HAL archives-ouvertes.fr

Abstract Algebra, Mathematical Structuralism and Semiotics

Thomas Hausberger

▶ To cite this version:

Thomas Hausberger. Abstract Algebra, Mathematical Structuralism and Semiotics. 2014. <hal-01083768>

HAL Id: hal-01083768
https://hal.archives-ouvertes.fr/hal-01083768

Submitted on 17 Nov 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ABSTRACT ALGEBRA, MATHEMATICAL STRUCTURALISM AND SEMIOTICS

Thomas Hausberger

Université Montpellier 2

We report in this paper on our attempt to help students reflect on the axiomatic method in mathematics and structuralist thinking through a didactically-engineered activity, as a lever to tackle the issue of the learning of abstract algebra. It sheds light into the cognitive processes involved in the conceptualization of an abstract algebraic structure, which are discussed in a semiotics framework.

INTRODUCTION

This article focuses on the teaching and learning of abstract algebra (the discipline dedicated to the study of algebraic structures, that is the investigation of logical consequences of specific systems of axioms involving composition laws, and the relationships among them) which is taught at Montpellier University at the third-year University level. The difficulties are acknowledged by several authors (Leron & Dubinsky 1995, Nardi 2000, Hausberger 2013) and reflect a "transition problem" (Guedet 2008) which, in the present case, occurs inside the University curriculum.

The epistemological analysis presented in Hausberger 2013 led us to a connection with the following epistemological transitions: "the systematization of the axiomatic method, after Hilbert, and the transition, after Noether, from thinking about operations on elements to thinking in terms of selected subsets and homomorphisms". Indeed, as emphasized by Cory (2007) :

> This image of the discipline turned the conceptual hierarchy of classical algebra upside-down. Groups, fields, rings and other related concepts, appeared now at the main focus of interest, based on the implicit realization that all these concepts are, in fact, instances of a more general, underlying idea: the idea of an algebraic structure.

In other words, this epistemological gap leads to the *vanishing of concrete mathematical objects in favor of hovering abstract structures*. In our view, this induces the following didactical problems: the teaching of abstract algebra tends to present a *semantic deficiency* regarding mathematical structures, which are defined by abstract axiomatic systems and whose syntaxic aspects prevail. For instance, how does the student build an "abstract group concept"? Indeed, what kind of representations can he rely on to do so when the purpose is to discard the particular nature of elements, in other words the mathematical context? Moreover, the investigation of the didactic transposition of the notion of structure shows that it is a *meta-concept* that is never mathematically defined in any course or textbook (and cannot be so):

> As a consequence, students are supposed to learn by themselves and by the examples what is meant by a structure whereas sentences like "a homomorphism is a structure-

preserving function" is supposed to help them make sense of a homomorphism (Hausberger 2013).

As announced in loc. cit., we have engineered an activity for students to reflect on the axiomatic method and structuralist thinking in a simple context (much less complex than group theory): the *theory of banquets*, a didactical invention that cannot be found in any algebra textbook. It aims at operating the fundamental *concrete-abstract and syntax-semantic dialectics* (see below) and at clarifying the concept of mathematical structure using the meta lever (Dorier & al. 2000), that is "the use, in teaching, of information or knowledge *about* mathematics. […]. This information can lead students to reflect, consciously or otherwise, both on their own learning activity in mathematics and the very nature of mathematics" (loc. cit. p. 151).

The purpose of this article is to present a few results that were obtained through the experimentation of the first part of the activity. It tackles the following questions: what kind of cognitive processes and reasoning do students use to make sense of a "discarnated" axiomatically-presented structure such that the banquet structure? How do they engage in the task of *classifying models* of the axiomatic system (and interpret the task: for instance, what kind of representations do they use, do they formalize a concept of isomorphism of banquets)? What kind of abstract banquet structure concept do they build through the completion of such a task? Similar to the context of classical algebra in secondary education, semiotics will give interesting tools to answer these questions. Yet, some adaptation needs to be made to reflect the context of abstract algebra and interpret structures in terms of semiotics, since these represent a higher level of organization compared to the classical mathematical objects that they formalize, generalize and unify.

