of G . But this lower bound is inefficient for groups whose first integral homology group has large torsion, such as \mathbb{Z}_m when m is large. In converse simplicial complexity is quite sensitive to torsion elements, and using correspondence of Theorem 1.2 we are able to prove that for any positive ε

$$\sigma(G) \geq (\ln |\text{Tors } H_1(G, \mathbb{Z})|)^{1-\varepsilon}$$

for groups with large torsion in homology. In Theorem 5.1 we complete the study of $\kappa(G)$ for abelian groups in terms of two parameters: the number of elements in G and the number of its invariant factors (which coincides with its minimal number of generators). Using this estimate we conclude that

$$(1.5) \quad (\log_2 m)^{1-\varepsilon} \leq \sigma(\mathbb{Z}_m) \leq 1.43 \log_2 m$$

for any positive ε provided m is large enough. In comparison, inequality (1.3) implies that $\sigma(\mathbb{Z}_m) \rightarrow \infty$ for large values of m but gives no information about the asymptotic behaviour of this sequence.

In literature can be found two other invariants that measure the complexity of finitely presented groups: the T -invariant of Delzant [Del96] and the c -complexity of Matveev & Pervova [MP01, PP08]. Delzant's T -invariant is additive for free product, this nice property being the main reason of its introduction. Nevertheless it is not sensitive to 2-torsion which makes it not pertinent for systolic considerations on groups. The c -complexity was introduced to measure the complexity of 3-manifolds using their fundamental group. We compare in subsection 2.2 simplicial complexity with T -invariant and c -complexity. We show in particular that simplicial complexity and c -complexity agree up to some universal constants. We therefore reprove some of classical results on c -complexity, like the lower bound in terms of the torsion or the estimate for abelian groups, compare with [PP08]. Nevertheless the c -complexity is not fit for systolic geometry, and the results we obtain here using simplicial complexity are always better than if c -complexity was used instead.

In the last section we present some applications of simplicial complexity to systolic geometry of higher dimensional spaces. One key result is the following estimate for the systolic volume of any $(2n + 1)$ -dimensional lens space L_m^{2n+1} with fundamental group \mathbb{Z}_m , see section 6 for definition.

Theorem 1.4. *There exists positive constants C_n , C'_n et D_n depending only on the dimension n such that for any integer $m \geq 2$*

$$(1.6) \quad C_n (\ln m)^{1 - \frac{C'_n}{\sqrt{\ln \ln m}}} \leq \mathfrak{S}(L_m^{2n+1}) \leq D_n m^n.$$

While the lower bound is of the same kind that for $\sigma(\mathbb{Z}_m)$, the best known upper bound is thus polynomial of degree n in m . Observe that this degree is half less than the degree in the trivial upperbound $\approx m^{2n}$ given by the round metric. In particular round metrics on lens spaces are not systolically extremal for large m . Determining the asymptotic behaviour of $\mathfrak{S}(L_m^{2n+1})$ in terms of both m and n is still an open question.

2. SIMPLICIAL COMPLEXITY

In this section we introduce the definition of simplicial complexity and give some of its basic properties. We then compare this new complexity with the two other standard complexities, namely the T -invariant of Delzant [Del96] and the c -complexity of Matveev & Pervova [MP01]. Next we show a central lower bound for the simplicial complexity in terms of the 1-torsion of the group.

2.1. Definition and examples. Given a finite simplicial complex P we denote by $s_k(P)$ the number of its k -simplices.

Definition 2.1. Let G be a group. We define its *simplicial complexity* $\kappa(G)$ by the following formula:

$$\kappa(G) := \inf_{\pi_1(X)=G} s_2(X),$$

the infimum being taken over all simplicial 2-complexes X with fundamental group G . A 2-complex X is then said *minimal for G* if $\pi_1(X) = G$, $s_2(X) = \kappa(G)$ and each vertex is incident to at least 2 edges. If G is of zero free index the last condition is equivalent to the one that each vertex is incident to a face.

Simplicial complexity satisfies the following properties:

1. $\kappa(G) = 0$ if and only if G is a free group.
2. The free product of two groups G_1 and G_2 satisfies

$$(2.1) \quad \kappa(G_1 * G_2) \leq \kappa(G_1) + \kappa(G_2).$$

But simplicial complexity is not additive with respect to free product: if $\kappa(G_1)$ and $\kappa(G_2)$ are both positive, inequality (2.1) can be strengthened by

$$\kappa(G_1 * G_2) < \kappa(G_1) + \kappa(G_2).$$

For this fix two 2-complexes X_1 and X_2 which are minimal for G_1 and G_2 respectively and glue them together by identifying one 2-simplex of X_1 with another 2-simplex of X_2 (the choice of these two 2-simplices being not relevant).

3. For a simplicial complex P , its *simplicial height* $h(P)$ is the total number of its simplices of any dimension. This invariant was introduced in [Gro96] and satisfies

$$h(P) \geq \kappa(\pi_1(P)).$$

Example 1. Even for groups whose structure is simple, the exact value of κ seems hard to compute. For small values up to 17 the following table describes the situation, see [Bul14]. Here K_2 denotes the fundamental group of the Klein bottle, while the annotation $(*)$ means that the corresponding minimal complex is unique and $?$ that there might be some others groups with the same simplicial complexity.

$\kappa(G)$	10	14	16	17
G	$\mathbb{Z}_2(*)$	$\mathbb{Z} \oplus \mathbb{Z}(*)$	K_2	$\mathbb{Z}_3, ?$

For instance there might be several groups with simplicial complexity 17 as $17 \leq \kappa(\mathbb{Z}_2 * \mathbb{Z}_2) \leq 18$ according to [Bul14]. The unique minimal complex for \mathbb{Z}_2 is the quotient of the icosahedron by the central symmetry, see Figure 1.

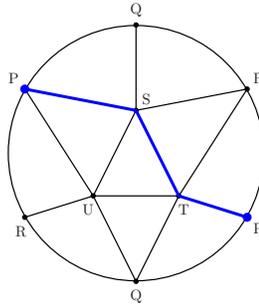


FIGURE 1. Minimal complex for \mathbb{Z}_2 .

For $\mathbb{Z} \oplus \mathbb{Z}$ the minimal complex is also unique and is given by the minimal triangulation of the 2-torus whose fundamental domain is depicted in Figure 2.

These two minimal complexes will be used in the sequel.

Example 2. For surface groups with large genus, the exact computation of their complexity remains an open problem. We can nevertheless give some bounds in terms of their genus.

Let $\pi_1(S_l)$ be the fundamental group of an orientable surface of genus $l \geq 1$. By elementary algebraic and combinatorial considerations

$$(2.2) \quad \kappa(\pi_1(S_l)) \geq \frac{4}{3}l.$$

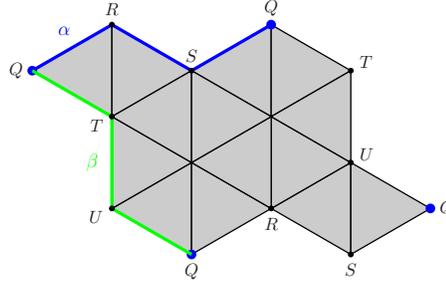


FIGURE 2. Minimal complex for $\mathbb{Z} \oplus \mathbb{Z}$.

Besides

$$(2.3) \quad \kappa(\pi_1(S_l)) \leq 4(l-1) + 2 \left\lceil \frac{7 + \sqrt{1 + 48l}}{2} \right\rceil$$

by a result of Jungerman & Ringel [JR80]. Here $\lceil a \rceil$ denotes the integer part of $a + 1$ if a is not an integer and a for integers. Strictly speaking, the upper bound is only available for $l \neq 2$. For $l = 2$ we have to replace the upper bound by 24, see [JR80]. Observe that the upper bound (2.3) is sharp for $\pi_1(S_1) = \mathbb{Z} \oplus \mathbb{Z}$. Because $\kappa(\mathbb{Z} \oplus \mathbb{Z}) = 14$ we can easily derive that the free abelian group A_n of rank n satisfies

$$\frac{1}{2}n(n-1) \leq \kappa(A_n) \leq 7n(n-1).$$

More precisely, the lower bound is given by the second Betti number and the upper bound is proved by induction. The precise computation for $\kappa(A_n)$ remains open.

Subadditive property (2.1) implies that for any group G ,

$$(2.4) \quad \kappa(G * \mathbb{Z}) \leq \kappa(G).$$

As for systolic area the question to know whether or not this inequality is actually an equality is open and fundamental. Because of (2.4) we will consider in the sequel only groups with zero free index.

If G is a zero free index group and X is a minimal complex for G , it is straightforward to check that

- (M₁) any edge of X is adjacent to at least two 2-simplices,
- (M₂) any vertex of X is adjacent at least to four 2-simplices.

