FINITE TIME EXTINCTION FOR NONLINEAR SCHRÖDINGER EQUATION IN 1D AND 2D

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ABSTRACT. We consider a nonlinear Schrödinger equation with power nonlinearity, either on a compact manifold without boundary, or on the whole space in the presence of harmonic confinement, in space dimension one and two. Up to introducing an extra superlinear damping to prevent finite time blow up, we show that the presence of a sublinear damping always leads to finite time extinction of the solution in 1D, and that the same phenomenon is present in the case of small mass initial data in 2D.

1. INTRODUCTION

In [9], the following equation was considered on a compact manifold without boundary:

$$i\partial_t u + \frac{1}{2}\Delta u = -ib\frac{u}{|u|^\alpha}, \quad t \geqslant 0,$$

for $b > 0$ and $\alpha \in (0,1]$. This sublinear damping leads to finite time extinction of the solution, that is $\|u(t)\|_{L^2} = 0$ for $t \geqslant T$, a phenomenon closely akin to the model involving such a damping is mechanics [1]. In the one-dimensional case, finite time extinction was proved for

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda|u|^{2\sigma}u - ib\frac{u}{|u|^\alpha}, \quad t \geqslant 0,$$

with $\lambda \in \mathbb{R}$ and $\sigma > 0$, provided that finite time blow-up does not occur in the case $b = 0$, that is, either $\sigma < 2$ or $\lambda \geqslant 0$. In this paper, we extend this study to several directions:

- The two-dimensional case is considered too.
- The space variable may belong to the whole space $\mathbb{R}^d$, provided that a confining potential is present.
- When finite time blow-up is present without damping, we introduce a superlinear damping in order to prevent blow-up.

This last point is related to some conclusion from [3]: a nonlinear damping term whose power is larger than that of a focusing nonlinearity always prevents finite time blow-up.

We consider the equation

$$i\partial_t u + \frac{1}{2}\Delta u = V(x)u + \lambda|u|^{2\sigma_1}u - ia|u|^{2\sigma_2}u - ib\frac{u}{|u|^\alpha}, \quad t \geqslant 0,\ x \in M, \tag{1.1}$$

where $V \in C^\infty(M;\mathbb{R})$ is a smooth, real-valued potential, with initial datum

$$u_{|t=0} = u_0. \tag{1.2}$$

Throughout all this paper, we suppose that the following assumption is satisfied.

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Assumption 1.1. *The parameters of the equation are chosen as follows:* $\lambda \in \mathbb{R}$, $a \geqslant 0$, $b, \sigma_1, \sigma_2 > 0$, *and* $\alpha \in [0,1]$. *We suppose that* M *is* d*-dimensional, with* $d = 1$ *or* 2.

- *Either* M *is a* d*-dimensional compact manifold without boundary,*
- *or* $M = \mathbb{R}^d$*, and* V *is harmonic,*

$$V(x) = \sum_{j=1}^{d} \omega_j^2 x_j^2, \quad \omega_j > 0.$$

If $M = \mathbb{R}^2$*, we restrict the range for* α*:* $\alpha \in [0, \frac{1}{2}]$.

Remark 1.2. In the case where $M = \mathbb{R}^d$, we could consider more general potentials. Our proofs remain valid provided that V is at most quadratic in the sense of [12], that is:

$$V \in C^\infty(\mathbb{R}^d;\mathbb{R}), \quad \text{with} \quad \partial^\gamma V \in L^\infty(\mathbb{R}^d), \quad \forall \gamma \in \mathbb{N}^d,\ |\gamma| \geqslant 2.$$

This assumption is sufficient to construct a global weak solution to (1.1). We also need the potential energy to control lower Lebesgue norms (see Lemma 2.3), a requirement which is satisfied provided that there exist $C, \varepsilon > 0$ such that

$$V(x) \geqslant C|x|^{1+\varepsilon}, \quad \forall x \in \mathbb{R}^d,\ |x| \geqslant 1.$$

Among other properties, such potentials prevents global in time dispersion (they are confining potentials). It is not clear whether this assumption is really necessary or if it is a technical requirement, in order for the conclusions of the present paper to hold.

The initial datum satisfies $u_0 \in \Sigma$, where

$$\Sigma^k = \left\{ f \in H^k(M), \quad \|f\|_{\Sigma^k}^2 := \|f\|_{H^k(M)}^2 + \||x|^k f\|_{L^2(M)}^2 < \infty \right\},$$

and we denote $\Sigma = \Sigma^1$. Note that if M is compact, we simply have $\Sigma^k = H^k(M)$, and on $M = \mathbb{R}^d$, $\Sigma^k = H^k \cap \mathcal{F}(H^k)$, where $\mathcal{F}$ denotes the Fourier transform (whose normalization is irrelevant in this definition).

Definition 1.3 (Weak solution, case $0 \leqslant \alpha < 1$)**.** *Suppose* $0 \leqslant \alpha < 1$. *A (global) weak solution to* (1.1) *is a function* $u \in \mathcal{C}(\mathbb{R}_+; L^2(M)) \cap L^\infty(\mathbb{R}_+;\Sigma)$ *solving* (1.1) *in* $\mathcal{D}'(\mathbb{R}_+^* \times M)$.

Definition 1.4 (Weak solution, case $\alpha = 1$)**.** *Suppose* $\alpha = 1$. *A (global) weak solution to* (1.1) *is a function* $u \in \mathcal{C}(\mathbb{R}_+; L^2(M)) \cap L^\infty(\mathbb{R}_+;\Sigma)$ *solving*

$$i\partial_t u + \frac{1}{2}\Delta u = V(x)u + \lambda |u|^{2\sigma_1} u - ia|u|^{2\sigma_2}u - ibF$$

in $\mathcal{D}'(\mathbb{R}_+^* \times M)$*, where* F *is such that*

$$\|F\|_{L^\infty(\mathbb{R}_+ \times M)} \leqslant 1, \quad \text{and} \quad F = \frac{u}{|u|} \text{ if } u \neq 0.$$

Theorem 1.5. *Let* $u_0 \in \Sigma$. *In either of the following cases,*

- $\sigma_1 < 2/d$*,*
- *or* $\lambda \geqslant 0$*,*
- *or* $\lambda < 0$, $a > 0$ *and* $\sigma_2 > \sigma_1$*,*

the Cauchy problem (1.1)-(1.2) *has a unique, global, weak solution.*

Multiplying (1.1) by $\bar{u}$, integrating over M and taking the imaginary part, we obtain formally:

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 + 2a\int_M |u(t,x)|^{2\sigma_2+2}dx + 2b\int_M |u(t,x)|^{2-\alpha}dx = 0. \tag{1.3}$$

We will check in the course of the proof of Theorem 1.5 that the solution satisfies this relation indeed.

Corollary 1.6. *Let $d = 1$ and $\alpha > 0$ in Assumption 1.1, and $u_0 \in \Sigma$. In either of the cases considered in Theorem 1.5, there exists $T > 0$ such that the unique weak solution to* (1.1)-(1.2) *satisfies*

$$\text{for every } t \geqslant T, \quad \|u(t)\|_{L^2(M)} = 0.$$

Theorem 1.7. *Let $d = 2$ in Assumption 1.1, and $u_0 \in \Sigma$.*
(1) In either of the cases considered in Theorem 1.5, there exists $C > 0$ such that the solution to (1.1)-(1.2) *satisfies*

$$\|u(t)\|_{L^2(M)} \leqslant \|u_0\|_{L^2(M)}e^{-Ct}, \quad t \geqslant 0.$$

(2) If in addition $u_0 \in \Sigma^2$, then $u \in L^\infty(\mathbb{R}_+;\Sigma^2)$. If $1/2 \leqslant \sigma_1 \leqslant 3/2$, then for any $R > 0$, there exists $\eta_R > 0$ such that if $\|u_0\|_{\Sigma^2} \leqslant R$ and $\|u_0\|_{L^2} \leqslant \eta_R$, then there exists $T > 0$ such that for every $t \geqslant T$, $\|u(t)\|_{L^2(M)} = 0$.

Note that the above smallness assumption is automatically fulfilled as soon as $\|u_0\|_{\Sigma^2}$ is sufficiently small.

The proof of the second part of this theorem relies on Brézis-Gallouët inequality introduced in [6] (and recently revisited in [21]), which require higher energy estimates.

2. Existence result and a priori estimates

2.1. **Preliminary technical results.** We recall the standard Gagliardo-Nirenberg inequalities (see e.g. [11]):

Lemma 2.1. *Let M be as in Assumption 1.1. If $d = 1$, let $p \in [2,\infty]$, and if $d = 2$, let $p \in [2,\infty)$. There exists $C = C(p,d)$ such that for all $f \in H^1(M)$,*

$$\|f\|_{L^p(M)} \leqslant C\|f\|_{L^2(M)}^{1-\delta(p)}\|f\|_{H^1(M)}^{\delta(p)}, \quad \text{where} \quad \delta(p) = d\left(\frac{1}{2} - \frac{1}{p}\right).$$

If $M = \mathbb{R}^d$, then the inhomogeneous Sobolev norm $\|\cdot\|_{H^1(\mathbb{R}^d)}$ can be replaced by the homogeneous norm $\|\cdot\|_{\dot{H}^1(\mathbb{R}^d)}$.