EPISTEMOLOGICAL AND DIDACTICAL FRAMEWORKS

Abstraction

The French verb *abstraire* has three differents meanings : 1. to discard ("faire abstraction de") 2. to isolate (from a context) 3. to construct (a concept). Although these are three different actions, they may take place in order to reach a common goal as is the case in abstract algebra: mathematicians disregard the particular nature of elements and isolate relations to build the structure as an abstract concept.

The "principle of abstraction" as a process to create concepts has been used by Frege to define cardinal numbers (Frege 1884). To introduce the reader to this revolutionary idea, Frege gives the enlightening example of the direction of a line which is defined as the class of all lines that are parallel to the given line. The principle was formalized later on by Russell (1903): to say that "things are equal because they have some property in common" and reduce a class to a single element, the relation that traduces this property should be symmetric and transitive (an equivalence relation).

Semiotics

Just as a language is compulsory to express any idea, mathematical objects are accessed through mathematical signs. We will use Frege's semiotics and make the distinction between the sense and the denotation of a sign (Frege 1892). The denotation of the sign is the object it refers to whereas the sense is related to the "mode of presentation" of the object. Mathematical signs are often polysemic but the context is meant to determine the reference uniquely. Conversely, different signs may represent the same object, thus having a different sense but a common denotation. In this way, different representations may bring light to different aspects of an object; they are acknowledged as denoting one and the same object through the realization that a particular processing or conversion of semiotic register of representation (Duval 1995) allows to transform one representation into the other, and reciprocally. In other words, as stated and illustrated by Winslow (2004, p.4), we may "think of objects as *signs modulo object preserving transformations* (OPT)":

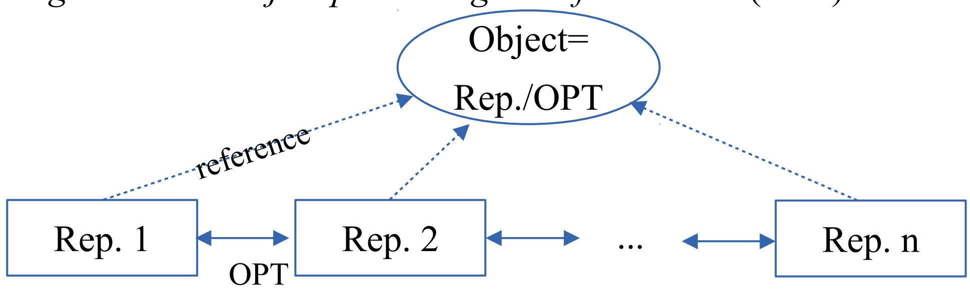


Figure 1

Syntax and semantic

Mathematical signs are organized within sentences and formulae that are built according to strict syntaxic rules. On a logical point of view, a definition is an "open sentence" that may be satisfied or not when the variables of the sentence are assigned in a suitable universe of discourse: this is the semantic conception of truth introduced by Tarski (1944; see also Durand-Guerrier 2003 for a more detailed account and didactical applications). In this respect, a data that satisfies the definition of a mathematical structure (which involves a set of axioms that forms its syntaxic content) may be called a *model* of the structure (in the given universe of discourse or hosting theory). The models will be regarded as the semantic content of the axiomatically-defined mathematical structure, its extension as a concept.

