Hybrid High-Order Methods for Variable-Diffusion Problems on General Meshes

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Abstract
We extend the Hybrid High-Order method introduced by the authors for the Poisson problem to problems with heterogeneous/anisotropic diffusion. The cornerstone is a local discrete gradient reconstruction from element- and face-based polynomial degrees of freedom. Optimal error estimates are proved.

Résumé
Méthodes hybrides d’ordre élevé pour des problèmes à diffusion variable sur des maillages généraux. Nous étendons la méthode hybride d’ordre élevé conçue par les auteurs pour le problème de Poisson à des problèmes de diffusion hétérogène/anisotrope. La pierre angulaire est une reconstruction locale du gradient discret à partir des degrés de liberté polynomiaux sur les éléments et les faces. On établit des estimations d’erreur optimales.

1. Introduction
Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, denote an open, bounded, polytopic domain. Let $f \in L^2(\Omega)$ and, for a subset $X \subset \Omega$, denote by $(\cdot, \cdot)_X$ and $\|\cdot\|_X$ the inner product and norm in $L^2(X)$, respectively. We focus on the following variable-diffusion problem: Find $u \in U_0 := H^1_0(\Omega)$ such that

$$(\kappa \nabla u, \nabla v)_\Omega = (f, v)_\Omega \quad \forall v \in U_0, \tag{1}$$

where $\kappa$ is a bounded, tensor-valued function in $\Omega$, taking symmetric values with lowest eigenvalue uniformly bounded from below away from zero. Owing to the Lax–Milgram Lemma, problem (1) is well-posed.

The approximation of diffusive problems on general polytopic meshes has received an increasing attention lately. Several low-order methods have been developed; see, e.g., [1, 2] and references therein. Recently, high-order methods have also become available; we mention the high-order Mimetic Finite Difference (MFD) schemes [3, 4], the Virtual Element Method [5], the Mixed High-Order method [6], and the Hybrid High-Order (HHO) methods [7, 8]. For the latter, the degrees of freedom (DOFs) are scalar-valued polynomials at mesh elements and faces up to some degree $k \geq 0$ (as for the MFD schemes in [4]), and the construction hinges on (i) a local discrete gradient reconstruction of order $k$ and (ii) a least-squares local penalty that weakly enforces the matching between element- and face-based DOFs while preserving the order of the gradient reconstruction. This design leads to optimal energy- and $L^2$-norm error estimates; cf. [7] for the Poisson problem ($\kappa$ being the identity tensor in (1)) and [8] for (quasi-incompressible) linear elasticity.

The purpose of the present work is to extend the HHO method of [7] to the variable-diffusion problem (1). The key idea is to modify the gradient reconstruction so as to account for the diffusion tensor $\kappa$. Then, adapting the ideas of [7], we prove stability of the discrete problem and derive optimal error estimates. We make the reasonable assumption that there is a partition $P_\Omega$ of $\Omega$ so that $\kappa$ is piecewise Lipschitz. For simplicity of exposition, we also assume that $\kappa$ is a piecewise polynomial; otherwise, an additional quadrature error has to be accounted for. In applications from the geosciences, $\kappa$ can often be taken piecewise constant.

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2. Discrete setting and local gradient reconstruction