We recall the standard compactness result (see e.g. [17]):

Lemma 2.2. *Let $M = \mathbb{R}^d$, $d = 1$ or 2. If $d = 1$, let $p \in [2,\infty]$, and if $d = 2$, let $p \in [2,\infty)$. The embedding $\Sigma \hookrightarrow L^p(\mathbb{R}^d)$ is compact.*

If M is a compact manifold without boundary, Hölder inequality readily yields, for $1 \leqslant p < q \leqslant \infty$,

$$\|f\|_{L^p(M)} \leqslant |M|^{1/p-1/q}\|f\|_{L^q(M)}, \quad \forall f \in L^q(M).$$

On the whole space $\mathbb{R}^d$, an analogous inequality is provided by the control of momenta, which can be viewed as dual to the Gagliardo-Nirenberg inequalities (see e.g. [10]):

Lemma 2.3. *Let $M = \mathbb{R}^d$, $d = 1$ or 2. If $d = 1$, let $p \in [2, \infty]$, and if $d = 2$, let $p \in [2, \infty)$. There exists $C = C(p, d)$ such that for all $f \in \mathcal{F}(H^1(\mathbb{R}^d))$,*

$$\|f\|_{L^{p'}(\mathbb{R}^d)} \leqslant C\|f\|^{1-\delta(p)}_{L^2(\mathbb{R}^d)}\|xf\|^{\delta(p)}_{L^2(\mathbb{R}^d)}, \quad \textit{where} \quad \delta(p) = d\left(\frac{1}{2} - \frac{1}{p}\right).$$

2.2. **Approximate solution.** Following the same strategy as in [9], we modify (1.1) by regularizing the sublinear nonlinearity:

$$i\partial_t u^\delta + \frac{1}{2}\Delta u^\delta = V(x)u^\delta + \lambda|u^\delta|^{2\sigma_1}u^\delta - ia|u^\delta|^{2\sigma_2}u^\delta - ib\frac{u^\delta}{(|u^\delta|^2+\delta)^{\alpha/2}}. \tag{2.1}$$

We keep the same initial datum (1.2). Since the external potential V is at most quadratic, local in time Strichartz inequalities are available for the Hamiltonian $-\frac{1}{2}\Delta + V$. With $d \leqslant 2$, all the nonlinearities are energy-subcritical, and we infer (see e.g. [11]):

Lemma 2.4. *Let $\delta > 0$, and $u_0 \in \Sigma$. There exist $T > 0$ and a unique solution*

$$u^\delta \in C([0,T];\Sigma) \cap L^{\frac{4\sigma_1+4}{d\sigma_1}}([0,T];L^{2\sigma_1+2}(M)) \cap L^{\frac{4\sigma_2+4}{d\sigma_2}}([0,T];L^{2\sigma_2+2}(M))$$

to the Cauchy problem (2.1)-(1.2). *In addition, for all $t \in [0,T]$, it satisfies*

$$\|u^\delta(t)\|^2_{L^2(M)} + 2b\int_0^t\int_M \frac{|u^\delta(\tau,x)|^2}{(|u^\delta(\tau,x)|^2+\delta)^{\alpha/2}}dxd\tau \leqslant \|u_0\|^2_{L^2(M)}. \tag{2.2}$$

To prove that the solution to (2.1) is actually global in the future (the equation is irreversible), denote by

$$E_0^\delta(t) = \|\nabla u^\delta(t)\|^2_{L^2} + 2\int_M V(x)|u^\delta(t,x)|^2dx + \frac{2\lambda}{\sigma_1+1}\|u^\delta(t)\|^{2\sigma_1+2}_{L^{2\sigma_1+2}}, \tag{2.3}$$

and, following the approach introduced in [5], for $k > 0$, set

$$E_k^\delta(t) := E_0^\delta(t) + k\|u^\delta(t)\|^{2\sigma_2+2}_{L^{2\sigma_2+2}}. \tag{2.4}$$

The energy E_0^δ involves the Hamiltonian part of (2.1), and E_k^δ consists of the artificial introduction of the extra nonlinearity $|u|^{2\sigma_2}u$, as if it were Hamiltonian instead of a damping term.

Proposition 2.5. *(1) Assume that $\sigma_1 < 2/d$ or $\lambda \geqslant 0$. There exists a $C = C(\|u_0\|_{L^2}) \geqslant 0$ independent of $\delta \in (0,1]$ such that*

$$E_0^\delta(t) \leqslant E_0^\delta(0) + C(\|u_0\|_{L^2}) \quad \forall t \in [0,T],$$

where $T > 0$ is a local existence time in Σ.
(2) If $\lambda < 0$, assume that $a > 0$, $0 < k < \frac{2a}{\sigma_2(\sigma_2+1)}$, and $\sigma_2 > \sigma_1$. There exists a $C = C(\|u_0\|_{L^2}) \geqslant 0$ independent of $\delta \in (0,1]$ such that

$$E_k^\delta(t) \leqslant E_k^\delta(0) + C(\|u_0\|_{L^2}) \quad \forall t \in [0,T],$$

where $T > 0$ is a local existence time in Σ.

Proof. Denote by

$$f_\delta(v) = \frac{v}{(|v|^2+\delta)^{\alpha/2}}.$$

Since

$$\Delta u^\delta = -2i\partial_t u^\delta + 2Vu^\delta + 2\lambda|u^\delta|^{2\sigma_1}u^\delta - 2ia|u^\delta|^{2\sigma_2}u^\delta - 2ibf_\delta(u^\delta),$$

we compute, with $(f \mid g) = \int_M f\bar{g}$,

$$\begin{aligned}
\frac{d}{dt}\|\nabla u^\delta(t)\|_{L^2}^2 &= -2\,\mathrm{Re}\left(\partial_t u^\delta \mid \Delta u^\delta\right) \\
&= -2\,\mathrm{Re}\left(\partial_t u^\delta \mid -2i\partial_t u^\delta + 2Vu^\delta + 2\lambda|u^\delta|^{2\sigma_1}u^\delta - 2ia|u^\delta|^{2\sigma_2}u^\delta - 2ibf_\delta(u^\delta)\right) \\
&= -\frac{d}{dt}\left(2\int_M V|u^\delta|^2 + \frac{2\lambda}{\sigma_1+1}\|u^\delta\|_{L^{2\sigma_1+2}}^{2\sigma_1+2}\right) + 4a\,\mathrm{Im}\left(\partial_t u^\delta \mid |u^\delta|^{2\sigma_2}u^\delta\right) \\
&\quad + 4b\,\mathrm{Im}\left(\partial_t u^\delta \mid f_\delta(u^\delta)\right).
\end{aligned}$$

Since

$$\partial_t u^\delta = \frac{i}{2}\Delta u^\delta - iVu^\delta - i\lambda|u^\delta|^{2\sigma_1}u^\delta - a|u^\delta|^{2\sigma_2}u^\delta - bf_\delta(u^\delta),$$

we have

$$\begin{aligned}
\mathrm{Im}\left(\partial_t u^\delta \mid |u^\delta|^{2\sigma_2}u^\delta\right) &= \frac{1}{2}\,\mathrm{Re}\left(\Delta u^\delta \mid |u^\delta|^{2\sigma_2}u^\delta\right) - \int_M V|u^\delta|^{2\sigma_2+2} - \lambda\int_M |u^\delta|^{2\sigma_1+2\sigma_2+2} \\
&= -\frac{1}{2}\int_M |u^\delta|^{2\sigma_2}|\nabla u^\delta|^2 - \sigma_2\int_M |u^\delta|^{2\sigma_2}\left|\nabla|u^\delta|\right|^2 - \int_M V|u^\delta|^{2\sigma_2+2} \\
&\quad - \lambda\int_M |u^\delta|^{2\sigma_1+2\sigma_2+2},
\end{aligned}$$

where for the last equality, we have used the identity

$$\Delta|u^\delta|^2 = 2\,\mathrm{Re}\left(\bar{u}^\delta\Delta u^\delta\right) + 2|\nabla u^\delta|^2.$$

On the other hand, we have

$$\mathrm{Im}\left(\partial_t u^\delta \mid f_\delta(u^\delta)\right) = \frac{1}{2}\,\mathrm{Re}\left(\Delta u^\delta \mid f_\delta(u^\delta)\right) - \int_M V\frac{|u^\delta|^2}{(|u^\delta|^2+\delta)^{\alpha/2}} - \lambda\int_M \frac{|u^\delta|^{2\sigma_1+2}}{(|u^\delta|^2+\delta)^{\alpha/2}}.$$