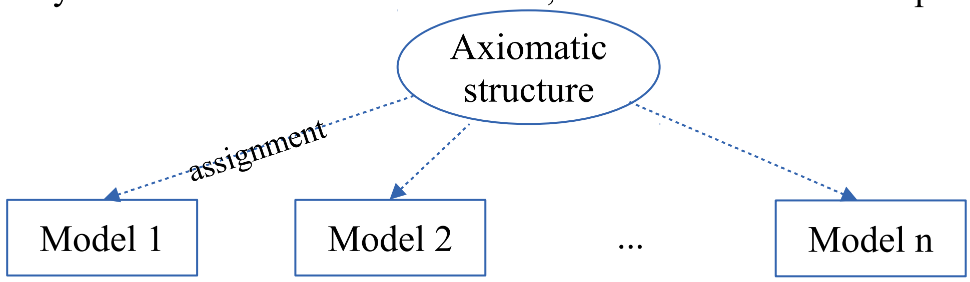


Figure 2

Tarski also defined the notion of logical consequence on a semantic point of view. We will use it below to show that a given axiom A_1 cannot be deduced from other axioms A_i: it amounts to showing the existence of a model satisfying the A_i's but not A_1. This contrasts with syntaxic methods which consists in deduction by application of valid rules of inference.

Structural objects

In a famous dispute with Hilbert, Frege argues against the legitimacy of abstract definitions by systems of axioms. One argument concerns the intrinsic polysemy of such definitions: in semiotic terms, an axiomatic definition as a sign has multiple references, the models of the axiomatic. In abstract definitions, the context doesn't inform on the denotation simply because it is abstracted (in meaning 1 of the verb).

In order to build an abstract structure (group, ring, banquet, etc.) concept, and therefore give a more adequate (yet still polysemic) semantic to the set of axioms as a sign, one needs to use structure preserving applications (SPT), the so-called isomorphisms, which are defined as relation-preserving bijections (all models may be viewed as sets endowed with additional data which defines relations and satisfy the axioms). This allows us to associate to an axiomatic structure its "structural objects" (our terminology), the isomorphism classes of models or models modulo SPT, in the same manner as mathematical objects were built from representations modulo OPT, by means of the principle of abstraction:

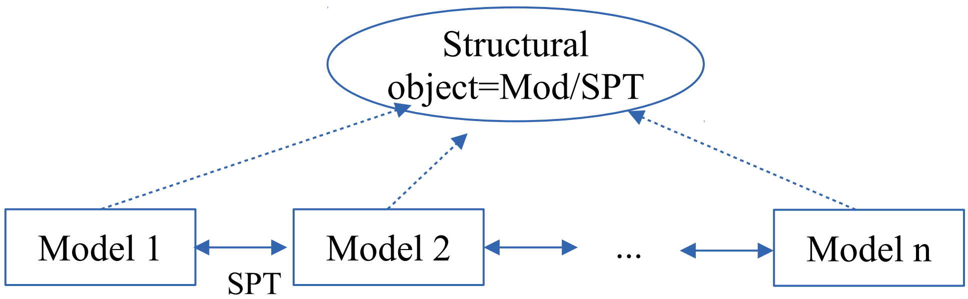


Figure 3

It should be pointed out that, compared to Winslow's diagram, dotted arrows do not represent the denotation of a sign but only "quotient maps". Since models are accessed through signs, the preceding diagram should in fact be reprinted, to reflect semiotic views, replacing each model by one of its representation and SPT by its semiotic version SOPT (structural object preserving transformations). Dotted arrows may then represent denotation when the context indicates a structural perspective: for instance, Z/2Zx Z/2Z and "the symmetry group of a rectangle" may both refer to the Klein 4-group V_4, as an abstract group concept. One may also write $V_4 = \langle a,b;a^2=b^2=(ab)^2=1\rangle$ (presentation by generators and relations) for a more syntaxical description. Nevertheless, the abstract structure (for instance of a group) remains more a concept than an object: since mathematicians take care of making the difference between a class and one representative, many authors would prefer to use the sign V_4 to denote the group Z/2Zx Z/2Z and say that it is isomorphic to the symmetry group of a rectangle or to the quotient of the free group on two generators by the relations $a^2=b^2=(ab)^2=1$. The idea behind structural objects is, following Sfard, that some kind of reification must occur for concept building: “Reification is defined as an ontological shift - a sudden ability to see something familiar in a totally new light” (Sfard 1991). For this to happen, we should need a plurality of models borrowed from different mathematical domains and represented in different semiotic registers. Similar to Winslow's context, the coordination of these representations (through SOPT) should be crucial to obtain a conceptual schema of the structural object. It should open the possibility to abstract from “templates” a “pattern” (Resnik 1997). Nevertheless, unlike Winslow's context, a representation of a model as a sign may now refer to both the model and the structural object (in a context where both appear), whereas a mathematical distinction must be kept. Solving this issue would require a more direct mediation of the structural object by a new adequate (to be precised) sign.

