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VOGAN CLASSES AND CELLS IN THE UNEQUAL PARAMETER CASE

by

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***Abstract.* —** Kazhdan and Lusztig proved that Vogan classes are unions of cells in the equal parameter case. We extend this result to the unequal parameter case.

Let (W,S) be a Coxeter system and let $\varphi : S \to \mathbb{Z}_{>0}$ be a weight function. To this datum is associated a partition of W into left, right and two-sided cells. Determining these partitions is a difficult problem, with deep connections (whenever W is a finite or an affine Weyl group) with representations of reductive groups, singularities of Schubert cells, geometry of unipotent classes.

In their original paper, Kazhdan and Lusztig described completely this partition whenever W is the symmetric group in terms of the Robinson-Schensted correspondence. Their main tool is the so-called $*$-operation. It is defined in any Coxeter group (whenever there exists s, $t \in S$ such that st has order 3): they proved that it provides some extra-properties of cells, *whenever* φ *is constant*. Our aim in this paper is to prove that Kazhdan-Lusztig result relating cells and the $*$-operation holds in full generality (this result has also been proved independently and simultaneously by M. Geck [**Ge2**]).

Commentary. In [**Bon**], the author used improperly the $*$-operation in the unequal parameter context. The present paper justifies *a posteriori* what was, at that time, a big mistake!

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Notation. We fix in this paper a Coxeter system (W,S) and a totally ordered abelian group Γ. We use an exponential notation for the group algebra $A = \mathbb{Z}[\Gamma]$:

$$A = \bigoplus_{\gamma \in \Gamma} \mathbb{Z} v^{\gamma},$$

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with $v^\gamma v^{\gamma'} = v^{\gamma+\gamma'}$ for all $\gamma, \gamma' \in \Gamma$. If $\gamma_0 \in \Gamma$, we set

$$\Gamma_{\geqslant \gamma_0} = \{\gamma \in \Gamma \mid \gamma \geqslant \gamma_0\},$$
$$\Gamma_{>\gamma_0} = \{\gamma \in \Gamma \mid \gamma > \gamma_0\},$$
$$A_{\geqslant \gamma_0} = \bigoplus_{\gamma \geqslant \gamma_0} \mathbb{Z} v^\gamma,$$
$$A_{>\gamma_0} = \bigoplus_{\gamma > \gamma_0} \mathbb{Z} v^\gamma$$

and similarly for $\Gamma_{\leqslant \gamma_0}$, $\Gamma_{<\gamma_0}$, $A_{\leqslant \gamma_0}$ and $A_{<\gamma_0}$. We denote by $\overline{} : A \to A$ the involutive automorphism such that $\overline{v^\gamma} = v^{-\gamma}$.

1. Preliminaries

Hypothesis and notation. *In this section, and only in this section, we fix an A-module $\mathcal{M}$ and we assume that:*

(P1) *$\mathcal{M}$ admits an A-basis $(m_x)_{x\in X}$, where X is a poset. We set*

$$\mathcal{M}_{>0} = \oplus_{x\in X} A_{>0} m_x.$$

(P2) *$\mathcal{M}$ admits a semilinear involution $\overline{} : \mathcal{M} \to \mathcal{M}$. We set*

$$\mathcal{M}_{\text{skew}} = \{m \in \mathcal{M} \mid m + \overline{m} = 0\}.$$

(P3) *If $x \in X$, then $\overline{m}_x \equiv m_x \mod \left(\bigoplus_{y<x} A m_y\right)$*

(P4) *If $x \in X$, then the set $\{y \in X \mid y \leqslant x\}$ is finite.*

Proposition 1.1. — *The $\mathbb{Z}$-linear map*

$$\begin{array}{ccc} \mathcal{M}_{>0} & \longrightarrow & \mathcal{M}_{\text{skew}} \\ m & \longmapsto & m - \overline{m} \end{array}$$

is an isomorphism.

Proof. — First, note that the corresponding result for the A-module A itself holds. In other words,

(1.2) *The map $A_{>0} \to A_{\text{skew}}$, $a \mapsto a - \overline{a}$ is an isomorphism.*

Indeed, if $a \in A_{\text{skew}}$, write $a = \sum_{\gamma\in\Gamma} r_\gamma v^\gamma$, with $r_\gamma \in R$. Now, if we set $a_+ = \sum_{\gamma>0} r_\gamma v^\gamma$, then $a = a_+ - \overline{a}_+$. This shows the surjectivity, while the injectivity is trivial.