We have

$$\begin{aligned}
\mathrm{Re}\left(\Delta u^\delta \mid f_\delta(u^\delta)\right) &= -\,\mathrm{Re}\left(\nabla u^\delta \mid \nabla f_\delta(u^\delta)\right) \\
&= -\int_M \frac{|\nabla u^\delta|^2}{(|u^\delta|^2+\delta)^{\alpha/2}} + \alpha\,\mathrm{Re}\int_M \bar{u}^\delta\nabla u^\delta\cdot\frac{\mathrm{Re}(\bar{u}^\delta\nabla u^\delta)}{(|u^\delta|^2+\delta)^{\alpha/2+1}} \\
&= -\int_M \left(|u^\delta|^2+\delta\right)\frac{|\nabla u|^2}{(|u^\delta|^2+\delta)^{\alpha/2+1}} + \alpha\int_M \frac{\left|\mathrm{Re}(\bar{u}^\delta\nabla u^\delta)\right|^2}{(|u^\delta|^2+\delta)^{\alpha/2+1}} \\
&= -\delta\int_M \frac{|\nabla u|^2}{(|u^\delta|^2+\delta)^{\alpha/2+1}} - \int_M \frac{\left|\mathrm{Im}(\bar{u}^\delta\nabla u^\delta)\right|^2}{(|u^\delta|^2+\delta)^{\alpha/2+1}} \\
&\quad - (1-\alpha)\int_M \frac{\left|\mathrm{Re}(\bar{u}^\delta\nabla u^\delta)\right|^2}{(|u^\delta|^2+\delta)^{\alpha/2+1}}.
\end{aligned}$$

From the above computations, we have:

$$\begin{aligned}\frac{d}{dt}E_0^\delta &= -2a\int_M |u^\delta|^{2\sigma_2}|\nabla u^\delta|^2 - 4a\sigma_2\int_M |u^\delta|^{2\sigma_2}\left|\nabla |u^\delta|\right|^2 - 4a\int_M V|u^\delta|^{2\sigma_2+2}\\
&\quad - 4a\lambda\int_M |u^\delta|^{2\sigma_1+2\sigma_2+2} - 4b\int_M V\frac{|u^\delta|^2}{\left(|u^\delta|^2+\delta\right)^{\alpha/2}} - 4b\lambda\int_M \frac{|u^\delta|^{2\sigma_1+2}}{\left(|u^\delta|^2+\delta\right)^{\alpha/2}}\\
&\quad - \delta\int_M \frac{|\nabla u|^2}{\left(|u^\delta|^2+\delta\right)^{\alpha/2+1}} - \int_M \frac{\left|\mathrm{Im}(\bar u^\delta\nabla u^\delta)\right|^2}{\left(|u^\delta|^2+\delta\right)^{\alpha/2+1}}\\
&\quad - 2b\delta\int_M \frac{|\nabla u|^2}{\left(|u^\delta|^2+\delta\right)^{\alpha/2+1}} - 2b\int_M \frac{\left|\mathrm{Im}(\bar u^\delta\nabla u^\delta)\right|^2}{\left(|u^\delta|^2+\delta\right)^{\alpha/2+1}}\\
&\quad - 2b(1-\alpha)\int_M \frac{\left|\mathrm{Re}(\bar u^\delta\nabla u^\delta)\right|^2}{\left(|u^\delta|^2+\delta\right)^{\alpha/2+1}}.\end{aligned}$$

If $\lambda \geqslant 0$ (defocusing case), E_0^δ, defined in (2.3), is non-increasing. If $\sigma_1 < 2/d$, we conclude as in the standard case presented for instance in [11].

To treat the focusing case $\lambda < 0$, with $\sigma_1 \geqslant 2/d$ (finite time blow-up is possible in the case $a = b = 0$), we follow the strategy adopted in [5] and generalized in [3], relying on E_k^δ, defined in (2.4). The following computation is valid for any $p > 2$:

$$\begin{aligned}\frac{d}{dt}\|u^\delta(t)\|_{L^p}^p &= p\,\mathrm{Re}\left(\partial_t u^\delta \mid |u^\delta|^{p-2}u^\delta\right)\\
&= -\frac{p}{2}\,\mathrm{Im}\left(\Delta u^\delta \mid |u^\delta|^{p-2}u^\delta\right) - ap\int_M |u^\delta|^{2\sigma_2+p} - bp\int_M \frac{|u^\delta|^p}{\left(|u^\delta|^2+\delta\right)^{\alpha/2}}\\
&= \frac{p}{2}\int_M \nabla|u^\delta|^{p-2}\cdot\mathrm{Im}\left(\bar u^\delta\nabla u^\delta\right) - ap\int_M |u^\delta|^{2\sigma_2+p} - bp\int_M \frac{|u^\delta|^p}{\left(|u^\delta|^2+\delta\right)^{\alpha/2}}.\end{aligned}$$

As in [3], we use the polar factorisation introduced in [15, 2] (see also [4, 8]), to show that

$$\int_M \nabla|u^\delta|^{p-2}\cdot\mathrm{Im}\left(\bar u^\delta\nabla u^\delta\right) = (p-2)\int_M |u^\delta|^{p-2}\,\mathrm{Re}(\bar\phi\nabla u^\delta)\cdot\mathrm{Im}(\bar\phi\nabla u^\delta),$$

where ϕ is the polar factor related to u^δ,

$$\phi(t,x) := \begin{cases} |u^\delta(t,x)|^{-1}u^\delta(t,x) & \text{if } u^\delta(t,x)\neq 0,\\ 0 & \text{if } u^\delta(t,x) = 0.\end{cases}$$

In view of the identity

$$2\,\mathrm{Re}(\bar\phi\nabla u^\delta)\cdot\mathrm{Im}(\bar\phi\nabla u^\delta) = -\left|\mathrm{Re}(\bar\phi\nabla u^\delta) - \mathrm{Im}(\bar\phi\nabla u^\delta)\right|^2 + |\nabla u^\delta|^2,$$

we obtain:

$$\begin{aligned}\frac{d}{dt}\|u^\delta(t)\|_{L^p}^p &= -\frac{p(p-2)}{4}\int_M |u^\delta|^{p-2}\left|\mathrm{Re}(\bar\phi\nabla u^\delta) - \mathrm{Im}(\bar\phi\nabla u^\delta)\right|^2\\
&\quad + \frac{p(p-2)}{4}\int_M |u^\delta|^{p-2}|\nabla u^\delta|^2 - ap\int_M |u^\delta|^{2\sigma_2+p} - bp\int_M \frac{|u^\delta|^p}{\left(|u^\delta|^2+\delta\right)^{\alpha/2}}.\end{aligned}$$

We finally have:

$$\begin{aligned}\frac{d}{dt}E_k^\delta \leqslant -2a\int_M |u^\delta|^{2\sigma_2}|\nabla u^\delta|^2 - 4a\lambda\int_M |u^\delta|^{2\sigma_1+2\sigma_2+2} - 4b\lambda\int_M \frac{|u^\delta|^{2\sigma_1+2}}{(|u^\delta|^2+\delta)^{\alpha/2}}\\ + k\sigma_2(\sigma_2+1)\int_M |u^\delta|^{2\sigma_2}|\nabla u^\delta|^2 - ak(2\sigma_2+2)\int_M |u^\delta|^{4\sigma_2+2}\\ - bk(2\sigma_2+2)\int_M \frac{|u^\delta|^{2\sigma_2+2}}{(|u^\delta|^2+\delta)^{\alpha/2}}.\end{aligned}$$

If $0<k\sigma_2(\sigma_2+1)<2a$, and since $\lambda<0$, we come up with:

$$\begin{aligned}\frac{d}{dt}E_k^\delta \leqslant 4a|\lambda|\int_M |u^\delta|^{2\sigma_1+2\sigma_2+2} + 4b|\lambda|\int_M \frac{|u^\delta|^{2\sigma_1+2}}{(|u^\delta|^2+\delta)^{\alpha/2}}\\ - ak(2\sigma_2+2)\int_M |u^\delta|^{4\sigma_2+2} - bk(2\sigma_2+2)\int_M \frac{|u^\delta|^{2\sigma_2+2}}{(|u^\delta|^2+\delta)^{\alpha/2}}.\end{aligned}$$

If $\sigma_2>\sigma_1$ (the superlinear damping is "stronger" than the focusing term), then the negative terms on the right hand side control the positive terms (since $\|u^\delta\|_{L^2}$ is non-increasing), hence the result. □

2.3. **Convergence of the approximation.** We now follow the strategy introduced in [13], and resumed in [9].

