MÖBIUS FUNCTION OF SEMIGROUP POSETS THROUGH HILBERT SERIES

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Abstract. In this paper, we investigate the Möbius function $\mu_{\mathcal{S}}$ associated to a (locally finite) poset arising from a semigroup $\mathcal{S}$ of $\mathbb{Z}^m$. For, we introduce and develop a new approach to study $\mu_{\mathcal{S}}$ by using the Hilbert series associated to $\mathcal{S}$. The latter allow us to provide formulas for $\mu_{\mathcal{S}}$ when $\mathcal{S}$ is a semigroup with unique Betti element, and when $\mathcal{S}$ is a complete intersection numerical semigroup with three generators. We also give a characterization for a finite locally poset to be isomorphic to a semigroup poset. With this in hand, we are able to calculate the Möbius function of certain posets (for instance the classical arithmetic Möbius function) by computing the Möbius function of the corresponding semigroup poset.

1. Introduction

The *Möbius function* is an important concept associated to (*locally finite*) posets. Möbius function can be considered as a generalization of the classical Möbius arithmetic function on the integers (given by the Möbius function of the poset obtained from the positive integers partially ordered by the divisibility). Möbius function has been extremely useful to investigate many different problems. For instance, the inclusion-exclusion principle can be retrieved by considering the set of all subsets of a finite set partially ordered by inclusion. We refer the reader to [19] for a large number of applications of the Möbius function. In this paper, we investigate the Möbius function associated to posets arising naturally from subsemigroups of $\mathbb{Z}^m$ as follows. Let $a_1, \ldots, a_n$ be nonzero vectors of $\mathbb{Z}^m$ and let $\mathcal{S} = \langle a_1, \ldots, a_n \rangle$ denote the semigroup generated by $a_1, \ldots, a_n$, that is,

$$\mathcal{S} = \langle a_1, \ldots, a_n \rangle = \{x_1 a_1 + \cdots + x_n a_n \,|\, x_1, \ldots, x_n \in \mathbb{N}\}.$$

We say that $\mathcal{S}$ is *pointed* if $\mathcal{S} \cap (-\mathcal{S}) = \{0\}$, where $-\mathcal{S} := \{-x \,|\, x \in \mathcal{S}\}$. Whenever $\mathcal{S}$ is pointed, $\mathcal{S}$ induces on $\mathbb{Z}^m$ a structure of poset whose partial order $\leqslant_{\mathcal{S}}$ is defined by $x \leqslant_{\mathcal{S}} y \iff y - x \in \mathcal{S}$ for all x and y in $\mathbb{Z}^m$. This (locally finite) poset will be denoted by $(\mathbb{Z}^m, \leqslant_{\mathcal{S}})$. We denote by $\mu_{\mathcal{S}}$ the Möbius function associated to $(\mathbb{Z}^m, \leqslant_{\mathcal{S}})$. As far as we are aware, $\mu_{\mathcal{S}}$ has only been investigated when $\mathcal{S}$ is a *numerical semigroup*, i.e., when $\mathcal{S} \subset \mathbb{N}$ and $\gcd\{a_1, \ldots, a_n\} = 1$. Moreover, the only known results concerning $\mu_{\mathcal{S}}$ is an old theorem due to Deddens [5] that determines the value of $\mu_{\mathcal{S}}$ when $\mathcal{S}$ has exactly two generators, and a recent paper due to Chappelon and Ramírez Alfonsín [4] where the authors investigate $\mu_{\mathcal{S}}$ when $\mathcal{S} = \langle a, a+d, \ldots, a+kd \rangle$ with $a, k, d \in \mathbb{Z}^+$. In both papers the authors approach the problem by a thorough study of the intrinsic properties of each semigroup. It is worth pointing out that the results and techniques developed in this paper are deeply inspired by those of [4].