Now, let $\Lambda : \mathcal{M}_{>0} \to \mathcal{M}_{\text{skew}}$, $m \mapsto m - \overline{m}$. For $\mathcal{X} \subset X$, we set $\mathcal{M}^{\mathcal{X}} = \oplus_{x\in\mathcal{X}} A\, m_x$ and $\mathcal{M}^{\mathcal{X}}_{>0} = \oplus_{x\in\mathcal{X}} A_{>0}\, m_x$. Assume that, for all $x \in \mathcal{X}$ and all $y \in X$ such that $y \leqslant x$, then $y \in \mathcal{X}$. By hypothesis, $\mathcal{M}^{\mathcal{X}}$ is stabilized by the involution $\overline{}$. Since X is the union of such finite $\mathcal{X}$ (by hypothesis), it shows that we may, and we will, assume that X is finite. Let us write $X = \{x_0, x_1, \dots, x_n\}$ in such a way that, if $x_i \leqslant x_j$, then $i \leqslant j$

(this is always possible). For simplifying notation, we set $m_{x_i} = m_i$. Note that, by hypothesis,

$$(*) \qquad \overline{m}_i \in m_i + \Big(\bigoplus_{0 \leqslant j<i} A\, m_j\Big).$$

In particular, $\overline{m}_0 = m_0$.

Now, let $m \in \mathcal{M}_{>0}$ be such that $\overline{m} = m$ and assume that $m \neq 0$. Write $m = \sum_{i=0}^r a_i m_i$, with $r \leqslant n$, $a_i \in A_{>0}$ and $a_r \neq 0$. Then, by hypothesis,

$$\overline{m} \equiv \overline{a}_r m_r \mod \Big(\bigoplus_{0 \leqslant j<i} A m_j\Big).$$

Since $\overline{m} = m$, this forces $\overline{a}_r = a_r$, which is impossible (because $a_r \in A_{>0}$ and $a_r \neq 0$). So Λ is injective.

Let us now show that Λ is surjective. So, let $m \in \mathcal{M}_{\text{skew}}$, and assume that $m \neq 0$ (for otherwise there is nothing to prove). Write $m = \sum_{i=0}^r a_i m_{x_i}$, with $r \leqslant n$, $a_i \in A$ and $a_r \neq 0$. We shall prove by induction on r that there exists $\mu \in \mathcal{M}$ such that $m = \mu - \overline{\mu}$. If $r = 0$, then the result follows from (1.2) and the fact that $\overline{m}_0 = m_0$. So assume that $r > 0$. By hypothesis,

$$m + \overline{m} \equiv (a_r + \overline{a}_r) m_r \mod \mathcal{M}^{\mathcal{X}_{r-1}},$$

where $\mathcal{X}_j = \{x_0, x_1, \ldots, x_j\}$. Since $m + \overline{m} = 0$, this forces $a_r \in A_{\text{skew}}$. So, by (1.2), there exists $a \in A_{>0}$ such that $a - \overline{a} = a_r$. Now, let $m' = m - a m_r + \overline{a m_r}$. Then $m' + \overline{m}' = 0$ and $m' \in \oplus_{0 \leqslant j<r} A m_j$. So, by the induction hypothesis, there exists $\mu' \in \mathcal{M}_{>0}$ such that $m' = \mu' - \overline{\mu}'$. Now, set $\mu = a m_r + \mu'$. Then $\mu \in \mathcal{M}_{>0}$ and $m = \mu - \overline{\mu} = \Lambda(\mu)$, as desired. ☐

Corollary 1.3. — *Let $m \in \mathcal{M}$. Then there exists a unique $M \in \mathcal{M}$ such that*

$$\begin{cases} \overline{M} = M, \\ M \equiv m \mod \mathcal{M}_{>0}. \end{cases}$$

Proof. — Setting $M = m + \mu$, the problem is equivalent to find $\mu \in \mathcal{M}_{>0}$ such that $\overline{m + \mu} = m + \mu$. This is equivalent to find $\mu \in \mathcal{M}_{>0}$ such that $\mu - \overline{\mu} = \overline{m} - m$: since $\overline{m} - m \in \mathcal{M}_{\text{skew}}$, this problem admits a unique solution, thanks to Theorem 1.1. ☐

The Corollary 1.3 can be applied to the A-module A itself. However, in this case, its proof becomes obvious: if $a_\circ = \sum_{\gamma \in \Gamma} a_\gamma v^\gamma$, then $a = \sum_{\gamma \leqslant 0} a_\gamma v^\gamma + \sum_{\gamma > 0} a_{-\gamma} v^\gamma$ is the unique element of A such that $\overline{a} = a$ and $a \equiv a_\circ \mod A_{>0}$.

Corollary 1.4. — *Let $\mathcal{X}$ be a subset of X such that, if $x \leqslant y$ and $y \in \mathcal{X}$, then $x \in \mathcal{X}$. Let $M \in \mathcal{M}$ be such that $\overline{M} = M$ and $M \in \mathcal{M}^{\mathcal{X}} + \mathcal{M}_{>0}$. Then $M \in \mathcal{M}^{\mathcal{X}}$.*

Proof. — Let $M_0 \in \mathcal{M}^{\mathcal{X}}$ be such that $M \equiv M_0 \mod \mathcal{M}_{>0}$. From the existence statement of Corollary 1.3 applied to $\mathcal{M}^{\mathcal{X}}$, there exists $M' \in \mathcal{M}^{\mathcal{X}}$ such that $\overline{M}' = M'$ and $M' \equiv M_0 \mod \mathcal{M}_{>0}^{\mathcal{X}}$. The fact that $M = M' \in \mathcal{M}^{\mathcal{X}}$ now follows from the uniquenes statement of Corollary 1.3. □

Corollary 1.5. — *Let $x \in X$. Then there exists a unique element $M_x \in \mathcal{M}$ such that*

$$\begin{cases} \overline{M}_x = M_x, \\ M_x \equiv m_x \mod \mathcal{M}_{>0}. \end{cases}$$

Moreover, $M_x \in m_x + \oplus_{y<x} A_{>0} m_y$ and $(M_x)_{x\in X}$ is an A-basis of $\mathcal{M}$.