We consider admissible mesh sequences in the sense of [9, Sect. 1.4]. Each mesh \( T_h \) in the sequence is a finite collection \( \{ T \} \) of nonempty, disjoint, open, polytopic elements such that \( \Omega = \bigcup_{T \in T_h} T \) and \( h = \text{max}_{T \in T_h} h_T \) (with \( h_T \) the diameter of \( T \)), and there is a matching simplicial submesh of \( T_h \) with locally equivalent mesh size and which is shape-regular in the usual sense. For all \( T \in T_h \), the faces of \( T \) are collected in the set \( \mathcal{F}_T \). In an admissible mesh sequence, \( \text{card}(\mathcal{F}_T) \) is uniformly bounded, the usual discrete and multiplicative trace inequalities hold on element faces, and the \( L^2 \)-orthogonal projector onto polynomial spaces enjoys optimal approximation properties on each mesh element. Let a polynomial degree \( k \geq 0 \) be fixed. For all \( T \in T_h \), we define the local space of DOFs as \( U_T^k := \mathbb{P}^k_d(T) \times \{ \times_{F \in \mathcal{F}_T} \mathbb{P}^k_{d-1}(F) \} \), where \( \mathbb{P}^k_d(T) \) (resp., \( \mathbb{P}^k_{d-1}(F) \)) is spanned by the restrictions to \( T \) (resp., \( F \)) of \( d \)-variate (resp., \( (d-1) \)-variate) polynomials of total degree \( \leq k \). In what follows, \( A \lesssim B \) denotes the inequality \( A \leq CB \) with positive constant \( C \) independent of the polynomial degree \( k \), the meshsize \( h \), and the diffusion tensor \( \kappa \). We assume that each mesh \( T_h \) in the sequence is compatible with the partition \( P_T \) associated with the diffusion tensor. We denote by \( \kappa_T^2 \) and \( \kappa_T^4 \) the lowest and largest eigenvalue of \( \kappa \) in \( T \), respectively, and we introduce the local heterogeneity/anisotropy ratio \( \rho_T := \kappa_T^2/\kappa_T^4 \geq 1 \). In what follows, we explicitly track the dependency of the bounds on the ratio \( \rho_T \). To avoid the proliferation of symbols, we assume that for all \( T \in T_h \), the Lipschitz constant of \( \kappa \) in \( T \), say \( L^\kappa_T \), satisfies \( L^\kappa_T \lesssim \kappa_T^2 \).

For all \( T \in T_h \), we define the local gradient reconstruction operator \( G_T^k : U_T^k \to \nabla \mathbb{P}^k_{d+1}(T) \) such that, for all \( v := (\nabla_T, (v_F)_{F \in \mathcal{F}_T}) \in U_T^k \) and all \( w \in \nabla \mathbb{P}^k_{d+1}(T) \),

\[
(\kappa \nabla G_T^k v, \nabla w)_T = (\kappa \nabla v_T, \nabla w)_T + \sum_{F \in \mathcal{F}_T} (v_F - v_T, \nabla w \cdot \kappa T \mathbf{1}_F)_F,
\]

which can be computed by solving a local (well-posed) Neumann problem in \( \mathbb{P}^k_{d+1}(T) \). We next introduce the potential reconstruction operator \( p_T^k : U_T^k \to \mathbb{P}^k_{d+1}(T) \) such that, for all \( v \in U_T^k \), \( \nabla p_T^k v := G_T^k v \) and \( \int_T p_T^k v := \int_T v_T \) (\( p_T^k v \) is well-defined since \( G_T^k v \in \nabla \mathbb{P}^k_{d+1}(T) \)). Finally, we define the local interpolation operator \( I_T^k : H^1(T) \to U_T^k \) such that, for all \( v \in H^1(T) \), \( I_T^k v := (\pi_T^k v, (\pi_T^k v)_F)_{F \in \mathcal{F}_T} \), where \( \pi_T^k \) and \( \pi_F \) are the \( L^2 \)-orthogonal projectors onto \( \mathbb{P}^k_d(T) \) and \( \mathbb{P}^k_{d-1}(F) \), respectively.

**Lemma 2.1** (Approximation properties for \( p_T^k I_T^k \)). The following holds for all \( v \in H^{k+1}(T) \) with \( \alpha = 1/2 \):

\[
\| v - p_T^k I_T^k v \|_T + h_T^{1/2} \| v - p_T^k I_T^k v \|_{\partial T} + h_T \| \nabla (v - p_T^k I_T^k v) \|_T + h_T^{3/2} \| \nabla (v - p_T^k I_T^k v) \|_{\partial T} \lesssim \rho_T^\alpha h_T^{k+2} \| v \|_{H^{k+2}(T)}. \tag{3}
\]

**Proof.** Let \( v \in H^{k+2}(T) \). A direct calculation using (2), the definitions of \( p_T^k \) and \( I_T^k \), and integration by parts shows that, for all \( w \in \nabla \mathbb{P}^k_{d+1}(T) \),

\[
(\kappa \nabla (v - p_T^k I_T^k v), \nabla w)_T = ((\kappa - \bar{\kappa}_T) \nabla (v - \pi_T^k v), \nabla w)_T - \sum_{F \in \mathcal{F}_T} (\pi_T^k v - \pi_F^k v, \nabla w \cdot (\kappa - \bar{\kappa}_T) \mathbf{1}_F)_F,
\]