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In this work we introduce and develop a different and more general approach to the study of $\mu_{\mathcal{S}}$ by means of the *Hilbert series of the semigroup* $\mathcal{S}$. As a consequence of this point of view we are able to provide formulas for $\mu_{\mathcal{S}}$ when $\mathcal{S}$ belongs to some families of semigroups. Finally, we study when a locally finite poset is isomorphic to a semigroup poset and, thus, its Möbius function can be computed by means of the techniques introduced in this work.

This paper is organized as follows. In the next section, we review some classic notions of the Möbius function and apply them to the particular case of semigroup posets. In Section 3, we recall some definitions and results about Hilbert series and prove two general results relating the Möbius function of $(\mathbb{Z}^m, \leqslant_{\mathcal{S}})$ and the Hilbert series of $\mathcal{S}$ for every pointed semigroup $\mathcal{S}$. The latter are the key results that will be used in Section 4 to provide explicit formulas for $\mu_{\mathcal{S}}$ when $\mathcal{S}$ belongs to certain families of semigroups; namely, when $\mathcal{S} = \mathbb{N}^m$, when $\mathcal{S}$ is a *semigroup with a unique Betti element* and when $\mathcal{S} = \langle a_1, a_2, a_3 \rangle \subset \mathbb{N}$ is a *complete intersection numerical semigroup*. Finally, in Section 5, we provide a criterion for determining when a locally finite poset is isomorphic to a semigroup poset and, thus, its Möbius function can be computed by means of the Möbius function of its corresponding semigroup poset. In particular, we are able to recover the classical arithmetic Möbius function and the Möbius function of the set of all subsets of a finite set partially ordered by inclusion.

2. Möbius function associated to a semigroup poset

Let $(\mathcal{P}, \leqslant)$ be a partially ordered set, or *poset* for short. The *strict partial order* $<_{\mathcal{P}}$ is the reduction of $\leqslant_{\mathcal{P}}$ given by $a <_{\mathcal{P}} b$ if and only if $a \leqslant_{\mathcal{P}} b$ and $a \neq b$. For any a and b in the poset $\mathcal{P}$, the *segment* between a and b is defined by

$$[a, b]_{\mathcal{P}} := \{c \in \mathcal{P} \mid a \leqslant_{\mathcal{P}} c \leqslant_{\mathcal{P}} b\}.$$

A poset is said to be *locally finite* if every segment has finite cardinality. In this paper, we only consider locally finite posets.

Let a and b be elements of the poset $\mathcal{P}$. A *chain* of length $l \geqslant 0$ between a and b is a subset of $[a, b]_{\mathcal{P}}$ containing a and b, with cardinality $l+1$ and totally ordered by $<_{\mathcal{P}}$, that is $\{a_0, a_1, \ldots, a_l\} \subset [a, b]_{\mathcal{P}}$ such that

$$a = a_0 <_{\mathcal{P}} a_1 <_{\mathcal{P}} a_2 <_{\mathcal{P}} \cdots <_{\mathcal{P}} a_{l-1} <_{\mathcal{P}} a_l = b.$$

For any nonnegative integer l, we denote by $C_l(a, b)$ the set of all chains of length l between a and b. The cardinality of $C_l(a, b)$ is denoted by $c_l(a, b)$. This number always exists because the poset $\mathcal{P}$ is supposed to be locally finite. For instance, the number of chains $c_2(2, 12)$, where the poset is the set $\mathbb{N}$ partially ordered by divisibility, is equal to 2. Indeed, there are exactly 2 chains of length 2 between 2 and 12 in $[2, 12]_{\mathbb{N}} = \{2, 4, 6, 12\}$, which are $\{2, 4, 12\}$ and $\{2, 6, 12\}$.

For any locally finite poset $\mathcal{P}$, the *Möbius function* $\mu_{\mathcal{P}}$ is the integer-valued function on $\mathcal{P} \times \mathcal{P}$ defined by

$$\mu_{\mathcal{P}}(a, b) = \sum_{l \geqslant 0} (-1)^l c_l(a, b), \tag{1}$$

for all elements a and b of the poset P. One can remark that this sum is always finite because, for a and b given, there exists a maximal length of a possible chain between a and b since the segment $[a, b]_{\mathcal{P}}$ has finite cardinality.

