Hybridization of mixed high-order methods on general meshes and application to the Stokes equations

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Abstract

In this work we study the hybridization of the Mixed High-Order methods for diffusion problems recently introduced in [18]. As for classical mixed finite element methods, hybridization proceeds in two steps: (i) first, the continuity of flux unknowns at interfaces is enforced via Lagrange multipliers and (ii) second, flux variables are locally eliminated. As a result, we identify a primal coercive problem which is equivalent to the original mixed problem, and whose numerical solution is less expensive. New error estimates are derived, and the resulting primal hybrid method for diffusion is used as a basis to design an arbitrary-order method for the Stokes problem on general meshes. Implementation aspects are thoroughly discussed, and numerical validation is provided.
Keywords Diffusion, Stokes, general meshes, mixed high-order methods, hybridization

1 Introduction

Approximation methods on general polygonal or polyhedral meshes are an active field of research. The interest in handling general meshes can be prompted, e.g., by the desire to adapt the element shape to the qualitative features of the solution (use of elongated hexahedral elements in boundary layers vs. tetrahedra in the interior of the domain) and by nonconforming or agglomerative [5] mesh adaptation.

A wide range of numerical methods have recently been proposed that handle general polygonal or polyhedral discretizations, usually obtained by replicating at the discrete level key features of the continuum problem. Among others, we can cite, e.g., the Mimetic Finite Difference (MFD) [14], the Hybrid Finite Volume (HFV) [26], and the Mixed Finite Volume (MFV) [23] methods, whose intimate connection has been studied in [24]. Other examples are Compatible Discrete Operator (CDO) schemes [11, 12], where the formal link with the continuum operators is expressed in terms of Tonti diagrams [34, 35], and the generalized

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Crouzeix–Raviart and penalized nonconforming methods of [16, 21], where generalizations of the traditional finite element counterparts are studied.

All the above-mentioned methods are low-order. More recently, an increasing attention has been put into the design of higher-order discretizations capable of handling comparably general meshes. In the context of traditional finite element methods, we can recall here the Extended Finite Element method of [32, 33] based on the use of nonpolynomial shape functions. High-order MFD schemes have been studied in [8]; see also [30] for recent developments. We cite here also the Virtual Element Method (VEM) introduced in [7], broadening the ideas underpinning the nodal MFD approach, and possibly allowing the underlying virtual functional space to embed higher-order continuity conditions between neighboring elements [9]. Roughly speaking, VEM can be interpreted as a generalization of the conforming (Lagrange, Hermite) finite elements. Generalizations of traditional mixed (e.g., Raviart–Thomas) and non-conforming (Crouzeix–Raviart) elements that support general meshes and high order have been considered in [18] and [20] with application to the diffusion equation and therein labeled Mixed and Hybrid High-Order methods; see also [19] for an application to linear elasticity and [36] for another perspective on mixed methods on general meshes.

In this work, we focus on the Mixed High-Order (MHO) method of [18]. It has been long known [2, 1] that classical mixed methods can be efficiently implemented by *hybridization*, which allows to replace the original saddle-point problem by a coercive one (a classical textbook on mixed method is [13], very recently revised and expanded [10]). Hybridization allows one to *locally* eliminate the flux variable in two steps: first, Lagrange multipliers are introduced to enforce the continuity of flux unknowns located at interfaces; second, flux variables are locally eliminated in favour of potential unknowns and Lagrange multipliers. Our goals are here

(i) to show how hybridization applies to the MHO method of [18] and investigate the relation with the Hybrid High-Order (HHO) method of [20];

(ii) to derive additional error estimates that allow, in particular, to recover the traditional interpretation of Lagrange multipliers as traces of the potential;

(iii) based on the latter property, to design and analyze a HHO method for the Stokes equations which will serve as a basis for more advanced applications that we have in view.

The interpretation of the Lagrange multipliers is used in the design of the method for the Stokes equations to define a discrete divergence operator that realizes an inf-sup stable velocity-pressure coupling. We note here that a similar construction for the Stokes problem would have been possible starting from the HHO method of [20]. Numerical results are presented to validate the theoretical predictions for the Stokes problem (for the Poisson problem, numerical evidence was already given in [18], cf. also [20]).

The paper is organized as follows. In Section 2 we briefly recall the main assumptions on the mesh in the spirit of [17] as well as some basic results on broken functional spaces used in the analysis. In Section 3 we carry out hybridization for a family of MHO methods based on design assumptions for the penalty term that are similar to those of [18, Section 4.1] but do not require the construction studied in Section 2.4.2 therein. More specifically, in Section 3.1 we introduce the notion of flux reconstruction and provide the design assumptions; Section 3.2

contains a reformulation of the MHO method where the continuity of flux unknowns located at interfaces is enforced by means of Lagrange multipliers (the *mixed hybrid* formulation); in Section 3.3 we state the *primal hybrid* problem obtained after local elimination of the flux variables and expressed in terms of a discrete gradient operator. The adjective "hybrid" refers to the fact that the problem is expressed in terms of potential unknowns and Lagrange multipliers; in Section 3.4 we show how to carry out an error analysis based on the primal hybrid formulation (as opposed to [18], where the mixed formulation is used). The application to the Stokes problem is considered in Section 4, where numerical validation is also provided. Finally, implementation aspects are thoroughly discussed in Section 5.

2 Setting

In this section we briefly recall the notion of admissible mesh sequence introduced in [17, Chapter 1] as well as some basic results for broken functional spaces.

2.1 Admissible mesh sequences

Throughout the rest of the paper, Ω denotes an open, connected, bounded polygonal or polyhedral domain in $\mathbb{R}^d$, $d \geqslant 1$. For any open, connected subset $X \subset \overline{\Omega}$ with non-zero Lebesgue measure, the standard inner product and norm of the Lebesgue space $L^2(X)$ are denoted by $(\cdot,\cdot)_X$ and $\|\cdot\|_X$, respectively, with the convention that the index is omitted if $X = \Omega$.

Denoting by $\mathcal{H} \subset \mathbb{R}_*^+$ a countable set of *meshsizes* having 0 as its unique accumulation point, we consider mesh sequences $(\mathcal{T}_h)_{h\in\mathcal{H}}$ where, for all $h \in \mathcal{H}$, $\mathcal{T}_h = \{T\}$ is a finite collection of nonempty disjoint open polyhedra T (called *elements* or *cells*) such that $\overline{\Omega} = \bigcup_{T\in\mathcal{T}_h} \overline{T}$ and $h = \max_{T\in\mathcal{T}_h} h_T$ (h_T stands for the diameter of T).

A hyperplanar closed connected subset F of $\overline{\Omega}$ is called a *face* if it has positive $(d-1)$-dimensional measure and (i) either there exist $T_1, T_2 \in \mathcal{T}_h$ such that $F \subset \partial T_1 \cap \partial T_2$ (and F is an *interface*) or (ii) there exists $T \in \mathcal{T}_h$ such that $F \subset \partial T \cap \partial\Omega$ (and F is a *boundary face*). The set of interfaces is denoted by $\mathcal{F}_h^{\rm i}$, the set of boundary faces by $\mathcal{F}_h^{\rm b}$, and we let $\mathcal{F}_h := \mathcal{F}_h^{\rm i} \cup \mathcal{F}_h^{\rm b}$. The diameter of a face $F \in \mathcal{F}_h$ is denoted by h_F

For all $T \in \mathcal{T}_h$, we let $\mathcal{F}_T := \{F \in \mathcal{F}_h \mid F \subset \partial T\}$ denote the set of faces lying on the boundary of T. Symmetrically, for all $F \in \mathcal{F}_h$, $\mathcal{T}_F := \{T \in \mathcal{T}_h \mid F \subset \partial T\}$ is the set of the one (if F is a boundary face) or two (if F is an interface) elements sharing F.

For all $F \in \mathcal{F}_T$, we denote by $\boldsymbol{n}_{TF}$ the normal to F pointing out of T. For every interface $F \subset \partial T_1 \cap \partial T_2$, we adopt the following convention: an orientation is fixed once and for all by means of a unit normal vector $\boldsymbol{n}_F$, and the elements T_1 and T_2 are numbered so that $\boldsymbol{n}_F := \boldsymbol{n}_{T_1F}$.

We assume throughout the rest of this work that the mesh sequence $(\mathcal{T}_h)_{\mathcal{H}}$ is *admissible* in the sense of [17, Chapter 1], i.e., for all $h \in \mathcal{H}$, $\mathcal{T}_h$ admits a matching simplicial submesh $\mathfrak{T}_h$ and the following properties hold for all $h \in \mathcal{H}$ with mesh regularity parameter $\varrho > 0$ independent of h: (i) for all simplex $S \in \mathfrak{T}_h$ of diameter h_S and inradius r_S, $\varrho h_S \leqslant r_S$ and (ii) for all

$T \in \mathcal{T}_h$, and all $S \in \mathfrak{T}_T := \{S \in \mathfrak{T}_h \mid S \subset T\}$, $\varrho h_T \leqslant h_S$. For an admissible mesh sequence, it is known from [17, Lemma 1.41] that the number of faces of one element can be bounded uniformly in h, i.e., it holds that

$$\forall h \in \mathcal{H}, \qquad \max_{T \in \mathcal{T}_h} \left\{ \mathfrak{N}_T := \mathrm{card}(\mathcal{F}_T) \right\} \leqslant \mathfrak{N}_\partial, \tag{1}$$

for an integer $(d+1) \leqslant \mathfrak{N}_\partial < +\infty$ depending on ϱ but independent of h. Furthermore, for all $h \in H$, $T \in \mathcal{T}_h$ and $F \in \mathcal{F}_T$, h_F is comparable to h_T in the following sense (cf. [17, Lemma 1.42]):

$$\rho^2 h_T \leqslant h_F \leqslant h_T. \tag{2}$$

2.2 Basic results on broken functional spaces

We next state some basic results that hold for admissible mesh sequences $(\mathcal{T}_h)_{h \in \mathcal{H}}$ concerning the functional spaces that we consider in the rest of the paper.

Let an integer $k \geqslant 0$ be fixed. For all $T \in \mathcal{T}_h$ and all $v \in \mathbb{P}_d^k(T)$ ($\mathbb{P}_d^k(T)$ is spanned by the restriction to T of d-variate polynomial functions of total degree $\leqslant k$), the following trace and inverse inequalities hold:

$$\|v\|_F \leqslant C_{\mathrm{tr}} h_F^{-1/2} \|v\|_T \qquad \forall F \in \mathcal{F}_T, \tag{3}$$

$$\|\boldsymbol{\nabla} v\|_T \leqslant C_{\mathrm{inv}} h_T^{-1} \|v\|_T, \tag{4}$$

with real numbers C_{tr} and C_{inv} that are independent of $h \in \mathcal{H}$ (but depending on ϱ), cf. [17, Lemmata 1.44 and 1.46].

Using [17, Lemma 1.40] together with the results of [25], one can prove the existence of a real number C_{app} depending on ϱ but independent of h such that, for all $T \in \mathcal{T}_h$, the L^2-orthogonal projector π_T^k on $\mathbb{P}_d^k(T)$ satisfies: For all $s \in \{0, \dots, k+1\}$, and all $v \in H^s(T)$,

$$|v - \pi_T^k v|_{H^m(T)} \leqslant C_{\mathrm{app}} h_T^{s-m} |v|_{H^s(T)} \qquad \forall m \in \{0, \dots, s\}. \tag{5}$$

This will be our reference convergence rate for approximation of the solutions to (isotropic) diffusion problems. In what follows we also need the L^2-orthogonal operator π_h^k on the broken polynomial space

$$\mathbb{P}_d^k(\mathcal{T}_h) := \left\{ v \in L^2(\Omega) \mid v_{|T} \in \mathbb{P}_d^k(T) \quad \forall T \in \mathcal{T}_h \right\}. \tag{6}$$

Clearly, for all $v \in L^2(\Omega)$, and all $T \in \mathcal{T}_h$, it holds that $\pi_T^k v_{|T} = (\pi_h^k v)_{|T}$, and optimal approximation properties for π_h^k follow from (5).

We also recall the following Poincaré inequality valid for all $T \in \mathcal{T}_h$:

$$\|v - \overline{v}\|_T \leqslant C_{\mathrm{P}} h_T \|\boldsymbol{\nabla} v\|_T, \qquad \forall v \in H^1(T), \tag{7}$$

where $\overline{v} = \pi_T^0 v$ and C_{P} is independent of h but possibly depends on ϱ ($C_{\mathrm{P}} = \pi^{-1}$ for convex elements [6]).

For the sake of conciseness, in what follows we often abbreviate by $a \lesssim b$ the inequality $a \leqslant Cb$ with generic constant $C > 0$ independent of h but possibly depending on ϱ.

3 Hybridization of Mixed High-Order methods

We consider in this section the Laplace equation supplemented with homogeneous Dirichlet boundary conditions

$$\begin{aligned} -\triangle u &= f && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{8}$$

where $f \in L^2(\Omega)$ denotes the forcing term. Letting

$$\boldsymbol{\Sigma} := \boldsymbol{H}(\text{div};\Omega), \qquad U := L^2(\Omega), \tag{9}$$

the mixed variational formulation of problem (8) reads: Find $(\boldsymbol{s}, u) \in \boldsymbol{\Sigma} \times U$ such that

$$\begin{aligned} (\boldsymbol{s}, \boldsymbol{t}) + (u, \boldsymbol{\nabla}\cdot\boldsymbol{t}) &= 0 && \forall \boldsymbol{t} \in \boldsymbol{\Sigma}, \\ -(\boldsymbol{\nabla}\cdot\boldsymbol{s}, v) &= (f, v) && \forall v \in U. \end{aligned} \tag{10}$$

The unknowns $\boldsymbol{s}$ and u will be henceforth referred to as the *flux* and *potential*, respectively. Let

$$W := H_0^1(\Omega). \tag{11}$$

The primal formulation of problem (8) consists in seeking $u \in W$ such that it holds

$$(\boldsymbol{\nabla} u, \boldsymbol{\nabla} v) = (f, v) \qquad \forall v \in W. \tag{12}$$

At the continuous level, the primal formulation (12) can be obtained from the mixed formulation (10) when $u \in W$ by eliminating the additional unknown $\boldsymbol{s}$. A discrete counterpart of this procedure (the so-called hybridization) is studied here for a family of MHO discretizations of (10) designed along the ideas of [18]. This procedure can be easily adapted to the more general heterogeneous anisotropic diffusion problem considered therein.

3.1 A family of Mixed High-Order methods

In this section we provide a framework for MHO methods inspired by [18], we derive a mixed hybrid formulation introducing Lagrange multipliers to enforce the continuity of interface unknowns, and we show how to obtain from it a coercive primal hybrid problem. New error estimates based on the latter are derived.

3.1.1 Degrees of freedom

Given any fixed integer $k \geqslant 0$, the flux degrees of freedom (DOFs) for the mixed method are defined as

$$\mathbb{T}_T^k := \boldsymbol{\nabla}\mathbb{P}_d^{k,0}(T) \quad \forall T \in \mathcal{T}_h, \qquad \mathbb{F}_F^k := \mathbb{P}_{d-1}^k(F) \quad \forall F \in \mathcal{F}_h, \tag{13}$$

where for $l \geqslant 0$, $\mathbb{P}_d^{l,0}(T)$ is spanned by scalar-valued polynomial functions of total degree $\leqslant l$ with zero average on T. Note that, in the lowest-order case $k = 0$, $\mathbb{T}_T^k$ has dimension zero, which reflects the fact that cell DOFs are unnecessary.