This properties of minimal complexes will be usefull in the sequel.

2.2. Comparison with other complexities. There exist two other numerical invariants for finitely presentable groups which measure their complexity. First recall that given a presentation

$$\mathcal{P} = \langle a_1, \dots, a_n \mid r_1, \dots, r_m \rangle$$

of a group G , its length is the number

$$\ell(\mathcal{P}) = \sum_{i=1}^m |r_i|$$

where $|\cdot|$ denotes the word length associated to the system of generators $\{a_1, \dots, a_n\}$.

The two other types of complexity for finitely presentable groups are the following:

- the c -complexity introduced by Matveev & Pervova [MP01] and defined as the minimal length $c(G)$ of a finite presentation,
- the T -invariant introduced by Delzant [Del96] and defined by

$$T(G) := \min_{\mathcal{P}=\langle a_1, \dots, a_n \mid r_1, \dots, r_m \rangle} \sum_{i=1}^m \max\{|r_i| - 2, 0\}.$$

Any finitely presentable group can be presented with relations of length either 2 or 3, such a representation being called *triangular*. The T -invariant is nothing else than the minimal number of relations of length 3 for a triangular representation.

One fundamental property of the T -invariant is its linearity with respect to free products:

$$T(G_1 * G_2) = T(G_1) + T(G_2).$$

But of course this invariant is not sensitive to torsion elements of order 2, for instance

$$T(\mathbb{Z}_2 * \dots * \mathbb{Z}_2) = 0.$$

In [PP08](see also [KS05]) it is shown that T -invariant and c -complexity satisfy the following relations:

- $T(G) \leq c(G)$,
- $c(G) \leq 9T(G)$ if G does not admit free factor isomorphic to \mathbb{Z}_2 ,
- $c(G) \leq 3T(G)$ if G has no torsion element of order 2.

On the other hand c -complexity and simplicial complexity coincide up to some universal constants:

Proposition 2.1. *For any group G we have*

$$(2.5) \quad \frac{1}{6} (\kappa(G) + h) \leq c(G) \leq 3 (\kappa(G) - 4),$$

where h is the minimal number of relations over finite presentations of G .

Proof. To prove the left-hand side we start with any finite presentation

$$\mathcal{P} = \langle a_1, \dots, a_n \mid r_1, \dots, r_m \rangle$$

of G . We can suppose this presentation contains no relation of length 1 and that every relation of length 2 is of the form $r_j = a_s^{\pm 2}$. Associated to \mathcal{P} we construct a 2-simplicial complex X with $\pi_1(X) = G$ and $s_2(X) \leq 6\ell(\mathcal{P}) - m$. Start with $Y = \vee_{i=1}^k S_i^1$ the wedge sum of k circles whose base point is denoted by P , each circle corresponding to a generator of the presentation \mathcal{P} . For each $j = 1, \dots, m$ we glue a 2-disk D_j^2 by identifying its boundary with the loop described by the relation r_j . The topological space thus obtained is denoted by X and can be triangulated as follows. First divide each circle into three edges, the base point P corresponding to one of the vertices. If $|r_j| \geq 3$ we triangulate the disk D_j^2 according to the parity of $|r_j|$. If $|r_j| = 2k$, we triangulate D_j^2 as in Figure 3 and get a contribution of $6|r_j| - 2$ triangles (remark the blocks made of nine triangles). If $|r_j| = 2k + 1$, we triangulate D_j^2 as in Figure 4 and get a contribution of $6|r_j| - 1$ triangles.

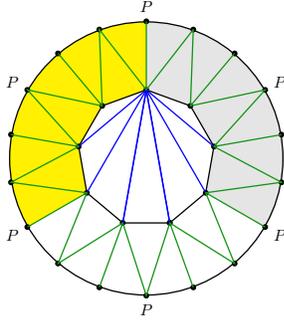


FIGURE 3. Triangulation of D_j^2 for $|r_j|$ even ($|r_j| = 6$).

If $|r_j| = 2$ we triangulate the corresponding projective plane like in Figure 1 and get a contribution of $10 = 5|r_j|$ simplices of dimension 2.

Finally the triangulation of X thus defined satisfies

$$s_2(X) \leq 6 \sum_{j=1}^m |r_j| - m \leq 6\ell(\mathcal{P}) - m.$$

Because we start with any finite presentation \mathcal{P} we conclude that $\kappa(G) + h \leq 6c(G)$.

The right-hand side of the inequality is proved as follows. Consider a simplicial complex X minimal for G and having a minimal number of edges. Take a maximal tree of the 1-dimensional skeleton of X and let p be a root of this tree. We first contract every

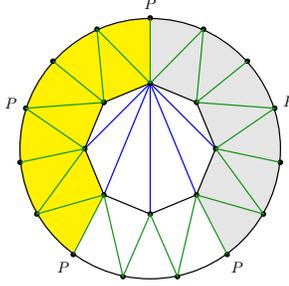


FIGURE 4. Triangulation of D_j^2 for $|r_j|$ odd ($|r_j| = 5$).

2-simplex adjacent to p and then the remaining part of the maximal tree into a point. This gives a finite cell complex \tilde{X} of dimension 2 whose 2-cells are glued along at most 3 cells of dimension 1. Because X is minimal, the root p was adjacent to at least four 2-simplices, so the number of 2-cells of \tilde{X} is at most $s_2(X) - 4$. This topological space being homotopy equivalent to X we get a presentation of $\pi_1(\tilde{X}) = G$ whose length is at most $3(s_2(X) - 4)$. \square

Remark 2.1. If G has no torsion element of order 2, the proof implies the following inequality:

$$\kappa(G) \leq 5c(G) + T(G) \leq 13T(G).$$

2.3. Lower bound in terms of 1-torsion. The simplicial complexity $\kappa(G)$ is quite sensitive to the number of torsion elements in $H_1(G, \mathbb{Z})$. The next proposition will be used several times in the sequel.

Proposition 2.2. *Let X be a simplicial complex of dimension 2. Then*

$$s_2(X) \geq 2 \log_3 |\text{Tors } H_1(X, \mathbb{Z})|.$$

In particular, any group G satisfies the inequality

$$\kappa(G) \geq 2 \log_3 |\text{Tors } H_1(G, \mathbb{Z})|.$$

Proof. Consider the complex of simplicial cochains

$$(2.6) \quad C^1(X, \mathbb{Z}) \xrightarrow{d^1} C^2(X, \mathbb{Z}) \xrightarrow{d^2} 0.$$

The universal coefficient theorem implies a duality between homology torsion and cohomology torsion, and we have (see [Hat02, Corollary 3.3] for instance)

$$(2.7) \quad \text{Tors } H_1(X, \mathbb{Z}) \simeq \text{Tors } H^2(X, \mathbb{Z}).$$

This implies that $|\text{Tors } H_1(X, \mathbb{Z})| = |\text{Tors}(C^2(X, \mathbb{Z})/\text{Im } d^1)|$.

We endow $C^i(X, \mathbb{Z})$ with the basis dual to the simplicial basis of $C_i(X, \mathbb{Z})$. Let D denote the matrix of d^1 with respect to these bases. The matrix D has $s_2(X)$ rows, and each row has exactly three non zero elements whose value is either 1 or -1 . It follows that every row vector of D has euclidian length $\sqrt{3}$. If we interpret the determinant of a square matrix V of order k as the volume of the parallelotope generated by its row vectors, we see that for any smaller square matrix V of D of order k

$$(2.8) \quad |\det V| \leq (\sqrt{3})^k.$$

Assume that the rank of D is equal to d , and denote by $t(D)$ the greatest common divisor of all minors of order d of D . By (2.8),

$$(2.9) \quad t(D) \leq (\sqrt{3})^d \leq (\sqrt{3})^{s_2(X)}.$$

Furthermore, it is obvious that $t(D)$ is invariant under change of basis of $C^1(X, \mathbb{Z})$ or of $C^2(X, \mathbb{Z})$. By a general result on free \mathbb{Z} -modules (see [VdW71]), there exist a basis $e^1, \dots, e^{s_1(X)}$ of $C^1(X, \mathbb{Z})$ and a basis $f^1, \dots, f^{s_2(X)}$ of $C^2(X, \mathbb{Z})$ such that $d^1(e^i) = m_i \cdot f^i$ for $1 \leq i \leq d$ and $d^1(e^i) = 0$ for $i > d$ (note that the m_i 's can be chosen such that m_i divides m_{i+1} albeit we will not need this). On the one hand, we have

$$|\text{Tors } H_1(X, \mathbb{Z})| = |\text{Tors}(C^2(X, \mathbb{Z})/\text{Im } d^1)| = \prod_{i=1}^d m_i.$$

On the other hand, the matrix of d^1 in the new bases $(e^i)_{i=1}^{s_1(X)}$ and $(f^j)_{j=1}^{s_2(X)}$ is a $s_1(X) \times s_2(X)$ -matrix of the following structure: the left square $s_1(X) \times s_1(X)$ sub-matrix is the diagonal matrix $\text{Diag}(m_1, \dots, m_d, 0, \dots, 0)$, all other entries are equal to zero. A straightforward computation of the minors of order d of this matrix gives

$$t(D) = \prod_{i=1}^d m_i.$$

Together with inequality (2.9) it completes the proof. \square

Remark 2.2. It is proved in [PP08] that for any finitely presentable group G

$$c(G) \geq 3 \log_3 |\text{Tors } H_1(G, \mathbb{Z})|.$$

This combined with the right-hand side of inequality (2.5) leads to a weaker estimate of $\kappa(G)$ than the one obtained in Proposition 2.2.