The concept of Möbius function for a locally finite poset $(\mathcal{P}, \leqslant)$ was introduced by Rota in [19]. In this paper, Rota proves the following property of the Möbius function: for all $(a, b) \in \mathcal{P} \times \mathcal{P}$,

$$\mu_{\mathcal{P}}(a, a) = 1 \quad \text{and} \quad \sum_{c \in [a,b]_{\mathcal{P}}} \mu_{\mathcal{P}}(a, c) = 0. \tag{2}$$

In this work we will consider posets associated to semigroups of $\mathbb{Z}^m$. We will begin by summarizing some generalities on semigroups that are useful for the understanding of this work. We refer the readers to [3] or, more generally, to [15] for further details.

Let $\mathcal{S} := \langle a_1, \ldots, a_n \rangle \subset \mathbb{Z}^m$ denote the subsemigroup of $\mathbb{Z}^m$ generated by $a_1, \ldots, a_n \in \mathbb{Z}^m$, i.e.,

$$\mathcal{S} := \langle a_1, \ldots, a_n \rangle = \{x_1 a_1 + \cdots + x_n a_n \mid x_1, \ldots, x_n \in \mathbb{N}\}.$$

The semigroup $\mathcal{S}$ induces the binary relation $\leqslant_{\mathcal{S}}$ on $\mathbb{Z}^m$ given by

$$x \leqslant_{\mathcal{S}} y \Longleftrightarrow y - x \in \mathcal{S}.$$

It turns out that $\leqslant_{\mathcal{S}}$ is an order if and only if $\mathcal{S}$ is pointed. Moreover, whenever $\mathcal{S}$ is pointed the poset $(\mathbb{Z}^m, \leqslant_{\mathcal{S}})$ is locally finite.

We denote by $\mu_{\mathcal{S}}$ the Möbius function associated to $(\mathbb{Z}^m, \leqslant_{\mathcal{S}})$. It is easy to see that $\mu_{\mathcal{S}}$ can be considered as a univariate function of $\mathbb{Z}^m$. Indeed, for all $x, y \in \mathbb{Z}^m$ and for all $l \geqslant 0$, we have that

$$c_l(x, y) = c_l(0, y - x). \tag{3}$$

The above follows since the set $C_l(x, y)$ is in bijection with $C_l(0, y - x)$. Indeed the map that assigns the chain $\{x_0, x_1, \ldots, x_l\} \in C_l(x, y)$ to the chain $\{0, x_1 - x_0, \ldots, x_l - x_0\} \in C_l(0, y - x)$ is clearly a bijection. Thus, by definition of $\mu_{\mathcal{S}}$ and equality (3) we obtain

$$\mu_{\mathcal{S}}(x, y) = \mu_{\mathcal{S}}(0, y - x)$$

for all $x, y \in \mathbb{Z}$.

In the sequel of this paper we shall only consider the reduced Möbius function $\mu_{\mathcal{S}} : \mathbb{Z}^m \longrightarrow \mathbb{Z}$ defined by

$$\mu_{\mathcal{S}}(x) := \mu_{\mathcal{S}}(0, x), \quad \text{for all } x \in \mathbb{Z}^m.$$

Thus, the formula given by (2) can be more easily presented when the locally finite poset is $(\mathbb{Z}^m, \leqslant_{\mathcal{S}})$.