A straightforward consequence from (2.2) and Proposition 2.5 is that for $u_0\in\Sigma$ fixed, the sequence $(u^\delta)_{0<\delta\leqslant 1}$ is uniformly bounded in $L^\infty(\mathbb{R}_+,\Sigma)\cap L^{2-\alpha}(\mathbb{R}_+\times M)$. We deduce the existence of $u\in L^\infty(\mathbb{R}_+,\Sigma)$ and of a subsequence u^{δ_n} such that

$$u^{\delta_n}\rightharpoonup u,\quad \text{in } w*\ L^\infty(\mathbb{R}_+,\Sigma), \tag{2.5}$$

with, in view of (2.2) and Proposition 2.5,

$$\|u\|_{L^\infty(\mathbb{R}_+,H^1(M))}\leqslant \|u_0\|_{H^1(M)}+C(\|u_0\|_{L^2(M)}).$$

Moreover, $\frac{u^\delta}{(|u^\delta|^2+\delta)^{\alpha/2}}$ is uniformly bounded in $L^\infty(\mathbb{R}_+,L^{\frac{2}{1-\alpha}}(M))$ (with $2/(1-\alpha)=\infty$ if $\alpha=1$), such that up to the extraction of an other subsequence, there is $F\in L^\infty(\mathbb{R}_+,L^{\frac{2}{1-\alpha}}(M))$ such that

$$\frac{u^{\delta_n}}{(|u^{\delta_n}|^2+\delta_n)^{\alpha/2}}\rightharpoonup F,\quad \text{in } w*\ L^\infty(\mathbb{R}_+,L^{\frac{2}{1-\alpha}}(M)). \tag{2.6}$$

Moreover, $\|F\|_{L^\infty(\mathbb{R}_+,L^{\frac{2}{1-\alpha}}(M))}\leqslant \|u_0\|^{1-\alpha}_{L^2(M)}$. In view of Lemma 2.2 (whose analogue is obvious in the case where M is compact),

$$|u^{\delta_n}|^{2\sigma_j}u^{\delta_n}\underset{n\to\infty}{\longrightarrow}|u|^{2\sigma_j}u\quad \text{in } L^1_{\rm loc}(\mathbb{R}_+\times M),\quad j=1,2.$$

Let $\theta \in \mathcal{C}_c^\infty(\mathbb{R}_+^* \times M)$. Then

$$\begin{aligned}
&\left\langle -ib\frac{u^{\delta_n}}{(|u^{\delta_n}|^2+\delta_n)^{\alpha/2}},\theta\right\rangle\\
&= \left\langle i\partial u^{\delta_n} + \frac{1}{2}\Delta u^{\delta_n} - Vu^{\delta_n} - \lambda|u^{\delta_n}|^{2\sigma_1}u^{\delta_n} + ia|u^{\delta_n}|^{2\sigma_2}u^{\delta_n},\theta\right\rangle\\
&= \left\langle u^{\delta_n}, -i\frac{\partial\theta}{\partial t} + \frac{1}{2}\Delta\theta\right\rangle + \left\langle -Vu^{\delta_n} - \lambda|u^{\delta_n}|^{2\sigma_1}u^{\delta_n} + ia|u^{\delta_n}|^{2\sigma_2}u^{\delta_n},\theta\right\rangle\\
&\mathop{\longrightarrow}_{n\to\infty} \left\langle u, -i\frac{\partial\theta}{\partial t} + \frac{1}{2}\Delta\theta\right\rangle + \left\langle -Vu - \lambda|u|^{2\sigma_1}u + ia|u|^{2\sigma_2}u,\theta\right\rangle\\
&= \left\langle i\frac{\partial u}{\partial t} + \frac{1}{2}\Delta u - Vu - \lambda|u|^{2\sigma_1}u + ia|u|^{2\sigma_2}u,\theta\right\rangle,
\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ stands for the distribution bracket on $\mathbb{R}_+^*\times M$. Thus, we deduce

$$i\partial_t u + \frac{1}{2}\Delta u = V(x)u + \lambda|u|^{2\sigma_1}u - ia|u|^{2\sigma_2}u - ibF, \quad \text{in } \mathcal{D}'(\mathbb{R}_+^*\times M).$$

We next show that $F = u/|u|^\alpha$ where the right hand side is well defined, that is if $\alpha < 1$, or $\alpha = 1$ and $u \neq 0$. We first suppose that $u_0 \in H^s(M)$ with s large. Let us fix $t' \in \mathbb{R}_+$ and $\delta > 0$. Thanks to (2.2), we infer, for any $t \in \mathbb{R}_+$,

$$\frac{d}{dt}\|u^\delta(t) - u^\delta(t')\|_{L^2}^2 \leqslant \frac{d}{dt}\big(-2\,\mathrm{Re}\left(u^\delta(t)\mid u^\delta(t')\right)\big),$$

where $(\cdot\mid\cdot)$ denotes the scalar product in $L^2(M)$. In view of (2.1), the right hand side is equal to

$$-2\,\mathrm{Re}\left(\frac{i}{2}\Delta u^\delta(t) - iVu^\delta - i\lambda|u^\delta|^{2\sigma_1}u^\delta - a|u^\delta|^{2\sigma_2}u^\delta - \frac{bu^\delta(t)}{(|u^\delta(t)|^2+\delta)^{\alpha/2}}\middle| u^\delta(t')\right).$$

By integration, we deduce

$$\begin{aligned}
\|u^\delta(t) - u^\delta(t')\|_{L^2(M)}^2 \leqslant 2|t-t'|\Big(&\frac{1}{2}\|\Delta u^\delta\|_{L^\infty(\mathbb{R}_+;H^{-1})}\|u^\delta\|_{L^\infty(\mathbb{R}_+;H^1)}\\
&+ \|Vu\|_{L^\infty(\mathbb{R}_+;L^2)}^2 + |\lambda|\|u^\delta\|_{L^\infty(\mathbb{R}_+;L^{2\sigma_1+2})}^{2\sigma_1+2}\\
&+ a\|u^\delta\|_{L^\infty(\mathbb{R}_+;L^{2\sigma_2+2})}^{2\sigma_2+2} + b\|u^\delta\|_{L^\infty(\mathbb{R}_+,L^{2-\alpha}(M))}^{2-\alpha}\Big).
\end{aligned} \tag{2.7}$$

From the continuity of the flow map $\Sigma \ni u_0 \mapsto u^\delta \in \mathcal{C}(\mathbb{R}_+,\Sigma)$ in Lemma 2.4, we deduce that (2.7) also holds if we only have $u_0 \in \Sigma$. Next, since $(u^\delta)_{0<\delta\leqslant 1}$ is uniformly bounded in $L^\infty(\mathbb{R}_+,\Sigma)$ and either M is compact or we may invoke Lemma 2.3 (recall that on $\mathbb{R}^2$, we assume $\alpha \leqslant 1/2$ in Assumption 1.1), (2.7) gives the existence of a positive constant C such that for every $t,t' \in \mathbb{R}_+$,

$$\|u^\delta(t) - u^\delta(t')\|_{L^2(M)} \leqslant C|t-t'|^{1/2}.$$

In particular, for any $T>0$, $(u^\delta)_{0<\delta\leqslant 1}$ is a bounded sequence in $\mathcal{C}([0,T],L^2(M))$ which is uniformly equicontinuous from $[0,T]$ to $L^2(M)$. Moreover, the compactness of the embedding $\Sigma \subset L^2(M)$ ensures that for every $t \in [0,T]$, the set $\{u^\delta(t)|\delta \in (0,1]\}$ is relatively compact in $L^2(M)$. As a result, Arzelà–Ascoli Theorem implies that $(u^{\delta_n})_n$ is relatively compact in $\mathcal{C}([0,T],L^2(M))$. On the other hand, we already know from (2.5) that

$$u^{\delta_n} \rightharpoonup u \quad \text{in } w*\ L^\infty(\mathbb{R}_+,L^2(M)).$$

Therefore, we infer that u is the unique accumulation point of the sequence $(u^{\delta_n})_n$ in $\mathcal{C}([0,T],L^2(M))$. Thus

$$u^{\delta_n} \to u \quad \text{in } \mathcal{C}([0,T],L^2(M)),$$

which implies in particular $u \in \mathcal{C}([0,T],L^2(M))$ as well as $u(0) = u^{\delta_n}(0) = u_0$. This is true for any $T > 0$, therefore

$$u \in \mathcal{C}(\mathbb{R}_+, L^2(M)).$$

Finally, up to the extraction of an other subsequence, $u^{\delta_n}(t,x) \to u(t,x)$ for almost every $(t,x) \in \mathbb{R}_+ \times M$. Therefore, for almost every $(t,x) \in \mathbb{R}_+ \times M$ such that $u(t,x) \neq 0$, we have

$$\frac{u^{\delta_n}}{(|u^{\delta_n}|^2 + \delta_n)^{\alpha/2}}(t,x) \to \frac{u}{|u|^\alpha}(t,x).$$

By comparison with (2.6), we deduce that up to a change of F on a set with zero measure,

$$F(t,x) = \frac{u}{|u|^\alpha}(t,x) \quad \text{(only if } u(t,x) \neq 0 \text{ in the case } \alpha = 1\text{)},$$

which completes the proof of the existence part of Theorem 1.5.