2.4. Stabilization for free products. Given a group G we denote by

$$G_{(n)} = \underbrace{G * \dots * G}_n$$

the free product of n copies of G . As the function $\kappa(G_{(n)})$ is sublinear in n according to inequality (2.1), we can define the *stable simplicial complexity* by

$$\kappa_\infty(G) := \lim_{n \rightarrow \infty} \frac{\kappa(G_{(n)})}{n}.$$

In this section we show that $\kappa_\infty(G) > 0$ for any unfree group G . That is, albeit simplicial complexity is not additive, its asymptotic behaviour for free products of a same group is essentially linear. The analogous question for systolic area is completely open.

Proposition 2.3. *Any unfree group G satisfies*

$$\kappa(G) - 1 \geq \kappa_\infty(G) \geq \begin{cases} 2 \log_3 2 & \text{if } G \text{ decomposes as } G = G' * \mathbb{Z}_2, \\ \frac{T(G)}{3} & \text{if } G \text{ does not admit such a decomposition.} \end{cases}$$

Proof. We start with a minimal complex X for G . Choose a triangle $\Delta \subset X$ and consider the complex

$$Y_n = \underbrace{X \cup_{\Delta} \dots \cup_{\Delta} X}_n.$$

We get

$$\kappa(G_{(n)}) \leq s_2(Y_n) = n(s_2(X) - 1) + 1 = n(\kappa(G) - 1) + 1$$

which implies the upper bound. Now if $G = G' * \mathbb{Z}_2$ then

$$|\text{Tors}H_1(G_{(n)}, \mathbb{Z})| \geq |\text{Tors}H_1((\mathbb{Z}_2)_{(n)}, \mathbb{Z})| = 2^n.$$

Using Proposition 2.2 we derive

$$\kappa_\infty(G) \geq 2 \log_3 2.$$

If G does not decompose as $G' * \mathbb{Z}_2$, because $T(G) \leq c(G) \leq 3(\kappa(G) - 4)$ we get the result as T -invariant is additive for free products. \square

3. SIMPLICIAL COMPLEXITY AND SYSTOLIC AREA

This section is completely devoted to the proof of Theorem 1.2 given in the introduction.

For the left-hand side inequality we consider a minimal simplicial complex X of dimension 2 with fundamental group G . Endow X with the metric h such that any edge is of length $\frac{2\pi}{3}$ and any face is the round hemisphere of radius 1. Because X is minimal, $s_2(X) = \kappa(G)$ and so

$$\text{area}(X, h) = 2\pi\kappa(G).$$

The definition of the metric h implies that any systolic geodesic can be homotoped to the 1-skeleton without increasing its length. Such a curve passes through at least three edges and thus $\text{sys}(X, h) \geq 2\pi$. This implies that

$$\sigma(G) \leq \sigma(X, h) \leq \frac{\kappa(G)}{2\pi}.$$

To proof the right-hand side inequality we argue as in [Gro83, 5.3.B]. In the sequel, if $B := B(p, R)$ denotes the metric ball centered at p of radius R , we denote by $|B|$ its area for the metric g , and nB the concentric ball $B(p, nR)$ for any positive integer n .

Set $R_0 = \frac{1}{25}$ and

$$(3.1) \quad \alpha = 25 \exp \left(\sqrt{\ln(62500 \cdot \sigma(G))} \right).$$

By [RS08, Theorem 1.4],

$$\sigma(G) \geq \frac{\pi}{16},$$

so α is well defined and satisfies $\alpha > 5$. Fix some positive ε small enough such that

$$(3.2) \quad \log_5 \frac{R_0}{5\varepsilon} \cdot \ln \frac{\alpha}{5} \geq \ln(62500 \cdot \kappa(G)).$$

By [RS08, Theorem 3.5 and Lemma 4.2], there exists a simplicial complex P of dimension 2 endowed with a metric g such that

- $\pi_1(P) = G$;
- $\text{sys}(P, g) = 1$;
- $\sigma(P, g) = \text{area}(P, g) < \sigma(G) + \varepsilon$;
- any ball $B(p, R) \subset P$ of radius $R \in [\varepsilon, \frac{1}{2}]$ centered at any point $p \in P$ satisfies the inequality

$$|B(p, R)| \geq \frac{1}{4} R^2.$$

Following Gromov (see [Gro83, Theorem 5.3.B]), we introduce the following definition. A ball $B(p, R)$ with $\varepsilon \leq R \leq R_0$ is said to be α -admissible if

- $|B(p, 5R)| \leq \alpha \cdot |B(p, R)|$;
- $\forall R' \in]R, R_0], \alpha \cdot |B(p, R')| \leq |B(p, 5R')|$.

If there exists a point $p \in P$ such that for any $R \in [\varepsilon, R_0]$ the ball $B(p, R)$ is never α -admissible, then, if $r \in \mathbb{N}$ denotes the unique integer such that $\frac{R_0}{5^{r+1}} \leq \varepsilon < \frac{R_0}{5^r}$,

$$|B(p, R_0)| \geq \alpha^r |B(p, \frac{R_0}{5^r})| \geq \frac{1}{4} \alpha^r \varepsilon^2 \geq \frac{1}{100} \left(\frac{\alpha}{5} \right)^r R_0^2 \geq \frac{1}{62500} \left(\frac{\alpha}{5} \right)^{\log_5 \frac{R_0}{5\varepsilon}}.$$

Thus

$$\sigma(P, g) \geq \kappa(G)$$

according to the inequality (3.2) and the result is proved in this case.

So we can assume that for any $p \in P$ there exists $R_p \in [\varepsilon, R_0]$ such that $B(p, R_p)$ is α -admissible. Denote by A the area of (P, g) .

Lemma 3.1. *Let $B(p, R)$ be an α -admissible ball. Then*

$$|B(p, R)| \geq A(\alpha) := \frac{1}{100} \left(\frac{1}{25} \right)^{\frac{\ln \frac{100A}{R_0^2}}{\ln \frac{\alpha}{25}}} R_0^2.$$

Proof of the lemma. Let $r \in \mathbb{N}$ be the unique integer such that $\frac{R_0}{5^{r+1}} \leq R < \frac{R_0}{5^r}$. We have

$$A = \text{area}(P, g) \geq |B(p, R_0)| \geq \alpha^r |B(p, R)| \geq \frac{1}{4} \alpha^r R^2 \geq \frac{1}{100} \left(\frac{\alpha}{25} \right)^r R_0^2,$$

and so

$$r \leq r(\alpha) := \frac{\ln \frac{100A}{R_0^2}}{\ln \frac{\alpha}{25}}.$$

This implies

$$|B(p, R)| \geq \frac{1}{4}R^2 \geq \frac{1}{100} \left(\frac{1}{25}\right)^r R_0^2 \geq A(\alpha).$$

□

We now construct a family $\{B_i\}_{i=1}^N$ of α -admissible balls of P in the following way. We first choose an α -admissible ball $B_1 := B(p_1, R_1)$ with $R_1 := \max\{R_p \mid p \in P\}$. At each step $i \geq 2$, we construct B_i using the data of $\{B_j\}_{j < i}$ as follows. Let R_i be the maximal radius of an α -admissible ball centered at a point in the complement of the union of the balls $\{2B_j\}_{j < i}$ and let $B_i := B(p_i, R_i)$ be such an α -admissible ball. By construction, B_i is disjoint from the other balls B_j as $R_i \leq R_j$. The process ends in a finite N number of steps when the balls $\{2B_i\}_{i=1}^N$ cover P , as $R_i \geq \varepsilon$ for every $i = 1, \dots, N$.

Consider \mathcal{N} the corresponding nerve of this cover. In general, if X is a paracompact topological space and \mathcal{U} a locally finite cover of X , there exists a canonical map Φ from X to the nerve $\mathcal{N}(\mathcal{U})$ of the cover \mathcal{U} defined as follows. If $\{\phi_V\}_{V \in \mathcal{U}}$ denotes a partition of unity associated with \mathcal{U} ,

$$\begin{aligned} \Phi : X &\rightarrow \mathcal{N}(\mathcal{U}) \\ x &\mapsto \sum_{V \in \mathcal{U}} \phi_V(x)V. \end{aligned}$$

This map is uniquely defined up to homotopy. In our case, $\mathcal{U} = \{2B_i\}_{i=1}^N$ and Φ associates the center of such a ball to the corresponding vertex of \mathcal{N} .