Proof. — The existence and uniqueness of M_x follow from Corollary 1.3. The statement about the base change follows by applying this existence and uniqueness to $\mathcal{M}^{X_x}$, where $X_x = \{y \in X \mid y \leqslant x\}$.

Finally, the fact that $(M_x)_{x\in X}$ is an A-basis of $\mathcal{M}$ follows from the fact that the base change from $(m_x)_{x\in X}$ to $(M_x)_{x\in X}$ is unitriangular. □

Corollary 1.5 gathers in a single general statement the argument given by Lusztig [**Lus1**] for the construction of the *Kazhdan-Lusztig basis* of a Hecke algebra (which is different from the argument contained in the original paper by Kazhdan and Lusztig [**KaLu**]) and the construction, still due to Lusztig [**Lus2**, Theorem 3.2], of the *canonical basis* associated with quantum groups.

2. The main result

2.A. Kazhdan-Lusztig basis. — We fix in this paper a *weight function* $\varphi : S \to \Gamma_{>0}$ (i.e. $\varphi(s) = \varphi(t)$ whenever s and t are conjugate in W). We denote by $\mathcal{H}$ the Hecke algebra associated with (W, S, φ): as an A-module, $\mathcal{H}$ admits an A-basis $(T_w)_{w\in W}$ and the multiplication is completely determined by the following rules:

$$\begin{cases} T_w T_{w'} = T_{ww'} & \text{if } \ell(ww') = \ell(w) + \ell(w'), \\ (T_s - v^{\varphi(s)})(T_s + v^{-\varphi(s)}) = 0, & \text{if } s \in S. \end{cases}$$

Here, $\ell : W \to \mathbb{Z}_{\geqslant 0}$ denote the *length function* on W.

We denote by $\overline{} : \mathcal{H} \to \mathcal{H}$ the involutive antilinear automorphism of $\mathcal{H}$ such that

$$\overline{T}_w = T_{w^{-1}}^{-1}.$$

The triple $(\mathcal{H}, (T_w)_{w\in W}, \overline{})$ satisfies the properties (P1), (P2), (P3) and (P4) of the previous section. Therefore, if $w \in W$, there exists a unique element $C_w \in \mathcal{H}$ such that

$$C_w \equiv T_w \mod \mathcal{H}_{>0}$$

(see Corollary 1.5) and $(C_w)_{w\in W}$ is an A-basis of $\mathcal{H}$ (see Corollary 1.4), called the Kazhdan-Lusztig basis.

2.B. Cells. — In this context, we define the preorders $\leqslant_L$, $\leqslant_R$, $\leqslant_{LR}$ and the equivalence relations $\sim_L$, $\sim_R$ and $\sim_{LR}$ as in **[KaLu]** or **[Lus3]**. If C is a left cell (i.e. an equivalence class for $\sim_L$) of W, we set

$$\mathcal{H}^{\leqslant_L C} = \bigoplus_{w\leqslant_L C} A\, C_w, \qquad \mathcal{H}^{<_L C} = \bigoplus_{w<_L C} A\, C_w$$

and
$$M(C) = \mathcal{H}^{\leqslant_L C}/\mathcal{H}^{<_L C}.$$

By the very definition of the preorder $\leqslant_L$, $\mathcal{H}^{\leqslant_L C}$ and $\mathcal{H}^{<_L C}$ are left ideals of $\mathcal{H}$, so $M(C)$ inherits a structure of $\mathcal{H}$-module. If $w\in C$, we denote by c_w the image of C_w in $M(C)$: then $(c_w)_{w\in C}$ is an A-basis of $M(C)$.

2.C. Parabolic subgroups. — If $I\subset S$, we set $W_I=\langle I\rangle$: it is a standard parabolic subgroup of W and (W_I,I) is a Coxeter system. We also set

$$\mathcal{H}_I = \bigoplus_{w\in W_I} A\, T_w.$$

It is a subalgebra of $\mathcal{H}$, naturally isomorphic to the Hecke algera associated with (W_I,I,φ_I), where $\varphi_I : I\to\mathbb{Z}_{>0}$ denotes the restriction of φ.

We denote by X_I the set of elements $x\in W$ which have minimal length in xW_I: it is well-known that the map $X_I\to W/W_I$, $x\mapsto xW_I$ is bijective and that

$$\begin{aligned} X_I &= \{x\in W_I \mid \forall\ s\in I,\ \ell(xs)>\ell(x)\} \\ &= \{x\in W_I \mid \forall\ w\in W_I,\ \ell(xw)=\ell(x)+\ell(w)\}. \end{aligned}$$

As a consequence, the right $\mathcal{H}_I$-module $\mathcal{H}$ is free (hence flat) with basis $(T_x)_{x\in X_I}$. This remark has the following consequence (in the next lemma, if E is a subset of $\mathcal{H}$, then $\mathcal{H}E$ denotes the left ideal generated by E):

Lemma 2.1. — *If $\mathfrak{I}$ and $\mathfrak{I}'$ are left ideals of $\mathcal{H}_I$ such that $\mathfrak{I}\subset\mathfrak{I}'$, then:*

(a) $\mathcal{H}\mathfrak{I} = \oplus_{x\in X_I} T_x \mathfrak{I}$.