Proposition 2.1. *Let $\mathcal{S}$ be a pointed semigroup and let $x \in \mathbb{Z}^m$. Then,*

$$\sum_{b \in \mathcal{S}} \mu_{\mathcal{S}}(x - b) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Firstly, from (1) we observe that $\mu_{\mathcal{S}}(b) = 0$ for all $b \notin \mathcal{S}$. Since $\mathcal{S}$ is pointed, if $b \in \mathcal{S}$ then $-b \notin \mathcal{S}$ and, hence, $\sum_{b \in \mathcal{S}} \mu_{\mathcal{S}}(0 - b) = \mu_{\mathcal{S}}(0) = 1$. Take $x \neq 0$, then from (2)

$$0 = \sum_{b \in [0,x]_{\mathbb{Z}^m}} \mu_{\mathcal{S}}(b) = \sum_{\substack{b \in \mathcal{S} \\ x - b \in \mathcal{S}}} \mu_{\mathcal{S}}(b) = \sum_{\substack{b \in \mathcal{S} \\ x - b \in \mathcal{S}}} \mu_{\mathcal{S}}(x - b) = \sum_{b \in \mathcal{S}} \mu_{\mathcal{S}}(x - b).$$

□

The formula presented in Proposition 2.1 will be very useful to get most or our results.

3. The Hilbert and Möbius series

In this section we present two results (Theorem 3.2 and Theorem 3.3), both relating the Möbius function of the poset $(\mathbb{Z}^m, \leqslant_{\mathcal{S}})$ with the Hilbert series of the semigroup $\mathcal{S}$. Before proving these theorems we recall some basic notions of multivariate Hilbert series. For a thorough study of multivariate Hilbert series we refer the reader to the book by Kreuzer and Robbiano [13].

Let k be any field and let $\mathcal{S} = \langle a_1, \ldots, a_n \rangle$ be a subsemigroup of $\mathbb{Z}^m$. The semigroup $\mathcal{S}$ induces a grading in the ring of polynomials $k[x_1, \ldots, x_n]$ by assigning $\deg_{\mathcal{S}}(x_i) := a_i$ for all $i \in \{1, \ldots, n\}$. Then the *$\mathcal{S}$-degree* of the monomial $m := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is $\deg_{\mathcal{S}}(m) := \sum \alpha_i a_i$; we say that a polynomial is *$\mathcal{S}$-homogeneous* if all its monomials have the same $\mathcal{S}$-degree. For all $b \in \mathbb{Z}^m$, we denote by $k[x_1, \ldots, x_n]_b$ the k-vector space formed by all polynomials $\mathcal{S}$-homogeneous of $\mathcal{S}$-degree b.

Consider $I \subset k[\mathbf{x}]$ an ideal generated by $\mathcal{S}$-homogeneous polynomials, then for all $b \in \mathbb{Z}^m$ we denote by I_b the k-vector space formed by the $\mathcal{S}$-homogeneous polynomials of I of $\mathcal{S}$-degree b. Note that I_b is a k-vector subspace of $k[x_1, \ldots, x_n]_b$. The quotient ring $k[x_1, \ldots, x_n]/I$ is also $\mathcal{S}$-graded by taking $(k[x_1, \ldots, x_n]/I)_b := k[x_1, \ldots, x_n]_b/I_b$ for all $b \in \mathbb{Z}^m$.

Whenever $\mathcal{S}$ is pointed the k-vector space $k[x_1, \ldots, x_n]_b$ has finite dimension for all $b \in \mathbb{Z}^m$ by [13, Proposition 4.1.19]. Hence, one can define the *multigraded Hilbert function* of $M := k[x_1, \ldots, x_n]/I$ as

$$HF_M : \mathbb{Z}^m \longrightarrow \mathbb{N},$$

where $HF_M(b) := \dim_k(M_b) = \dim_k(k[x_1, \ldots, x_n]_b) - \dim_k(I_b)$ for al $b \in \mathbb{Z}^m$.