Lemma 3.2. *The map $\Phi : P \rightarrow \mathcal{N}$ induces an isomorphism of fundamental groups.*

Proof of the lemma. Denote by $\mathcal{N}^{(k)}$ the k -skeleton of \mathcal{N} . We will construct a map $\Psi : \mathcal{N}^{(2)} \rightarrow X$ such that the induced map

$$\Psi_{\#} : \pi_1(\mathcal{N}) \simeq \pi_1(\mathcal{N}^{(2)}) \rightarrow \pi_1(P)$$

is the inverse of $\Phi_{\#} : \pi_1(P) \rightarrow \pi_1(\mathcal{N})$.

We have denoted the set of centers of the balls of the covering \mathcal{U} by $\{p_i\}_{i=1}^N$ and set $v_i = \Phi(p_i)$. We first define Ψ on $\mathcal{N}^{(0)}$ by

$$\Psi(v_i) = p_i.$$

If two vertices v_i and v_j are connected by an edge $[v_i, v_j]$, we join p_i and p_j in P by any minimizing geodesic denoted by $\gamma_{i,j}$. The map Ψ is then defined on the edge $[v_i, v_j]$ to the arc $\gamma_{i,j}$ in the obvious way

$$\Psi : [v_i, v_j] \longrightarrow \gamma_{i,j}.$$

This defines Ψ on the 1-skeleton $\mathcal{N}^{(1)}$. Remark that $l_g(\gamma_{i,j}) \leq 4 \cdot R_0$ (v_i and v_j are connected by an edge if and only if $2B_i \cap 2B_j \neq \emptyset$).

Next we consider any 2-simplex $\tau = [v_i, v_j, v_k]$ of \mathcal{N} . The concatenation $\gamma_{i,j} \star \gamma_{j,k} \star \gamma_{k,i}$ is a closed curve of P of length at most $12 \cdot R_0 < 1$. So it is contractible and any contraction of this curve into a point gives rise to an extension of the map Ψ to τ . We get this way a map

$$\Psi : \mathcal{N}^{(2)} \rightarrow P.$$

Observe that the restriction of Ψ to $\mathcal{N}^{(1)}$ is unique up to homotopy.

By construction, $\Phi(p_i) = v_i$ for any $i = 1, \dots, N$, and if $[v_i, v_j]$ denotes an edge of \mathcal{N} and p belongs to the corresponding geodesic γ_{ij} , $\Phi(p) \in \text{St}([v_i, v_j])$ where $\text{St}([v_i, v_j])$ denotes the star of $[v_i, v_j]$. This implies that $\Phi \circ \Psi : [v_i, v_j] \rightarrow \mathcal{N}$ is homotopically equivalent to the identity relatively to $\{v_i, v_j\}$. So $\Phi \circ \Psi : \mathcal{N}^{(1)} \rightarrow \mathcal{N}$ is homotopically equivalent to the identity relatively to $\mathcal{N}^{(0)}$. From this, we get that the induced morphism $\Phi_{\#} \circ \Psi_{\#} : \pi_1(\mathcal{N}) \rightarrow \pi_1(\mathcal{N})$ is the identity and so $\Psi_{\#} : \pi_1(\mathcal{N}) \rightarrow \pi_1(P)$ is onto.

It remains to prove that $\Psi_{\#}$ is onto. Consider a geodesic loop δ based at the center p_1 of the ball B_1 and whose length is minimal in its own homotopy class. We complete p_1 into a finite family $\{p_{i_j}\}_{j \in \mathbb{Z}_n}$ of points of P such that

- each p_{i_j} is the center of some ball B_{i_j} of \mathcal{U} ;
- the family $\{2B_{i_j}\}_{j \in \mathbb{Z}_n}$ covers δ ;
- $2B_{i_j} \cap 2B_{i_{j+1}} \neq \emptyset$.

For each $j \in \mathbb{Z}_n$, fix any point $x_j \in 2B_{i_j} \cap \delta$ and denote by δ_j the part of the loop δ joining x_j and x_{j+1} and contained in $2B_{i_j} \cup 2B_{i_{j+1}}$. By construction, $l_g(\delta_j) \leq 8 \cdot R_0$. Fix a minimizing geodesic β_j joining p_{i_j} and x_j . The concatenation

$$\gamma_{i_j, i_{j+1}} \star \beta_{j+1} \star (\delta_j)^{-1} \star (\beta_j)^{-1}$$

is a closed curve of length at most $24 \cdot R_0 < 1$ thus contractible. So δ is homotopic to $\gamma_{1,2} \star \gamma_{2,3} \star \dots \star \gamma_{n,1}$ with based point p_1 fixed. This proves the surjectivity of $\Psi_{\#}$ and completes the proof. \square

As $\pi_1(\mathcal{N}) \simeq G$, we deduce the lower bound

$$s_2(\mathcal{N}) \geq \kappa(G).$$

We now estimate the number $s_2(\mathcal{N})$ by the systolic volume of (P, g) . First of all,

$$A = \text{area}(P, g) \geq \sum_{i=1}^N |B_i| \geq \frac{1}{\alpha} \sum_{i=1}^N |5B_i|,$$

as the balls B_i are pairwise disjoint and α -admissible. If B_i belongs to exactly F_i distinct 2-simplexes of \mathcal{N} , the ball $5B_i$ contains at least F_i pairwise disjoint balls of $\{B_i\}_{i=1}^N$, and so

$$|5B_i| \geq F_i \cdot A(\alpha).$$

From the equality $\sum_{i=1}^N F_i = 3 \cdot s_2(\mathcal{N})$, we deduce that

$$\kappa(G) \leq \frac{\alpha A}{3A(\alpha)}.$$

As

$$A(\alpha) = \frac{1}{100} \left(\frac{1}{25} \right)^{\frac{\ln \frac{100A}{R_0^2}}{\ln \frac{9}{25}}} R_0^2,$$

we get

$$\kappa(G) \leq \frac{62500}{3} \cdot 25^{\frac{\ln \frac{100A}{R_0^2}}{\ln \frac{9}{25}}} \cdot \alpha A.$$

as $R_0 = \frac{1}{25}$. From the equality

$$\ln \frac{\alpha}{25} = \sqrt{\ln(62500 \cdot \sigma(G))},$$

we then compute that

$$\kappa(G) \leq \frac{62500}{3} \cdot e^{\ln 25 \cdot \frac{\ln(62500 \cdot A)}{\sqrt{\ln(62500 \cdot \sigma(G))}}} \cdot 25 e^{\sqrt{\ln(62500 \cdot \sigma(G))}} \cdot A.$$

Now observe that $A = \text{area}(P, g) = \sigma(P, g) < \sigma(G) + \varepsilon$. This finally implies the result for $C = \frac{62500 \cdot 25}{3} = \frac{2^2 \cdot 5^8}{3}$ and $C' = 1 + \ln 25$.

4. FINITUDE RESULTS FOR SIMPLICIAL COMPLEXITY

We focus in this section on finitude problems for the invariant κ : how estimate the (obviously finite) number of isomorphism classes of groups whose simplicial complexity is at most T ? Recall that given a positive integer T the set $\mathcal{G}_\kappa(T)$ is defined as the isomorphism classes of groups G with free index zero such that $\kappa(G) \leq T$ while $\mathcal{A}_\kappa(T)$ denotes the subset corresponding to finite abelian groups.

4.1. Upper bound for $|\mathcal{G}_\kappa(T)|$. We start with the proof of the upper bound contained in Theorem 1.3, namely

$$|\mathcal{G}_\kappa(T)| \leq 2^{6T \log_2 T}.$$

Start with any simplicial complex X of dimension 2 minimal for G . Because G is of zero free index and the simplicial complex X is minimal recall that

(M_1) : any edge of X is adjacent to at least two 2-simplices,

(M_2) : any vertex of X is adjacent at least to four 2-simplices.

We derive from (M_1) and from (M_2) the following upper bounds on the number of 0- and 1-simplices:

$$s_0(X) \leq \frac{3T}{4} \text{ and } s_1(X) \leq \frac{3T}{2}.$$

Consider the barycentric subdivision $\text{sd}(X)$ of X and color the vertices as follows.

- The original vertices of X are colored in black,
- barycenters of edges of X are colored in green,
- barycenters of faces of X are colored in red.