where \( \bar{\kappa}_T \) denotes the mean-value of \( \kappa \) in \( T \). Note that the right-hand side vanishes if \( \kappa \) is piecewise constant. In the general case, owing to the assumptions on \( \kappa \) and using the approximation properties of the \( L^2 \)-orthogonal projectors along with a discrete trace inequality for \( \| \kappa^{1/2} \nabla w \|_F \), we infer that

\[
| (\kappa \nabla (v - p_T^k I_T^k v), \nabla w)_T | \lesssim L^\kappa_T h_T h_T^{k} \| v \|_{H^{k+1}(T)} \| \nabla w \|_T \lesssim \kappa_T^{1/2} h_T^{k+1} \| v \|_{H^{k+1}(T)} \| \nabla w \|_T. \tag{4}
\]

We now observe that

\[
\| \kappa^{1/2} \nabla (v - p_T^k I_T^k v) \|_T = (\kappa \nabla (v - p_T^k I_T^k v), \nabla (v - \pi_T^k v))_T + (\kappa \nabla (v - p_T^k I_T^k v), \nabla (\pi_T^k v - v_T^k I_T^k v))_T. \tag{5}
\]

Denote by \( \mathcal{S}_1 \) and \( \mathcal{S}_2 \) the addends on the right-hand side of (5). Using the Cauchy–Schwarz inequality and the approximation properties of \( \pi_T^{k+1} \), we obtain \( |\mathcal{S}_1| \lesssim \| \kappa^{1/2} \nabla (v - p_T^k I_T^k v) \|_T \| \kappa_T^k \|_{H^{k+1}(T)} \| \nabla w \|_T \).
When \( \kappa \) is piecewise constant, \( T_2 \) vanishes, so that using Young's inequality yields \( \| \nabla (v - p_T h_k^T v) \|_T \leq (\kappa_T^{-\frac{1}{2}}) \| \kappa^{1/2} \nabla (v - p_T h_k^T v) \|_T \leq \rho_T^{-1/2} h_k^{1/2} \| v \|_{H^{1/2}(T)} \). In the general case, using (4) with \( w = (\pi_{T}^{k+1} v - p_T h_k^T v) \) and since \( \| \nabla ((\pi_{T}^{k+1} v - p_T h_k^T v)) \|_T = \| \nabla \pi_{T}^{k+1} (v - p_T h_k^T v) \|_T \leq \| \nabla (v - p_T h_k^T v) \|_T \) owing to the \( H^1 \)-stability of the projector \( \pi_{T}^{k+1} \), we infer that \( |T_2| \leq \rho_T^{-1/2} (\kappa^{1/2})^{1/2} h_k^{1/2} \| v \|_{H^{1/2}(T)} \| \kappa^{1/2} \nabla (v - p_T h_k^T v) \|_T \), which leads to the estimate on \( \| \nabla (v - p_T h_k^T v) \|_T \) in (3). The other terms in (3) are then bounded as in [7, Lemma 3].

Remark 1 (\( \alpha = 0 \)). It is also possible to take \( \alpha = 0 \) whenever, for all \( T \in T_h \), the eigenvectors of \( \kappa|T \) are constant and its eigenvalues satisfy, with obvious notation, \( |\lambda(x) - \lambda_T| \lesssim h_T \lambda(x) \) for all \( x \in T \).

3. Discrete problem and stability

For all \( T \in T_h \), we introduce the local bilinear forms \( a_T \) and \( s_T \) on \( U^k_T \times U^k_T \) such that
\[
a_T(u, v) := (\kappa G^k_T u, G^k_T v)_T + s_T(u, v), \quad s_T(u, v) := \frac{K_F}{h_T} (\pi_F(u - P^k_T u), \pi_F(v - P^k_T v))_F,
\]
with \( \kappa_F := \| n_{F,T} \cdot \kappa n_{F,T} \|_{L^\infty(F)} \) and the local potential reconstruction \( P^k_T : U^k_T \rightarrow \mathbb{R}^{k+1}(T) \) such that \( P^k_T v := v_T + (P^k_T v - \rho_T^{-1} L_T v) \). We define the global space reconstruction \( U^k_h := \left\{ \chi \in \mathbb{P}^{k+1}_d(T) \right\} \times \left\{ f_T \in \mathbb{P}^{k+1}_d(F) \right\} \), and for all \( T \in T_h \), we denote by \( L_T : U^k_T \rightarrow \mathbb{R}^{k+1}_T \) the restriction operator that maps the global DOFs in \( u^k_h \) to the corresponding local DOFs in \( u^k_T \). The discrete problem consists in seeking \( u_h \in U^k_{h,0} := \{ v_h = (v_T)_{T \in T_h}, (v_F)_{F \in F_h} \} \in U^k_h \) such that
\[
a_h(u_h, v_h) := \sum_{T \in T_h} a_T (L_T u_h, L_T v_h) = \sum_{T \in T_h} (f, v_T)_T := : l_h(v_h) \quad \forall v_h \in U^k_{h,0}.
\]
(7)