(b) *The natural map $\mathcal{H}\otimes_{\mathcal{H}_I}\mathfrak{I}\to\mathcal{H}\mathfrak{I}$ is an isomorphism of $\mathcal{H}$-modules.*

(c) *The natural map $\mathcal{H}\otimes_{\mathcal{H}_I}(\mathfrak{I}'/\mathfrak{I})\to\mathcal{H}\mathfrak{I}'/\mathcal{H}\mathfrak{I}$ is an isomorphism.*

We will now recall results from Geck **[Ge1]** about the parabolic induction of cells. First, it is clear that $(C_w)_{w\in W_I}$ is the Kazhdan-Lusztig basis of $\mathcal{H}_I$. We can then define a preorder $\leqslant_L^I$ and its associated equivalence class $\sim_L^I$ on W_I in the same way as $\leqslant_L$

and $\sim_L$ are defined for W. If $w \in W$, then there exists a unique $x \in X_I$ and a unique $w' \in W_I$ such that $w = xw'$: we then set

$$G_w^I = T_x C_{w'}.$$

Finally, if C is a left cell in W_I, then we define the left $\mathcal{H}_I$-module $M^I(C)$ similarly as $M(C')$ was defined for left cells C' of W.

***Theorem 2.2* (Geck).** — *Let E be a subset of W_I such that, if $x \in E$ and if $y \in W_I$ is such that $y \leqslant_L^I x$, then $y \in E$. Let $\mathfrak{I} = \oplus_{w \in E} A\, C_w$. Then*

$$\mathcal{H}\mathfrak{I} = \bigoplus_{w \in X_I \cdot E} A\, G_w^I = \bigoplus_{w \in X_I \cdot E} A\, C_w.$$

Moreover, the transition matrix between the A-basis $(C_w)_{w \in X_I \cdot E}$ and the A-basis $(G_w^I)_{w \in X_I \cdot E}$ is unitriangular (for the Bruhat order) and its non-diagonal entries belong to $A_{>0}$.

***Corollary 2.3* (Geck).** — *We have:*

(a) *$\leqslant_L^I$ and $\sim_L^I$ are just the restriction of $\leqslant_L$ and $\sim_L$ to W_I (and so we will use only the notation $\leqslant_L$ and $\sim_L$).*

(b) *If C is a left cell in W_I, then $X_I \cdot C$ is a union of left cells of W.*

3. Generalized $*$-operation

Hypothesis and notation. *We fix in this section, and only in this section, a subset I of S, two left cells C_1 and C_2 of W_I, and we assume that:*

(V1) *There exists a bijection $\sigma : C_1 \to C_2$ such that the A-linear map $M^I(C_1) \to M^I(C_2)$, $c_w \mapsto c_{\sigma(w)}$ is in fact $\mathcal{H}_I$-linear.*

(V2) *If $\{i,j\} = \{1,2\}$, then $\{w \in W_I \mid w \in C_i$ and $w <_L C_j\} = \varnothing$.*

We set $E_0 = \{w \in W_I \mid w <_L C_1$ or $w <_L C_2\}$, $E_i = X_0 \dot\cup C_i$ for $i \in \{1,2\}$ and

$$\mathcal{H}_I^{(i)} = \bigoplus_{w \in E_i} AC_w$$

for $i \in \{0,1,2\}$.

REMARK - If W is finite and if we assume that Lusztig's Conjectures [**Lus3**, Conjectures P1 to P15] hold for (W_I, I, φ_I), then (V2) is a consequence of (V1). ■

Note that $\mathcal{H}_I^{(i)}$ is a left $\mathcal{H}_I$-module for $i \in \{0,1,2\}$. By (V2), $\mathcal{H}_I^{(i)}/\mathcal{H}_I^{(0)}$ is a left $\mathcal{H}_I$-module isomorphic to $M^I(C_i)$ (for $i \in \{1,2\}$) so it admits an A-basis $(c_w)_{w\in C_i}$. By (V1), the map σ induces an isomorphism of $\mathcal{H}_I$-modules

$$(\clubsuit) \qquad \mathcal{H}_I^{(1)}/\mathcal{H}_I^{(0)} \xrightarrow{\sim} \mathcal{H}_I^{(2)}/\mathcal{H}_I^{(0)}.$$

Let $\sigma^L : X_I \cdot C_1 \xrightarrow{\sim} X_I \cdot C_2$ denote the bijection induced by σ (i.e. $\sigma^L(xw) = x\sigma(w)$ if $x \in X_I$ and $w \in C_1$). By Lemma 2.1 and Theorem 2.2,

$$(\diamondsuit) \qquad \mathcal{H}\mathcal{H}_I^{(i)} = \bigoplus_{w\in X_I\cdot E_i} A\, G_w^I = \bigoplus_{w\in X_I\cdot E_i} A\, C_w.$$