Consider the 1-skeleton of $\text{sd}(X)$ and erase the edges joining red and black vertices, see Figure 5. We denote by b , g and r the number of respectively black, green and red vertices. The 1-dimensional simplicial complex thus obtained is a 3-colored graph which satisfies the following properties:

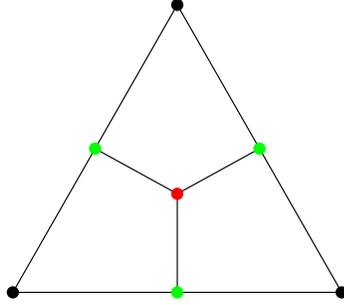


FIGURE 5. The 3-colored graph associated to a face.

- (P_1) : $b \leq \frac{3T}{4}$, $g \leq \frac{3T}{2}$ and $r \leq T$,
- (P_2) : any green vertex is adjacent to exactly two black vertices,
- (P_3) : any red vertex is adjacent to exactly three green vertices,
- (P_4) : no pair of red and black vertices are adjacent.

It is straightforward to check that to any minimal complex of a group with zero free index and simplicial complexity at most T corresponds a unique 3-colored graph satisfying properties (P_1) to (P_4) . Observe that there exist 3-colored graphs satisfying properties (P_1) to (P_4) which does not correspond to any minimal complex, and even does not correspond to any simplicial complex.

The number of 3-colored graphs satisfying properties (P_1) to (P_4) can be estimated using their incidence matrix which has the following form

$$\begin{pmatrix} 0 & A^t & 0 \\ A & 0 & B^t \\ 0 & B & 0 \end{pmatrix}$$

where A is a $g \times b$ matrix with each row containing exactly two non zero coefficients equal to 1, and B is a $r \times g$ matrix for which each row contains exactly three non zero coefficient equal to 1. Thus the number of such matrices A is at most

$$\left(\frac{b(b-1)}{2} \right)^g$$

and the number of such matrices B at most

$$\left(\frac{g(g-1)(g-2)}{6} \right)^r.$$

From this we compute that the number of 3-colored graphs satisfying properties (P_1) to (P_4) is at most

$$\frac{9T^3}{8} \cdot \left(\frac{1}{2} \cdot \frac{3T}{4} \cdot \left(\frac{3T}{4} - 1 \right) \right)^{\frac{3T}{2}} \cdot \left(\frac{1}{6} \cdot \frac{3T}{2} \cdot \left(\frac{3T}{2} - 1 \right) \cdot \left(\frac{3T}{2} - 2 \right) \right)^T$$

which is less than T^{6T} for $T \geq 2$. This concludes the proof.

4.2. Simplicial complexity for finite abelian groups. In this subsection we shall see that the subset $\mathcal{A}_\kappa(T)$ of *finite abelian* groups with simplicial complexity bounded by T is already large.

Recall that a finite abelian group G decomposes in a direct sum

$$G = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \dots \oplus \mathbb{Z}_{n_s}$$

where $n_1 | n_2 | \dots | n_s$. These numbers are called the *invariant factors* of G and are uniquely defined by the group. This decomposition can be used to estimate the simplicial complexity of G as follows.

Theorem 4.1. *Any finite abelian group G satisfies the double inequality*

$$2 \log_3 |G| \leq \kappa(G) \leq 14 \log_2 |G| + 7s^2 - 4s,$$

where s is the number of invariant factors of G .

Remark 4.1. In particular

$$\kappa(G) \leq 7(\log_2 |G|)^2 + 10 \log_2 |G|$$

for any finite abelian group. The order of this upper bound is asymptotically sharp as shown by the following example. If

$$G = \bigoplus_{i=1}^s (\mathbb{Z}_2)_i,$$

we see that $|G| = 2^s$ and $H_2(G, \mathbb{Z}_2)$ is a \mathbb{Z}_2 -vector space of dimension $\frac{s(s+1)}{2}$. So

$$\kappa(G) \geq \frac{s(s+1)}{2} \geq \frac{1}{2}(\log_2 |G|)^2.$$

From Theorem 4.1 we directly derive that for any $m \geq 2$

$$2 \log_3 m \leq \kappa(\mathbb{Z}_m) \leq 14 \log_2 m + 3.$$

In particular we get the lower bound announced in Theorem 1.3:

Corollary 4.1. *For any positive T*

$$|\mathcal{A}_\kappa(T)| \geq \left\lfloor 2^{\frac{T-3}{14}} \right\rfloor$$

where $\lfloor x \rfloor$ denotes the integral part of the number x .

The rest of the subsection is devoted to the proof of Theorem 4.1.

Proof of Theorem 4.1. From Proposition 2.2

$$\kappa(G) \geq 2 \log_3 |\text{tors}H_1(G, \mathbb{Z})|$$

which gives the left-hand side inequality of Theorem 4.1 as $|\text{tors}H_1(G, \mathbb{Z})| = |G|$ for a finite abelian group.

The proof of the right-hand side inequality of Theorem 4.1 relies on the following estimate for the simplicial complexity of \mathbb{Z}_m :

Lemma 4.1.

$$\kappa(\mathbb{Z}_m) \leq 14 \log_2 m + 3.$$

Proof of the Lemma. Start with a Moebius strip denoted by \mathcal{M} and fix (see Figure 6)

- a point P of its boundary $\partial\mathcal{M}$,
- a simple loop γ based at P such that $\gamma \setminus \{P\}$ lies in the interior of \mathcal{M} .

In particular

$$\{\partial\mathcal{M}\} = 2\{\gamma\} \in \pi_1(\mathcal{M})$$

and the class of γ generates the fundamental group of \mathcal{M} .

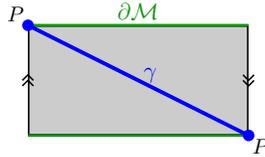


FIGURE 6. The Moebius strip.

We now define a *Moebius telescope* \mathcal{T}_n of height n as follows. Let $\{\mathcal{M}_k\}_{k \in \mathbb{N}}$ be an infinite number of copies of the Moebius strip \mathcal{M} . Start with $\mathcal{T}_1 = \mathcal{M}_0$ and then define by induction

$$\mathcal{T}_{n+1} = \mathcal{T}_n \cup_{\phi_n} \mathcal{M}_n$$

where ϕ_n is a homeomorphism between γ_n and $\partial\mathcal{M}_{n-1}$ that sends P_n to P_{n-1} . All gluing homeomorphisms will be chosen to be piecewise linear in the sequel.

Observe that

- $\mathcal{T}_1 \subset \dots \subset \mathcal{T}_{n-1} \subset \mathcal{T}_n$,
- all points $P_i \in \mathcal{M}_i$ for $i = 0, \dots, n-1$ glues onto a same point denoted P ,
- γ_0 is a deformation retract of \mathcal{T}_n , and thus $\pi_1(\mathcal{T}_n) = \mathbb{Z}$,
- $\{\gamma_i\} = 2^i\{\gamma_0\}$ for $i = 0, \dots, n-1$.

Fix any integer m and define the smallest integer n such that $m < 2^{n+1}$. The dyadic decomposition of m writes $m = 2^{n_1} + \dots + 2^{n_s}$ for some integers $0 \leq n_1 < \dots < n_s = n$. Let $\xi(m) = \gamma_{n_1} \star \gamma_{n_2} \star \dots \star \gamma_{n_{s-1}} \star \partial\mathcal{M}_{n-1} \in \mathcal{T}_n$ be the loop based at P (\star denoting the concatenation operation for based loops). Consider the 2-cell complex

$$X_m = \mathcal{T}_n \cup_{\xi(m)} D^2$$

where the boundary of the 2-disk D^2 is glued along the curve $\xi(m)$. Because $\{\xi(m)\} = m\{\gamma_0\}$ we get $\pi_1(X_m) = \mathbb{Z}_m$.

We shall construct an economic triangulation for \mathcal{T}_n , and then for X_m . Start with the minimal triangulation of $\mathbb{R}P^2$ which consists of 10 triangles (see Figure 1 and compare with Figure 7). We fix a vertex P and choose a simplicial loop γ based at P which generates $\pi_1(\mathbb{R}P^2)$ such as in Figure 7. By deleting the interior of a triangle one of whose vertices is P but which is not adjacent to any edge of the loop γ , we obtain a triangulation of \mathcal{M} with 9 triangles, see Figure 7.

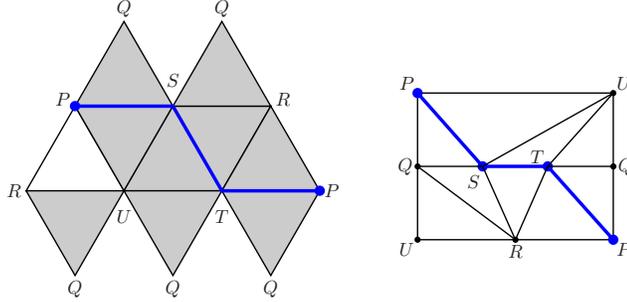


FIGURE 7. A special triangulation of the Moebius strip.