For every $b = (b_1, \ldots, b_m) \in \mathbb{Z}^m$, we denote by $\mathbf{t}^b$ the monomial $t_1^{b_1} \cdots t_m^{b_m}$ in the Laurent polynomial ring $\mathbb{Z}[t_1, \ldots, t_m, t_1^{-1}, \ldots, t_m^{-1}]$. We define the *multivariate Hilbert series of M* as the following formal power series in $\mathbb{Z}[[t_1, \ldots, t_m, t_1^{-1}, \ldots, t_m^{-1}]]$:

$$\mathcal{H}_M(\mathbf{t}) := \sum_{b \in \mathbb{Z}^m} HF_M(b)\, \mathbf{t}^b.$$

For every $\mathcal{S}$-homogeneous ideal I, the Hilbert series of $M = k[x_1, \ldots, x_n]/I$ can be expressed as a quotient of polynomials in the Laurent polynomial ring in the following way (see [13, Theorem 5.8.19]):

$$\mathcal{H}_M(\mathbf{t}) = \frac{\mathbf{t}^{\alpha}\ h(t_1, \ldots, t_m)}{(1 - \mathbf{t}^{a_1}) \cdots (1 - \mathbf{t}^{a_n})},$$

where $\alpha \in \mathbb{Z}^m$ and $h(t_1, \ldots, t_m) \in \mathbb{Z}[t_1, \ldots, t_m]$.

Denote by $I_{\mathcal{S}}$ the *toric ideal of* $\mathcal{S}$, i.e., the kernel of the k-homomorphism

$$\varphi : k[x_1, \ldots, x_n] \longrightarrow k[t_1, \ldots, t_m, t_1^{-1}, \ldots, t_m^{-1}]$$

induced by $\varphi(x_i) = \mathbf{t}^{a_i}$ for all $i \in \{1, \ldots, n\}$.

It is well known that $I_{\mathcal{S}}$ is $\mathcal{S}$-homogeneous (see [20, Corollary 4.3]). Therefore, it makes sense to study the multivariate Hilbert series of $k[x_1, \ldots, x_n]/I_{\mathcal{S}}$ with respect to the grading induced by $\mathcal{S}$.

Proposition 3.1. *Let $\mathcal{S}$ be a pointed semigroup and $M := k[x_1, \ldots, x_n]/I_{\mathcal{S}}$. Then,*

$$\mathcal{H}_M(\mathbf{t}) = \textstyle\sum_{b \in \mathcal{S}} \mathbf{t}^b.$$

Proof. Take $b \in \mathbb{Z}^m$, then $k[x_1, \ldots, x_n]_b = \{0\}$ and $HF_M(b) = 0$ whenever $b \notin \mathcal{S}$. Let us prove that $HF_M(b) = 1$ for all $b \in \mathcal{S}$. Indeed, φ induces an isomorphism of k-vector spaces between M_b and $k[\mathbf{t}^b]$, for all $b \in \mathcal{S}$. Hence, $HF_M(b) = \dim_k(M_b) = \dim_k(k[\mathbf{t}^b]) = 1$. □

From now on, we denote by $\mathcal{H}_\mathcal{S}(\mathbf{t})$ the multivariate Hilbert series of $k[x_1, \ldots, x_n]/I_\mathcal{S}$ and we call it the *Hilbert series of* $\mathcal{S}$. We may now state and prove Theorems 3.2 and 3.3 that relate $\mu_\mathcal{S}$ with the Hilbert series of $\mathcal{S}$. The proofs of both results rely on Proposition 2.1.

Theorem 3.2. *Let $c_1, \ldots, c_k$ be nonzero vectors of $\mathbb{Z}^m$ and denote*

$$(1 - \mathbf{t}^{c_1}) \cdots (1 - \mathbf{t}^{c_k})\ \mathcal{H}_\mathcal{S}(\mathbf{t}) = \sum_{b \in \mathbb{Z}^m} f_b\, \mathbf{t}^b \in \mathbb{Z}[[t_1, \ldots, t_m, t_1^{-1}, \ldots, t_m^{-1}]].$$

Then,

$$\textstyle\sum_{b \in \mathbb{Z}^m} f_b\, \mu_\mathcal{S}(x - b) = 0 \text{ for all } x \notin \{\sum_{i \in A} c_i \,|\, A \subset \{1, \ldots, k\}\}.$$