Now, if $i \in \{1,2\}$ and $w \in X_I \cdot C_i$, we denote by g_w^I (respectively $\mathbf{c}_w$) the image of G_w^I (respectively C_w) in the quotient $\mathcal{H}\mathcal{H}_I^{(i)}/\mathcal{H}\mathcal{H}_I^{(0)}$. By Lemma 2.1, the isomorphism ($\clubsuit$) induces an isomorphism of $\mathcal{H}$-modules

$$\sigma_* : \mathcal{H}\mathcal{H}_I^{(1)}/\mathcal{H}\mathcal{H}_I^{(0)} \xrightarrow{\sim} \mathcal{H}\mathcal{H}_I^{(2)}/\mathcal{H}\mathcal{H}_I^{(0)}$$

which is defined by

$$\sigma_*(g_w^I) = g_{\sigma^L(w)}^I$$

for all $w \in X_I \cdot C_1$. The key result of this section if the following one:

Theorem 3.1. — *If $w \in X_I \cdot C_1$, then $\sigma_*(\mathbf{c}_w) = \mathbf{c}_{\sigma^L(w)}$.*

Proof. — Let $i \in \{1,2\}$. For simplification, we set $M[i] = \mathcal{H}\mathcal{H}_I^{(i)}/\mathcal{H}\mathcal{H}_I^{(0)}$. By ($\diamondsuit$), $(g_w^I)_{w\in X_I\cdot C_i}$ and $(\mathbf{c}_w)_{w\in X_I\cdot C_i}$ are both A-bases of $M[i]$. We set

$$M[i]_{>0} = \bigoplus_{w\in X_I\cdot C_i} A_{>0}\, g_w^I.$$

By Geck's Theorem,

$$(\heartsuit) \qquad \mathbf{c}_w \equiv g_w^I \mod M[i]_{>0}.$$

Moreover, the antilinear involution $\overline{}$ on $\mathcal{H}$ stabilizes $\mathcal{H}\mathcal{H}_I^{(i)}$ and $\mathcal{H}\mathcal{H}_I^{(0)}$ so it induces an antilinear involution, still denoted by $\overline{}$, on $M[i]$. It is also clear that the isomorphism $\sigma_* : M[1] \to M[2]$, $g_w^I \mapsto g_{\sigma^L(w)}^I$ satisfies $\sigma_*(\overline{m}) = \overline{\sigma_*(m)}$ and $\sigma_*(M[1]_{>0}) = M[2]_{>0}$. Therefore, it follows from ($\heartsuit$) that, if $w \in X_I \cdot C_1$, then

$$\begin{cases} \overline{\sigma_*(\mathbf{c}_w)} = \sigma_*(\mathbf{c}_w), \\ \sigma_*(\mathbf{c}_w) \equiv g_{\sigma^L(w)}^I \mod M[2]_{>0}. \end{cases}$$

But it follows again from Geck's Theorem that the datum $(M[i], (g_w^I)_{w\in X_I\cdot C_i}, \overline{})$ satisfies the properties (P1), (P2), (P3) and (P4) of §1. So $\sigma_*(\mathbf{c}_w) = \mathbf{c}_{\sigma^L(w)}$ by Corollary 1.5. □

We can now state the main consequence of Theorem 3.1. We first need a notation: if $C_1 \neq C_2$ (which is the interesting case...), we extend σ^L to an involution of the set W, by setting

$$\sigma^L(w) = \begin{cases} w & \text{if } w \notin X_I \cdot C_1 \dot{\cup} X_I \cdot C_2, \\ \sigma^L(w) & \text{if } w \in X_I \cdot C_1, \\ (\sigma^L)^{-1}(w) & \text{if } w \in X_I \cdot C_2. \end{cases}$$

Note that $\sigma^L : W \to W$ is an involution.

Theorem 3.2. — *Let w, $w' \in W$. Then $w \sim_L w'$ if and only if $\sigma^L(w) \sim \sigma^L(w')$.*

Proof. — First, let us write

$$C_x C_y = \sum_{z \in W} h_{x,y,z} C_z,$$

where $h_{x,y,z} \in A$.

Now, assume that $w \sim_L w'$. According to Corollary 2.3(b), there exists a unique cell C in W_I such that w, $w' \in X_I \cdot C$. If $C \notin \{C_1, C_2\}$, then $\sigma^L(w) = w$ and $\sigma^L(w') = w'$, so $\sigma^L(w) \sim_L \sigma^L(w')$. So we may assume that $C \in \{C_1, C_2\}$. Since σ^L is involutive, we may assume that $C = C_1$. Therefore, w, $w' \in X_I \cdot C_1$.