Remark 1 (Degrees of freedom)**.** *Note that we rely on a generalized notion of DOFs. The actual algebraic unknowns of the discrete problem are the coefficients of the expansion of functions with respect to a given basis for the local spaces* $\mathbb{T}_T^k$ *and* $\mathbb{F}_F^k$ *defined by* (13)*, cf. Section 5. How such basis is chosen has no impact on the theoretical developments below, but is important in practice since it influences the conditioning of local and global problems. A possible choice is discussed in Section 5.*

The local and global DOF spaces for the flux approximation in the mixed method are, respectively,

$$\boldsymbol{\Sigma}_T^k := \mathbb{T}_T^k \times \left\{ \bigtimes_{F \in \mathcal{F}_T} \mathbb{F}_F^k \right\} \quad \forall T \in \mathcal{T}_h, \qquad \check{\boldsymbol{\Sigma}}_h^k := \bigtimes_{T \in \mathcal{T}_h} \boldsymbol{\Sigma}_T^k. \tag{14}$$

We introduce the following patched version of $\check{\boldsymbol{\Sigma}}_h^k$:

$$\boldsymbol{\Sigma}_h^k := \left\{ \boldsymbol{\tau}_h = (\boldsymbol{\tau}_T, (\tau_{TF})_{F \in \mathcal{F}_T})_{T \in \mathcal{T}_h} \in \check{\boldsymbol{\Sigma}}_h^k \mid \textstyle\sum_{T \in \mathcal{T}_F} \tau_{TF} = 0 \quad \forall F \in \mathcal{F}_h^{\mathrm{i}} \right\}. \tag{15}$$

We equip $\boldsymbol{\Sigma}_T^k$ with the following norm, *which is slightly different from that in [18]*:

$$\forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_T^k, \qquad |||\boldsymbol{\tau}|||_T^2 := \|\boldsymbol{\tau}_T\|_T^2 + \sum_{F \in \mathcal{F}_T} h_F \|\tau_{TF}\|_F^2. \tag{16}$$

For all $T \in \mathcal{T}_h$, we denote by $R_{\boldsymbol{\Sigma},T}^k : \check{\boldsymbol{\Sigma}}_h^k \to \boldsymbol{\Sigma}_T^k$ the restriction operator which realizes the mapping between global and local flux DOFs, and we equip $\check{\boldsymbol{\Sigma}}_h^k$ (thus also its vector subspace $\boldsymbol{\Sigma}_h^k$) with the norm

$$\forall \boldsymbol{\tau}_h \in \check{\boldsymbol{\Sigma}}_h^k, \qquad |||\boldsymbol{\tau}_h|||^2 := \sum_{T \in \mathcal{T}_F} |||R_{\boldsymbol{\Sigma},T}^k \boldsymbol{\tau}_h|||_T^2. \tag{17}$$

Let, for a fixed $s > 2$,

$$\boldsymbol{\Sigma}^+(T) := \{\boldsymbol{t} \in L^s(T)^d \mid \boldsymbol{\nabla}\cdot\boldsymbol{t} \in L^2(T)\} \quad \boldsymbol{\Sigma}^+ := \{\boldsymbol{t} \in L^2(\Omega)^d \mid \boldsymbol{t}_{|T} \in \boldsymbol{\Sigma}^+(T), \forall T \in \mathcal{T}_h\}.$$

We define the local interpolator $I_{\boldsymbol{\Sigma},T}^k : \boldsymbol{\Sigma}^+(T) \to \boldsymbol{\Sigma}_T^k$ such that, for all $\boldsymbol{t} \in \boldsymbol{\Sigma}^+(T)$, $I_{\boldsymbol{\Sigma},T}^k \boldsymbol{t} = (\boldsymbol{\tau}_T, (\tau_{TF})_{F \in \mathcal{F}_T})$ with

$$\boldsymbol{\tau}_T = \varpi_T^k \boldsymbol{t}, \qquad \tau_{TF} = \pi_F^k(\boldsymbol{t}\cdot\boldsymbol{n}_{TF}) \quad \forall F \in \mathcal{F}_T,$$

where π_F^k is the L^2-orthogonal projector on $\mathbb{F}_F^k$ and ϖ_T^k denotes the L^2-orthogonal projector on $\mathbb{T}_T^k$ (in fact, an elliptic projector on $\mathbb{P}_d^{k+1,0}(T)$) such that

$$(\varpi_T^k \boldsymbol{t}, \boldsymbol{w})_T = (\boldsymbol{t}, \boldsymbol{w})_T \qquad \forall \boldsymbol{w} \in \mathbb{T}_T^k.$$

The global interpolator $I_{\boldsymbol{\Sigma},h}^k : \boldsymbol{\Sigma}^+ \to \check{\boldsymbol{\Sigma}}_h^k$ is such that, for all $\boldsymbol{t} \in \boldsymbol{\Sigma}^+$,

$$R_{\boldsymbol{\Sigma},T}^k I_{\boldsymbol{\Sigma},h}^k \boldsymbol{t} = I_{\boldsymbol{\Sigma},T}^k \boldsymbol{t}_{|T} \qquad \forall T \in \mathcal{T}_h. \tag{18}$$

Remark 2 (Restriction of $I_{\boldsymbol{\Sigma},h}^k$ to $\boldsymbol{\Sigma}^+ \cap \boldsymbol{H}(\mathrm{div};\Omega)$)**.** *An important remark is that functions in* $\boldsymbol{\Sigma}^+ \cap \boldsymbol{H}(\mathrm{div};\Omega)$ *are mapped by* $I_{\boldsymbol{\Sigma},h}^k$ *to elements of the patched space* $\boldsymbol{\Sigma}_h^k$*, cf.* (15)*.*

The local and global DOF spaces for the potential are given by, respectively,

$$U_T^k := \mathbb{P}_d^k(T) \quad \forall T \in \mathcal{T}_h, \qquad U_h^k := \bigtimes_{T\in\mathcal{T}_h} U_T^k. \tag{19}$$

In the following, we identify when needed the space U_h^k with the broken polynomial space $\mathbb{P}_d^k(\mathcal{T}_h)$ defined by (6). The spaces U_T^k and U_h^k are naturally endowed with the L^2-norm topology.

3.1.2 Discrete divergence operator

Let $T \in \mathcal{T}_h$. The MHO method of [18] mainly relies on the local discrete divergence operator $D_T^k : \mathbf{\Sigma}_T^k \to \mathbb{P}_d^k(T)$ such that, for all $\boldsymbol{\tau} = (\boldsymbol{\tau}_T, (\tau_{TF})_{F\in\mathcal{F}_T}) \in \mathbf{\Sigma}_T^k$,

$$(D_T^k\boldsymbol{\tau}, v)_T = -(\boldsymbol{\tau}_T, \boldsymbol{\nabla} v)_T + \sum_{F\in\mathcal{F}_T} (\tau_{TF}, v)_F, \qquad \forall v \in \mathbb{P}_d^k(T). \tag{20}$$

The operator D_T^k is designed so as to satisfy the following commuting diagram property (cf. [18, Proposition 1] for a proof):

$$D_T^k(I_{\mathbf{\Sigma},T}^k\boldsymbol{t}) = \pi_T^k(\boldsymbol{\nabla}\cdot\boldsymbol{t}) \qquad \forall \boldsymbol{t} \in \mathbf{\Sigma}^+(T). \tag{21}$$

Since the norm defined by (17) is not the same as in [18], we give a proof of the following result.

Proposition 3 (Continuity of D_T^k)**.** *There exists a real number $C > 0$ independent of h but depending on ϱ such that, for all $T \in \mathcal{T}_h$ and all $\boldsymbol{\tau} \in \mathbf{\Sigma}_T^k$,*

$$\|D_T^k\boldsymbol{\tau}\|_T \leqslant C h_T^{-1} |\!|\!| \boldsymbol{\tau} |\!|\!|_T. \tag{22}$$

Proof. Recalling (20) we have, for all $\boldsymbol{\tau} \in \mathbf{\Sigma}_T^k$,

$$\|D_T^k\boldsymbol{\tau}\|_T = \sup_{v\in\mathbb{P}_d^k(T), \|v\|_T=1} \left\{ -(\boldsymbol{\tau}_T, \boldsymbol{\nabla} v)_T + \sum_{F\in\mathcal{F}_T} (\tau_{TF}, v)_F \right\}. \tag{23}$$

Using the Cauchy–Schwarz inequality followed by the discrete inverse inequality (4), it is inferred that $|(\boldsymbol{\nabla} v, \boldsymbol{\tau}_T)_T| \lesssim h_T^{-1}\|v\|_T\|\boldsymbol{\tau}_T\|_T$. Again the Cauchy–Schwarz inequality together with the discrete trace inequality (3) yield, for all $F \in \mathcal{F}_T$, $|(v, \tau_{TF})_F| \lesssim h_F^{-1}\|v\|_T h_F^{1/2}\|\tau_{TF}\|_F$. The result then follows using the discrete Cauchy–Schwarz inequality together with the above bounds and (2) to estimate the right-hand side of (23) and recalling the definition (16) of the $|\!|\!|\cdot|\!|\!|_T$-norm. □

To close this section, we define the global divergence operator $D_h^k : \check{\mathbf{\Sigma}}_h^k \to \mathbb{P}_d^k(\mathcal{T}_h)$ such that, for all $\boldsymbol{\tau}_h \in \check{\mathbf{\Sigma}}_h^k$ and all $T \in \mathcal{T}_h$,

$$D_h^k\boldsymbol{\tau}_{h|T} = D_T^k R_{\mathbf{\Sigma},T}^k \boldsymbol{\tau}_h. \tag{24}$$

Using the definition (18) of the global interpolator $I_{\mathbf{\Sigma},h}^k$ together with the commuting diagram property (21) for the local divergence operator, the following global commuting diagram property follows ($\boldsymbol{\nabla}_h\cdot$ denotes here the broken divergence operator on $\mathcal{T}_h$):

$$D_h^k(I_{\mathbf{\Sigma},h}^k\boldsymbol{t}) = \pi_h^k(\boldsymbol{\nabla}_h\cdot\boldsymbol{t}), \qquad \forall \boldsymbol{t} \in \mathbf{\Sigma}^+. \tag{25}$$

3.1.3 Flux reconstruction operator

Following [18, Section 2.4], we introduce a flux reconstruction operator $\mathfrak{C}_T^k : \boldsymbol{\Sigma}_T^k \to \boldsymbol{\nabla}\mathbb{P}_d^{k+1,0}(T)$ defined for all $\boldsymbol{\tau} = (\boldsymbol{\tau}_T, (\tau_{TF})_{F\in\mathcal{F}_T}) \in \boldsymbol{\Sigma}_T^k$ such that, for all $w \in \mathbb{P}_d^{k+1,0}(T)$,

$$(\mathfrak{C}_T^k\boldsymbol{\tau}, \boldsymbol{\nabla}w)_T = -(D_T^k\boldsymbol{\tau}, w)_T + \sum_{F\in\mathcal{F}_T}(\tau_{TF}, w)_F \tag{26a}$$

$$= (\boldsymbol{\tau}_T, \boldsymbol{\nabla}\pi_T^k w)_T + \sum_{F\in\mathcal{F}_T}(\tau_{TF}, \pi_F^k w - \pi_T^k w)_F, \tag{26b}$$

where we have used $D_T^k\boldsymbol{\tau} \in \mathbb{P}_d^k(T)$ together with (20) to pass from the first to the second line. Observe that computing $y \in \mathbb{P}_d^{k+1,0}(T)$ such that $\mathfrak{C}_T^k\boldsymbol{\tau} = \boldsymbol{\nabla}y$ and (26) holds requires to solve a well-posed Neumann problem (in practice, other conditions than y having zero average on T can be considered to obtain a well-posed local problem, cf. the discussion in Section 5). Additionally observing that both sides vanish when w is constant (which can be interpreted as a compatibility condition for the discrete Neumann problem), we infer that (26) holds in fact for all $w \in \mathbb{P}_d^{k+1}(T)$, and we deduce the following consistency property for $\mathfrak{C}_T^k$:

$$\mathfrak{C}_T^k I_{\boldsymbol{\Sigma},T}^k\boldsymbol{\nabla}w = \boldsymbol{\nabla}w, \qquad \forall w \in \mathbb{P}_d^{k+1}(T). \tag{27}$$

The following result follows from [18, Lemma 3] using Proposition 3.
Lemma 4 (Stability and continuity for $\mathfrak{C}_T^k$)**.** *For all $T \in \mathcal{T}_h$, the flux reconstruction operator $\mathfrak{C}_T^k$ satisfies, for all $\boldsymbol{\tau} = (\boldsymbol{\tau}_T, (\tau_{TF})_{F\in\mathcal{F}_T}) \in \boldsymbol{\Sigma}_T^k$*

$$\|\boldsymbol{\tau}_T\|_T \leqslant \|\mathfrak{C}_T^k\boldsymbol{\tau}\|_T \leqslant \|\!|\boldsymbol{\tau}|\!\|_T. \tag{28}$$

3.1.4 Design conditions and mixed formulation

We provide here design conditions for a mixed approximation of (8) based on the flux reconstruction defined in the previous section that are essentially equivalent to those proposed in [18, Section 4.1] but do not require the introduction of the construction studied in Section 2.4.2 therein. We let H denote a global bilinear form on $\check{\boldsymbol{\Sigma}}_h^k \times \check{\boldsymbol{\Sigma}}_h^k$ assembled element-wise from local contributions, i.e., such that, for all $\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h \in \check{\boldsymbol{\Sigma}}_h^k$,

$$H(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) := \sum_{T\in\mathcal{T}_h} H_T(R_{\boldsymbol{\Sigma},T}^k\boldsymbol{\sigma}_h, R_{\boldsymbol{\Sigma},T}^k\boldsymbol{\tau}_h),$$

where, for all $T \in \mathcal{T}_h$, the bilinear form H_T on $\boldsymbol{\Sigma}_T^k \times \boldsymbol{\Sigma}_T^k$ is given by, for all $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \boldsymbol{\Sigma}_T^k$,

$$H_T(\boldsymbol{\sigma}, \boldsymbol{\tau}) := (\mathfrak{C}_T^k\boldsymbol{\sigma}, \mathfrak{C}_T^k\boldsymbol{\tau})_T + J_T(\boldsymbol{\sigma}, \boldsymbol{\tau}). \tag{29}$$

For further use, we also define the global stabilization bilinear form J on $\boldsymbol{\Sigma}_T^k \times \boldsymbol{\Sigma}_T^k$ such that

$$J(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) := \sum_{T\in\mathcal{T}_h} J_T(I_{\boldsymbol{\Sigma},T}^k\boldsymbol{\sigma}_h, I_{\boldsymbol{\Sigma},T}^k\boldsymbol{\tau}_h). \tag{30}$$

The requirements on J_T are summarized in what follows.

Assumption 1 (Stabilization bilinear form J_T)**.** *We assume the following design conditions:*

(H1) Symmetry, semi-definiteness, and consistency. *The bilinear form J_T is symmetric, positive semi-definite, and satisfies*

$$\forall w \in \mathbb{P}_d^{k+1}(T), \qquad J_T(I_{\boldsymbol{\Sigma},T}^k \boldsymbol{\nabla} w, \boldsymbol{\tau}) = 0 \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_T^k, \tag{31}$$

(H2) Stability and continuity. *There exists a real number $\eta > 0$ independent of T and of h (but depending on ϱ) such that H_T is coercive on $\ker(D_T^k)$ and continuous on $\boldsymbol{\Sigma}_T^k$:*

$$\eta |||\boldsymbol{\tau}|||_T^2 \leqslant H_T(\boldsymbol{\tau}, \boldsymbol{\tau}) \qquad \forall \boldsymbol{\tau} \in \ker(D_T^k)\,, \tag{32a}$$

$$H_T(\boldsymbol{\tau}, \boldsymbol{\tau}) \leqslant \eta^{-1} |||\boldsymbol{\tau}|||_T^2 \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_T^k\,. \tag{32b}$$

Several remarks are of order.