Proof. Firstly, from Proposition 3.1, we obtain that

$$f_b = \sum_{\substack{A \subset \{1, \ldots, k\} \\ b - \sum_{i \in A} c_i \in \mathcal{S}}} (-1)^{|A|},$$

for all $b \in \mathbb{Z}^m$. Set $\Delta := \{\sum_{i \in A} c_i \,|\, A \subset \{1, \ldots, k\}\}$. By Proposition 2.1, for all $x \notin \Delta$ and $A \subset \{1, \ldots, k\}$ we have that

$$\sum_{b \in \mathcal{S}} \mu_\mathcal{S} \left(x - \sum_{i \in A} a_i - b \right) = 0.$$

Hence, for all $x \notin \Delta$ it follows that

$$\sum_{b \in \mathbb{Z}^m} \alpha_b\, \mu_\mathcal{S}(x - b) = \sum_{b \in \mathcal{S}} \sum_{A \subset \{1, \ldots, k\}} (-1)^{|A|} \mu_\mathcal{S} \left(x - \sum_{i \in A} c_i - b \right) = 0,$$

where $\alpha_b = \sum_{\substack{A \subset \{1, \ldots, k\} \\ b - \sum_{i \in A} c_i \in \mathcal{S}}} (-1)^{|A|} = f_b$, which completes the proof. □

We notice that the formula $(1 - \mathbf{t}^{c_1}) \cdots (1 - \mathbf{t}^{c_k})\ \mathcal{H}_\mathcal{S}(\mathbf{t}) = \sum_{b \in \mathbb{Z}^m} f_b\, \mathbf{t}^b$ might have an infinite number of terms. Nevertheless, for every $x \in \mathbb{Z}^m$ the formula $\sum_{b \in \mathbb{Z}^m} f_b\, \mu_\mathcal{S}(x - b) = 0$ only involves a finite number of nonzero summands due to the fact that $\mu_\mathcal{S}(x - b) \neq 0$ implies that $x - b \in \mathcal{S}$ and $\mathcal{S}$ is pointed.

As a consequence of this result, whenever we know an explicit expression of $\mathcal{H}_\mathcal{S}(\mathbf{t})$ as

$$\mathcal{H}_\mathcal{S}(\mathbf{t}) = \frac{f(\mathbf{t})}{(1 - \mathbf{t}^{c_1}) \cdots (1 - \mathbf{t}^{c_k})},$$

where $f(\mathbf{t}) \in \mathbb{Z}[t_1, \ldots, t_m, t_1^{-1}, \ldots, t_m^{-1}]$ for some $c_1, \ldots, c_k \in \mathbb{Z}^m$, we can derive a recursive formula for the Möbius function. Although this will be done for some semigroups, unfortunately we cannot do it systematically. For instance, in the case of the so called almost arithmetic semigroups, i.e., subsemigroups $\mathcal{S} = \langle a_1, \ldots, a_n \rangle \subset \mathbb{N}$ such that all but one of the a_i form an ordinary arithmetic progression. In [17, Theorem 3] the authors find an expression of the Hilbert series of these semigroups as a quotient of two polynomials needing some previously computed parameters. In this case, the recursive formula for the Möbius function might be very difficult to compute. We rather prefer to avoid this long and tedious calculation.

Now, we consider the *Möbius series* $\mathcal{G}_\mathcal{S}$, the generating function of the Möbius function, which is the power series in $\mathbb{Z}[[t_1,\ldots,t_m,t_1^{-1},\ldots,t_m^{-1}]]$ defined by

$$\mathcal{G}_\mathcal{S}(\mathbf{t}) := \sum_{b\in\mathbb{Z}^m} \mu_\mathcal{S}(b)\,\mathbf{t}^b.$$