THE THEORY OF BANQUETS

As a didactical production, the theory of banquets was engineered on the basis of the epistemological analysis presented in Hausberger 2013 to cover the three usage contexts of the meta-concept “structure”: 1. the structure as defined by a system of axioms 2. the abstract structure (of a given group or banquet) 3. a 'structure-theorem' (which describes the way an object can be reconstructed from simpler objects of the same type). It is filled with meta-discourse, as is already visible in the worksheet title: “The theory of banquets: a mini-theory to reflect on structuralist thinking”, and begins with two alexandrins from the poem *Palais* (palace) by Appolinaire to materialize a change of contract and appeal to their creativity in “bringing the meat and flesh” to the banquet while *Le palais don du ciel comme un roi nu s'élève*[1] (the rising of the “discarnated” structure). The interested reader will be asked to email the author for a copy of the complete worksheet, since it is too long to be reproduced here in full details and has not been published yet.

The activity is divided into three parts : 1. logical investigation of the axiomatic system and classification of models 2. elaboration of an abstract theory of tables (this is the other way round: students are asked to formalize the disposition of guests around a round table) and structure-theorem for banquets (a banquet is the disjoint union of tables) 3. Connection with the theory of permutations (a reinterpretation of the banquet theory that permits to see the structure-theorem as a direct consequence of the well-known theorem of canonical cycle-decomposition of a permutation).

Part 1 and 2 clearly bring-in a concrete-abstract dialectic. We have chosen a top-bottom approach in part 1 for two reasons: as it is the standard exposition in manuals, we want to know how students make sense of such a definition and discuss with them the extension of the concept; moreover, part 1 will suitably enrich the didactical *milieu* for students to be able to model the situation given in part 2. Nevertheless, part

1 The palace gift of heaven as a naked king rises, our translation.

1 is already dialectical in itself: it amounts for students to move on from the still abstract and syntaxical conception of a structure exemplified by figure 2 to the more concrete and semantic conception of figure 3. The expected result of the abstract-concrete and syntax-semantic dialectics is a formulation of an abstract and syntaxic characterization of structural objects (question 2 d of the worksheet, see below).

The definition of a banquet is as follows: it is a set E (the objects) endowed with a binary relation R (encoding the relations between objects) which satisfies the following axioms: A1. No object fulfills xRx A2. If xRy and xRz then $y=z$ A3. If yRx and zRx then $y=z$. A4. For all x, there exists at least one y such that xRy.

Students were asked in part 1 the following questions:

1 a. Coherence: is it as valid mathematical theory, meaning are the axioms non-contradictory? In other words, does there exist a model?
1 b. Independence: is one of the axioms a logical consequence of others or are all axioms mutually independent?
2 a. Classify all banquets of order n≤3
2 b. Classify banquets of order 4
2 c. What can you say about **Z**/4**Z** endowed with $x R y \Leftrightarrow y = x + 1$?
2 d. How to characterize abstractly the preceding banquet (meaning its abstract structure of banquet among all the different classes of banquet, in fact how to characterize its class)?