2.4. **Uniqueness.** If u and v are two solutions to (1.1), then by subtracting the two equations, multiplying by $\overline{u-v}$, integrating over M and taking the imaginary part, we obtain:

$$\begin{aligned} &\frac{d}{dt}\|u-v\|_{L^2}^2 + 2a \operatorname{Re} \int_M \left(|u|^{2\sigma_2}u - |v|^{2\sigma_2}v\right)\overline{u-v} \\ &\quad + 2b \operatorname{Re} \int_M \left(\frac{u}{|u|^\alpha} - \frac{v}{|v|^\alpha}\right)\overline{u-v} = 2\lambda \operatorname{Im} \int_M \left(|u|^{2\sigma_1}u - |v|^{2\sigma_1}v\right)\overline{u-v}. \end{aligned} \tag{2.8}$$

Extending Lemma 3.1 from [9], we have

Lemma 2.6. *Let $\sigma \geqslant -1$. For all $z_1, z_2 \in \mathbb{C}$,*

$$\operatorname{Re}\left(\left(|z_1|^\sigma z_1 - |z_2|^\sigma z_2\right)\left(\overline{z_1 - z_2}\right)\right) \geqslant 0.$$

Proof. Using polar coordinates, write $z_j = \rho_j e^{i\theta_j}$, $\rho_j \geqslant 0$, $\theta_j \in \mathbb{R}$. The quantity involved in the statement is

$$\rho_1^{\sigma+2} + \rho_2^{\sigma+2} - \rho_1^{\sigma+1}\rho_2 \cos(\theta_1 - \theta_2) - \rho_2^{\sigma+1}\rho_1 \cos(\theta_1 - \theta_2).$$

Since the cosine function is bounded by one, the above quantity is bounded from below by

$$\rho_1^{\sigma+2} + \rho_2^{\sigma+2} - \rho_1^{\sigma+1}\rho_2 - \rho_2^{\sigma+1}\rho_1 = \left(\rho_1^{\sigma+1} - \rho_2^{\sigma+1}\right)(\rho_1 - \rho_2).$$

If $\sigma = -1$, the above quantity is identically zero. If $\sigma > -1$, then we conclude by observing that both factors on the right hand side always have the same sign. □

If $d = 1$, (2.8) and the above lemma yield

$$\begin{aligned} \frac{d}{dt}\|u(t) - v(t)\|_{L^2}^2 &\leqslant 2|\lambda| \int_M \left|\left(|u|^{2\sigma_1}u - |v|^{2\sigma_1}v\right)\overline{u-v}\right| \\ &\leqslant C\left(\|u\|_{L^\infty H^1}^{2\sigma_1} + \|v\|_{L^\infty H^1}^{2\sigma_1}\right)\|u(t) - v(t)\|_{L^2}^2, \end{aligned}$$

and Gronwall lemma shows that there is at most one (global) weak solution to (1.1).

When $d = 2$, in order to overcome the absence of control in $L^\infty(M)$, we invoke the argument introduced by Yudovitch [14], and resumed in the context of nonlinear Schrödinger equations in [19, 20], and by Burq, Gérard and Tzvetkov [7] in the case of three-dimensional domains. Since their argument readily works in the present context, we simply recall it.

Denote by $\epsilon(t) = \|u(t) - v(t)\|^2_{L^2(M)}$. For p finite and large, (2.8), Lemma 2.6 and Hölder inequality yield

$$\dot\epsilon(t) \leqslant C \int_M \left(|u(t,x)|^{2\sigma_1} + |v(t,x)|^{2\sigma_1}\right) |u(t,x) - v(t,x)|^2 dx$$
$$\leqslant C \left(\|u(t)\|^{2\sigma_1}_{L^{2p\sigma_1}} + \|v(t)\|^{2\sigma_1}_{L^{2p\sigma_1}}\right) \|u(t) - v(t)\|^2_{L^{2p'}},$$

where the constant C does not depend on p. By interpolation,

$$\|u(t) - v(t)\|_{L^{2p'}} \leqslant \|u(t) - v(t)\|^{1-3/2p}_{L^2} \|u(t) - v(t)\|^{3/2p}_{L^6},$$

hence, in view of the boundedness of the $L^\infty_t H^1_x$ norm of u and v, and of Sobolev embedding $H^1(M) \hookrightarrow L^6(M)$,

$$\dot\epsilon(t) \leqslant C \left(\|u(t)\|^{2\sigma_1}_{L^{2p\sigma_1}} + \|v(t)\|^{2\sigma_1}_{L^{2p\sigma_1}}\right) \epsilon(t)^{1-3/2p}.$$

Gagliardo–Nirenberg inequality implies

$$\|u(t)\|^{2\sigma_1}_{L^{2p\sigma_1}} + \|v(t)\|^{2\sigma_1}_{L^{2p\sigma_1}} \leqslant C\left([p]!\right)^{1/p} \left(\|u(t)\|^{2\sigma_1}_{H^1} + \|v(t)\|^{2\sigma_1}_{H^1}\right),$$

with another constant C, still independent of p (see e.g. [22]). Therefore, using Stirling formula for p large,

$$\dot\epsilon(t) \leqslant Cp\epsilon(t)^{1-3/2p}.$$

By integration in time, under the assumption $\epsilon(0) = 0$, we come up with

$$\epsilon(t)^{3/2p} \leqslant Ct,$$

for some constant C independent of p. Choosing t sufficiently small and letting $p \to \infty$, we see that $\epsilon = 0$ on some interval $[0, t_0]$ for some universal constant t_0, hence $\epsilon \equiv 0$ by induction.

Therefore, there is at most one (global) weak solution to (1.1)–(1.2). In addition, by considering $v = 0$ in (2.8), we see that this solution satisfies (1.3).

3. Finite time extinction in 1D and exponential decay in 2D

The following lemma follows from inequalities on $\mathbb{R}^d$, adapted from the Nash inequality [18] (see [9]):

Lemma 3.1. *Let M be as in Assumption 1.1. Let $\alpha \in]0,1]$. There exists $C > 0$ such that*

$$\|f\|^{\alpha d+4-2\alpha}_{L^2(M)} \leqslant C\left(\|f\|^{2-\alpha}_{L^{2-\alpha}(M)}\right)^2 \|f\|^{\alpha d}_{H^1(M)}, \quad \forall f \in H^1(M). \tag{3.1}$$

$$\|f\|^{\alpha d+8-4\alpha}_{L^2(M)} \leqslant C\left(\|f\|^{2-\alpha}_{L^{2-\alpha}(M)}\right)^4 \|f\|^{\alpha d}_{H^2(M)}, \quad \forall f \in H^2(M). \tag{3.2}$$

If $M = \mathbb{R}^d$, then the inhomogeneous Sobolev norm $\|\cdot\|_{H^s(\mathbb{R}^d)}$ can be replaced by the homogeneous norm $\|\cdot\|_{\dot H^s(\mathbb{R}^d)}$.

3.1. **Proof of Corollary 1.6.** Suppose that $d = 1$ in Theorem 1.5. In view of (1.3), we have

$$\frac{d}{dt}\|u(t)\|^2_{L^2} + 2b\int_M |u(t,x)|^{2-\alpha}dx \leqslant 0.$$

Theorem 1.5 and Lemma 3.1 yield

$$\frac{d}{dt}\|u(t)\|^2_{L^2} + Cb\|u(t)\|^{2-\alpha/2}_{L^2} \leqslant 0,$$

where C is proportional to $\|u\|_{L^\infty(\mathbb{R}_+;H^1)}^{-\alpha/2}$. By integration, we deduce, as long as $\|u(t)\|_{L^2}$ is not zero,

$$\|u(t)\|_{L^2} \leqslant \left(\|u_0\|_{L^2}^{\alpha/2} - Cbt\right)^{2/\alpha}.$$

Corollary 1.6 then follows.

3.2. **First part of Theorem 1.7.** Suppose now that $d = 2$ in Theorem 1.5. In view of (1.3), we have

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 + 2b\int_M |u(t,x)|^{2-\alpha}dx \leqslant 0.$$

Theorem 1.5 and Lemma 3.1 yield

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 + Cb\|u(t)\|_{L^2}^2 \leqslant 0,$$

where C is proportional to $\|u\|_{L^\infty(\mathbb{R}_+;H^1)}^{-\alpha}$. By integration, we deduce the first part of Theorem 1.7, that is, the exponential decay of $\|u(t)\|_{L^2(M)}$.

4. Higher order estimates

As in [9], the exponential decay in 2D obtained in the previous section can be improved to get finite time extinction provided that we invoke the Nash in equality (3.2) rather than merely (3.1). This requires of course to control the H^2-norm of u. In order to obtain bounds in Σ^2, we resume the idea due to Kato [16] (see also [11]): to obtain estimates of order two in space, it suffices to obtain estimates of order one in time, and to use the equation to relate these quantities.

4.1. **Evolution of the time derivative.** Using directly (1.1), for a global weak solution provided by Theorem 1.5, we obtain

$$\begin{aligned}\frac{d}{dt}\|\partial_t u\|_{L^2}^2 &= 2\,\mathrm{Re}\int_M \partial_t\bar{u}\partial_t^2 u\\ &= 2\lambda\,\mathrm{Im}\int_M \partial_t\bar{u}\partial_t\left(|u|^{2\sigma_1}u\right) - 2a\,\mathrm{Re}\int_M \partial_t\bar{u}\partial_t\left(|u|^{2\sigma_2}u\right)\\ &\quad - 2b\,\mathrm{Re}\int_M \partial_t\bar{u}\partial_t\left(\frac{u}{|u|^\alpha}\right).\end{aligned}$$

For the first term of the right hand side, we use the identity

$$\mathrm{Im}\int_M \partial_t\bar{u}\partial_t\left(|u|^{2\sigma_1}u\right) = \frac{d}{dt}\left(\mathrm{Im}\int_M |u|^{2\sigma_1}u\partial_t\bar{u}\right) - \mathrm{Im}\int_M |u|^{2\sigma_1}u\partial_t^2\bar{u}. \tag{4.1}$$