To analyze the stability of the discrete problem, we introduce the following seminorm on \( U^k_T \):
\[
\| v \|_{k,T}^2 := \| \kappa^{1/2} \nabla v \|_T^2 + \sum_{F \in F_T} \frac{K_F}{h_F} \| v_F - v_T \|_F^2,
\]
and we set \( \| v \|_{k,T}^2 := \sum_{T \in T_h} \rho_T^{-1} \| L_T v \|_{k,T}^2 \) for all \( v \in U^k_T \). Observe that \( \| \cdot \|_{k,T} \) is a norm on \( U^k_{h,0} \).

Lemma 3.1 (Stability). The following inequalities hold for all \( v \in U^k_T \):
\[
\rho_T^{-1} \| v \|_{k,T}^2 \leq a_T(v, v) \leq \rho_T \| v \|_{k,T}^2.
\]
Consequently, \( \| v \|_{k,h}^2 \leq a_h(v_h, v_h) \) for all \( v_h \in U^k_{h,0} \) and problem (7) is well-posed.
Proof. We adapt the proof of [7, Lemma 4]. Concerning the face terms, we obtain
\[
\sum_{F \in F_T} \frac{K_F}{h_F} \| v_F - v_T \|_F^2 \leq s_T(v, v) + \rho_T \| \kappa^{1/2} \nabla v \|_T^2,
\]
(8)

and the proof of the bound on \( \| \nabla (v - p_T h_k^T v) \|_T \) follows similarly as in (7).

To compare \( \| \kappa^{1/2} G^k_T v \|_T \) and \( \| \kappa^{1/2} \nabla v \|_T \), we observe that, for all \( w \in \mathbb{P}^{k+1}_d(T) \) and all \( F \in F_T \),
\[
\| \nabla w \kappa n_{TF} \|_T^2 \leq \| n_{TF} \cdot \kappa n_{TF} \|_T \| \nabla w \kappa \nabla w \|_F \leq \frac{K_F}{h_F} \| \kappa^{1/2} \nabla w \|_T^2,
\]
(11)

where we have used the Cauchy–Schwarz inequality for \( \kappa \), the definition of \( K_F \), and the discrete trace inequality. Taking \( w = v_T \) in the definition of \( G^k_T v \) yields \( \| \kappa^{1/2} \nabla v_T \|_T^2 = (\kappa^{1/2} G^k_T v, \nabla v_T) - \sum_{F \in F_T} (v_F - v_T, \nabla v_T \kappa n_{TF})_F \). Hence, using (11), a discrete trace inequality for \( \| \kappa^{1/2} \nabla v_T \|_F \), the first bound in (10), \( \rho_T \geq 1 \), and Young's inequality yields
\[
\| \kappa^{1/2} \nabla v_T \|_F^2 \leq \| \kappa^{1/2} G^k_T v \|_T^2 + \sum_{F \in F_T} \frac{K_F}{h_F} \| v_F - v_T \|_F^2 \leq \rho_T \| \kappa^{1/2} G^k_T v \|_T^2 + s_T(v, v).
\]

Moreover, since \( \| \kappa^{1/2} G^k_T v \|_T = \sup_{w \in \mathbb{P}^{k+1}_d(T)} \frac{\| \kappa^{1/2} G^k_T w \|_T}{\| \kappa^{1/2} \nabla w \|_T} \) and proceeding similarly leads to \( \| \kappa^{1/2} G^k_T v \|_T \leq \| v \|_{k,T} \). Combining the above bounds yields (9), and the rest of the proof is straightforward. \( \square \)
4. Error analysis