Remark 5 (Condition (31))**.** *Condition* (31) *will be used in eq.* (56) *below to establish a link between the consistent gradient operator $\boldsymbol{G}_T^k$ defined by* (54) *and the bilinear form H_T, a necessary step to prove a consistency result used in the proof of Theorem 14.*

Remark 6 (Condition (32b))**.** *In view of* (29) *and of the second inequality in* (28)*, and since J_T is symmetric and positive semi-definite owing to* **(H1)***, condition* (32b) *holds if and only if there is a real number $C > 0$ independent of h such that, for all $T \in \mathcal{T}_h$,*

$$J_T(\boldsymbol{\tau}, \boldsymbol{\tau}) \leqslant C |||\boldsymbol{\tau}|||_T^2 \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_T^k. \tag{33}$$

Letting $f_h := \pi_h^k f$, the mixed formulation for the discrete problem reads: Find $(\boldsymbol{\sigma}_h, u_h) \in \boldsymbol{\Sigma}_h^k \times U_h^k$ such that

$$H(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + (u_h, D_h^k \boldsymbol{\tau}_h) = 0 \qquad \forall \boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h^k, \tag{34a}$$

$$-(D_h^k \boldsymbol{\sigma}_h, v_h) = (f_h, v_h) \qquad \forall v_h \in U_h^k. \tag{34b}$$

The well-posedness of problem (34) can be proved as in [18, Lemma 8], to which we refer for the details (the main ingredients are also recalled in the proof of Lemma 8 below).

3.1.5 An example of stabilizing bilinear form

A possible choice for J_T originally proposed in [18, Section 4.1] is to let, for all $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \boldsymbol{\Sigma}_T^k$,

$$J_T(\boldsymbol{\sigma}, \boldsymbol{\tau}) = \sum_{F \in \mathcal{F}_T} h_F (\mathfrak{C}_T^k \boldsymbol{\sigma} \cdot \boldsymbol{n}_{TF} - \sigma_{TF}, \mathfrak{C}_T^k \boldsymbol{\tau} \cdot \boldsymbol{n}_{TF} - \tau_{TF})_F. \tag{35}$$

Proposition 7 (Stabilization bilinear form (35))**.** *The bilinear form J_T defined by* (35) *satisfies properties* **(H1)***–***(H2)***.*

Proof. (i) *Proof of* **(H1)***.* J_T is clearly symmetric and positive semi-definite. To prove the consistency property (31), let $w \in \mathbb{P}_d^{k+1}(T)$ and set $\boldsymbol{\sigma} := I_{\boldsymbol{\Sigma},T}^k \boldsymbol{\nabla} w$. Owing to (27), it holds $\mathfrak{C}_T^k \boldsymbol{\sigma} = \boldsymbol{\nabla} w$ and, since $(\boldsymbol{\nabla} w \cdot \boldsymbol{n}_F)_{|F} \in \mathbb{P}_{d-1}^k(F)$ for all $F \in \mathcal{F}_T$, $\sigma_{TF} = \pi_F^k(\boldsymbol{\nabla} w \cdot \boldsymbol{n}_{TF}) = \boldsymbol{\nabla} w \cdot \boldsymbol{n}_{TF}$. As a consequence, $\mathfrak{C}_T^k \boldsymbol{\sigma} \cdot \boldsymbol{n}_{TF} - \sigma_{TF} \equiv 0$ for all $F \in \mathcal{F}_T$, yielding (31). (ii) *Proof* *of* **(H2)***.* A stronger form of stability than (32a) will be proved for J_T in Proposition 9. To prove the continuity condition (32b), let $\boldsymbol{\tau} \in \boldsymbol{\Sigma}_T^k$ and observe that, for all $F \in \mathcal{F}_T$, it holds

$$h_F^{1/2} \| \mathfrak{C}_T^k \boldsymbol{\tau} \cdot \boldsymbol{n}_{TF} - \tau_{TF} \|_F \lesssim \| \mathfrak{C}_T^k \boldsymbol{\tau} \|_T + h_F^{1/2} \| \tau_{TF} \|_F \leqslant \| \! | \boldsymbol{\tau} | \! \|_T + h_F^{1/2} \| \tau_{TF} \|_F, \tag{36}$$

where we have used the triangular and discrete trace (3) inequalities and concluded using the second inequality in (28). We then have, using (36) and concluding with the bound (1) on the number of faces $\mathfrak{N}_T$ of T,

$$J_T(\boldsymbol{\tau}, \boldsymbol{\tau}) = \sum_{F \in \mathcal{F}_T} h_F \| \mathfrak{C}_T^k \boldsymbol{\tau} \cdot \boldsymbol{n}_{TF} - \tau_{TF} \|_F^2 \lesssim \mathfrak{N}_T \| \! | \boldsymbol{\tau} | \! \|_T^2 + \sum_{F \in \mathcal{F}_T} h_F \| \tau_{TF} \|_F^2 \lesssim \| \! | \boldsymbol{\tau} | \! \|_T^2,$$

which yields (33) and, as a consequence, (32b) (cf. Remark 6). □

3.2 Mixed hybrid formulation

To hybridize (34) in the spirit of [2], we use the unpatched space $\check{\boldsymbol{\Sigma}}_h^k$ defined by (14) in place of the subspace $\boldsymbol{\Sigma}_h^k$ defined by (15), and we enforce the continuity of flux DOFs located at interfaces via Lagrange multipliers. Letting

$$\Lambda_h^k := \bigtimes_{F \in \mathcal{F}_h} \Lambda_F^k, \qquad \Lambda_F^k := \begin{cases} \mathbb{P}_{d-1}^k(F) & \text{if } F \in \mathcal{F}_h^{\mathrm{i}}, \\ \{0\} & \text{if } F \in \mathcal{F}_h^{\mathrm{b}}, \end{cases} \tag{37}$$

the resulting mixed hybrid method reads: Find $(\overline{\boldsymbol{\sigma}}_h, \overline{u}_h, \lambda_h) \in \check{\boldsymbol{\Sigma}}_h^k \times U_h^k \times \Lambda_h^k$ such that

$$H(\overline{\boldsymbol{\sigma}}_h, \boldsymbol{\tau}_h) + (\overline{u}_h, D_h^k \boldsymbol{\tau}_h) - \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} (\lambda_F, \tau_{TF})_F = 0 \qquad \forall \boldsymbol{\tau}_h \in \check{\boldsymbol{\Sigma}}_h^k, \tag{38a}$$

$$-(D_h^k \overline{\boldsymbol{\sigma}}_h, v_h) = (f_h, v_h) \qquad \forall v_h \in U_h^k, \tag{38b}$$

$$\sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} (\overline{\sigma}_{TF}, \mu_F)_F = 0 \qquad \forall \mu_h \in \Lambda_h^k, \tag{38c}$$

where we have used the notations $\lambda_h = (\lambda_F)_{F \in \mathcal{F}_h}$ and $\mu_h = (\mu_F)_{F \in \mathcal{F}_h}$. Defining the following local and global hybrid DOF spaces (U_T^k and U_h^k are defined by (19)):

$$W_T^k := U_T^k \times \left\{ \bigtimes_{F \in \mathcal{F}_T} \Lambda_F^k \right\} \quad \forall T \in \mathcal{T}_h, \qquad W_h^k := U_h^k \times \Lambda_h^k, \tag{39}$$

and introducing the bilinear form B on $\check{\boldsymbol{\Sigma}}_h^k \times W_h^k$ such that, for all $(\boldsymbol{\tau}_h, z_h) \in \check{\boldsymbol{\Sigma}}_h^k \times W_h^k$ with $z_h = (v_h, \mu_h)$, recalling (20) and (24) to infer the second equality,

$$B(\boldsymbol{\tau}_h, z_h) := (v_h, D_h^k \boldsymbol{\tau}_h) - \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} (\mu_F, \tau_{TF})_F \tag{40a}$$

$$= \sum_{T \in \mathcal{T}_h} \left\{ -(\boldsymbol{\nabla} v_T, \boldsymbol{\tau}_T)_T + \sum_{F \in \mathcal{F}_T} (v_T - \mu_F, \tau_{TF})_F \right\}, \tag{40b}$$

problem (38) reformulates as follows: Find $\overline{\boldsymbol{\sigma}}_h \in \check{\boldsymbol{\Sigma}}_h^k$ and $\overline{w}_h := (\overline{u}_h, \lambda_h) \in W_h^k$ such that,

$$H(\overline{\boldsymbol{\sigma}}_h, \boldsymbol{\tau}_h) + B(\boldsymbol{\tau}_h, \overline{w}_h) = 0 \qquad \forall \boldsymbol{\tau}_h \in \check{\boldsymbol{\Sigma}}_h^k, \tag{41a}$$

$$-B(\overline{\boldsymbol{\sigma}}_h, z_h) = (f_h, v_h) \qquad \forall z_h = (v_h, \mu_h) \in W_h^k. \tag{41b}$$

The following result establishes a link between the solutions of (34) and (38) (or, equivalently, (41)).

Lemma 8 (Relation between (34) and (38), (41))**.** *The following inf-sup condition holds with $C > 0$ independent of h but depending on ϱ:*

$$C\|z_h\|_{0,h} \leqslant \sup_{\boldsymbol{\tau}_h \in \check{\boldsymbol{\Sigma}}_h^k, |||\boldsymbol{\tau}_h|||=1} B(\boldsymbol{\tau}_h, z_h). \tag{42}$$

where, for all $z_h = (v_h, \mu_h) \in W_h^k$, $\|z_h\|_{0,h}^2 := \|v_h\|^2 + \sum_{T\in\mathcal{T}_h} \sum_{F\in\mathcal{F}_T} h_F^{-1} \|\mu_F - v_T\|_F^2$. Additionally, problem (38) (or, equivalently, (41)) has a unique solution $(\overline{\boldsymbol{\sigma}}_h, \overline{u}_h, \lambda_h) \in \check{\boldsymbol{\Sigma}}_h^k \times U_h^k \times \Lambda_h^k$. Finally, denoting by $(\boldsymbol{\sigma}_h, u_h) \in \boldsymbol{\Sigma}_h^k \times U_h^k$ the unique solution to problem (34), it holds $(\overline{\boldsymbol{\sigma}}_h, \overline{u}_h) = (\boldsymbol{\sigma}_h, u_h)$. In view of this result, we drop the bar in what follows.

Proof. Let $z_h = (v_h, \mu_h) \in W_h^k$. Following, e.g., [13], there is $\boldsymbol{t}_v \in \boldsymbol{\Sigma}^+ \cap \boldsymbol{H}(\text{div};\Omega)$ such that $\boldsymbol{\nabla}\cdot\boldsymbol{t}_v = v_h$. Letting $\boldsymbol{\tau}_{h,1} := I_{\boldsymbol{\Sigma},h}^k \boldsymbol{t}_v \in \boldsymbol{\Sigma}_h^k$ (recall Remark 2), using the continuity of $I_{\boldsymbol{\Sigma},h}^k$ and of $v_h \to \boldsymbol{t}_v$, it holds $|||\boldsymbol{\tau}_{h,1}||| \lesssim \|\boldsymbol{t}_v\|_{\boldsymbol{\Sigma}^+} \lesssim \|v_h\|$ and, owing to the commuting property (25), $\|v_h\|^2 = (v_h, D_h^k \boldsymbol{\tau}_{h,1}) = B(\boldsymbol{\tau}_{h,1}, z_h)$ since $\sum_{T\in\mathcal{T}_h} \sum_{F\in\mathcal{F}_T} (\mu_F, \boldsymbol{\tau}_{TF,1})_F = 0$ again as a consequence of having $\boldsymbol{\tau}_{h,1} \in \boldsymbol{\Sigma}_h^k$. Moreover, letting $\boldsymbol{\tau}_{h,2} \in \check{\boldsymbol{\Sigma}}_h^k$ be such that $\boldsymbol{\tau}_{T,2} \equiv \boldsymbol{0}$ and $\tau_{TF,2} = h_F^{-1}(v_T - \mu_F)$ for all $T \in \mathcal{T}_h$ and $F \in \mathcal{F}_T$, one has, recalling (40b), $B(\boldsymbol{\tau}_{h,2}, z_h) = \sum_{T\in\mathcal{T}_h} \sum_{F\in\mathcal{F}_T} h_F^{-1} \|\mu_F - v_T\|_F^2$. Additionally, it clearly holds that $|||\boldsymbol{\tau}_{h,2}||| \lesssim \|z_h\|_{0,h}$. As a result, using the linearity of B in its first argument, one has, denoting by \$ the supremum in the right-hand side of (42),

$$\|z_h\|_{0,h}^2 = B(\boldsymbol{\tau}_{h,1} + \boldsymbol{\tau}_{h,2}, z_h) \leqslant \$ |||\boldsymbol{\tau}_{h,1} + \boldsymbol{\tau}_{h,2}||| \lesssim \$ \|z_h\|_{0,h},$$

and (42) follows. The well-posedness of problem (38) is a consequence of (42) together with the coercivity of the bilinear form H in the kernel of D_h^k, cf. (32a), see e.g. [13]. To prove the last part of the statement, we observe that the triplet $(\boldsymbol{\sigma}_h, u_h, 0)$ is clearly a solution to problem (38) since (38a) with $\lambda_h = 0$ and (38b) follow from (34a) and (34b), respectively, while (38c), which enforces, $\sum_{T\in\mathcal{T}_F} \overline{\sigma}_{TF} = 0$ for all $F \in \mathcal{F}_h^{\text{i}}$, holds true if $\overline{\boldsymbol{\sigma}}_h = \boldsymbol{\sigma}_h \in \boldsymbol{\Sigma}_h^k$, cf. (15). On the other hand, since problem (38) is well-posed, it must hold that $(\overline{\boldsymbol{\sigma}}_h, \overline{u}_h) = (\boldsymbol{\sigma}_h, u_h)$, which concludes the proof. □

3.3 Discrete gradient operator and primal hybrid formulation

In this section we equip the space of hybrid potential DOFs with a H_0^1-like discrete norm and introduce a discrete gradient operator that will be used to reformulate the mixed hybrid problem (38) as a coercive primal hybrid problem.

3.3.1 Hybrid DOF space

Recall the definition (39) of the hybrid DOF spaces, and, for all $T \in \mathcal{T}_h$, denote by $R^k_{W,T} : W^k_h \mapsto W^k_T$ the restriction operator that maps global to local DOFs. We equip W^k_h with the H^1_0-like norm such that, for all $z_h = (v_h, \mu_h) \in W^k_h$,

$$\|z_h\|^2_{1,h} := \sum_{T \in \mathcal{T}_h} \|R^k_{W,T} z_h\|^2_{1,T}, \tag{43}$$

with local norm such that, for all $z = (v_T, (\mu_F)_{F \in \mathcal{F}_T}) \in W^k_T$,

$$\|z\|^2_{1,T} := \|\boldsymbol{\nabla} v_T\|^2_T + \sum_{F \in \mathcal{F}_T} h_F^{-1} \|\mu_F - v_T\|^2_F \qquad \forall T \in \mathcal{T}_h. \tag{44}$$

One can easily prove that the map defined by (43) is a norm on W^k_h following a reasoning analogous to that of [19, Proposition 5] based on the fact that Lagrange multipliers are zero at boundary faces, cf. (37). Let

$$W(T) := \{v \in H^1(T) \mid v_{|\partial T \cap \partial\Omega} \equiv 0\}. \tag{45}$$

We introduce the local interpolator $I^k_{W,T} : W(T) \to W^k_T$ such that, for all $v \in W(T)$, $I^k_{W,T} v = (v_T, (\mu_F)_{F \in \mathcal{F}_T})$ with

$$v_T = \pi^k_T v, \qquad \mu_F = \pi^k_F v \quad \forall F \in \mathcal{F}_T. \tag{46}$$

The corresponding global interpolator is $I^k_{W,h} : W \to W^k_h$ (with W defined by (11)) such that, for all $v \in W$, $I^k_{W,h} v = (v_h, \mu_h)$ with $v_h = (v_T)_{T \in \mathcal{T}_h}$, $\mu_h = (\mu_F)_{F \in \mathcal{F}_h}$, and

$$v_T = \pi^k_T v \quad \forall T \in \mathcal{T}_h, \qquad \mu_F = \pi^k_F v \quad \forall F \in \mathcal{F}_h. \tag{47}$$

3.3.2 Potential lifting operator

A first step towards identifying a discrete gradient operator consists in defining local and global operators which allow, given a set of potential DOFs, to identify the corresponding flux DOFs. We assume from this point on that a stronger (but very usual) assumption than (32a) holds, namely:

$$\eta \|\!|\boldsymbol{\tau}|\!\|^2_T \leqslant H_T(\boldsymbol{\tau}, \boldsymbol{\tau}) \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}^k_T, \tag{\textbf{H2}$^+$}$$

so that H_T (resp. H) is actually an inner-product on $\boldsymbol{\Sigma}^k_T$ (resp. $\check{\boldsymbol{\Sigma}}^k_h$), defining a norm $\|\cdot\|_{H,T}$ (resp. $\|\cdot\|_H$) equivalent to $\|\!|\cdot|\!\|_T$ (resp. $\|\!|\cdot|\!\|$).