Theorem 3.3. *Let $\mathcal{S}$ be a pointed semigroup. Then,*

$$\mathcal{H}_\mathcal{S}(\mathbf{t}).\mathcal{G}_\mathcal{S}(\mathbf{t}) = 1.$$

Proof. From the definitions of $\mathcal{H}_\mathcal{S}(\mathbf{t})$ and $\mathcal{G}_\mathcal{S}(\mathbf{t})$, we obtain that

$$\mathcal{H}_\mathcal{S}(\mathbf{t}).\mathcal{G}_\mathcal{S}(\mathbf{t}) = \left(\sum_{b\in\mathcal{S}}\mathbf{t}^b\right)\left(\sum_{b\in\mathbb{Z}^m}\mu_\mathcal{S}(b)\mathbf{t}^b\right) = \sum_{b\in\mathbb{Z}^m}\left(\sum_{c\in\mathcal{S}}\mu_\mathcal{S}(b-c)\right)\mathbf{t}^b.$$

The result follows by Proposition 2.1. □

Therefore, whenever we can explicitly compute the inverse of $\mathcal{H}_\mathcal{S}(\mathbf{t})$ we obtain $\mu_\mathcal{S}$. This is the case of Corollary 3.4 that explains how to obtain $\mu_\mathcal{S}$ when the Hilbert series $\mathcal{H}_\mathcal{S}(t)$ has a certain form.

Let $B = (b_1, b_2, \ldots, b_k)$ be a k-tuple of nonzero vectors of $\mathbb{Z}^m$ such that the semigroup $\mathcal{T} := \langle b_1,\ldots,b_k\rangle$ is pointed and take $b\in\mathbb{Z}^m$. We denote by $d_B(b)$ the *denumerant*, the number of non-negative integer representations of b by $b_1,\ldots,b_k$, that is, the number of solutions of the form $b=\sum_{i=1}^k x_i b_i$, where x_i is a nonnegative integer for all i. As $\mathcal{T}$ is pointed, then $d_B(b)$ is finite for all $b\in\mathbb{Z}^m$ and $d_B(0)=1$. It is well known (see, e.g., [13, Theorem 5.8.15]) that its generating function is given by

$$\sum_{b\in\mathbb{Z}^m} d_B(b)\,\mathbf{t}^b = \frac{1}{(1-\mathbf{t}^{b_1})(1-\mathbf{t}^{b_2})\cdots(1-\mathbf{t}^{b_k})}.$$

Corollary 3.4. *Let $\mathcal{S}$ be a pointed semigroup such that its Hilbert series is of the form*

$$\mathcal{H}_\mathcal{S}(\mathbf{t}) = \frac{(1-\mathbf{t}^{b_1})(1-\mathbf{t}^{b_2})\cdots(1-\mathbf{t}^{b_k})}{\sum_{i=1}^r c_i\,\mathbf{t}^{d_i}},$$

where $b_1,\ldots,b_k\in\mathbb{Z}^m$ are nonzero, $d_1,\ldots,d_r\in\mathbb{Z}^m$ and $c_1,\ldots,c_r$ are integers. Then,

$$\mu_\mathcal{S}(b) = \sum_{i=1}^r c_i.d_B(b-d_i),$$

for all $b\in\mathbb{Z}^m$, where $B=(b_1,\ldots,b_k)$.

Proof. From Theorem 3.3 and the definition of the denumerant, we have

$$\begin{aligned}\mathcal{G}_\mathcal{S}(\mathbf{t}) &= \frac{1}{\mathcal{H}_\mathcal{S}(\mathbf{t})} = \frac{\sum_{i=1}^r c_i\,\mathbf{t}^{d_i}}{(1-\mathbf{t}^{b_1})(1-\mathbf{t}^{b_2})\cdots(1-\mathbf{t}^{b_k})} = \left(\sum_{i=1}^r c_i\,\mathbf{t}^{d_i}\right)\sum_{b\in\mathbb{Z}^m} d_A(b)\mathbf{t}^b \\ &= c_1\sum_{b\in\mathbb{Z}^m} d_A(b)\,\mathbf{t}^{m+d_1} + \cdots + c_r\sum_{b\in\mathbb{Z}^m} d_A(b)\,\mathbf{t}^{b+d_r} \\ &= c_1\sum_{b\in\mathbb{Z}^m} d_A(b-d_1)\,\mathbf{t}^b + \cdots + c_r\sum_{b\in\mathbb{Z}^m} d_A(b-d_r)\,\mathbf{t}^b \\ &= \sum_{b\in\mathbb{Z}^m}\left(\sum_{i=1}^r c_i.d_A(b-d_i)\right)\mathbf{t}^b.\end{aligned}$$