Solving these questions amount to solving the following tasks and subtasks:

T1. Construct a model by suitable assignment of variables
T2. Classify banquets of a given order:
  ST2a. Define a notion of isomorphism
  ST2b. Give a list that covers representatives of all possible classes
  ST2c. Show that two elements of the list are non-isomorphic
T3. Show that 2 models are isomorphic by explicit construction of an isomorphism
T4. Characterize abstractly an isomorphism class

Note that answering question 1 b amounts to solving T1 through the semantic point of view of logical consequence (see above) and negation of an axiom. In doing so, the boundaries of the banquet concept will be marked out. We will need to focus first on T1 and give a list of available domains of interpretation for the axiomatic system and corresponding semiotic registers, since available representations greatly impact the other tasks.

Empirical interpretation: the name banquet may suggest by itself (or by reading the entire worksheet) guests around tables, so we define xRy if and only if x sits on the right of y. Note that proving that this universe of discourse can serve to interpret the whole banquet theory reduces to proving the structure-theorem. We could also imagine a rectangle table and pick up guest sitting face to face, as a particular model.

Set theory: the set E is described by naming its elements and the binary relation is represented by its graph inside E^2. This straightforward representation is not very interesting since it doesn't "encode" much structure.
Matrix theory: a binary relation may be seen as a function from E^2 to {0,1} (true/false), and therefore be represented by a double-entry table, in other words a matrix. In this interpretation, the axioms say that the diagonal contains only zeros, that there is exactly one 1 in each row and at least one in each column. In finite dimension, we can easily prove that there is exactly one 1 per row and column, hence it is a permutation matrix.
Graph theory: *xRy* if and only if vertices *x* and *y* are connected by an edge directed from *x* to *y*. The axioms say that from a vertex originates exactly one edge and terminates at most one; therefore, unlike general graph theory, is it easy to see when two drawings define the same graph.
Function theory: According to axioms A2 and A4, $x R y \Leftrightarrow y = f(x)$ defines a function *f* and the other axioms say that it is injective and has no fixed point. When the set *E* is finite, then *f* is a permutation without fixed points and we may use the standard semiotic representations for these (including cycle-decomposition).

It is in fact quite amazing to see the diversity of interpretations and models, which certainly reflects the unity and creativity of mathematics. Models may be represented in a mixed or purely symbolic register. When the graphical register is used, it may be a personal idealization of people around tables or an institutional representation borrowed from graph theory. Of course, we cannot expect students to connect to all these theories: for, instance, we bet that students won't translate the problem fully in the function setting and identify the connection with permutation theory (which would ruin part 3 of the activity). But we can wonder about the importance they may give to representations connected to everyday life (empirical setting). Moreover, models may be more or less "generic": compare *(E,f)*, *E*=**N** and *f(x)=x+1*, with a matrix that may serve to represent any binary operation. Students may also think that a model should be given by a mathematical formulae (like the example given in question 2 c), and restrict themselves to concrete examples in function theory, whereas a generic representation of *R* is necessary to complete the other tasks.

To give an idea about processes and conversions from one setting to the other, one should notice that a representation such as points marked on circles (empirical) is easily transformed into a graph by adding arrows clockwise between points; we may then associate to the graph its adjacency matrix, from which the function is soon reconstructed by reading the positions of the ones. When *E* is finite, the algorithm of cycle-decomposition of the permutation gives the tables (one per cycle) and the length of the cycle gives the number of people around each table, thus coming back to the empirical setting.

The pertinence of the setting (choice of a domain of interpretation) depends on the task: graph theory may easily suggests a model that verifies all the axioms except A2; matrix theory is quite pertinent for ST2b (a complete list of all possible relations), yet, graph theory again (or even better, a cyclic representation as obtained in the empirical setting) is best to decide if two banquets are isomorphic, as it gives a visual representation that makes common pattern visible and illustrates the etymology of isomorphism as a form-preserving mapping.