Proposition 9 (Property (**H2**$^+$) for the stabilization bilinear form (35))**.** *The stabilization bilinear form J_T defined by* (35) *satisfies* (**H2**$^+$).

Proof. Recalling the first inequality in (28) to infer $\|\boldsymbol{\tau}_T\|_T \leqslant \|\mathfrak{C}^k_T \boldsymbol{\tau}\|_T$, and introducing the quantity $\mathfrak{C}^k_T \boldsymbol{\tau} \cdot \boldsymbol{n}_{TF}$ in the second term in the right-hand side of (16), one has, for all $\boldsymbol{\tau} \in \boldsymbol{\Sigma}^k_T$,

$$\begin{aligned}\|\!|\boldsymbol{\tau}|\!\|^2_T &\lesssim \|\mathfrak{C}^k_T \boldsymbol{\tau}\|^2_T + \sum_{F \in \mathcal{F}_T} h_F \|\mathfrak{C}^k_T \boldsymbol{\tau} \cdot \boldsymbol{n}_{TF} - \tau_{TF}\|^2_F + \sum_{F \in \mathcal{F}_T} h_F \|\mathfrak{C}^k_T \boldsymbol{\tau} \cdot \boldsymbol{n}_{TF}\|^2_F \\ &\lesssim \|\mathfrak{C}^k_T \boldsymbol{\tau}\|^2_T + J_T(\boldsymbol{\tau}, \boldsymbol{\tau}) = H_T(\boldsymbol{\tau}, \boldsymbol{\tau}),\end{aligned}$$

where we have used the definition (35) of J_T together with the discrete trace inequality (3) and the bound (1) on $\mathfrak{N}_T$ to pass from the first to the second line, plus the definition (29) of the bilinear form H_T to conclude. $\square$

For all $T \in \mathcal{T}_h$, a local potential lifting operator $\boldsymbol{\varsigma}_T^k : W_T^k \to \boldsymbol{\Sigma}_T^k$ can be naturally defined such that, for all $z = (v_T, (\mu_F)_{F\in\mathcal{F}_T}) \in W_T^k$, it holds

$$0 = H_T(\boldsymbol{\varsigma}_T^k z, \boldsymbol{\tau}) + (v_T, D_T^k \boldsymbol{\tau})_T - \sum_{F\in\mathcal{F}_T} (\mu_F, \tau_{TF})_F \quad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_T^k, \tag{48}$$

insofar as this yields a well-posed problem for $\boldsymbol{\varsigma}_T^k z$ in view of (**H2**$^+$). On recalling the definition (20) of the discrete divergence operator, (48) also rewrites

$$H_T(\boldsymbol{\varsigma}_T^k z, \boldsymbol{\tau}) = (\boldsymbol{\nabla} v_T, \boldsymbol{\tau}_T)_T + \sum_{F\in\mathcal{F}_T} (\mu_F - v_T, \tau_{TF})_F \quad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_T^k. \tag{49}$$

We also define the global lifting operator $\boldsymbol{\varsigma}_h^k : W_h^k \to \breve{\boldsymbol{\Sigma}}_h^k$ such that, for all $z_h \in W_h^k$,

$$R_{\boldsymbol{\Sigma},T}^k \boldsymbol{\varsigma}_h^k z_h = \boldsymbol{\varsigma}_T^k R_{W,T}^k z_h \qquad \forall T \in \mathcal{T}_h.$$

An important remark is that, as a consequence of (48), $\boldsymbol{\varsigma}_h^k z_h$ satisfies

$$\forall z_h \in W_h^k, \qquad H(\boldsymbol{\varsigma}_h^k z_h, \boldsymbol{\tau}_h) = -B(\boldsymbol{\tau}_h, z_h) \qquad \forall \boldsymbol{\tau}_h \in \breve{\boldsymbol{\Sigma}}_h^k, \tag{50}$$

with bilinear form B defined by (40a).

Lemma 10 (Stability and continuity for $\boldsymbol{\varsigma}_T^k$)**.** *For all $T \in \mathcal{T}_h$ and all $z \in W_T^k$, it holds, denoting by $\|\cdot\|_{H,T}$ the norm defined by H_T on $\boldsymbol{\Sigma}_T^k$,*

$$\eta^{1/2} \|z\|_{1,T} \leqslant \|\boldsymbol{\varsigma}_T^k z\|_{H,T} \leqslant \eta^{-1/2} \|z\|_{1,T}. \tag{51}$$

Thus, for all $z_h \in W_h^k$, we have, with $\|\cdot\|_H$ denoting the norm defined by H on $\breve{\boldsymbol{\Sigma}}_h^k$,

$$\eta^{1/2} \|z_h\|_{1,h} \leqslant \|\boldsymbol{\varsigma}_h^k z_h\|_H \leqslant \eta^{-1/2} \|z_h\|_{1,h}. \tag{52}$$

Proof. Let $z = (v_T, (\mu_F)_{F\in\mathcal{F}_T}) \in W_T^k$. Letting $\boldsymbol{\tau}_z = (\boldsymbol{\nabla}(v_T - \pi_T^0 v_T), (h_F^{-1}(\mu_F - v_T))_{F\in\mathcal{F}_T}) \in \boldsymbol{\Sigma}_T^k$ so that $|||\boldsymbol{\tau}_z|||_T = \|z\|_{1,T}$, one has, using (49) with $\boldsymbol{\tau} = \boldsymbol{\tau}_z$ followed by (32b),

$$H_T(\boldsymbol{\varsigma}_T^k z, \boldsymbol{\tau}_z) = \|z\|_{1,T}^2 = |||\boldsymbol{\tau}_z|||_T \|z\|_{1,T} \geqslant \eta^{1/2} \|\boldsymbol{\tau}_z\|_{H,T} \|z\|_{1,T}$$

Hence, to prove the first inequality in (51), observe that

$$\eta^{1/2} \|z\|_{1,T} \leqslant \frac{H_T(\boldsymbol{\varsigma}_T^k z, \boldsymbol{\tau}_z)}{\|\boldsymbol{\tau}_z\|_{H,T}} \leqslant \sup_{\boldsymbol{\tau}\in\boldsymbol{\Sigma}_T^k\backslash\{\boldsymbol{0}\}} \frac{H_T(\boldsymbol{\varsigma}_T^k z, \boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{H,T}} = \|\boldsymbol{\varsigma}_T^k z\|_{H,T}$$

since H_T defines an inner-product. On the other hand, it holds for all $\boldsymbol{\tau} \in \boldsymbol{\Sigma}_T^k$, bounding the right-hand side of (49) with the Cauchy-Schwarz inequality, and recalling the definitions (44) of the $\|\cdot\|_{1,T}$-norm and (16) of the $|||\cdot|||_T$-norm,

$$\begin{aligned} H_T(\boldsymbol{\varsigma}_T^k z, \boldsymbol{\tau}) &\leqslant \left\{ \|\boldsymbol{\nabla} v_T\|_T^2 + \sum_{F\in\mathcal{F}_T} \frac{1}{h_F} \|\mu_F - v_T\|_F^2 \right\}^{1/2} \times \left\{ \|\boldsymbol{\tau}_T\|_T^2 + \sum_{F\in\mathcal{F}_T} h_F \|\tau_{TF}\|_F^2 \right\}^{1/2} \\ &= \|z\|_{1,T} |||\boldsymbol{\tau}|||_T \leqslant \eta^{-1/2} \|z\|_{1,T} \|\boldsymbol{\tau}\|_{H,T}, \end{aligned} \tag{53}$$

where we have used $(\mathbf{H2}^+)$ to conclude. The second inequality in (51) then follows observing that $\|\boldsymbol{\varsigma}_T^k z\|_{H,T} = \sup_{\boldsymbol{\tau}\in\boldsymbol{\Sigma}_T^k} \left\{ H_T(\boldsymbol{\varsigma}_T^k z, \boldsymbol{\tau}) / \|\boldsymbol{\tau}\|_{H,T} \right\}$ and (53) allows one to bound the supremum. Finally, (52) can be proved squaring (51) and summing over $T \in \mathcal{T}_h$. □

3.3.3 Discrete gradient and potential reconstruction operators

Let us next define the *consistent gradient* reconstruction operator

$$\boldsymbol{G}_T^k := \mathfrak{C}_T^k \circ \boldsymbol{\varsigma}_T^k, \tag{54}$$

with $\mathfrak{C}_T^k$ and $\boldsymbol{\varsigma}_T^k$ defined by (26) and (48), respectively. The consistent gradient satisfies the following remarkable property: For all $z = (v_T, (\mu_F)_{F\in\mathcal{F}_T}) \in W_T^k$,

$$(\boldsymbol{G}_T^k z, \boldsymbol{\nabla} w)_T = (\boldsymbol{\nabla} v_T, \boldsymbol{\nabla} w)_T + \sum_{F\in\mathcal{F}_T} (\mu_F - v_T, \boldsymbol{\nabla} w \cdot \boldsymbol{n}_{TF})_F \quad \forall w \in \mathbb{P}_d^{k+1,0}(T). \tag{55}$$

To prove (55), let $w \in \mathbb{P}_d^{k+1,0}(T)$ be fixed, make $\boldsymbol{\tau} := I_{\boldsymbol{\Sigma},T}^k \boldsymbol{\nabla} w$ in (49), and use the fact that $\mathfrak{C}_T^k \boldsymbol{\tau} = \boldsymbol{\nabla} w$ owing to (27) and that $\mathfrak{C}_T^k \boldsymbol{\varsigma}_T^k z = \boldsymbol{G}_T^k z$ and $J_T(\boldsymbol{\varsigma}_T^k z, \boldsymbol{\tau}) = J_T(\boldsymbol{\varsigma}_T^k z, I_{\boldsymbol{\Sigma},T}^k \boldsymbol{\nabla} w) = 0$ owing to (54) and (31), respectively, to infer from the definition (29) of H_T that

$$H_T(\boldsymbol{\varsigma}_T^k z, \boldsymbol{\tau}) = (\mathfrak{C}_T^k \boldsymbol{\varsigma}_T^k z, \mathfrak{C}_T^k \boldsymbol{\tau}) + J_T(\boldsymbol{\varsigma}_T^k z, \boldsymbol{\tau}) = (\boldsymbol{G}_T^k z, \boldsymbol{\nabla} w)_T. \tag{56}$$

It is also useful to compare (55) with (40b) when $\boldsymbol{\tau} = I_{\boldsymbol{\Sigma},T}^k \boldsymbol{\nabla} w$.

Remark 11 (Discrete gradient operator $\boldsymbol{G}_T^k$)**.** *Equation* (55) *shows that the discrete gradient operator defined by* (54) *is in fact analogous to the one defined in [20, eq. (11)] in the framework of HHO methods. Correspondingly, equation* (55) *defines a well-posed Neumann problem in* $y \in \mathbb{P}_d^{k+1,0}(T)$ *such that* $\boldsymbol{G}_T^k z = \boldsymbol{\nabla} y$*. Finally, we observe that the right-hand side vanishes for test functions in* $\mathbb{P}_d^0(T)$*, hence* (55) *holds in fact for all* $w \in \mathbb{P}_d^{k+1}(T)$*.*

The continuity of $\boldsymbol{G}_T^k$ on W_T^k equipped with the $\|\cdot\|_{1,T}$-norm is a consequence of (54) together with the continuity of $\mathfrak{C}_T^k$ and $\boldsymbol{\varsigma}_T^k$, but the stability is limited by what can be controlled by $\mathfrak{C}_T^k$:

$$\|(\boldsymbol{\varsigma}_T^k z)_T\|_T \leqslant \|\boldsymbol{G}_T^k z\|_T \leqslant \|\boldsymbol{\varsigma}_T^k z\|_{H,T} \leqslant \eta^{-1/2} \|z\|_{1,T} \quad \forall z \in W_T^k, \tag{57}$$

where we have denoted by $(\boldsymbol{\varsigma}_T^k z)_T \in \mathbb{T}_T^k$ the cell DOFs for $\boldsymbol{\varsigma}_T^k z \in \boldsymbol{\Sigma}_T^k$, we have used the first inequality in (28) followed by (54) and (29) plus the fact that J_T is positive semi-definite owing to **(H1)** to infer $\|\boldsymbol{G}_T^k z\|_T \leqslant \|\boldsymbol{\varsigma}_T^k z\|_{H,T}$, and we have concluded using the second inequality in (51).

For all $T \in \mathcal{T}_h$, we also define from $\boldsymbol{G}_T^k$ the local potential reconstruction operator $r_T^k : W_T^k \to \mathbb{P}_d^{k+1}(T)$ such that, for all $z = (v_T, (\mu_F)_{F\in\mathcal{F}_T}) \in W_T^k$,

$$\boldsymbol{\nabla} r_T^k z = \boldsymbol{G}_T^k z, \qquad \int_T r_T^k z = \int_T v_T. \tag{58}$$

Recalling that there exists $y \in \mathbb{P}_d^{k+1,0}(T)$ such that $\boldsymbol{G}_T^k z = \boldsymbol{\nabla} y$, $r_T^k z$ can be simply obtained as $y + \pi_T^0 v_T$.

Remark 12 (Relation with cell centered Galerkin methods for $k = 0$)**.** *Let $T \in \mathcal{T}_h$. For $k = 0$, one obtains from (58), for all $z = (v_T, (\mu_F)_{F\in\mathcal{F}_T}) \in W_T^k$ and all $\boldsymbol{x} \in T$,*

$$r_T^k z(\boldsymbol{x}) = v_T + \boldsymbol{G}_T^0 z\cdot(\boldsymbol{x} - \boldsymbol{x}_T),$$

where $\boldsymbol{x}_T$ denotes the barycenter of T. This potential reconstruction has been used in [16] to design a cell centered Galerkin method with least-square penalty of interface jumps.

In view of Remark 11, the following approximation result for r_T^k is simply a restatement of [20, Lemma 3], (the proof relies on the fact that $r_T^k I_{W,T}^k$ is an elliptic projector on $\mathbb{P}_{k+1}^d(T)$).