If each Moebius strip of the Moebius telescope of height n is endowed with this triangulation, we get a triangulation of \mathcal{T}_n with $9n$ triangles. Observe that the loop $\xi(m)$ consists of exactly $3s$ edges. Now triangulate the 2-disk D^2 by using at most $5s + 3$ triangles as in Figure 8 (compare with the proof of Proposition 2.1).

The triangulation of X_m thus obtained satisfies

$$s_2(X_m) \leq 9n + 5s + 3 \leq 14n + 3 \leq 14 \log_2 m + 3$$

which concludes the proof of the Lemma. \square

Remark 4.2. The group \mathbb{Z}_m can be realized as the fundamental group of a 2-cell complex with only one 2-cell. It can be shown that this complex can not be triangulated with less than $3m$ triangles, the boundary of the 2-cell being mapped into the 1-skeleton by a PL-map of degree m . The Moebius telescope thus shows that to get an economic simplicial

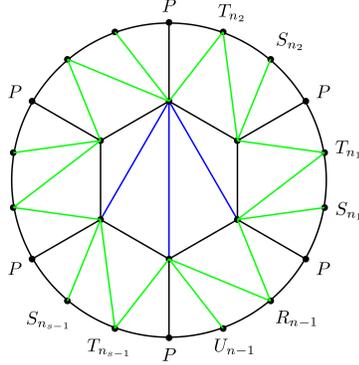


FIGURE 8. Triangulation of D^2 for $s = 6$.

complex whose fundamental group is \mathbb{Z}_m we first have to start with a 2-cell complex with roughly $\log_2 m$ cells of dimension 2.

Remark 4.3. For $m = 2^n + 1$, the proof of Lemma 4.1 implies that

$$\kappa(\mathbb{Z}_m) \leq 14 \log_2(m - 1) + 3.$$

For $m = 3$ this upperbound is sharp (see the table in Example 1) and the Moebius telescope gives thus the minimal complex for \mathbb{Z}_3 . It is natural to ask whether $\kappa(\mathbb{Z}_m) = 14 \log_2(m - 1) + 3$ for $m = 2^n + 1$.

We now prove the general upper bound of Theorem 4.1. Consider the decomposition

$$G = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \dots \oplus \mathbb{Z}_{n_s}$$

where $n_1 | n_2 | \dots | n_s$.

For $k = 1, \dots, s$ take the economic 2-simplicial complex X_{n_k} constructed in the proof of Lemma 4.1 with fundamental group \mathbb{Z}_{n_k} . By gluing all the points $P_k \in X_{n_k}$ for $k = 1, \dots, s$ into the same point P , we obtain a 2-simplicial complex

$$\bigvee_{k=1}^s X_{n_k}$$

with at most

$$\sum_{k=1}^s (14 \log_2 n_k + 3) = 14 \log_2 |G| + 3s$$

2-simplices.

The fundamental group of this wedge sum is the group

$$\mathbb{Z}_{n_1} * \dots * \mathbb{Z}_{n_s}.$$

In order to get a simplicial complex whose fundamental group is G we fix for each $k = 1, \dots, s$ a loop $\alpha_k \subset X_{n_k}$ based at P , consisting of tree edges and generating the fundamental group $\pi_1(X_{n_k}) \simeq \mathbb{Z}_{n_k}$. For each $1 \leq k < l \leq s$ we glue a minimal triangulated 2-torus to our 2-simplicial complex by identifying the pair of loops (α_k, α_l) with a pair of loops (α', α'') of the minimal 2-torus as depicted in Figure 2. We thus get a new 2-simplicial complex X_G with fundamental group G as each pair (α_k, α_l) now commutes. Because we add $7s(s-1)$ triangles (14 triangles for each minimal 2-torus) we have that

$$s_2(X_G) \leq 14 \log_2 |G| + 3s + 7s(s-1) = 14 \log_2 |G| + 7s^2 - 4s,$$

which concludes the proof of Theorem 4.1. \square

5. FROM SIMPLICIAL COMPLEXITY TO SYSTOLIC AREA FOR GROUPS

In this section we first prove Theorem 1.1. Then we show how to derive inequality (1.5).

5.1. Proof of Theorem 1.1. We start with the proof of the right-hand side inequality in Theorem 1.1.

By Theorem 1.2 we know that

$$\kappa(G) \leq C \sigma(G)^{1 + \frac{C'}{\sqrt{\log_2(\sigma(G))}}}$$

with $C = \frac{2^2 \cdot 5^8}{3}$ and $C' = 1 + \ln 25$. This implies the inclusion

$$\mathcal{G}_\sigma(T) \subset \mathcal{G}_\kappa \left(CT^{1 + \frac{C'}{\sqrt{\log_2 T}}} \right).$$

This together with the right-hand side inequality in Theorem 1.3 gives that

$$|\mathcal{G}_\sigma(T)| \leq 2^{6K \log_2 K},$$

where $K = CT^{1 + \frac{C'}{\sqrt{\log_2 T}}}$. For $T \geq 2$ we compute that

$$\begin{aligned} 6K \log_2 K &= 6CT^{1 + \frac{C'}{\sqrt{\log_2 T}}} (\log_2 T + C' \sqrt{\log_2 T} + \log_2 C) \\ &\leq 6CT^{1 + \frac{C'}{\sqrt{\log_2 T}}} (25 \log_2 T) \\ &\leq 6CT^{1 + \frac{B'}{\sqrt{\log_2 T}}} \end{aligned}$$

where $6C = 2^3 \cdot 5^8$ and $B' = 9$. This implies the right-hand side inequality in Theorem 1.1 with $B = 2^{6C} = 2^{2^3 \cdot 5^8}$ and $B' = 9$.

We now turn to the proof of the right-hand side inequality in Theorem 1.1.

The first strategy we can try is the following. According to Theorem 1.2 we have

$$\sigma(G) \leq \frac{\kappa(G)}{2\pi}$$

which implies the inclusion

$$\mathcal{A}_\kappa(T) \subset \mathcal{A}_\sigma\left(\frac{T}{2\pi}\right).$$

Thus for any positive T

$$\left|\mathcal{A}_\sigma\left(\frac{T}{2\pi}\right)\right| \geq |\mathcal{A}_\kappa(T)| \geq \left\lceil 2^{\frac{T-3}{14}} \right\rceil$$

according to Corollary 4.1. But this lower bound is not as good as the left-hand side of the inequality stated in Theorem 1.1. To improve our estimate we proceed as follows.

We construct a metric version of the Moebius telescope. Indeed consider the Moebius strip endowed with the Riemannian metric of curvature 1 given by the quotient of a spherical strip by the central symmetry like in Figure 9.

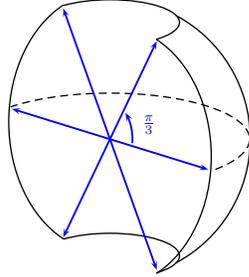


FIGURE 9. Spherical strip.

The height of this spherical strip is chosen to be $2\pi/3$ and its area is thus equal to $2\pi\sqrt{3}$. The equatorial curve and the boundary have the same length π . The metric Moebius telescope is constructed in the same way as in Lemma 4.1 using this special metric on each Moebius strip. Then in order to obtain a metric version of the space X_m we glue along the loop $\xi(m)$ a disk of constant curvature 1 and radius $\frac{\pi}{2}$ having at its center a conical singularity. The angle of the conical singularity is chosen such that the length of the boundary equals the length of the curve $\xi(m)$. Recall that if n is the smallest integer such that $m < 2^{n+1}$, the dyadic decomposition of m writes $m = 2^{n_1} + \dots + 2^{n_s}$ for some integers $0 \leq n_1 < \dots < n_s = n$. In particular, the length of $\xi(m)$ is at most $\pi s \leq \pi n$ and the area of the 2-disk is thus also at most πn . The systole of this metric on X_m equals π which implies that

$$\sigma(\mathbb{Z}_m) \leq \frac{1 + 2\sqrt{3}}{\pi} \log_2 m.$$

We thus obtain the right-hand side of inequality (1.5), and the left-hand side inequality in Theorem 1.1:

$$|\mathcal{A}_\sigma(T)| \geq \left\lceil 2^{\frac{\pi}{1+2\sqrt{3}}T} \right\rceil.$$

5.2. Systolic area of \mathbb{Z}_m . For a finite abelian group G , the behaviour of $\sigma(G)$ in terms of the number $|G|$ of its elements and the number of its invariant factors can be described by combining Theorems 1.2 and 4.1. First we easily derive the following.