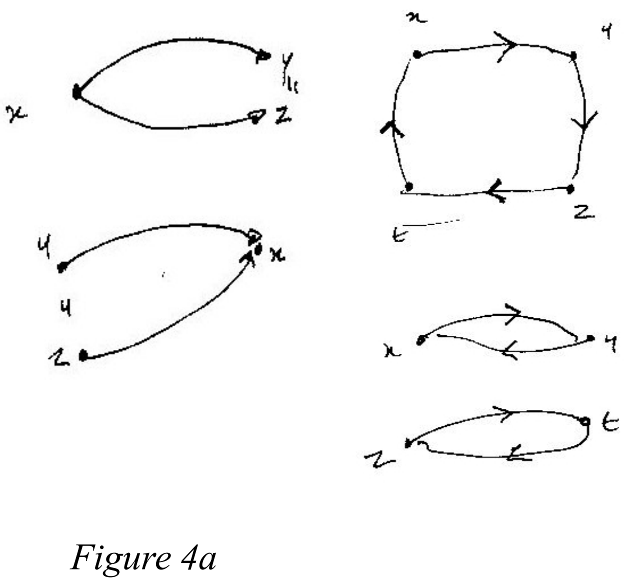
We will now present students' productions. Tasks 2 to 4 will be discussed in greater details while analyzing these.

EMPIRICAL DATA

The full banquet activity has been experienced during academic year 2013-14 with third year University students with a background in group theory before teaching ring and field theory. They worked in small groups of 4-5 students and were asked to keep a research notebook that was collected before each phase of institutionalization. In parallel, in more laboratory sessions, we videotaped two pairs of more advanced students (having a master's degree).

Due to lack of space, we will only give an account of the laboratory session with one of the pairs. Nevertheless, this will already be enough to give an idea of some interesting phenomena that could be observed by using our theoretical framework.

The pair of students tried to recognize the banquet structure as a pattern in known mathematical objects and theories ("what is it, what's this structure?"). Unlike in the classroom experiment, they didn't bring in wedding banquets; they first thought about the order relation, then analyzed the example $B_4=(\mathbf{Z}/4\mathbf{Z},R)$ of question 2c as a "kind of a shift" and generalized it ($E=\mathbf{Z}$, f(x)=x+1 or x-1). Semiotic representations of the semantic of axioms A2 and A3 in the graphical register (figure 4a) led them to build models in graph theory which they used for tasks T1b and T2. Recognition of cyclic patterns suggested permutations, as a common representation: they performed conversions of registers (but didn't connect to the function setting), producing the following classification which comprises 9 banquets of order 4 (figure 4b).



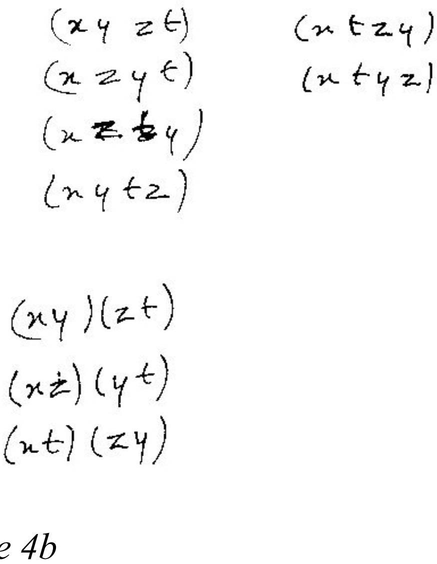


Figure 4a *Figure 4b*

Student A: Here, we are doing with what we know, yet we speak about a structure

Student B: Wait, we can always number the elements […]

Interviewer: For you, this is an abstract classification because you didn't consider particular relations and you can always rename elements x,y,z,t

A: So there would be 2 classes up to isomorphism?

B: Here Z/4Z and there Z/2Z x Z/2Z

I: You are thinking about the classification of groups […] So there are 2 types of objects and (x y z t) and (x z y t) would be the same?

B: Not the same, of the same type

The student B couldn't define what he meant by a type, he just made a connection between the word used by the interviewer and the notion of type of a permutation. The word bijection finally appeared but students found it difficult to define what "structure-preserving" meant. They drew the graph for (x z y t) but obtained crossing edges which confused them even more (both are identical as graphs but not as drawings). On the contrary, converting to a graph the example B_4 allowed connection to (x y z t) (obvious congruence of drawings). They didn't realize that abstracting the nature of elements simply meant forgetting letters, a mental process that make the recognition of isomorphism classes in the representation as cycle products automatic.