The full derivative will be incorporated into the first higher energy, so we focus on the last term. From the equation,

$$\begin{aligned}-\mathrm{Im}\int_M |u|^{2\sigma_1}u\partial_t^2\bar{u} &= -\mathrm{Im}\int_M |u|^{2\sigma_1}u\partial_t\left(-\frac{i}{2}\Delta\bar{u} + iV\bar{u} + i\lambda|u|^{2\sigma_1}\bar{u} - a|u|^{2\sigma_2}\bar{u} - b\frac{\bar{u}}{|u|^\alpha}\right)\\ &= \frac{1}{2}\,\mathrm{Re}\int_M |u|^{2\sigma_1}u\partial_t\Delta\bar{u} - \frac{1}{2(\sigma_1+1)}\frac{d}{dt}\int_M V|u|^{2\sigma_1+2}\\ &\quad - \frac{\lambda}{2}\frac{d}{dt}\int_M |u|^{4\sigma_1+2} + a\,\mathrm{Im}\int_M |u|^{2\sigma_1}u\partial_t\left(|u|^{2\sigma_2}\bar{u}\right)\\ &\quad + b\,\mathrm{Im}\int_M |u|^{2\sigma_1}u\partial_t\left(\frac{\bar{u}}{|u|^\alpha}\right)\end{aligned}$$

For the first term, we invoke the fact that the Laplacian is self-adjoint,and use the identity

$$\Delta\left(|u|^{2\sigma_1}u\right) = u\Delta|u|^{2\sigma_1} + 2\nabla u\cdot\nabla|u|^{2\sigma_1} + |u|^{2\sigma_1}\Delta u.$$

We also compute

$$\Delta|u|^{2\sigma_1} = \sigma_1(\sigma_1-1)|u|^{2\sigma_1-4}\left|\nabla|u|^2\right|^2 + 2\sigma_1|u|^{2\sigma_1-2}\left(|\nabla u|^2 + \mathrm{Re}\left(\bar u\Delta u\right)\right).$$

Therefore,

$$\begin{aligned}\frac{1}{2}\mathrm{Re}\int_M u\partial_t\bar u\Delta|u|^{2\sigma_1} &= \frac{\sigma_1(\sigma_1-1)}{2}\mathrm{Re}\int_M u\partial_t\bar u|u|^{2\sigma_1-4}\left|\nabla|u|^2\right|^2\\ &\quad+\sigma_1\mathrm{Re}\int_M u\partial_t\bar u\left(|\nabla u|^2+\mathrm{Re}\left(\bar u\Delta u\right)\right)|u|^{2\sigma_1-2}.\end{aligned}$$

The first two terms can be factored out in a more concise way in order to emphasize an exact time derivative:

$$\frac{\sigma_1(\sigma_1-1)}{2}\mathrm{Re}\int_M u\partial_t\bar u|u|^{2\sigma_1-4}\left|\nabla|u|^2\right|^2 = \frac{\sigma_1}{4}\int_M \partial_t|u|^{2\sigma_1-2}\left|\nabla|u|^2\right|^2,$$

and

$$\sigma_1\mathrm{Re}\int_M u\partial_t\bar u|\nabla u|^2|u|^{2\sigma_1-2} = \frac{1}{2}\int_M\partial_t|u|^{2\sigma_1}|\nabla u|^2.$$

We compute $\mathrm{Re}(\bar u\Delta u)$ by using (1.1):

$$\mathrm{Re}(\bar u\Delta u) = 2\,\mathrm{Im}\left(\bar u\partial_t u\right) + 2V|u|^2 + 2\lambda|u|^{2\sigma_1+2},$$

and we end up with

$$\begin{aligned}\frac{1}{2}\mathrm{Re}\int_M u\partial_t\bar u\Delta|u|^{2\sigma_1} &= \frac{\sigma_1}{4}\int_M\partial_t|u|^{2\sigma_1-2}\left|\nabla|u|^2\right|^2 + \frac{1}{2}\int_M\partial_t|u|^{2\sigma_1}|\nabla u|^2\\ &\quad+\int_M\partial_t|u|^{2\sigma_1}\mathrm{Im}\left(\bar u\partial_t u\right)\\ &\quad+\frac{\sigma_1}{\sigma_1+1}\frac{d}{dt}\int_M V|u|^{2\sigma_1+2} + \frac{\lambda\sigma_1}{2\sigma_1+1}\frac{d}{dt}\int_M|u|^{4\sigma_1+2}.\end{aligned}$$

We also note that

$$\int_M\partial_t|u|^{2\sigma_1}\mathrm{Im}\left(\bar u\partial_t u\right) = \mathrm{Im}\int_M\partial_t u\partial_t\left(|u|^{2\sigma_1}\bar u\right) = -\mathrm{Im}\int_M\partial_t\bar u\partial_t\left(|u|^{2\sigma_1}u\right),$$

so that we recover the left hand side of (4.1), with the opposite sign. Therefore, we have

$$\begin{aligned}2\,\mathrm{Im}\int_M\partial_t\bar u\partial_t\left(|u|^{2\sigma_1}u\right) &= \frac{d}{dt}\left(\mathrm{Im}\int_M|u|^{2\sigma_1}u\partial_t\bar u + \frac{2\sigma_1-1}{2\sigma_1+2}\int_M V|u|^{2\sigma_1+2}\right)\\ &\quad-\frac{d}{dt}\left(\frac{\lambda}{4\sigma_1+2}\int_M|u|^{4\sigma_1+2}\right)\\ &\quad+\frac{\sigma_1}{4}\int_M\partial_t|u|^{2\sigma_1-2}\left|\nabla|u|^2\right|^2 + \frac{1}{2}\int_M\partial_t|u|^{2\sigma_1}|\nabla u|^2\\ &\quad+\mathrm{Re}\int_M\partial_t\bar u\nabla u\cdot\nabla|u|^{2\sigma_1} + \frac{1}{2}\mathrm{Re}\int_M|u|^{2\sigma_1}\partial_t\bar u\Delta u.\end{aligned}$$

For the last term, we use (1.1) to substitute Δu:

$$\begin{aligned}\frac{1}{2}\operatorname{Re}\int_M |u|^{2\sigma_1}\partial_t\bar{u}\Delta u &= \operatorname{Re}\int_M |u|^{2\sigma_1}\partial_t\bar{u}\left(Vu+\lambda|u|^{2\sigma_1}u - ia|u|^{2\sigma_2}u - ib\frac{u}{|u|^\alpha}\right)\\ &= \frac{1}{2\sigma_1+2}\frac{d}{dt}\int_M V|u|^{2\sigma_1+2} + \frac{\lambda}{4\sigma_1+2}\frac{d}{dt}\int_M |u|^{4\sigma_1+2}\\ &\quad + a\operatorname{Im}\int_M |u|^{2\sigma_1+2\sigma_2}u\partial_t\bar{u} + b\int_M |u|^{2\sigma_1-\alpha}\operatorname{Im}(u\partial_t\bar{u}).\end{aligned}$$

At this stage, we infer

$$\begin{aligned}&\frac{d}{dt}\left(\|\partial_t u\|_{L^2}^2 - \lambda\int_M |u|^{2\sigma_1}\operatorname{Im}(u\partial_t\bar{u}) - \frac{\lambda\sigma_1}{\sigma_1+1}\int_M V|u|^{2\sigma_1+2}\right) =\\ &\quad\frac{\lambda\sigma_1}{4}\int_M \partial_t|u|^{2\sigma_1-2}\left|\nabla|u|^2\right|^2 + \frac{\lambda}{2}\int_M \partial_t|u|^{2\sigma_1}|\nabla u|^2 + \lambda\operatorname{Re}\int_M \partial_t\bar{u}\nabla u\cdot\nabla|u|^{2\sigma_1}\\ &+\lambda a\int_M |u|^{2\sigma_1+2\sigma_2}\operatorname{Im}(u\partial_t\bar{u}) + \lambda b\int_M |u|^{2\sigma_1-\alpha}\operatorname{Im}(u\partial_t\bar{u})\\ &-2a\operatorname{Re}\int_M \partial_t\bar{u}\partial_t\left(|u|^{2\sigma_2}u\right) - 2b\operatorname{Re}\int_M \partial_t\bar{u}\partial_t\left(\frac{u}{|u|^\alpha}\right).\end{aligned}$$

The final simplification consists in developing the last two terms in the following fashion:

$$\begin{aligned}\operatorname{Re}\int_M \partial_t\bar{u}\partial_t\left(|u|^p u\right) &= \left(\frac{p}{2}+1\right)\int_M |u|^p|\partial_t u|^2 + \frac{p}{2}\int_M |u|^{p-2}\operatorname{Re}(u\partial_t\bar{u})^2\\ &= \left(\frac{p}{2}+1\right)\int_M |u|^p|\partial_t u|^2\\ &\quad + \frac{p}{2}\int_M |u|^{p-2}\left((\operatorname{Re}u\partial_t\bar{u})^2 - (\operatorname{Im}u\partial_t\bar{u})^2\right)\\ &= \left(\frac{p}{2}+1\right)\int_M |u|^{p-2}\left((\operatorname{Re}u\partial_t\bar{u})^2 + (\operatorname{Im}u\partial_t\bar{u})^2\right)\\ &\quad + \frac{p}{2}\int_M |u|^{p-2}\left((\operatorname{Re}u\partial_t\bar{u})^2 - (\operatorname{Im}u\partial_t\bar{u})^2\right)\\ &= (p+1)\int_M |u|^{p-2}(\operatorname{Re}u\partial_t\bar{u})^2 + \int_M |u|^{p-2}(\operatorname{Im}u\partial_t\bar{u})^2.\end{aligned}$$