Theorem 5.1. *Let G be a finite abelian group with s invariant factors. Then*

$$\sigma(G) \leq \frac{1}{2\pi} (14 \log_2 |G| + 7s^2 - 4s).$$

Now recall that given such a finite abelian group G we have

$$\kappa(G) \geq 2 \log_3 |G|$$

and

$$\kappa(G) \leq C \sigma(G)^{1 + \frac{C'}{\sqrt{\log_2 \sigma(G)}}}$$

where $C = \frac{2^2 \cdot 5^8}{3}$ and $C' = 1 + \ln 25$ according to Theorems 1.2 and 4.1. We get

$$\log_2 \left(\frac{2 \log_3 |G|}{C} \right) \leq C' \sqrt{\log_2 \sigma(G)} + \log_2 \sigma(G)$$

which implies that

$$\sqrt{\log_2 \sigma(G)} \geq \frac{\sqrt{C'^2 + 4 \log_2 \left(\frac{2 \log_3 |G|}{C} \right)} - C'}{2}.$$

Now set $\varphi : [1, \infty[\rightarrow \mathbb{R}$ the decreasing function given by the formula

$$\phi(x) = \frac{2}{1 + \sqrt{1 + \frac{4 \log_2 x}{(C')^2}}}.$$

Theorem 5.2. *A finite abelian group G satisfies the following inequality:*

$$\sigma(G) \geq \left(\frac{2 \log_3 (|G|)}{C} \right)^{1 - \phi \left(\frac{2 \log_3 (|G|)}{C} \right)}.$$

Because $\phi(x) \sim \frac{C'}{\sqrt{\log_2 x}}$ for large values of x , and that $C' \simeq 4.22$, we obtain the following almost logarithmic lower bound on $\sigma(\mathbb{Z}_m)$ for large m .

Corollary 5.1. *Provided m is large enough*

$$\sigma(\mathbb{Z}_m) \geq \frac{2 \log_3 2}{C} (\log_2 m)^{1 - \frac{5}{\sqrt{\log_2 (\log_2 m)}}}$$

where $C = \frac{2^2 \cdot 5^8}{3}$.

6. APPLICATIONS TO SYSTOLIC VOLUME OF HOMOLOGY CLASSES

In this section we first recall the definition of systolic volume associated to a homology class of a group, and then explain how to derive an interesting lower bound for the systolic volume in terms of the 1-torsion of the group using the notion of simplicial complexity.

Let $\mathbf{a} \in H_n(G, \mathbb{Z})$ be a n -dimensional homology class of a group G where n denotes some positive integer. A *geometric cycle* (X, f) representing the class \mathbf{a} is a pair (X, f) where X is an orientable pseudomanifold X of dimension n and $f : X \rightarrow K(G, 1)$ a continuous map such that $f_*[X] = \mathbf{a}$ where $[X]$ denotes the fundamental class of X and $K(G, 1)$ the Eilenberg-MacLane space of G . The representation is said to be *normal* if in addition the induced map $f_{\#} : \pi_1(X) \rightarrow G$ is an epimorphism. Given a geometric cycle (X, f) and a piecewise smooth metric g on X , the *relative homotopic systole* $\text{sys}_f(X, g)$ is defined as the least length of a loop γ of X such that $f \circ \gamma$ is not contractible. The *systolic volume* of (X, f) is then the number

$$\mathfrak{S}_f(X) := \inf_g \frac{\text{vol}(X, g)}{\text{sys}_f(X, g)^n},$$

where the infimum is taken over all piecewise smooth metrics g on X and $\text{vol}(X, g)$ denotes the n -dimensional volume of X . When $f : X \rightarrow K(\pi_1(X), 1)$ is the classifying map (induced by an isomorphism between the fundamental groups), we simply denote by $\mathfrak{S}(X)$ the systolic volume of the pair (X, f) . From [Gro83, Section 6] we know that for any dimension n

$$\mathfrak{S}_n := \inf_{(X, f)} \mathfrak{S}_f(X) > 0,$$

the infimum being taken over all geometric cycles (X, f) representing a non trivial homology class of dimension n . The following notion was introduced by Gromov in [Gro83, Section 6].

Definition 6.1. The *systolic volume* of a homology class $\mathbf{a} \in H_n(G, \mathbb{Z})$ is defined as the number

$$\mathfrak{S}(\mathbf{a}) := \inf_{(X, f)} \mathfrak{S}_f(X)$$

where the infimum is taken over all geometric cycles (X, f) representing the class \mathbf{a} .

Observe that for any homology class $\mathbf{a} \in H_2(G, \mathbb{Z})$ we have

$$\mathfrak{S}(\mathbf{a}) \geq \sigma(G).$$

Recently the systolic volume of homology classes has been extensively studied in [BB10] where the reader can find numerous results on this invariant.

6.1. A lower bound of systolic volume by 1-torsion. We now define the 1-torsion of a homology class and explain how to use it to bound from below its systolic volume.

Definition 6.2. The *1-torsion* of a homology class $\mathbf{a} \in H_n(G, \mathbb{Z})$ is defined as the number

$$t_1(\mathbf{a}) = \inf_{(X, f)} |\text{Tors } H_1(X, \mathbb{Z})|$$

where the infimum is taken over the set of geometric cycles (X, f) representing the class \mathbf{a} and $|\text{Tors } H_1(X, \mathbb{Z})|$ denotes the number of torsion elements in the first integral homology group of X .

We now present the main result of this section.

Theorem 6.1. *Let G be a finitely presentable group and $\mathbf{a} \in H_n(G, \mathbb{Z})$. Then*

$$\mathfrak{S}(\mathbf{a}) \geq C_n (\ln t_1(\mathbf{a}))^{1 - \frac{c'_n}{\sqrt{\ln(\ln t_1(\mathbf{a}))}}},$$

where C_n and c'_n are two positive numbers depending only on n .

In particular,

$$\mathfrak{S}(\mathbf{a}) \geq (\ln t_1(\mathbf{a}))^{1-\varepsilon}$$

for any $\varepsilon > 0$ provided $t_1(\mathbf{a})$ is large enough. It is important to remark that there is no hope in dimension ≥ 3 to prove a universal lower bound of the type

$$\mathfrak{S}(G, \mathbf{a}) \geq C \ln t_1(\mathbf{a})$$

for some positive constant C . Indeed, for any positive integer s , the Eilenberg-MacLane space of the group

$$G_s := \underbrace{\mathbb{Z}_2 * \dots * \mathbb{Z}_2}_s$$

is the complex $\bigvee_{i=1}^s \mathbb{R}P_i^\infty$. If n is a positive integer and $\mathbb{R}P_i^{2n+1} \subset \mathbb{R}P_i^\infty$ denotes the skeleton of odd dimension $2n+1$ of the i -th component, we consider the sequence of homology classes

$$\mathbf{a}_s = \sum_{i=1}^s [\mathbb{R}P_i^{2n+1}] \in H_{2n+1}(G_s, \mathbb{Z}).$$

We can see that $t_1(\mathbf{a}_s) = 2^s$ as $|\text{Tors } H_1(X, \mathbb{Z})| \geq 2^s$ for any representation (X, f) of \mathbf{a} . According to [BB10, Theorem 5.4] this implies that

$$\mathfrak{S}(G_s, \mathbf{a}_s) \leq \mathfrak{S}(\#_{i=1}^s \mathbb{R}P_i^{2n+1}) \leq C \cdot \frac{\ln t_1(\mathbf{a}_s)}{\ln \ln t_1(\mathbf{a}_s)}$$

for some positive constant C . For even dimension, we consider the sequence of classes $\tilde{\mathbf{a}}_s = [S^1] \times \mathbf{a}_s \in H_{2n+2}(\mathbb{Z} \times G_s, \mathbb{Z})$ for which the same upper bound holds.

Proof of Theorem 6.1. Let G be a finitely presentable group and $\mathbf{a} \in H_n(G, \mathbb{Z})$ a homology class. Recall that the *simplicial height* $h(\mathbf{a})$ of a homology class \mathbf{a} is the minimum number of simplexes (of any dimension) of a geometric cycle representing the class \mathbf{a} . According to [Gro83, 6.4.C"] (see also [Gro96, 3.C.3]) there exists two positive numbers c_n and c'_n depending only on the dimension n such that

$$\mathfrak{S}(\mathbf{a}) \geq c_n \cdot \frac{h(\mathbf{a})}{\exp(c'_n \sqrt{\ln h(\mathbf{a})})}.$$

We conclude using Proposition 2.2 which asserts that

$$h(\mathbf{a}) \geq 2 \log_3 t_1(\mathbf{a}).$$

□

6.2. Application to lens spaces. In general the 1-torsion of a class is difficult to compute. In the case of $G = \mathbb{Z}_m$, we can estimate from below the 1-torsion of any generator by the number m as follows.