Theorem 4.1 (Energy-error estimate). Let \( u \in U_0 \) solve (1) and let \( u_h \in U_{k,h,0} \) solve (7). Assume that \( u|_T \in H^{k+2}(T) \) for all \( T \in T_h \). Then, letting \( \tilde{u}_h := \left( (\pi_T^k u)|_{T \in T_h}, (\pi_T^k u)|_{F \in F_T} \right) \in U_{k,h,0} \) and, recalling the definition of \( \alpha \) from Lemma 2.1, the following holds with consistency error \( \mathcal{E}_h(v_h) := a_h(\tilde{u}_h,v_h) - l_h(v_h) \):

\[
\| \tilde{u}_h - u_h \|_{k,h} \lesssim \sup_{v_h \in U_{k,h,0}} \frac{\mathcal{E}_h(v_h)}{\|v_h\|_{k,h}} \lesssim \left\{ \sum_{T \in T_h} \kappa_T^k \rho_T^{1+2\alpha} h_T^{2(k+1)} \|u\|_{H^{k+2}(T)}^2 \right\}^{1/2}.
\]  

(12)

Proof. We adapt the proof of [7, Theorem 8]. The first inequality in (12) is an immediate consequence of Lemma 3.1. Proceeding as in [7] with \( \tilde{u}_T := p_T^k L_T \tilde{u}_h = p_T^k L_T(u|_T) \) and \( v_h \in U_{k,h,0} \) with \( \|v_h\|_{k,h} = 1 \) leads to

\[
\mathcal{E}_h(v_h) = \sum_{T \in T_h} (a_T(\tilde{u}_T - \tilde{u}^T), \tilde{v}_T) + \sum_{T \in T_h} \sum_{F \in F_T} (\nabla v_T - \tilde{v}_T, (\nabla \tilde{u}_T - \nabla \tilde{u}) \cdot \kappa_{TF})_F + \sum_{T \in T_h} s_T(L_T \tilde{u}_h, L_T v_h).
\]

Denote by \( \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \) the three terms on the right-hand side. Combining the results of Lemmas 2.1 and 3.1, we infer that \( |\mathcal{T}_1 + \mathcal{T}_2|^2 \leq \sum_{T \in T_h} \kappa_T^k \rho_T^{1+2\alpha} h_T^{2(k+1)} \|v\|_{H^{k+2}(T)}^2 \). Moreover, since \( s_T(L_T \tilde{u}_h, L_T v_h) \leq s_T(L_T \tilde{u}_h, L_T \tilde{u}_h)^{1/2} s_T(L_T v_h, L_T v_h)^{1/2} \), proceeding as in [7] for the first factor, and using the second bound in (10) for the second factor yields

\[
|\mathcal{T}_3|^2 \leq \sum_{T \in T_h} \kappa_T^k \rho_T^{1+2\alpha} h_T^{2(k+1)} \|v\|_{H^{k+2}(T)}^2.
\]

Finally, adapting the proof of [7, Theorem 10] leads to the following \( L^2 \)-norm error estimate.

Theorem 4.2 (\( L^2 \)-error estimate). Assume elliptic regularity for problem (1) in the form \( \|z\|_{H^2(\Omega)} \lesssim \|g\|_\alpha \) for all \( g \in L^2(\Omega) \) and \( z \in U_0 \) solving (1) with data \( g \). Assume \( f \in H^{k+\delta}(\Omega) \) with \( \delta = 0 \) for \( k \geq 1 \) and \( \delta = 1 \) for \( k = 0 \). Then, using the same notation as in Theorem 4.1, and defining the piecewise polynomial functions \( \tilde{u}_h \) and \( u_h \) such that \( \tilde{u}_h|_T = \pi_T^k u \) and \( u_h|_T = u_T \) for all \( T \in T_h \), the following holds:

\[
\| \tilde{u}_h - u_h \|_\Omega \lesssim \left( \sum_{T \in T_h} \kappa_T^k \rho_T^{1+2\alpha} h_T^{2(k+1)} \|u\|_{H^{k+2}(T)}^2 \right)^{1/2} + h^{k+2} \|f\|_{H^{k+\delta}(\Omega)},
\]

where \( \left( \kappa_T^k \rho_T^{1+2\alpha} h_T \right)^{1/\alpha} := \max_{T \in T_h} (\kappa_T^k \rho_T^{1+2\alpha} h_T)^{1/\alpha} \).

References