We conclude:

Proposition 4.1. *Let $u_0\in\Sigma^2$. In either of the cases considered in Theorem 1.5, the global weak solution u satisfies:*

$$\begin{aligned}&\frac{d}{dt}\left(\|\partial_t u\|_{L^2}^2 - \lambda\operatorname{Im}\int_M |u|^{2\sigma_1}u\partial_t\bar{u} - \frac{\lambda\sigma_1}{\sigma_1+1}\int_M V|u|^{2\sigma_1+2}\right) =\\ &\quad\frac{\lambda\sigma_1}{4}\int_M \partial_t|u|^{2\sigma_1-2}\left|\nabla|u|^2\right|^2 + \frac{\lambda}{2}\int_M \partial_t|u|^{2\sigma_1}|\nabla u|^2 + \lambda\operatorname{Re}\int_M \partial_t\bar{u}\nabla u\cdot\nabla|u|^{2\sigma_1}\\ &+\lambda a\int_M |u|^{2\sigma_1+2\sigma_2}\operatorname{Im}(u\partial_t\bar{u}) + \lambda b\int_M |u|^{2\sigma_1-\alpha}\operatorname{Im}(u\partial_t\bar{u})\\ &-2a(2\sigma_2+1)\int_M |u|^{2\sigma_2-2}(\operatorname{Re}u\partial_t\bar{u})^2 - 2a\int_M |u|^{2\sigma_2-2}(\operatorname{Im}u\partial_t\bar{u})^2\\ &-2b(1-\alpha)\int_M |u|^{-2-\alpha}(\operatorname{Re}u\partial_t\bar{u})^2 - 2b\int_M |u|^{-2-\alpha}(\operatorname{Im}u\partial_t\bar{u})^2.\end{aligned}$$

4.2. **From order one in time to order two in space.** We rewrite the quantity involved in Proposition 4.1 in order to get rid of all time derivatives:

$$\|\partial_t u\|_{L^2}^2 - \lambda \operatorname{Im}\int |u|^{2\sigma_1} u\partial_t \bar u - \frac{\lambda\sigma_1}{\sigma_1+1}\int V|u|^{2\sigma_1+2} = \operatorname{Re}\int \left(\partial_t u + i\lambda |u|^{2\sigma_1}u\right)\partial_t \bar u - \frac{\lambda\sigma_1}{\sigma_1+1}\int V|u|^{2\sigma_1+2}$$

Leaving out the real part for one moment, the first integral on the right hand side is rewritten as

$$\int \left(\frac{i}{2}\Delta u - iVu - a|u|^{2\sigma_2}u - b\frac{u}{|u|^\alpha}\right)\left(-\frac{i}{2}\Delta \bar u + iV\bar u + i\lambda|u|^{2\sigma_1}\bar u - a|u|^{2\sigma_2}\bar u - b\frac{\bar u}{|u|^\alpha}\right),$$

whose real part is equal to:

$$\begin{aligned}&\frac{1}{4}\|\Delta u\|_{L^2}^2 - \operatorname{Re}\int V\bar u \Delta u - \frac{\lambda}{2}\operatorname{Re}\int |u|^{2\sigma_1}\bar u\Delta u + a\operatorname{Im}\int |u|^{2\sigma_2}\bar u \Delta u\\ &+ b\operatorname{Im}\int \frac{\bar u}{|u|^\alpha}\Delta u + \int V^2|u|^2 + \frac{\lambda}{\sigma_1+1}\int V|u|^{2\sigma_1+2} + a^2\int |U|^{4\sigma_2+2}\\ &+ 2ab\int |u|^{2\sigma_2+2-\alpha} + b^2\int |u|^{2-2\alpha}.\end{aligned}$$

Note that it is in order for the last term to belong to some reasonable Lebesgue space that we assume $\alpha \leqslant 1/2$ in the case where $M=\mathbb{R}^2$. By integration by parts, we can also write

$$\begin{aligned}-\operatorname{Re}\int V\bar u\Delta u &= \int V|\nabla u|^2 - \frac{1}{2}\int |u|^2\Delta V,\\ -\frac{\lambda}{2}\operatorname{Re}\int |u|^{2\sigma_1}\bar u \Delta u &= \lambda\frac{\sigma_1+1}{2}\int |u|^{2\sigma_1}|\nabla u|^2 + \lambda\frac{\sigma_1}{2}\operatorname{Re}\int |u|^{2\sigma_1-2}\bar u^2(\nabla u)^2,\\ a\operatorname{Im}\int |u|^{2\sigma_2}\bar u\Delta u &= -a\sigma_2\operatorname{Im}\int |u|^{2\sigma_2-2}\bar u^2(\nabla u)^2.\end{aligned}$$

Gathering all the terms together, this leads us to setting as a second order energy:

$$\begin{aligned}\mathcal{E}_2(t) := &\frac{1}{4}\|\Delta u\|_{L^2}^2 + \int V^2|u|^2 + \int V|\nabla u|^2\\ &+ a^2\int |u|^{4\sigma_2+2} + 2ab\int |u|^{2\sigma_2+2-\alpha} + b^2\int |u|^{2-2\alpha}\\ &- \frac{1}{2}\int |u|^2\Delta V + \lambda\frac{\sigma_1+1}{2}\int |u|^{2\sigma_1}|\nabla u|^2 + \lambda\frac{\sigma_1}{2}\operatorname{Re}\int |u|^{2\sigma_1-2}\bar u^2(\nabla u)^2\\ &- a\sigma_2\operatorname{Im}\int |u|^{2\sigma_2-2}\bar u^2(\nabla u)^2 + b\operatorname{Im}\int \frac{\bar u}{|u|^\alpha}\Delta u + \frac{\lambda}{\sigma_1+1}\int V|u|^{2\sigma_1+2}.\end{aligned}$$

Lemma 4.2. *Let u be given by Theorem 1.5.*

- *There exists $C>0$ such that for all $t\geqslant 0$,*

$$\frac{1}{C}\|u(t)\|_{\Sigma^2}^2 \leqslant \mathcal{E}_2(t) \leqslant C\|u(t)\|_{\Sigma^2}^2 + C.$$

- *There exists C such that for all $t\geqslant 0$, $\|\partial_t u(t)\|_{L^2}^2 \leqslant C\mathcal{E}_2(t)$.*

Proof. The first two terms in $\mathcal{E}_2$ correspond to the definition of $\|u(t)\|_{\Sigma^2}^2$, up to irrelevant multiplying constants. The third term is non-negative, and is controlled by $\|u(t)\|_{\Sigma^2}^2$, as shown by an integration by parts.

The three terms on the second line in the definition of $\mathcal{E}_2$ are non-negative. The first two terms are controlled by some power of $\|u(t)\|_{H^1}$, which is uniformly bounded from Theorem 1.5. The third term is controlled by the L^2-norm of u if M is compact, thanks to Lemma 2.3 if $M = \mathbb{R}^2$ and $\alpha < 1/2$. If $M = \mathbb{R}^2$ and $\alpha = 1/2$, it is easy to check that

$$\|f\|_{L^1(\mathbb{R}^2)} \leqslant C\|f\|_{L^2(\mathbb{R}^2)}^{1/2} \||x|^2 f\|_{L^2(\mathbb{R}^2)}^{1/2}, \quad \forall f \in \Sigma^2. \tag{4.2}$$

Since ΔV is bounded, $\int |u|^2 \Delta V$ is equivalent to $\|u\|_{L^2}^2$. The last two terms on the third line are both controlled as follows: for $0 < \varepsilon < 1$,

$$\int |u|^{2\sigma_1} |\nabla u|^2 \leqslant \|u\|_{L^{2/\varepsilon}}^{2\sigma_1} \|\nabla u\|_{L^{2/(1-\varepsilon)}}^2 \lesssim \|u\|_{H^1}^{2\sigma_1} \|\nabla u\|_{L^2}^{2(1-\varepsilon)} \|\Delta u\|_{L^2}^{2\varepsilon},$$

where we have used Gagliardo-Nirenberg inequality (Lemma 2.1) applied to ∇u for the last inequality. The first term of the fourth line is controlled in exactly the same fashion, by simply replacing σ_1 with σ_2.

By Cauchy-Schwarz inequality, we have

$$\left| \operatorname{Im} \int \frac{\bar{u}}{|u|^\alpha} \Delta u \right| \leqslant \|u\|_{L^{2-2\alpha}}^{1-\alpha} \|\Delta u\|_{L^2}.$$

If M is compact, we conclude by Hölder inequality,

$$\|u\|_{L^{2-2\alpha}} \leqslant |M|^{\alpha/(4-2\alpha)} \|u\|_{L^2}.$$

If $M = \mathbb{R}^2$, we proceed as above, by either invoking Lemma 2.3 if $\alpha < 1/2$, or (4.2) if $\alpha = 1/2$.