Lemma 6.1. *Let \mathbf{a} be a generator of $H_{2n+1}(\mathbb{Z}_m, \mathbb{Z})$. Then*

$$t_1(\mathbf{a}) \geq m.$$

Proof. Let (X, f) be a geometric cycle representing \mathbf{a} . As \mathbf{a} is a generator of $H_{2n+1}(\mathbb{Z}_m, \mathbb{Z}) \simeq \mathbb{Z}_m$, the map f induces an isomorphism

$$(6.1) \quad f^* : H^{2n+1}(\mathbb{Z}_m, \mathbb{Z}_m) \rightarrow H^{2n+1}(X, \mathbb{Z}_m).$$

Let

$$\beta : H^1(\mathbb{Z}_m, \mathbb{Z}_m) \rightarrow H^2(\mathbb{Z}_m, \mathbb{Z}).$$

denotes the Bockstein homomorphism and

$$j : H^2(\mathbb{Z}_m, \mathbb{Z}) \rightarrow H^2(\mathbb{Z}_m, \mathbb{Z}_m)$$

the morphism of reduction modulo m . In our case, j is an isomorphism. A generator of $H^{2n+1}(\mathbb{Z}_m, \mathbb{Z}_m)$ (not necessarily dual to \mathbf{a}) can be chosen as the element $\mathbf{u} \cup (j \circ \beta(\mathbf{u}))^n$ where $\mathbf{u} \in H^1(\mathbb{Z}_m, \mathbb{Z}_m)$ is some generator. Now consider $f^*(\beta(\mathbf{u})) = \beta(f^*(\mathbf{u})) \in H^2(X, \mathbb{Z})$. Taking into account the isomorphism (6.1) we see that $f^*(\mathbf{u}) \cup (f^*(j \circ \beta(\mathbf{u})))^n$ is an element of order m in $H^{2n+1}(X, \mathbb{Z}_m)$. This implies that the order of $f^*(j \circ \beta(\mathbf{u})) \in H^2(X, \mathbb{Z}_m)$ is m , and thus that the order of $\beta(f^*(\mathbf{u})) \in H^2(X, \mathbb{Z})$ is also m . By the duality (2.7) we get the result. □

Remark that the statement of this lemma as well as its proof holds in the case of a simplicial complex X representing the class \mathbf{a} . In this more general case note that the map (6.1) is only into.

Given two integers $n \geq 0$ and $m \geq 2$ let $L_n(m)$ denote a lens space of dimension $2n + 1$ with fundamental group \mathbb{Z}_m : there exist integers q_1, \dots, q_m coprime with m and an isometry A of order m of the form

$$A(z_1, \dots, z_n) = (e^{2\pi i \frac{q_1}{m}} z_1, \dots, e^{2\pi i \frac{q_n}{m}} z_n)$$

such that

$$L_n(m) := \{Z = (z_1, \dots, z_n) \in \mathbb{C}^n \mid \sum_{k=1}^n |z_k|^2 = 1\} / \sim_A \simeq S^{2n+1} / \mathbb{Z}_m,$$

where $Z \sim Z'$ if and only if $Z = A^k Z'$. Observe that the fundamental class of a lens space $L_n(m)$ realizes a generator \mathbf{a} of the homology group $H_{2n+1}(\mathbb{Z}_m, \mathbb{Z})$.

Combining Lemma 6.1 and Theorem 6.1 we derive the following result.

Theorem 6.2. *For any integer $m \geq 2$ we have*

$$\mathfrak{S}(L_n(m)) \geq C_n (\ln m)^{1 - \frac{C'_n}{\sqrt{\ln(\ln m)}}}$$

where C_n and C'_n are two positive numbers depending only on the dimension n .

We remark that this lower bound is of the same type as for $\sigma(\mathbb{Z}_m)$ in Corollary 5.1. Now we turn to the proof of the new upper bound for systolic volume of lens spaces.

Theorem 6.3. *For any integer $m \geq 2$ we have*

$$\mathfrak{S}(L_n(m)) \leq D_n m^n$$

where D_n is a positive number depending only on the dimension n .

As observed in the introduction, this polynomial upperbound is better than the one obtained by computing the systolic volume for the round metric (which is roughly $\approx m^{2n}$).

Proof. We first decompose the manifold $L_n(m)$ into $(2n + 1)$ -dimensional cubes, and then use this decomposition to construct a metric for which we control the systolic volume in terms of the number of these cubes, compare with [BB05] and [BB10].

We decompose $L_n(m)$ into $(2n + 1)$ -cubes as follows. Start with the standard cellular decomposition of $L_n(m)$ denoted by Θ , see [Hat02, p.145] generalizing for the case $n > 1$ the construction of [ST80]. This decomposition Θ has exactly one k -cell denoted by e^k in each dimension $k = 0, 1, \dots, 2n + 1$ and we denote by q_k its center. We subdivide the cellular decomposition Θ in such a way that each new cell admits a simplicial structure. We proceed by induction as follows. For $k = 0$ there is nothing to do but observe that $e^0 = \{q_0\}$. Because e^1 is attached to e^0 we subdivide e^1 into the two arcs denoted by e_0^1 and e_1^1 connecting q_0 and q_1 . Observe at this stage that the complex thus obtained is not a simplicial complex, as two simplices may share more than one face in common. The next step consists to first remark that e^2 is attached to $e^1 \cup e^0$ through a linear map $\phi_2 : \partial e^2 \rightarrow e^1 \cup e^0$ of degree m , and then take in e^2 the cone over the preimages by ϕ_1 of e_0^1 and e_1^1 with respect to the vertex q_2 . Doing so we subdivide e^2 into $2m$ new 2-cells with a natural structure of simplex. We then proceed that way by induction on the dimension following the structure of Θ_1 . More precisely at each step $k \geq 3$ we form the cone over the preimages by the attaching map of the $k - 1$ -dimensional new cells with respect to the center of the k -cell. The attaching maps

$$\phi_k : \partial e^k \longrightarrow \bigcup_{s=0}^{k-1} e^s$$

having degree m if k is even, and zero if k is odd, this gives a new decomposition Θ_1 with $2^{n+1}m^n$ simplices of dimension $2n + 1$. Remark that despite the fact that Θ_1 is not a simplicial decomposition this structure is coherent in the sense that any face of a simplex is a simplex of lower dimension.

Denote by Θ_2 the barycentric subdivision of Θ_1 . The structure Θ_2 is now simplicial with $2^{n+1} \cdot (2n + 1)! \cdot m^n$ simplices of dimension $2n + 1$. We decompose each $(2n + 1)$ -simplex

of Θ_2 into $2(n+1)$ cubes of dimension $2n+1$. This gives a decomposition Θ_3 of $L_n(m)$ into $2^{n+1} \cdot (2(n+1))! \cdot 2(n+1) \cdot m^n$ cubes of dimension $2n+1$. Endow each cube of Θ_3 with the Euclidean metric with side length 1. We thus get a polyhedral metric g on $L_n(m)$. This metric satisfies $\text{sys}(L_n(m, g)) \geq 2$ according to [BB10, Lemma 5.6]. Because $\text{vol}(L_n(m, g)) = 2^{n+1} [2(n+1)]! [2(n+1)] m^n$, this gives the result with $D_n = \frac{[2(n+1)]!(n+1)}{2^{n+1}}$. \square

6.3. Application to 3-manifolds. Let M be a closed manifold. For a covering space M' with k sheets of M , it is straightforward to check that

$$(6.2) \quad \mathfrak{S}(M) \geq \frac{1}{k} \mathfrak{S}(M').$$

Let M be a manifold of dimension 3 with finite fundamental group. Its universal cover is the sphere S^3 , and the action of the fundamental group $\pi_1(M)$ on S^3 is orthogonal. The list of finite groups which act orthogonally on S^3 can be found in [Mil57] for instance. An analysis of this list shows that $\pi_1(M)$ possesses a cyclic subgroup of index $k \leq 12$. Denote by M' the covering space corresponding to this subgroup. The manifold M' is a lens space $L_1(n)$ with $n \geq \frac{|\pi_1(M)|}{12}$. So by applying Theorem 6.2 and the inequality (6.2) with $k = 12$ we derive the last theorem of this paper.

Theorem 6.4. *There exists two positive constants a and b such that any manifold M of dimension 3 with finite fundamental group satisfies*

$$\mathfrak{S}(M) \geq a (\ln |\pi_1(M)|)^{1 - \frac{b}{\sqrt{\ln(\ln |\pi_1(M)|)}}},$$

where $|\pi_1(M)|$ denotes the number of elements in $\pi_1(M)$.

Remark that finite fundamental groups of 3-manifolds can have a large number of elements but a very small torsion in $H_1(\pi_1(M), \mathbb{Z})$. A direct estimate of $\mathfrak{S}(M)$ by the torsion of $H_1(\pi_1(M), \mathbb{Z})$ is interesting only if the manifold M is a lens space.

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