By the definition of $\leqslant_L$ and $\sim_L$, there exists four sequences $x_1, \dots, x_m$, $y_1, \dots, y_n$, $w_1, \dots, w_m$, $w'_1, \dots, w'_n$ such that:

$$\begin{cases} w_1 = w, w_m = w', \\ w'_1 = w', w'_n = w, \\ \forall\ i \in \{1, 2, \dots, m-1\},\ h_{x_i, w_i, w_{i+1}} \neq 0, \\ \forall\ j \in \{1, 2, \dots, n-1\},\ h_{y_j, w'_j, w'_{j+1}} \neq 0. \end{cases}$$

Therefore, $w = w_1 \leqslant_L w_2 \leqslant_L \cdots \leqslant_L w_m = w' = w'_1 \leqslant_L w'_2 \leqslant_L \cdots \leqslant_L w'_n = w$ and so $w = w_1 \sim_L w_2 \sim_L \cdots \sim_L w_m = w' = w'_1 \sim_L w'_2 \sim_L \cdots \sim_L w'_n = w$. Again by Corollary 2.3(b), w_i, $w'_j \in X_I \cdot C_1$. So it follows from Theorem 3.1 that $h_{x, \sigma^L(w_i), \sigma^L(w_{i+1})} = h_{x, w_i, w_{i+1}}$ and $h_{x, \sigma^L(w'_j), \sigma^L(w'_{j+1})} = h_{y_j, w'_j, w'_{j+1}}$ for all $x \in W$. Therefore,

$$\begin{cases} \forall\ i \in \{1, 2, \dots, m-1\},\ h_{x_i, \sigma^L(w_i), \sigma^L(w_{i+1})} \neq 0, \\ \forall\ j \in \{1, 2, \dots, n-1\},\ h_{y_j, \sigma^L(w'_j), \sigma^L(w'_{j+1})} \neq 0. \end{cases}$$

It then follows that

$$\begin{aligned} \sigma^L(w) = \sigma^L(w_1) \leqslant_L \sigma^L(w_2) \leqslant_L \cdots &\leqslant_L \sigma^L(w_m) = \sigma^L(w') = \sigma^L(w'_1) \\ &\leqslant_L \sigma^L(w'_2) \leqslant_L \cdots \leqslant_L \sigma^L(w'_n) = \sigma^L(w), \end{aligned}$$

and so $\sigma^L(w) \sim_L \sigma^L(w')$, as expected. □

Corollary 3.3. — *Let C be a left cell of W. Then $\sigma^L(C)$ is a left cell of W and the A-linear map $M(C) \to M(\sigma^L(C))$, $c_w \mapsto c_{\sigma^L(w)}$ is an isomorphism of $\mathcal{H}$-modules.*

Proof. — This follows immediately from Theorems 3.1 and 3.2. □

4. Generalized Vogan classes

4.A. Dihedral parabolic subgroups. — Let $\mathcal{E}_\varphi$ denote the set of pairs (s,t) of elements of S such that one of the following holds:

(O) st has odd order $\geqslant 3$; or

(E) $\varphi(s) < \varphi(t)$ and st has even order $\geqslant 4$.

We fix in this subsection a pair $(s,t) \in \mathcal{E}_\varphi$. Let $w_{s,t}$ denote the longest element of $W_{s,t}$. We then set:

$$R_s = \{w \in W_{s,t} \mid \ell(ws) < \ell(w) \text{ and } \ell(wt) > \ell(w)\} = (W_{s,t} \cap X_t) \setminus X_s$$

and $$R_t = \{w \in W_{s,t} \mid \ell(ws) > \ell(w) \text{ and } \ell(wt) < \ell(w)\} = (W_{s,t} \cap X_s) \setminus X_t.$$

If we are in the case (O), we then set

$$\Gamma_s = R_s \qquad \text{and} \qquad \Gamma_t = R_t$$

while, if we are in the case (E), we set

$$\Gamma_s = R_s \setminus \{s\} \qquad \Gamma_t = R_t \setminus \{w_{s,t}s\}.$$

Finally, if we are in the case (O), we $\bigstar_{s,t} : \Gamma_s \to \Gamma_t$, $w \mapsto w_{s,t}w$ while, in the case (E), we set $\bigstar_{s,t} : \Gamma_s \to \Gamma_t$, $w \mapsto ws$. Then, by **[Lus3**, §7], we have:

Lemma 4.1. — *If $(s,t) \in \mathcal{E}_\varphi$, then Γ_s and Γ_t are two left cells of $W_{s,t}$ and $\bigstar_{s,t} : \Gamma_s \to \Gamma_t$ is a bijection which satisfies the properties* (V1) *and* (V2) *of §2.*

So $\bigstar_{s,t}$ induces a bijection $\bigstar_{s,t}^L : W \to W$ and, according to Theorem 3.2, the following holds:

Corollary 4.2. — *If $(s,t) \in \mathcal{E}_\varphi$ and if C is a left cell of W, then $C' = \bigstar_{s,t}^L(C)$ is also a left cell and the map $M(C) \to M(C')$, $c_w \mapsto c_{\bigstar_{s,t}^L(w)}$ is an isomorphism of $\mathcal{H}$-modules.*

REMARK - The bijection $\bigstar_{s,t}^L$ is called the $*$-operation and is usually denoted by $w \mapsto m^*$. ■

4.B. Generalized Vogan classes. — Let $\mathcal{V}_\varphi$ be the group of bijections of W generated by all the $\bigstar^L_{s,t}$, where (s,t) runs over $\mathcal{E}_\varphi$. We will call it the *left Vogan group* (*associated with* φ). Let $\mathcal{P}(S)$ denotes the set of subsets of S and, if $w \in W$, we set

$$\mathcal{R}(w) = \{s \in S \mid \ell(ws) < \ell(w)\}.$$

It is called the *right descent set* of w. It is well-known that the map $\mathcal{R} : W \to \mathcal{P}(S)$ is constant on left cells [**Lus3**, Lemma 8.6].