Lemma 13 (Approximation properties for $r_T^k I_{W,T}^k$)**.** *There exists a real number $C > 0$, independent of h_T (but depending on ϱ) such that, for all $v \in W(T) \cap H^{k+2}(T)$,*

$$h_T\|\boldsymbol{\nabla}(v - r_T^k I_{W,T}^k v)\|_T + h_T^{3/2}\|\boldsymbol{\nabla}(v - r_T^k I_{W,T}^k v)\|_{\partial T} \\ + \|v - r_T^k I_{W,T}^k v\|_T + h_T^{1/2}\|v - r_T^k I_{W,T}^k v\|_{\partial T} \leqslant C h_T^{k+2}\|v\|_{H^{k+2}(T)}. \quad (59)$$

We close this section by defining global gradient and potential reconstructions as follows: For all $z_h \in W_h^k$,

$$\boldsymbol{G}_h^k z_{h|T} := \boldsymbol{G}_T^k R_{W,T}^k z_h, \qquad r_h^k z_{h|T} = r_T^k R_{W,T}^k z_h \qquad \forall T \in \mathcal{T}_h. \quad (60)$$

The operator $\boldsymbol{G}_h^k$ obviously inherits continuity and stability properties on W_h^k from (57).

3.3.4 Primal hybrid formulation

Denoting by $(\boldsymbol{\sigma}_h, w_h) \in \check{\boldsymbol{\Sigma}}_h^k \times W_h^k$ the solution to problem (41) (we have removed the bar from $\boldsymbol{\sigma}_h$ as a result of Lemma 8), it is readily inferred from (50) and (41a) that

$$\boldsymbol{\sigma}_h = \boldsymbol{\varsigma}_h^k w_h. \quad (61)$$

Then, using (61), equation (41b) can be rewritten for all $z_h = (v_h, \mu_h) \in W_h^k$ as

$$-B(\boldsymbol{\varsigma}_h^k w_h, z_h) = (f_h, v_h).$$

Define the bilinear form A on $W_h^k \times W_h^k$ such that, for all $w_h, z_h \in W_h^k$,

$$A(w_h, z_h) := H(\boldsymbol{\varsigma}_h^k w_h, \boldsymbol{\varsigma}_h^k z_h) = (\boldsymbol{G}_h^k w_h, \boldsymbol{G}_h^k z_h) + j(w_h, z_h), \quad (62)$$

where we have introduced the bilinear form j on $W_h^k \times W_h^k$ such that (J is defined by (30)),

$$j(w_h, z_h) := J(\boldsymbol{\varsigma}_h^k w_h, \boldsymbol{\varsigma}_h^k z_h). \quad (63)$$

The equality in (62) is a straightforward consequence of (29) together with (49). Then, recalling (50) and using the symmetry of the bilinear form H, it is inferred, for all $z_h \in W_h^k$,

$$-B(\boldsymbol{\varsigma}_h^k w_h, z_h) = H(\boldsymbol{\varsigma}_h^k w_h, \boldsymbol{\varsigma}_h^k z_h) = A(w_h, z_h),$$

and we conclude that problem (41) (or, equivalently, (38)) can be reformulated as follows: Find $w_h = (u_h, \lambda_h) \in W_h^k$ such that,

$$A(w_h, z_h) = (f, v_h) \qquad \forall z_h = (v_h, \mu_h) \in W_h^k, \quad (64)$$

and (61) holds. It follows from (52) that, for all $z_h \in W_h^k$, observing that $A(z_h, z_h) = \|\varsigma_h^k z_h\|_{H,T}^2$ as a consequence of (62),

$$\eta \|z_h\|_{1,h}^2 \leqslant A(z_h, z_h) := \|z_h\|_A^2 \leqslant \eta^{-1} \|z_h\|_{1,h}^2. \tag{65}$$

As a result, the bilinear form A is coercive, and the well-posedness of the new problem (64) follows directly from the Lax–Milgram lemma.

From a practical viewpoint, the primal hybrid problem (64) can be solved more efficiently than the mixed hybrid problem (34) (which has a saddle-point structure). We also observe that the discrete flux $\boldsymbol{\sigma}_h$ can be recovered according to (61) by an element-by-element post-processing.

3.4 Error analysis

In this section we briefly show how the error analysis can be carried out directly based on the primal formulation (64). The error analysis for the mixed formulation (34) can be found in [18]. A potential reconstruction of order $(k+2)$ is also identified.

3.4.1 Energy error estimate

Theorem 14 (Energy error estimate). *Let $u \in W$ and $w_h = (u_h, \lambda_h) \in W_h^k$ denote the unique solutions to* (12) *and* (64)*, respectively, and let $\widehat{w}_h = (\widehat{u}_h, \widehat{\lambda}_h) := I_{W,h}^k u$ with interpolator $I_{W,h}^k$ defined by* (47)*. Then, provided $u \in H^{k+2}(\mathcal{T}_h)$, the following estimate holds with real number C independent of h:*

$$\eta^{1/2} \|\widehat{w}_h - w_h\|_{1,h} \leqslant \|\widehat{w}_h - w_h\|_A \leqslant C h^{k+1} \|u\|_{H^{k+2}(\mathcal{T}_h)}, \tag{66}$$

with norm $\|\cdot\|_A$ defined by (65)*.*

Proof. Since the proof relies on the techniques introduced in [20, Theorem 8] and later used in [19, Theorem 9] for the analysis of a HHO method, we only sketch the main points and provide details only for the penalty term, for which a different treatment is required. The first inequality in (66) is an immediate consequence of the coercivity of A, cf. (65). Moreover, again recalling (65), it is readily inferred that

$$\|\widehat{w}_h - w_h\|_A \leqslant \eta^{-1/2} \frac{A(\widehat{w}_h - w_h, \widehat{w}_h - w_h)}{\|\widehat{w}_h - w_h\|_{1,h}} \leqslant \eta^{-1/2} \sup_{z_h \in W_h^k,\, \|z_h\|_{1,h}=1} A(\widehat{w}_h - w_h, z_h).$$

Owing to (64), we then infer that it holds

$$\|\widehat{w}_h - w_h\|_A \leqslant \eta^{-1/2} \sup_{z_h \in W_h^k,\, \|z_h\|_{1,h}=1} \mathcal{E}_h(z_h), \tag{67}$$

with consistency error defined, for all $z_h = (v_h, \mu_h) \in W_h^k$, by

$$\mathcal{E}_h(z_h) := A(\widehat{w}_h, z_h) - (f, v_h) = \Big\{ (\boldsymbol{G}_h^k \widehat{w}_h, \boldsymbol{G}_h^k z_h) + (\triangle u, v_h) \Big\} + j(\widehat{w}_h, z_h) := \mathfrak{T}_1 + \mathfrak{T}_2,$$

where we have used the fact that $f = -\triangle u$ a.e. in Ω. Letting $\breve{u}_T := r_T^k R_{W,T}^k \widehat{w}_h$ for all $T \in \mathcal{T}_h$ (with potential reconstruction r_T^k defined by (58)), we can proceed for the first term like in the proof of [20, Theorem 8]. Recalling that, by definition, $\boldsymbol{G}_T^k R_{W,T}^k \widehat{w}_h = \boldsymbol{\nabla} \breve{u}_T$, it is readily inferred from (55) that

$$\mathfrak{T}_1 = \sum_{T \in \mathcal{T}_h} \left\{ (\boldsymbol{\nabla}(\breve{u}_T - u), \boldsymbol{\nabla} v_T)_T + \sum_{F \in \mathcal{F}_T} (\boldsymbol{\nabla}(\breve{u}_T - u) \cdot \boldsymbol{n}_{TF}, \mu_F - v_T)_F \right\},$$

where we have integrated by parts element-wise the second term, and we have used the flux continuity across interfaces together with the fact that $\mu_F \equiv 0$ for all $F \in \mathcal{F}_h^{\mathrm{b}}$ to infer $\sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} (\boldsymbol{\nabla} u \cdot \boldsymbol{n}_{TF}, \mu_F)_F = 0$. Thus, from Lemmas 10 and 13 with Cauchy-Schwarz inequalities, one can derive the bound

$$|\mathfrak{T}_1| \lesssim h^{k+1} \|u\|_{H^{k+2}(\mathcal{T}_h)} |||\varsigma_h^k z_h|||. \tag{68}$$

On the other hand, we have for the second term, letting $\boldsymbol{\tau}_h := \varsigma_h^k z_h \in \breve{\boldsymbol{\Sigma}}_h^k$,

$$\begin{aligned}
\mathfrak{T}_2 &= H(\varsigma_h^k \widehat{w}_h, \boldsymbol{\tau}_h) - (\boldsymbol{G}_h^k \widehat{w}_h, \boldsymbol{G}_h^k z_h) && \text{eq. (62)} \\
&= \sum_{T \in \mathcal{T}_h} \left\{ (\boldsymbol{\nabla} \widehat{u}_T, \boldsymbol{\tau}_T)_T + \sum_{F \in \mathcal{F}_T} (\widehat{\lambda}_F - \widehat{u}_T, \tau_{TF})_F - (\boldsymbol{\nabla} \breve{u}_T, \mathfrak{C}_T^k R_{\boldsymbol{\Sigma},T}^k \boldsymbol{\tau}_h)_T \right\} && \text{eq. (49)} \\
&= \sum_{T \in \mathcal{T}_h} \left\{ (\boldsymbol{\nabla}(\widehat{u}_T - \pi_T^k \breve{u}_T), \boldsymbol{\tau}_T)_T + \sum_{F \in \mathcal{F}_T} (\widehat{\lambda}_F - \pi_F^k \breve{u}_T - \widehat{u}_T + \pi_T^k \breve{u}_T, \tau_{TF})_F \right\} && \text{eq. (26b)} \\
&= \sum_{T \in \mathcal{T}_h} \left\{ (\boldsymbol{\nabla} \pi_T^k (u - \breve{u}_T), \boldsymbol{\tau}_T)_T + \sum_{F \in \mathcal{F}_T} (\pi_F^k (u - \breve{u}_T) + \pi_T^k (\breve{u}_T - u), \tau_{TF})_F \right\}. && \text{eq. (47)}
\end{aligned}$$

Using the Cauchy–Schwarz, discrete inverse (4) and trace (3) inequalities for the terms involving π_T^k, and recalling that we had set $\boldsymbol{\tau}_h = \varsigma_h^k z_h$, it is inferred

$$\begin{aligned}
|\mathfrak{T}_2| &\lesssim \left\{ \sum_{T \in \mathcal{T}_h} \left[h_T^{-2} \|\pi_T^k (u - \breve{u}_T)\|_T^2 + \sum_{F \in \mathcal{F}_T} h_F^{-1} \|\pi_F^k (u - \breve{u}_T)\|_F^2 \right] \right\}^{1/2} |||\varsigma_h^k z_h||| \\
&\lesssim h^{k+1} \|u\|_{H^{k+2}(\mathcal{T}_h)} |||\varsigma_h^k z_h|||,
\end{aligned} \tag{69}$$

where we have concluded using the fact that π_T^k and π_F^k are bounded operators as projectors followed by the approximation properties (59) of the potential reconstruction. Using (68)–(69) to bound the consistency error in (67) together with (**H2**$^+$) and the second inequality in (52) to infer $|||\varsigma_h^k z_h||| \lesssim \|\varsigma_h^k z_h\|_H \lesssim \|z_h\|_{1,h}$ concludes the proof. □

Corollary 15 (Convergence of the gradient reconstruction)**.** *Under the assumptions of Theorem 14 it holds with real number $C > 0$ independent of h (but depending on ϱ),*

$$\|\boldsymbol{\nabla} u - \boldsymbol{G}_h^k w_h\| \leqslant C h^{k+1} \|u\|_{H^{k+2}(\mathcal{T}_h)}.$$

Proof. Use the triangular inequality to infer

$$\|\boldsymbol{\nabla} u - \boldsymbol{G}_h^k w_h\| \leqslant \|\boldsymbol{\nabla} u - \boldsymbol{G}_h^k \widehat{w}_h\| + \|\boldsymbol{G}_h^k (\widehat{w}_h - w_h)\|,$$

and estimate the first term in the right-hand side using Lemma 13 with $\boldsymbol{G}_h^k \widehat{w}_{h|T} = \boldsymbol{\nabla} r_T^k I_{W,T}^k u$ and the second using (66) after observing that $\|\boldsymbol{G}_h^k (\widehat{w}_h - w_h)\| \leqslant \|\widehat{w}_h - w_h\|_A$ as a result of (62) since the bilinear form j defined by (63) is positive semi-definite owing to **(H1)**. □

3.4.2 Error estimates with elliptic regularity

This section collects error estimates that hold under additional regularity assumptions on the problem. The following result allows to bound the norm of the Lagrange multipliers in terms of the potential DOFs and of the potential lifting defined by (48).

Proposition 16. *There exists a real number $C > 0$ independent of h (but depending on ϱ) such that, for all $T \in \mathcal{T}_h$ and all $z = (v_T, (\mu_F)_{F\in\mathcal{F}_T}) \in W_T^k$, the following inequality holds for all $F \in \mathcal{F}_T$:*

$$h_F^{-1}\|\mu_F\|_F^2 \leqslant C\left(h_T^{-2}\|v_T\|_T^2 + \|\varsigma_T^k z\|_T^2\right). \tag{70}$$

Additionally, for all $z_h = (v_h, \mu_h) \in W_h^k$, we have

$$|\mu_h|_{\mathrm{LM}}^2 := \sum_{F\in\mathcal{F}_h} h_F^{-1}\|\mu_F\|_F^2 \leqslant C \sum_{T\in\mathcal{T}_h}\left(h_T^{-2}\|v_T\|_T^2 + \|\varsigma_T^k R_{W,T}^k z_h\|_T^2\right). \tag{71}$$

Proof. Let an element $T \in \mathcal{T}_h$ and a face $F \in \mathcal{F}_T$ be fixed, and, for a given $z = (v_T, (\mu_F)_{F\in\mathcal{F}_T}) \in W_T^k$, let $\boldsymbol{\tau} = (\boldsymbol{\tau}_T, (\tau_{TF})_{F\in\mathcal{F}_T}) \in \boldsymbol{\Sigma}_T^k$ be such that $\tau_{TF} = h_F^{-1}\mu_F$, $\boldsymbol{\tau}_T \equiv \mathbf{0}$, and $\tau_{TF'} \equiv 0$ for all $F' \in \mathcal{F}_T\backslash\{F\}$. Using $\boldsymbol{\tau}$ as a test function in (48), it is inferred

$$\begin{aligned}
h_F^{-1}\|\mu_F\|_F^2 &= (v_T, D_T^k\boldsymbol{\tau})_T + H_T(\varsigma_T^k z, \boldsymbol{\tau}) \\
&\leqslant \|v_T\|_T\|D_T^k\boldsymbol{\tau}\|_T + \eta^{-1}\|\varsigma_T^k z\|_T\|\boldsymbol{\tau}\|_T && \text{Cauchy–Schwarz and eq. (32b)} \\
&\lesssim \left(h_T^{-2}\|v_T\|_T^2 + \|\varsigma_T^k z\|_T^2\right)^{1/2}\|\boldsymbol{\tau}\|_T, && \text{eq. (22)}
\end{aligned}$$

and (70) follows observing that, owing to (16), $\|\boldsymbol{\tau}\|_T = h_F^{-1/2}\|\mu_F\|_F$. Inequality (71) can be proved observing that $\sum_{F\in\mathcal{F}_h} h_F^{-1}\|\mu_F\|_F^2 \leqslant \sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T} h_F^{-1}\|\mu_F\|_F^2$ and using (70). □

Assume from this point on that elliptic regularity holds in the following form: For all $g \in L^2(\Omega)$, the unique solution $\zeta \in W$ to

$$(\boldsymbol{\nabla}\zeta, \boldsymbol{\nabla}v) = (g, v) \qquad \forall v \in W,$$

satisfies the a priori estimate $\|\zeta\|_{H^2(\Omega)} \leqslant C_{\mathrm{ell}}\|g\|$ with C_{ell} only depending on Ω. This holds true, for instance, when Ω is convex. Then, additional error estimates can be derived which are the counterpart of the classical results for the Raviart–Thomas mixed method proved in [27, 22, 2], thanks to the well-known Aubin-Nitsche trick [3], cf. also [19, 20] for its adaptation to HHO methods.