CONCLUSIONS AND PERSPECTIVES

This study clearly contributes to the recognition of the influence of semiotic representations in cognitive activities dedicated to the learning of abstract algebra. We have discussed the hypothesis that a logical investigation of an axiomatic system and the classification of its models up to isomorphism, in the paradigm of the "theory of banquets" which connects to group-theory, is a cognitive activity that could bring good conditions for learners to develop an appropriate conceptualization of an abstract structure, and in particular access what we called "structural objects". Our empirical data let us show an insufficient syntax-semantic dialectic and mental processes based on the recognition of (visual) patterns. Conversions of registers are operated by students in order to realize that two objects are isomorphic, which is successful when the congruence of representations is obvious, but they cannot handle the treatments inside a register since they cannot rely on a formal definition of an isomorphism or make this definition functional. This also traduces an incomplete understanding of abstraction as a process that leads to structural objects. Finally, many students tried to work out the analogy with group theory from which they borrowed directly or tried to adapt representations and concepts. As stated by Winslow (2004), "mathematical concepts are not learned one by one but as coherent patterns or structures", and this also happens at the superior level of structures themselves, thus gaining access to what we called level-2 unification (Hausberger 2012).

Our analysis of empirical data will be pursued in greater details in an expanded version of this article. It will also be interesting to experiment the banquet activity with students who have not yet been taught group theory and compare the results. We hope to gain by these investigations and refinements of our semiotic tools a better understanding of the students' difficulties in abstract algebra which are inherent in structuralist thinking, on a cognitive point of view.

REFERENCES:

Cory, L. (2007). History of Algebra. In *Encyclopaedia Britannica Online*.

Dorier, J-L., Robert, A., Robinet, J. et Rogalski, M. (2000). The meta lever, in Dorier (ed.) :*On the teaching of linear algebra*, Dordrecht : Kluwer Academic Publisher, 151-176.

Durand-Guerrier, V. (2003). Logic and mathematical reasoning from a didactical point of view: a model-theoretic approach. *Proceedings of CERME 3.*

Duval, R. (1995). Semiosis et pensée humaine. Registres sémiotiques et apprentissages intellectuels. Bern: Peter Lang.

Frege, G. (1884). Die Grundlagen der Arithmetic, Breslau: W. Koebner.

Frege, G. (1892). Über Sinn und Bedeutung, in *Zeitschrift für Philosophie und philosophische Kritik*, 100: 25–50.

Gueudet, G. (2008). Investigation the seconday-tertiary transition. *ESM* 67 (3), 237-254

Hausberger, T. (2012). Le challenge de la pensée structuraliste dans l'apprentissage de l'algèbre abstraite : une approche épistémologique. *Proceedings of EMF2012.*

Hausberger, T. (2013). On the concept of (homo)morphism: a key notion in the learning of abstract algebra. *Proceedings of CERME 9*, 2346-2355.

Leron, U., Dubinsky, E. (1995). An abstract algebra story. *American Mathematical Monthly* 102(3), 227-242.

Nardi, E. (2000). Mathematics Undergrates' Responses to Semantic Abbreviations, Geometric Images and Multi-level Abstractions in Group Theory. *ESM* 34, 169-189.

Resnik, M. (1997). Mathematics as a science of patterns. Oxford: Clarendon Press.

Russell, B. (1903). The Principles of Mathematics. Cambridge: Cambridge University Press.

Sfard, A.: 1991, 'On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin', Educational Studies in Mathematics 22, 1-36.

Tarski, A. (1944). The semantic conception of truth and the foundation of semantics, in *Philosophy and Phenomenological Research* 4, pp. 341-376.

Winsløw, C. (2004). Semiotics as an analytic tool for the didactics of mathematics. *Nordic studies in Mathematics Education* 9 (2), 81-100.