EXAMPLE - It can be checked by using computer computations in GAP that

$$|\mathcal{V}_\varphi| = 2^{40} \cdot 3^{20} \cdot 5^8 \cdot 7^4 \cdot 11^2$$

whenever (W,S) is of type H_4. ■

Now, let Maps$(\mathcal{V}_\varphi, \mathcal{P}(S))$ denote the set of maps $\mathcal{V}_\varphi \to \mathcal{P}(S)$. Then, to each $w \in W$, we associate the map $\tau_w^\varphi \in$ Maps$(\mathcal{V}_\varphi, \mathcal{P}(S))$ which is defined by

$$\tau_w^\varphi(\sigma) = \mathcal{R}(\sigma(w))$$

for all $\sigma \in \mathcal{V}_\varphi$. The fiber of the map $\tau^\varphi : W \to$ Maps$(\mathcal{V}_\varphi, \mathcal{P}(S))$ are called the *generalized Vogan left classes*. In other words, two elements x and y of W lie in the same generalized Vogan left class if and only if

$$\forall\, \sigma \in \mathcal{V}_\varphi,\ \mathcal{R}(\sigma(x)) = \mathcal{R}(\sigma(y)).$$

It follows from Corollary 4.2 that:

Theorem 4.3. — *Generalized Vogan left classes are unions of left cells.*

4.C. Knuth classes. — Let $s \in S$. We now define a permutation κ_s^φ of W as follows:

$$\kappa_s^\varphi(w) = \begin{cases} sw & \text{if there exists } t \in S \text{ such that } tw < w < sw < tsw \text{ and } \varphi(s) \leqslant \varphi(t), \\ sw & \text{if there exists } t \in S \text{ such that } tsw < sw < w < tw \text{ and } \varphi(s) \leqslant \varphi(t), \\ w & \text{otherwise.} \end{cases}$$

Then κ_s^φ is an involution of W. We denote by $\mathcal{K}_\varphi$ the group of permutations of W generated by the κ_s^φ, for $s \in S$. A *Knuth left class* is an orbit for the group $\mathcal{K}_\varphi$. The following result is well-known [**Lus1**]:

Proposition 4.4. — *Every left cell is a union of Knuth left classes.*

4.D. Knuth classes and Vogan classes. — If $w \in W$, we set

$$\mathscr{L}(w) = \{s \in S \mid \ell(sw) < \ell(w)\}.$$

It is called the *left descent set* of w. If $\sigma \in \mathscr{V}_\varphi$, then

$$\mathscr{L}(\sigma(w)) = \mathscr{L}(w). \tag{4.5}$$

Proof. — We only need to prove the result whenever $\sigma = \bigstar^L_{s,t}$ for some $(s,t) \in \mathscr{E}_\varphi$. Now, write $w = xw'$ with $w \in X_{s,t}$ and $w' \in W_{s,t}$ and let $u \in S$. Then $\sigma(w) = x\sigma(w')$. By Deodhar's Lemma, two cases may occur:

• If $ux \in X_I$, then $u \in \mathscr{L}(w)$ (or $\mathscr{L}(\sigma(w))$) if and only if $u \in \mathscr{L}(x)$. So $u \in \mathscr{L}(w)$ if and only if $u \in \mathscr{L}(\sigma(w))$, as desired.

• If $ux \notin X_I$, then $ux = xv$, for some $v \in \{s,t\}$. Therefore, $u \in \mathscr{L}(w)$ (respectively $u \in \mathscr{L}(w)$) if and only if $v \in \mathscr{L}(w')$ (respectively $v \in \mathscr{L}(\sigma(w'))$). But it is easy to check directly in the dihedral group $W_{s,t}$ that $\mathscr{L}(w') = \mathscr{L}(\sigma(w'))$. So again $u \in \mathscr{L}(w)$ if and only if $u \in \mathscr{L}(\sigma(w))$, as desired. □

Proposition 4.6. — *If C is a Knuth left class and if $\sigma \in \mathscr{V}_\varphi$, then $\sigma(C)$ is also a Knuth left class.*