Lemma 17 (Error estimate for the potential and the Lagrange multipliers)**.** *Under the assumptions of Theorem 14 and elliptic regularity, the following bounds hold for $w_h = (u_h, \lambda_h) \in W_h^k$ solution to (64), with $\widehat{u}_h \in U_h^k$ and $\widehat{\lambda}_h \in \Lambda_h^k$ defined as in Theorem 14 and $|\cdot|_{\mathrm{LM}}$ as in (71):*

$$\|u_h - \widehat{u}_h\| \leqslant Ch^{k+2}\|u\|_{H^{k+2}(\mathcal{T}_h)}, \tag{72a}$$

$$|\lambda_h - \widehat{\lambda}_h|_{\mathrm{LM}} \leqslant Ch^{k+1}\|u\|_{H^{k+2}(\mathcal{T}_h)}, \tag{72b}$$

where $C > 0$ is a real number independent of h (but depending on ϱ).

Proof. The bound (72a) can be proved for the primal hybrid formulation proceeding as in [20, Theorem 10] estimating the penalty term as in Theorem 14. To prove (72b), it suffices to use (71) followed by (72a) and (66). □

The estimate (72a) shows that the discrete potential u_h resulting from (64) essentially behaves as the L^2-orthogonal projection of the exact potential u on $\mathbb{P}_d^k(\mathcal{T}_h)$. As for classical mixed finite element methods [2, 31], we can improve on this result and finally exhibit a potential reconstruction that converges as h^{k+2}.

Lemma 18 (Potential reconstruction)**.** *Under the assumptions of Lemma 17, denoting by u and $w_h = (u_h, \lambda_h)$ the unique solutions to* (12) *and* (64)*, respectively, it holds with real number $C > 0$ independent of h (but depending on ϱ)*

$$\|u - r_h^k w_h\| \leqslant C h^{k+2} \|u\|_{H^{k+2}(T)},$$

where the potential reconstruction operator r_h^k is defined by (60)*.*

Proof. Recalling that, by definition, $\widehat{w}_h = I_{W,h}^k u$, and using the triangular inequality, one has

$$\|u - r_h^k w_h\| \leqslant \|u - r_h^k \widehat{w}_h\| + \|r_h^k(\widehat{w}_h - w_h)\| := \mathfrak{T}_1 + \mathfrak{T}_2.$$

As a result of Lemma 13 it is readily inferred $|\mathfrak{T}_1| \lesssim h^{k+2}\|u\|_{H^{k+2}(\mathcal{T}_h)}$. Additionally, one has

$$\begin{aligned}
\|r_h^k(\widehat{w}_h - w_h)\|^2 &= \sum_{T\in\mathcal{T}_h} \|r_T^k I_{W,T}^k(\widehat{w}_h - w_h)\|_T^2 \\
&\lesssim \sum_{T\in\mathcal{T}_h} \left\{ h_T^2 \|\nabla r_T^k I_{W,T}^k(\widehat{w}_h - w_h)\|_T^2 + \|\pi_T^0(\widehat{u}_T - u_T)\|_T^2 \right\} && \text{eq. (7)} \\
&\leqslant \sum_{T\in\mathcal{T}_h} \left\{ h_T^2 \|\boldsymbol{G}_T^k I_{W,T}^k(\widehat{w}_h - w_h)\|_T^2 + \|\widehat{u}_T - u_T\|_T^2 \right\}, && \text{eq. (58)}
\end{aligned}$$

where, in the last line, we have used the fact that π_T^0 is a bounded operator. Hence, using (66) together with (72a), we infer $|\mathfrak{T}_2| \lesssim h^{k+2}\|u\|_{H^{k+2}(\mathcal{T}_h)}$, and the conclusion follows recalling the bound for $\mathfrak{T}_1$. □

Remark 19 (Relation with the Hybrid High-Order Method)**.** *In [20], the authors study a HHO method based on the following bilinear form on $W_h^k \times W_h^k$, which only differs from the bilinear form A defined by* (64) *in the choice of the stabilization term:*

$$A_{\text{HHO}}(w_h, z_h) = (\boldsymbol{G}_h^k w_h, \boldsymbol{G}_h^k z_h) + J_{\text{HHO}}(w_h, z_h),$$

where, in comparison with (63)*, no link with a mixed hybrid method is used, but*

$$J_{\text{HHO}}(w_h, z_h) := \sum_{T\in\mathcal{T}_h} \sum_{F\in\mathcal{F}_T} (\pi_F^k(\mathfrak{r}_T^k I_{W,T}^k w_h - \lambda_F), \pi_F^k(\mathfrak{r}_T^k I_{W,T}^k z_h - \mu_F))_F,$$

where, for all $T \in \mathcal{T}_h$, the potential reconstruction operator $\mathfrak{r}_T^k : W_T^k \to \mathbb{P}_d^{k+1}(T)$ is such that, for all $z = (v_T, (\mu_F)_{F\in\mathcal{F}_T}) \in W_T^k$, $\mathfrak{r}_T^k z = \left(r_T^k z - \pi_T^k(r_T^k z)\right) + v_T$.

4 Application to the Stokes problem

In this section, we discuss an inf-sup stable discretization of the Stokes problem

$$-\triangle \boldsymbol{u} + \boldsymbol{\nabla} p = \boldsymbol{f} \qquad \text{in } \Omega, \tag{73a}$$
$$-\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0 \qquad \text{in } \Omega, \tag{73b}$$
$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on } \partial\Omega, \tag{73c}$$
$$(p, 1)_\Omega = 0, \tag{73d}$$

where $\boldsymbol{f} = (f_i)_{1\leqslant i\leqslant d} \in L^2(\Omega)^d$ denotes the volumetric body force. Letting

$$\boldsymbol{W} := H_0^1(\Omega)^d \qquad P := L_0^2(\Omega), \tag{74}$$

($L_0^2(\Omega)$ denotes the space of square-integrable functions with zero mean on Ω), the weak formulation of (73) reads: Find $(\boldsymbol{u}, p) \in \boldsymbol{W} \times P$ such that

$$(\boldsymbol{\nabla u}, \boldsymbol{\nabla v}) - (p, \boldsymbol{\nabla}\cdot\boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{W}, \tag{75a}$$
$$(\boldsymbol{\nabla}\cdot\boldsymbol{u}, q) = 0 \qquad \forall q \in P. \tag{75b}$$

The key idea is here to (i) discretize the diffusive term in the momentum conservation equation (75a) using the bilinear form A defined by (62) for each component of the discrete velocity field; (ii) realize the velocity-pressure coupling by means of a discrete divergence operator $\mathcal{D}_h^k$ designed in the same spirit as D_h^k (cf. (24)) and relying on the interpretation of the Lagrange multipliers as traces of the potential provided by the estimate (72b).

4.1 Degrees of freedom

Recalling the definition (39) of W_T^k and W_h^k, we define, for all $T \in \mathcal{T}_h$, the local DOF space for the velocity as

$$\boldsymbol{W}_T^k := (W_T^k)^d.$$

while we seek the pressure in $\mathbb{P}_d^k(T)$. Correspondingly, the global DOF spaces for the velocity and pressure are given by

$$\boldsymbol{W}_h^k := (W_h^k)^d, \qquad \mathcal{P}_h := \mathbb{P}_d^k \cap L_0^2(\Omega). \tag{76}$$

We also define the local and global velocity interpolators $I_{\boldsymbol{W},T}^k$ and $I_{\boldsymbol{W},h}^k$ obtained applying component-wise the interpolators $I_{W,T}^k$ and $I_{W,h}^k$ defined by (46) and (47), respectively. Finally, for all $T \in \mathcal{T}_h$, we denote by $R_{\boldsymbol{W},T}^k : \boldsymbol{W}_h^k \to \boldsymbol{W}_T^k$ the restriction operator that realizes the mapping between global and local velocity DOFs.

4.2 Velocity-pressure coupling

The local divergence operator $\mathcal{D}_T^k : \boldsymbol{W}_T^k \to \mathbb{P}_d^k(T)$ is such that, for all $\boldsymbol{z} = (v_{T,i}, (\mu_{F,i})_{F\in\mathcal{F}_T})_{1\leqslant i\leqslant d} \in \boldsymbol{W}_T^k$,

$$(\mathcal{D}_T^k \boldsymbol{z}, q)_T = \sum_{i=1}^d \left\{ -(v_{T,i}, \partial_i q)_T + \sum_{F\in\mathcal{F}_T} (\mu_{F,i} n_{TF,i}, q)_F \right\} \qquad q \in \mathbb{P}_d^k(T), \tag{77}$$

where ∂_i denotes the partial derivative with respect to the ith space variable. We record the following equivalence obtained integrating by parts the first term in (77):

$$(\mathcal{D}_T^k \boldsymbol{z}, q)_T = \sum_{i=1}^{d} \left\{ (\partial_i v_{T,i}, q)_T + \sum_{F \in \mathcal{F}_T} ((\mu_{F,i} - v_{T,i}) n_{TF,i}, q)_F \right\} \qquad \forall q \in \mathbb{P}_d^k(T). \tag{78}$$

We also define the global discrete divergence operator $\mathcal{D}_h^k : \boldsymbol{W}_h^k \to \mathbb{P}_d^k(\mathcal{T}_h)$ such that, for all $\boldsymbol{z}_h \in \boldsymbol{W}_h^k$,

$$(\mathcal{D}_h^k \boldsymbol{z}_h, q_h) = \sum_{T \in \mathcal{T}_h} (\mathcal{D}_T^k R_{\boldsymbol{W},T}^k \boldsymbol{z}_h, q_h)_T \qquad \forall q_h \in \mathbb{P}_d^k(\mathcal{T}_h). \tag{79}$$

The operator $\mathcal{D}_h^k$ defined by (79) can be regarded as the discrete counterpart of the divergence operator defined from $\boldsymbol{W}$ to P, cf. (74), as opposed to the operator D_h^k defined by (24), which discretizes the divergence operator from $\boldsymbol{\Sigma}$ to U, cf. (9).

The proof of the following result is analogous to that given in the context of linear elasticity in [19, Proposition 4], to which we refer for the details.

Proposition 20 (Commuting property for $\mathcal{D}_h^k$)**.** *The following commuting diagrams hold with* $\boldsymbol{W}(T) := W(T)^d$ *and* $W(T)$ *defined by* (45)*:*

$$\begin{array}{ccc} \boldsymbol{W}(T) & \xrightarrow{\boldsymbol{\nabla}\cdot} & L^2(T) \\ {\scriptstyle I_{\boldsymbol{W},T}^k}\downarrow & & \downarrow{\scriptstyle \pi_T^k} \\ \boldsymbol{W}_T^k & \xrightarrow{\mathcal{D}_T^k} & \mathbb{P}_d^k(T) \end{array} \qquad \begin{array}{ccc} \boldsymbol{W} & \xrightarrow{\boldsymbol{\nabla}\cdot} & P \\ {\scriptstyle I_{\boldsymbol{W},h}^k}\downarrow & & \downarrow{\scriptstyle \pi_h^k} \\ \boldsymbol{W}_h^k & \xrightarrow{\mathcal{D}_h^k} & \mathcal{P}_h \end{array}$$

4.3 Discrete problem

The discretization of the viscous term in (75a) hinges on the bilinear form $\mathcal{A}$ on $\boldsymbol{W}_h^k \times \boldsymbol{W}_h^k$ such that, for all $\boldsymbol{w}_h = (w_{h,i})_{1 \leqslant i \leqslant d}$ and $\boldsymbol{z}_h = (z_{h,i})_{1 \leqslant i \leqslant d}$ in $\boldsymbol{W}_h^k$,

$$\mathcal{A}(\boldsymbol{w}_h, \boldsymbol{z}_h) := \sum_{i=1}^{d} A(w_{h,i}, z_{h,i}), \tag{80}$$

with bilinear form A defined by (62). The coercivity and continuity of the bilinear form $\mathcal{A}$ follow from the corresponding properties (65) of the bilinear form A:

$$\eta \|\boldsymbol{z}_h\|_{1,h}^2 \leqslant \mathcal{A}(\boldsymbol{z}_h, \boldsymbol{z}_h) := \|\boldsymbol{z}_h\|_{\mathcal{A}}^2 \leqslant \eta^{-1} \|\boldsymbol{z}_h\|_{1,h}^2, \tag{81}$$

where we have set $\|\boldsymbol{z}_h\|_{1,h}^2 := \sum_{i=1}^{d} \|z_{h,i}\|_{1,h}^2$ and the scalar version of the $\|\cdot\|_{1,h}$-norm is defined by (43).

The source term in (75a) is discretized by means of the linear form L on $\boldsymbol{W}_h^k$ such that, for all $\boldsymbol{z}_h = (v_{h,i}, \mu_{h,i})_{1 \leqslant i \leqslant d}$,

$$L(\boldsymbol{z}_h) = \sum_{i=1}^{d} (f_i, v_{h,i}). \tag{82}$$

The discretization of problem (75) reads: Find $(\boldsymbol{w}_h, p_h) \in \boldsymbol{W}_h^k \times \mathcal{P}_h$ such that

$$\mathcal{A}(\boldsymbol{w}_h, \boldsymbol{z}_h) - (p_h, \mathcal{D}_h^k \boldsymbol{z}_h) = L(\boldsymbol{z}_h) \qquad \forall \boldsymbol{z}_h \in \boldsymbol{W}_h^k, \tag{83a}$$

$$(\mathcal{D}_h^k \boldsymbol{w}_h, q_h) = 0 \qquad \forall q_h \in \mathcal{P}_h. \tag{83b}$$

Since $\mathcal{A}$ is coercive and continuous on $\boldsymbol{W}_h^k$ (cf. (81)), the well-posedness of (83) hinges on the following result, which can be proved using classical techniques, cf. [13].

Lemma 21 (Well-posedness). *There exists $\beta > 0$ independent of h (but depending on ϱ) such that, for all $q_h \in \mathcal{P}_h$, the following inf-sup condition holds:*

$$\beta \|q_h\| \leqslant \sup_{\boldsymbol{z}_h \in \boldsymbol{W}_h^k \setminus \{\boldsymbol{0}\}} \frac{(\mathcal{D}_h^k \boldsymbol{z}_h, q_h)}{\|\boldsymbol{z}_h\|_{1,h}}. \tag{84}$$

Additionally, problem (83) *is well-posed.*

The proof of Lemma 21 is analogous to the corresponding one for the operator D_h^k.

4.4 Convergence analysis

Lemma 22 (Basic error estimate). *Let $(\boldsymbol{u}, p) \in \boldsymbol{W} \times P$ denote the unique solution to* (75)*, and let $(\widehat{\boldsymbol{w}}_h, \widehat{p}_h) := (I_{\boldsymbol{W},h}^k \boldsymbol{u}, \pi_h^k p)$. Then, denoting by $(\boldsymbol{w}_h, p_h) \in \boldsymbol{W}_h^k \times \mathcal{P}_h$ the unique solution to* (83)*, the following holds with $\|\cdot\|_{\mathcal{A}}$-norm defined by* (81)*:*

$$\max\left(\frac{\beta\eta^{1/2}}{2}\|p_h - \widehat{p}_h\|, \|\boldsymbol{w}_h - \widehat{\boldsymbol{w}}_h\|_{\mathcal{A}}\right) \leqslant \sup_{\boldsymbol{z}_h \in \boldsymbol{W}_h^k} \frac{\mathcal{E}_h(\boldsymbol{z}_h)}{\|\boldsymbol{z}_h\|_{\mathcal{A}}}, \tag{85}$$

where the consistency error is such that $\mathcal{E}_h(\boldsymbol{z}_h) = L(\boldsymbol{z}_h) + (\widehat{p}_h, \mathcal{D}_h^k \boldsymbol{z}_h) - \mathcal{A}(\widehat{\boldsymbol{w}}_h, \boldsymbol{z}_h)$.