Finally, Cauchy-Schwarz inequality and Sobolev embedding yield.

$$\int V|u|^{2\sigma_1+2} \leqslant \|Vu\|_{L^2} \|u\|_{L^{4\sigma_2+2}}^{2\sigma_2+1} \leqslant C\|u\|_{\Sigma}^{2\sigma_1+2},$$

hence the first point of the lemma.

For the second point, recall that we also have, by construction,

$$\mathcal{E}_2(t) = \|\partial_t u\|_{L^2}^2 - \lambda \operatorname{Im} \int |u|^{2\sigma_1} u \partial_t \bar{u} - \frac{\lambda\sigma_1}{\sigma_1+1} \int V|u|^{2\sigma_1+2}.$$

We have just seen that the last term is estimated as

$$\int V|u|^{2\sigma_1+2} \lesssim \|u\|_{\Sigma}^{2\sigma_1+2}.$$

For the second term, Cauchy-Schwarz inequality, Sobolev embedding and Young inequality yield

$$\begin{aligned} |\lambda| \left| \operatorname{Im} \int |u|^{2\sigma_1} u \partial_t \bar{u} \right| &\leqslant |\lambda| \|\partial_t u\|_{L^2} \|u\|_{L^{4\sigma_1+2}}^{2\sigma_1+1} \lesssim \|\partial_t u\|_{L^2} \|u\|_{\Sigma}^{2\sigma_1+1} \\ &\leqslant \varepsilon \|\partial_t u\|_{L^2}^2 + \frac{C}{\varepsilon} \|u\|_{\Sigma}^{4\sigma_1+2}, \end{aligned}$$

hence the second point of the lemma by choosing $\varepsilon = 1/2$. □

5. Finite time extinction in 2D

We recall the celebrated Brézis-Gallouët inequality, established in [6].

Lemma 5.1 (Brézis-Gallouët inequality)**.** *Let $d = 2$ in Assumption 1.1. There exists C such that for all $f \in H^2(M)$,*

$$\|f\|_{L^\infty(M)} \leqslant C\left(\|f\|_{H^1(M)}\sqrt{\ln\left(2+\|f\|_{H^2(M)}\right)}+1\right).$$

Recall that by construction, the time derivative of $\mathcal{E}_2$ is given by Proposition 4.1. Since the last two lines are non-negative, and noticing that all the terms in the second line can be estimated in a common fashion, we have:

$$\dot{\mathcal{E}}_2 \lesssim \int |u|^{2\sigma_1-1}|\partial_t u||\nabla u|^2 + \int |u|^{2\sigma_1+2\sigma_2+1}|\partial_t u| + \int |u|^{2\sigma_1-\alpha+1}|\partial_t u|. \tag{5.1}$$

The first term is controlled, up to a multiplicative constant, by

$$\|u\|_{L^\infty}^{2\sigma_1-1}\|\nabla u\|_{L^4}^2\|\partial_t u\|_{L^2} \lesssim \|u\|_{L^\infty}^{2\sigma_1-1}\|\Delta u\|_{L^2}\|\partial_t u\|_{L^2},$$

where we have used Gagliardo-Nirenberg inequality. Using Lemma 4.2, we infer

$$\int |u|^{2\sigma_1-1}|\partial_t u||\nabla u|^2 \lesssim \|u\|_{L^\infty}^{2\sigma_1-1}\mathcal{E}_2.$$

Brézis-Gallouët inequality implies:

$$\int |u|^{2\sigma_1-1}|\partial_t u||\nabla u|^2 \lesssim \left(\|u\|_\Sigma\sqrt{\ln(2+\mathcal{E}_2)}+1\right)^{2\sigma_1-1}\mathcal{E}_2. \tag{5.2}$$

The last two terms in (5.1) are estimated thanks to Cauchy-Schwarz inequality, Sobolev embedding and the second point in Lemma 4.2:

$$\int |u|^{2\sigma_1+2\sigma_2+1}|\partial_t u| + \int |u|^{2\sigma_1-\alpha+1}|\partial_t u| \lesssim \left(\|u\|_\Sigma^{2\sigma_1+2\sigma_2+1}+\|u\|_\Sigma^{2\sigma_1+1-\alpha}\right)\mathcal{E}_2.$$

Along with (5.2), (5.1) then yields

$$\dot{\mathcal{E}}_2 \leqslant K(\|u\|_\Sigma)\left(1+\ln(2+\mathcal{E}_2)\right)^{\sigma_1-1/2}\left(2+\mathcal{E}_2\right),$$

where $K(\cdot)$ denotes a continuous function. Integrating in time, we infer that

$$F(t) := \begin{cases} \left(1+\ln(2+\mathcal{E}_2(t))\right)^{3/2-\sigma_1} & \text{if} \quad \sigma_1 < \dfrac{3}{2}, \\ \ln\left(1+\ln(2+\mathcal{E}_2(t))\right) & \text{if} \quad \sigma_1 = \dfrac{3}{2}, \end{cases}$$

is controlled by $F(0)+tK(\|u_0\|_\Sigma)$, where we have used also Proposition 2.5 (after passing to the limit $\delta \to 0$), up to changing the continuous function K. In order to ease notations, we now denote by K_j any positive continuous function of $\|u_0\|_{\Sigma^j}$, which may change from line to line, but only finitely many times.

Case $\sigma_1 < 3/2$. In this case, the control on F yields, along with Lemma 4.2,

$$\|u(t)\|_{H^2} \leqslant K_2 e^{t^{\frac{2}{3-2\sigma_1}}K_1}.$$

Nash inequality (3.2) then implies

$$\|u(t)\|_{L^2} \leqslant K_2\|u(t)\|_{L^{2-\alpha}}^{2(2-\alpha)/(4-\alpha)} e^{t^{\frac{2}{3-2\sigma_1}}K_1}.$$

Let $\theta = 1-\alpha/4$. The above inequality and (1.3) yield

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 \leqslant -2\|u(t)\|_{L^{2-\alpha}}^{2-\alpha} \leqslant -K_2 e^{-t^{\frac{2}{3-2\sigma_1}}K_1}\|u(t)\|_{L^2}^{2\theta},$$

hence

$$\frac{d}{dt}\|u(t)\|_{L^2}^{2(1-\theta)} \leqslant -K_2 e^{-t^{\frac{2}{3-2\sigma_1}}K_1}.$$

By integration, we infer

$$\|u(t)\|_{L^2}^{2(1-\theta)} \leqslant \|u_0\|_{L^2}^{2(1-\theta)} - K_2\int_0^t e^{-\tau^{\frac{2}{3-2\sigma_1}}K_1}d\tau. \tag{5.3}$$

By changing variables in the integral, note that there exists a $\mathbf{K_2} = K(\|u_0\|_{\Sigma^2})$ such that

$$K_2\int_0^\infty e^{-\tau^{\frac{2}{3-2\sigma_1}}K_1}d\tau = \mathbf{K_2}\int_0^\infty e^{-\tau^{\frac{2}{3-2\sigma_1}}}d\tau,$$

where the integrals are obviously finite, and the last one is independent of u_0. We conclude that if

$$\|u_0\|_{L^2}^{2(1-\theta)} - \mathbf{K_2}\int_0^\infty e^{-\tau^{\frac{2}{3-2\sigma_1}}}d\tau < 0, \tag{5.4}$$

then there for t sufficiently large, the right hand side in (5.3) becomes zero. Therefore, there exists some finite time $T>0$ such that $\|u(T)\|_{L^2}=0$. Since (5.4) corresponds to a smallness assumption on $\|u_0\|_{L^2}$ when $\|u_0\|_{\Sigma^2}$ is fixed, the second point in Theorem 1.7 follows in the case $1/2\leqslant \sigma_1 < 3/2$.

Case $\sigma_1 = 3/2$**.** The control on F now leads to a control by a double exponential:

$$\|u(t)\|_{H^2} \leqslant \exp\left(K_2 e^{K_1 t}\right).$$

In the same fashion as above, we infer

$$\frac{d}{dt}\|u(t)\|_{L^2}^{2(1-\theta)} \leqslant -\exp\left(-K_2 e^{K_1 t}\right),$$

hence

$$\|u(t)\|_{L^2}^{2(1-\theta)} \leqslant \|u_0\|_{L^2}^{2(1-\theta)} - \int_0^t \exp\left(-K_2 e^{K_1\tau}\right)d\tau.$$

We have

$$\int_0^\infty \exp\left(-K_2 e^{K_1\tau}\right)d\tau = \frac{1}{K_1}\int_0^\infty \exp\left(-K_2 e^{\tau}\right)d\tau = \frac{1}{K_1}\int_{\ln K_2}^\infty \exp\left(-e^{\tau}\right)d\tau$$
$$\geqslant \frac{1}{K_1(R)}\int_{\ln K_2(R)}^\infty \exp\left(-e^{\tau}\right)d\tau,$$

that is, a constant which depends only on R provided that $\|u_0\|_{\Sigma^2}\leqslant R$. Finite time extinction then follows as soon as

$$\|u_0\|_{L^2}^{2(1-\theta)} < \frac{1}{K_1(R)}\int_{\ln K_2(R)}^\infty \exp\left(-e^{\tau}\right)d\tau.$$

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