Proof. — It is sufficient to show that, if $s \in S$, if $(t,u) \in \mathscr{E}_\varphi$ and if $w \in W$, then $\bigstar^L_{t,u}(w)$ and $\bigstar^L_{t,u}(\kappa^\varphi_s(w))$ are in the same Knuth left class. If $\kappa^\varphi_s(w) = w$, then this is obvious. So we may (and we will) assume that $\kappa_s(w) \neq w$. Therefore, there exists $s' \in S$ such that $s'w < w < sw < s'sw$ or $s'sw < sw < w < s'w$, and $\varphi(s) \leqslant \varphi(s')$. So $\kappa^\varphi_s(w) = sw$ and, by replacing if necessary w by sw, we may assume that $s'w < w < sw < s'sw$. We write $\sigma = \bigstar^L_{t,u}$ and $w = xw'$, with $w' \in W_I$. Two cases may occur:

First case: assume that $sx \in X_I$. Then $\sigma(sw) = s\sigma(w)$. By (4.5), we have

$$s'\sigma(w) < \sigma(w) < s\sigma(w) = \sigma(sw) < s'\sigma(sw) = s's\sigma(w).$$

So $\kappa_s(\sigma(w)) = \sigma(sw)$, so $\sigma(w)$ and $\sigma(\kappa^\varphi_s(w))$ are in the same Knuth left class.

Second case: assume that $sx \notin X_I$. Then $sx = xt'$ with $t' \in \{t,u\}$ by Deodhar's Lemma. Therefore,

$$s'xw' < xw' < xt'w' < s'xt'w'.$$

This shows that $w' < t'w'$ and $s'x \notin X_I$. Therefore, again by Deodhar's Lemma, we have $s'x = xu'$ for some $u' \in \{t,u\}$. Hence

$$u'w' < w' < t'w' < u't'w'$$

and $t' \neq u'$. So $\{t,u\} = \{t',u'\}$. Moreover, $\varphi(s) = \varphi(t')$ and $\varphi(s') = \varphi(u')$. In this situation, two cases may occur:

• Assume that tu has odd order. In this case, $\varphi(t')=\varphi(u')$ and so $\varphi(s)=\varphi(s')$. Moreover,

$$\sigma(w)=xw_{t,u}w' \qquad \text{and} \qquad \sigma(sw)=xw_{t,u}t'w'=xu'w_{t,u}w'=s'\sigma(w).$$

Therefore,

$$s\sigma(w)<\sigma(w)<s'\sigma(w)=\sigma(sw)<ss'\sigma(w),$$

and so $\sigma(\kappa_s^\varphi(w))=\kappa_{s'}^\varphi(w)$ since $\varphi(s)=\varphi(s')$. This shows again that $\sigma(w)$ and $\sigma(\kappa_s^\varphi(w))$ are in the same Knuth left cell.

• Assume that tu has even order. Since $\{t',u'\}=\{t,u\}$, $\varphi(t)<\varphi(u)$ and $\varphi(t')=\varphi(s)\leqslant\varphi(s')=\varphi(u')$, we have $t'=t$ and $u'=u$. In particular,

$$uw'<w'<tw'<utw'.$$

This shows that w', $tw'\in\Gamma_t\dot\cup\Gamma_u$, so $\sigma(w')=w't$ and $\sigma(tw')=tw't=t\sigma(w')$. Again, by (4.5),

$$u\sigma(w')<\sigma(w')<t\sigma(w')=\sigma(tw')<ut\sigma(w')$$

and so

$$s'\sigma(w)<\sigma(w)<s\sigma(w)=\sigma(sw)<s's\sigma(w)$$

and so $\sigma(sw)=\kappa_s^\varphi(\sigma(w))$, as desired. □

5. Commentaries

5.A. . — Since the map $W\to W$, $w\mapsto w^{-1}$ exchanges left cells and right cells, and exchanges left descent sets and right descent sets, all the results of this paper can be transposed to results about right cells.

5.B. . — As it has been seen in Type H_4, the group $\mathcal{V}_\varphi$ can become enormous, even in small rank, so it is not reasonable to compute generalized Vogan left classes by computing completely the map τ^φ. Computation can be performed by imitating the inductive definition of classical Vogan left classes. With our point-of-view, this amounts to start with the partition given by the fibers of the map $\mathcal{R}:W\to\mathcal{P}(S)$, and to refine it successively using the action of the generators of $\mathcal{V}_\varphi$, and to stop whenever the partition does not refine any more.

More precisely, let $\mathcal{V}_\varphi(k)$ denote the set of elements of $\mathcal{V}_\varphi$ which can be expressed as the product of at most k involutions of the form $\bigstar_{s,t}^L$, for $(s,t)\in\mathcal{E}_\varphi$ and let $\tau^\varphi(k)$: $W\to\mathrm{Maps}(\mathcal{V}_\varphi(k),\mathcal{P}(S))$ be the map obtained in a similar way as τ^φ. This map can be easily computed inductively for small values of k, and gives rise to a partition $\mathcal{C}(k)$ of W which is *a priori* coarser than the partition into generalized Vogan left classes.

However, when $\mathscr{C}(k)=\mathscr{C}(k+1)$, this means that $\mathscr{C}(k)$ coincides with the partition into generalized Vogan left classes.

For instance, in type H_4, this algorithm stops at $k=5$. Computing the generators of $\mathscr{V}_\varphi$ takes less than 4 minutes on a very basic computer, while the deduction of Vogan classes is then almost immediate.

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