Proof. We denote by \$ the supremum in the right-hand side of (85). Observe that $\mathcal{D}_T^k \boldsymbol{w}_h = \mathcal{D}_T^k \widehat{\boldsymbol{w}}_h = 0$ as a consequence of (83b) and the right commuting diagram in Proposition 20 together with (73b), respectively. As a result, making $\boldsymbol{z}_h = \boldsymbol{w}_h - \widehat{\boldsymbol{w}}_h$ in (83a), one can easily show that

$$\|\boldsymbol{w}_h - \widehat{\boldsymbol{w}}_h\|_{\mathcal{A}} \leqslant \$. \tag{86}$$

Let us now estimate the error on the pressure. Using (83a) together with the definition of the consistency error yields, for all $\boldsymbol{z}_h \in \boldsymbol{W}_h^k$,

$$(p_h - \widehat{p}_h, \mathcal{D}_h^k \boldsymbol{z}_h) = (p_h, \mathcal{D}_h^k \boldsymbol{z}_h) - (\widehat{p}_h, \mathcal{D}_h^k \boldsymbol{z}_h) = \mathcal{A}(\boldsymbol{w}_h - \widehat{\boldsymbol{w}}_h, \boldsymbol{z}_h) - \mathcal{E}_h(\boldsymbol{z}_h).$$

Using the inf-sup condition (84) for $q_h = p_h - \widehat{p}_h$ together with (86), the Cauchy–Schwarz inequality, and the second inequality in (81), it is inferred that

$$\beta\eta^{1/2}\|p_h - \widehat{p}_h\| \leqslant \sup_{\boldsymbol{z}_h \in \boldsymbol{W}_h^k \setminus \{\boldsymbol{0}\}} \frac{(p_h - \widehat{p}_h, \mathcal{D}_h^k \boldsymbol{z}_h)}{\eta^{-1/2}\|\boldsymbol{z}_h\|_{1,h}} \leqslant \|\boldsymbol{w}_h - \widehat{\boldsymbol{w}}_h\|_{\mathcal{A}} + \$ \leqslant 2\$. \tag{87}$$

The estimate (85) is an immediate consequence of (86)–(87). □

Theorem 23 (Convergence rate)**.** *Under the assumptions of Lemma 22, and assuming the additional regularity* $\boldsymbol{u} \in H^{k+2}(\mathcal{T}_h)^d$ *and* $p \in H^{k+1}(\mathcal{T}_h)$*, the following holds:*

$$\max\left(\frac{\beta\eta^{1/2}}{2}\|p_h - \widehat{p}_h\|, \|\boldsymbol{w}_h - \widehat{\boldsymbol{w}}_h\|_{\mathcal{A}}\right) \leqslant Ch^{k+1}\left(\|\boldsymbol{u}\|_{H^{k+2}(\mathcal{T}_h)^d} + \|p\|_{H^{k+1}(\mathcal{T}_h)}\right), \tag{88}$$

with $C > 0$ *independent of* h *(but depending on* ϱ*).*

Proof. For a given $\boldsymbol{z}_h = \left((v_{T,i})_{T\in\mathcal{T}_h}, (\mu_{F,i})_{F\in\mathcal{F}_h}\right)_{1\leqslant i\leqslant d} \in \boldsymbol{W}_h^k$, we introduce the vector-valued polynomial functions $\boldsymbol{v}_T := (v_{T,i})_{1\leqslant i\leqslant d}$ for all $T \in \mathcal{T}_h$ and $\boldsymbol{\mu}_F := (\mu_{F,i})_{1\leqslant i\leqslant d}$ for all $F \in \mathcal{F}_h$. We also let $\check{\boldsymbol{u}}_h = (\check{u}_{h,i})_{1\leqslant i\leqslant d}$ where, for all $1 \leqslant i \leqslant d$, $\check{u}_{h,i} := r_T^k \widehat{w}_{h,i}$ and r_T^k is the potential reconstruction operator defined by (58). Using the fact that $\boldsymbol{f} = -\triangle\boldsymbol{u} + \boldsymbol{\nabla} p$ a.e. in Ω, recalling the definitions (80) of the bilinear form $\mathcal{A}$ and (62) of the bilinear form A together with (78), and performing an element-by-element integration by parts on the linear form L defined by (82), we decompose the consistency error as follows:

$$\begin{aligned}\mathcal{E}_h(\boldsymbol{z}_h) = \sum_{T\in\mathcal{T}_h}\Big\{(\boldsymbol{\nabla}(\boldsymbol{u} - \check{\boldsymbol{u}}_h), \boldsymbol{\nabla}\boldsymbol{v}_T)_T + \sum_{F\in\mathcal{F}_T}(\boldsymbol{\nabla}(\boldsymbol{u} - \check{\boldsymbol{u}}_{h|T})\boldsymbol{n}_{TF}, \boldsymbol{\mu}_F - \boldsymbol{v}_T)_F\Big\} \\ + \sum_{T\in\mathcal{T}_h}\Big\{(\widehat{p}_h - p, \boldsymbol{\nabla}\cdot\boldsymbol{v}_T)_T + \sum_{F\in\mathcal{F}_T}(\widehat{p}_h - p, (\boldsymbol{\mu}_F - \boldsymbol{v}_T)\cdot\boldsymbol{n}_{TF})_F\Big\} + \sum_{i=1}^d j(\widehat{w}_{h,i}, z_{h,i}),\end{aligned} \tag{89}$$

where we have used continuity of the normal momentum flux across interfaces as well as the fact that the homogeneous Dirichlet boundary condition is embedded in $\boldsymbol{W}_h^k$ (cf. (76) and (39)) to introduce the term $\sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T}((\boldsymbol{\nabla}\boldsymbol{u} - pI_d)\boldsymbol{n}_{TF}, \boldsymbol{\mu}_F)_F$.

Denote by $\mathfrak{T}_1$, $\mathfrak{T}_2$, and $\mathfrak{T}_3$ the terms in the right-hand side of (89). Multiple uses of the Cauchy–Schwarz inequality followed by the approximation properties (59) of r_T^k and (5) of the L^2-orthogonal projector yield

$$\begin{aligned}|\mathfrak{T}_1| + |\mathfrak{T}_2| &\leqslant h^{k+1}\left(\|\boldsymbol{u}\|_{H^{k+2}(\mathcal{T}_h)^d} + \|p\|_{H^{k+1}(\mathcal{T}_h)}\right)\|\boldsymbol{z}_h\|_{1,h} \\ &\lesssim h^{k+1}\left(\|\boldsymbol{u}\|_{H^{k+2}(\mathcal{T}_h)^d} + \|p\|_{H^{k+1}(\mathcal{T}_h)}\right)\|\boldsymbol{z}_h\|_{\mathcal{A}},\end{aligned} \tag{90}$$

where to conclude we have used the first inequality in (52). Recalling (63), proceeding as for the second term in the proof of Theorem 14, and using (**H2**$^+$) together with the second inequality in (52) and the first inequality in (65) to infer $|||\boldsymbol{\varsigma}_h^k z_{h,i}||| \lesssim \|z_{h,i}\|_{1,h} \lesssim \|z_{h,i}\|_A$ for all $1 \leqslant i \leqslant d$, we have

$$|\mathfrak{T}_3| \lesssim h^{k+1}\|\boldsymbol{u}\|_{H^{k+2}(\mathcal{T}_h)^d}\left\{\sum_{i=1}^d |||\boldsymbol{\varsigma}_h^k z_{h,i}|||^2\right\}^{1/2} \lesssim h^{k+1}\|\boldsymbol{u}\|_{H^{k+2}(\mathcal{T}_h)^d}\|\boldsymbol{z}_h\|_{\mathcal{A}}, \tag{91}$$

where the conclusion follows from (81) observing that $\sum_{i=1}^d \|z_{h,i}\|_A^2 = \|\boldsymbol{z}_h\|_{\mathcal{A}}^2$. Finally, to prove the estimate (88), use (90)–(91) to bound the right-hand side of (89) and the resulting bound in (85). □

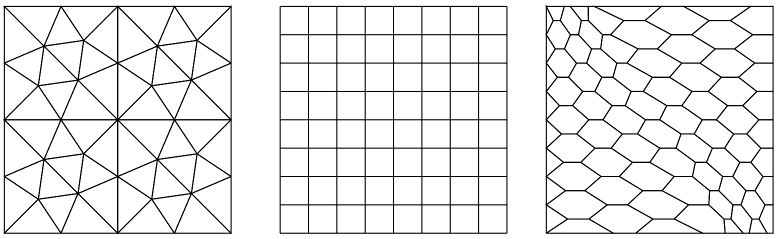


Figure 1: Triangular, Cartesian and hexagonal mesh families es for the numerical example of Section 4.5

4.5 Numerical example

We solve the Stokes problem (73) on the unit square $\Omega = (0,1)^2$ with $\boldsymbol{f} \equiv \mathbf{0}$ and Dirichlet boudary conditions inferred from the following exact solution:

$$\boldsymbol{u}(x,y) = \big(-\exp(x)(y\cos y + \sin y), \exp(x)(y\sin y)\big), \qquad p = 2\exp(x)\sin(y) - p_0,$$

where $p_0 \in \mathbb{R}$ is chosen so as to ensure $\int_\Omega p = 0$. We consider the three mesh families depicted in Figure 1. The triangular and Cartesian mesh families correspond, respectively, to the mesh families 1 and 2 of the FVCA5 benchmark [28], whereas the (predominantly) hexagonal mesh family was first introduced in [21].

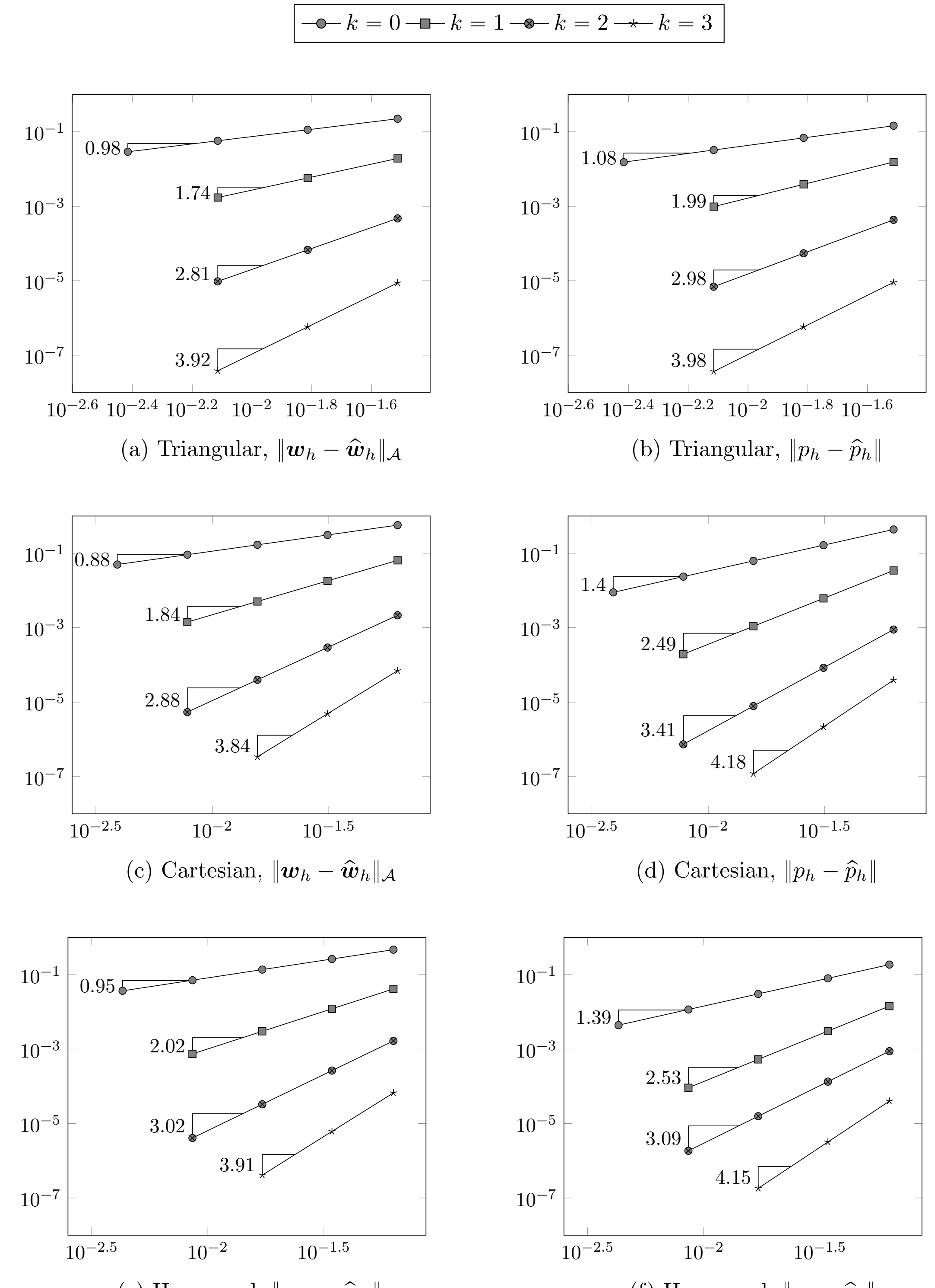
Figure 2 displays convergence results for the different meshes and polynomial degrees up to 3. Following (85), we display the $\|\cdot\|_{\mathcal{A}}$-norm of the error in the velocity as well as the L^2-norm of the in the pressure. In all the cases, the numerical results matches the order estimates predicted by the theory.

Local computations are based on the linear algebra facilities provided by the boost uBLAS library [29]. The local linear systems for the computation of the operators D_T^k, $\mathfrak{C}_T^k$, and the local contributions to the bilinear form A are solved using the Cholesky factorization available in uBLAS; cf. equations (95), (96), and (99) below. The global system (involving face unknowns only) is solved using SuperLU [15] through the PETSc 3.4 interface [4]. The tests have been run sequentially on a laptop computer powered by an Intel Core i7-3520 CPU clocked at 2.90 GHz and equipped with 8Gb of RAM.

5 Implementation

In this section we discuss the practical implementation of the primal hybrid method (64) for the Poisson problem. The implementation of the method (83) for the Stokes equations follows similar principles and is not detailed here for the sake of brevity.

An essential point consists in selecting appropriate bases for the polynomial spaces on elements and faces. Particular care is required to make sure that the resulting local problems are well-conditioned, since the accuracy of the local computations may affect the overall quality of the approximation. For a given polynomial degree $l \in \{k, k+1\}$, one possibility leading to a



(a) Triangular, $\|\boldsymbol{w}_h - \widehat{\boldsymbol{w}}_h\|_{\mathcal{A}}$

(b) Triangular, $\|p_h - \widehat{p}_h\|$

(c) Cartesian, $\|\boldsymbol{w}_h - \widehat{\boldsymbol{w}}_h\|_{\mathcal{A}}$

(d) Cartesian, $\|p_h - \widehat{p}_h\|$

(e) Hexagonal, $\|\boldsymbol{w}_h - \widehat{\boldsymbol{w}}_h\|_{\mathcal{A}}$

(f) Hexagonal, $\|p_h - \widehat{p}_h\|$

Figure 2: Convergence results for the numerical example of Section 4.5 on the mesh families of Figure 1. The notation is the same as in Theorem 23.

hierarchical basis for $\mathbb{P}_d^l(T)$, $T \in \mathcal{T}_h$, is to choose the following family of monomial functions:

$$\left\{\varphi_T = \prod_{i=1}^{d} \xi_{T,i}^{\alpha_i} \quad \middle| \quad \xi_{T,i} := \tfrac{x_i - x_{T,i}}{h_T} \quad \forall 1 \leqslant i \leqslant d, \quad \underline{\alpha} \in \mathbb{N}^d, \quad \|\underline{\alpha}\|_{l^1} \leqslant l\right\}, \tag{92}$$

where $\boldsymbol{x}_T$ denotes the barycenter of T. The idea is here (i) to express basis functions with respect to a reference frame local to one element, which ensures that the basis does not depend on the position of the element and (ii) to scale with respect to a local length scale. Choosing this length scale equal to h_T ensures that the basis functions take values in the interval $[-1, 1]$. For anisotropic elements, a better option would be to use the inertial frame of reference and, possibly, to perform orthonormalization, cf. [5]. Similarly, a hierarchical monomial basis can be defined for the spaces $\mathbb{P}_d^k(F)$, $F \in \mathcal{F}_h$, using the face barycenter $\boldsymbol{x}_F$ and the face diameter h_F.

Let, for a given polynomial degree $l \geqslant 0$ and a number of variables $n \geqslant 0$, $N_n^l := \dim(\mathbb{P}_n^l)$. For any element $T \in \mathcal{T}_h$, we assume for the sake of simplicity that a hierarchical basis $\mathcal{B}_T^{k+1} := \{\varphi_T^i\}_{0 \leqslant i < N_d^{k+1}}$ (not necessarily given by (92)) has been selected for $\mathbb{P}_d^{k+1}(T)$ so that φ_T^0 is the constant function on T and $(\varphi_T^i, \varphi_T^0)_T = 0$ for all $1 \leqslant i < N_d^{k+1}$. While this latter condition is not verified for general element shapes by the choice (92), one can obtain also in that case a well-posed local problem (26) for the computation of $\mathfrak{C}_T^k$ by removing φ_T^0, since the remaining functions vanish at $\boldsymbol{x}_T$. For more general choices, the zero-average condition can be enforced by a Lagrange multiplier constant over the element. Having assumed that $\mathcal{B}_T^{k+1}$ is hierarchical, a basis for $\mathbb{P}_d^k(T)$ is readily obtained by selecting the first N_d^k basis functions. Additionally, for any face $F \in \mathcal{F}_h$, we denote by $\mathcal{B}_F^k := \{\varphi_F^i\}_{0 \leqslant i < N_{d-1}^k}$ a basis for $\mathbb{P}_d^k(F)$ (not necessarily hierarchical in this case).

In the spirit of Remark 1, solving the primal hybrid problem (64) amounts to computing the coefficients $(u_T^i)_{0 \leqslant i < N_d^k}$ for all $T \in \mathcal{T}_h$ and $(\lambda_F^i)_{0 \leqslant i < N_{d-1}^k}$ for all $F \in \mathcal{F}_h$ of the following expansions for the local potential unknown $u_T \in U_T^k$ and the local Lagrange multiplier $\lambda_F \in \Lambda_F^k$, respectively:

$$u_T = \sum_{0 \leqslant i < N_d^k} u_T^i \varphi_T^i, \qquad \lambda_F = \sum_{0 \leqslant i < N_{d-1}^k} \lambda_F^i \varphi_F^i. \tag{93}$$

For all $T \in \mathcal{T}_h$, we also introduce as intermediate unknowns the algebraic flux DOFs $(\sigma_T^i)_{1 \leqslant i < N_d^k}$ and $(\sigma_{TF}^i)_{0 \leqslant i < N_{d-1}^k}$, $F \in \mathcal{F}_T$, corresponding to the coefficients of the following expansions for the components of the local flux unknown $(\boldsymbol{\sigma}_T, (\sigma_{TF})_{F \in \mathcal{F}_T}) \in \boldsymbol{\Sigma}_T^k$:

$$\mathbb{T}_T^k \ni \boldsymbol{\sigma}_T = \sum_{1 \leqslant i < N_d^k} \sigma_T^i \boldsymbol{\nabla} \varphi_T^i \qquad \mathbb{F}_F^k \ni \sigma_{TF} = \sum_{0 \leqslant i < N_{d-1}^k} \sigma_{TF}^i \varphi_F^i \quad \forall F \in \mathcal{F}_T. \tag{94}$$

where we have used the fact that $(\boldsymbol{\nabla} \varphi_T^i)_{1 \leqslant i < N_d^k}$ is a basis for the DOF space $\mathbb{T}_T^k$ defined by (13) (the sum starts from 1 to accomodate the zero-average constraint in the definition of $\mathbb{T}_T^k$). Clearly, the total number of local flux DOFs in $\boldsymbol{\Sigma}_T^k$ (cf. (14)) is

$$N_{\Sigma,T}^k := (N_d^k - 1) + \mathfrak{N}_T N_{d-1}^k,$$

with $\mathfrak{N}_T$ defined in (1).

For a given element $T \in \mathcal{T}_h$, the discrete operators $D_T^k, \mathfrak{C}_T^k, \varsigma_T^k$ act on and take values in finite dimensional spaces, hence they can be represented by matrices once the choice of the bases for the DOF spaces has been made. Their action on a vector of DOFs then results from right matrix-vector multiplication. In what follows, we show how to carry out the computation of such matrices in detail and how to use them to infer the local contribution to the bilinear form A stemming from the element T.

5.1 Discrete divergence operator

The discrete divergence operator D_T^k acting on $\mathbf{\Sigma}_T^k$ with values in $\mathbb{P}_d^k(T)$ can be represented by the matrix D of size $N_d^k \times N_{\Sigma,T}^k$ with block-structure $\left[\, \mathsf{D}_T \,\vdots\, (\mathsf{D}_F)_{F\in\mathcal{F}_T} \right]$ induced by the geometric items to which flux DOFs in $\mathbf{\Sigma}_T^k$ are associated. According to the definition (20) of D_T^k, the matrix D can be computed as the solution of the following linear system of size N_d^k with $N_{\Sigma,T}^k$ right-hand sides:

$$\mathsf{M}_D\mathsf{D} = \mathsf{R}_D, \tag{95}$$

with block form

$$N_d^k \left\{ \overbrace{\left[\,\mathsf{M}_D\,\right]}^{N_d^k} \quad \underbrace{\left[\, \overbrace{\mathsf{D}_T}^{N_d^k - 1} \,\vdots\, \overbrace{\mathsf{D}_{F_1}}^{N_{d-1}^k} \,\vdots\, \cdots \,\vdots\, \overbrace{\mathsf{D}_{F_{\mathfrak{N}_T}}}^{N_{d-1}^k} \,\right]}_{N_{\Sigma,T}^k} = \underbrace{\left[\, \overbrace{\mathsf{R}_{D,T}}^{N_d^k - 1} \,\vdots\, \overbrace{\mathsf{R}_{D,F_1}}^{N_{d-1}^k} \,\vdots\, \cdots \,\vdots\, \overbrace{\mathsf{R}_{D,F_{\mathfrak{N}_T}}}^{N_{d-1}^k} \,\right]}_{N_{\Sigma,T}^k} \right.$$

where the system matrix is $\mathsf{M}_D := \left[(\varphi_T^i, \varphi_T^j)_T\right]_{0\leqslant i,j<N_d^k}$, while the right-hand side is such that

$$\mathsf{R}_{D,T} := \left[(\nabla\varphi_T^i, \nabla\varphi_T^j)_T\right]_{0\leqslant i<N_d^k, 1\leqslant j<N_d^k} \qquad \mathsf{R}_{D,F} := \left[(\varphi_T^i, \varphi_F^j)_F\right]_{0\leqslant i<N_d^k, 0\leqslant j<N_{d-1}^k} \quad \forall F\in\mathcal{F}_T.$$

When considering orthonormal bases such as, e.g., the ones introduced in [5], the matrix M_D is unit diagonal and numerical resolution is unnecessary.

5.2 Consistent flux reconstruction operator

The consistent flux reconstruction operator $\mathfrak{C}_T^k$ acting on $\mathbf{\Sigma}_T^k$ with values in $\nabla\mathbb{P}_d^{k+1,0}(T)$ can be represented by the matrix C of size $(N_d^{k+1} - 1) \times N_{\Sigma,T}^k$ with the block-structure $\left[\, \mathsf{C}_T \,\vdots\, (\mathsf{C}_F)_{F\in\mathcal{F}_T} \right]$ induced by the geometric items to which flux DOFs in $\mathbf{\Sigma}_T^k$ are associated. According to definition (26a), this requires to solve a linear system of size $(N_d^{k+1}-1)$ with $N_{\Sigma,T}^k$ right-hand sides,

$$\mathsf{M}_C\mathsf{C} = \mathsf{Q}_C\mathsf{D} + \mathsf{R}_C := \widetilde{\mathsf{R}}_C. \tag{96}$$

The linear system (96) has the following block form:

$$N_d^{k+1}-1\left\{\underbrace{\boxed{\mathsf{M}_C}}_{N_d^{k+1}-1}\right.\;\underbrace{\left[\overbrace{\mathsf{C}_T}^{N_d^k-1} \,\Big\vert\, \overbrace{\mathsf{C}_{F_1}}^{N_{d-1}^k} \,\Big\vert\, \cdots \,\Big\vert\, \overbrace{\mathsf{C}_{F_{\mathfrak{N}_T}}}^{N_{d-1}^k}\right]}_{N_{\Sigma,T}^k} = \overbrace{\boxed{\mathsf{Q}_C}}^{N_d^k}\;\overbrace{\boxed{\mathsf{D}}}^{N_{\Sigma,T}^k} + \underbrace{\left[\overbrace{0}^{N_d^k-1} \,\Big\vert\, \overbrace{\mathsf{R}_{C,F_1}}^{N_{d-1}^k} \,\Big\vert\, \cdots \,\Big\vert\, \overbrace{\mathsf{R}_{C,F_{\mathfrak{N}_T}}}^{N_{d-1}^k}\right]}_{N_{\Sigma,T}^k}$$

with system matrix $\mathsf{M}_C := \left[(\boldsymbol{\nabla}\varphi_T^i, \boldsymbol{\nabla}\varphi_T^j)\right]_{1\leqslant i,j<N_d^{k+1}}$ and the matrix blocks appearing in the right-hand side in addition to the matrix D obtained solving (95) are given by

$$\mathsf{Q}_C := \left[-(\varphi_T^i, \varphi_T^j)_T\right]_{1\leqslant i<N_d^{k+1},\,0\leqslant j<N_d^k}, \qquad \mathsf{R}_{C,F} := \left[(\varphi_T^i, \varphi_F^j)_F\right]_{1\leqslant i<N_d^{k+1},\,0\leqslant j<N_{d-1}^k} \quad \forall F\in\mathcal{F}_T.$$

5.3 Bilinear form H_T

We are now ready to compute the matrix H of size $N_{\Sigma,T}^k \times N_{\Sigma,T}^k$ representing the local bilinear form H_T defined by (32) as

$$\mathsf{H} = \mathsf{C}^t\widetilde{\mathsf{R}}_C + \mathsf{J}, \tag{97}$$

where the factors appearing in the first term are defined in (96), while the matrix J representing the stabilization term J_T defined by (35) is given by (the block partitioning is the one induced by the geometric entity to which flux DOFs are attached):

$$\mathsf{J} = \sum_{F\in\mathcal{F}_T} \mathsf{C}^t\mathsf{Q}_{J,1,F}\mathsf{C} - \left[\,0 \,\vert\, (\mathsf{C}^t\mathsf{Q}_{J,2,F})_{F\in\mathcal{F}_T}\,\right] - \left[\,0 \,\vert\, (\mathsf{C}^t\mathsf{Q}_{J,2,F})_{F\in\mathcal{F}_T}\,\right]^t + h_F\begin{bmatrix} 0 & 0 \\ 0 & \operatorname{diag}(\mathsf{M}_F)_{F\in\mathcal{F}_T}\end{bmatrix},$$

where C is defined by (96) while, for all $F\in\mathcal{F}_T$, we have defined the auxiliary matrices

$$\mathsf{Q}_{J,1,F} := h_F\left[(\boldsymbol{\nabla}\varphi_T^i{\cdot}\boldsymbol{n}_{TF}, \boldsymbol{\nabla}\varphi_T^j{\cdot}\boldsymbol{n}_{TF})_F\right]_{1\leqslant i,j<N_d^{k+1}},$$
$$\mathsf{Q}_{J,2,F} := h_F\left[(\boldsymbol{\nabla}\varphi_T^i{\cdot}\boldsymbol{n}_{TF}, \varphi_F^j)_F\right]_{1\leqslant i<N_d^{k+1},\,0\leqslant j<N_{d-1}^k},$$

and face mass matrices

$$\mathsf{M}_F := \left[(\varphi_F^i, \varphi_F^j)_F\right]_{0\leqslant i,j<N_{d-1}^k}. \tag{98}$$

5.4 Hybridization

The first step to perform hybridization is to construct the matrix B representing the bilinear form B defined by (40a), which has the following block form corresponding to the geometric items to which DOFs in $\boldsymbol{\Sigma}_T^k$ (rows) and W_T^k (columns) are associated:

$$
\mathsf{B} = \underbrace{\left[\begin{array}{c|c|c|c}
 & 0 & \cdots & 0 \\ \cline{2-2}\cline{4-4}
 & \mathsf{M}_{F_1} & & \\ \cline{2-2}
\overbrace{\mathsf{R}_D^t}^{N_d^k} & 0 & \ddots & 0 \\ \cline{4-4}
 & \underbrace{\phantom{\mathsf{M}}}_{N_{d-1}^k} & & \underbrace{\mathsf{M}_{F_{\mathfrak{N}_T}}}_{N_{d-1}^k}
\end{array}\right]}_{N_{W,T}^k}
\begin{array}{l} \}\ N_d^k-1 \\ \}\ \mathfrak{N}_T N_{d-1}^k \end{array}
\Bigg\}\ N_{\Sigma,T}^k
$$

with matrix R_D as in (95), M_F defined by (98), and

$$N_{W,T}^k := N_d^k + \mathfrak{N}_T N_{d-1}^k,$$

corresponding to the number of DOFs in W_T^k.

The condition on the Lagrange multipliers in Λ_h^k on boundary faces $F \in \mathcal{F}_b$ (cf. (37)) is enforced via Lagrange multipliers in $\mathbb{P}_{d-1}^k(F)$. This choice is reflected by the fact that we include boundary faces in the definition of the matrix B.

The local contribution to the bilinear form A defined by (62) is finally given by

$$\mathsf{A} = \mathsf{B}^t \mathsf{H}^{-1} \mathsf{B}, \tag{99}$$

which requires the solution of a linear system involving the matrix H defined by (97). Observe that $\mathsf{H}^{-1}\mathsf{B}$ is in fact the matrix representation of the lifting operator ς_T^k defined by (48).

The matrix A has the following block structure induced by the geometric items to which DOFs in W_T^k are attached:

$$
\mathsf{A} = \left[\begin{array}{c|c}
\overbrace{\mathsf{A}_{TT}}^{N_d^k} & \overbrace{\mathsf{A}_{TF}}^{\mathfrak{N}_T N_{d-1}^k} \\ \hline
\mathsf{A}_{TF}^t & \mathsf{A}_{FF}
\end{array}\right]
\begin{array}{l} \}\ N_d^k \\ \}\ \mathfrak{N}_T N_{d-1}^k \end{array}
$$

Observing that cell DOFs for a given element T are only linked to the face DOFs (Lagrange multipliers) attached to the faces in $\mathcal{F}_T$, one can finally obtain a problem in the sole Lagrange multipliers by computing the Schur complement of A_{TT}. This requires the numerical inversion of the symmetric positive-definite matrix A_{TT} of size $N_d^k \times N_d^k$

Remark 24 (GPU implementation)**.** *We remark at this point that the dimensions of the symmetric positive definite matrices M_D, M_C, and A_{TT} are independent of the element shape. This, together with the fact that the solution of the linear systems* (95) *and* (96) *as well as the computation of the Schur complement of A_{TT} is a trivially parallel task without the need for shared memory between $T \neq T'$, prompts for the use of a GPU Cholesky solver. This topic will be the subject of a future work.*

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