This concludes the proof. □

This result will be particularly important in the following section when trying to obtain explicit formulas for the Möbius function $\mu_{\mathcal{S}}$ when $\mathcal{S}$ is a complete intersection semigroup. Recall that a pointed semigroup $\mathcal{S} = \langle a_1, \ldots, a_n \rangle$ is a complete intersection semigroup if its corresponding toric ideal $I_{\mathcal{S}}$ is a complete intersection. Moreover, $I_{\mathcal{S}}$ is a complete intersection if it is generated by $n - d$ $\mathcal{S}$-homogeneous polynomials, where d is the dimension of the $\mathbb{Q}$-vector space spanned by $a_1, \ldots, a_n$. Whenever $I_{\mathcal{S}}$ is a complete intersection generated $\mathcal{S}$-homogeneous polynomials of $\mathcal{S}$-degrees $b_1, \ldots, b_{n-d} \in \mathbb{Z}^m$, by [13, Page 341], we have that

$$\mathcal{H}_{\mathcal{S}}(\mathbf{t}) = \frac{(1 - \mathbf{t}^{b_1}) \cdots (1 - \mathbf{t}^{b_{n-d}})}{(1 - \mathbf{t}^{a_1}) \cdots (1 - \mathbf{t}^{a_n})}. \tag{4}$$

And so, we will be able to use Corollary 3.4.

For characterizations of complete intersection toric ideals we refer the reader to [2, 7].

4. Explicit formulas for the Möbius function

This section is devoted to obtain explicit formulas for the Möbius function $\mu_{\mathcal{S}}$ when $\mathcal{S}$ belongs to some families of subsemigroups of $\mathbb{Z}^m$. As far as we are aware, the only known results concerning $\mu_{\mathcal{S}}$ is an old theorem due to Deddens [5] that determines the values of $\mu_{\mathcal{S}}$ when $\mathcal{S} = \langle a, b \rangle \subset \mathbb{Z}^+$, and a recent paper due to Chappelon and Ramírez Alfonsín [4] where the authors investigate $\mu_{\mathcal{S}}$ when $\mathcal{S} = \langle a, a+d, \ldots, a+kd \rangle \subset \mathbb{Z}^+$ with $a, k, d \in \mathbb{Z}^+$ and obtain a semi-explicit formula when a is even and $k = 2$. In both papers the authors approach the problem by a thorough study of the intrinsic properties of each semigroup. Here we will first provide in Theorem 4.1 an explicit formula for $\mu_{\mathcal{S}}$ when $\mathcal{S} = \langle e_1, \ldots, e_m \rangle \subset \mathbb{N}^m$; this formula will be of interest in the last section. After that, we focus on two families of semigroups, namely, the so called semigroups with a unique Betti element and complete intersection numerical semigroups generated by three elements and provide formulas for the Möbius function of these semigroups in Theorems 4.2 and 4.5 respectively .

These families generalize the two mentioned above, indeed, if $\mathcal{S} = \langle a, b \rangle \subset \mathbb{Z}^+$ then $\mathcal{S}$ is a semigroup with a unique Betti element, and if $\mathcal{S} = \langle a, a+d, a+2d \rangle$ with a even and $\gcd\{a, d\} = 1$ then $\mathcal{S}$ is a complete intersection (see [15, Theorem 3.5]).

The results included in this section arise as consequences of Theorem 3.3 (for completeness, we also give a second proof of Theorem 4.5 by means of Theorem 3.2 as an appendix).

4.1. The semigroup $\mathbb{N}^m$.

Let $\{e_1, \ldots, e_m\}$ denote the canonical basis of $\mathbb{N}^m$. For $\mathcal{S} = \langle e_1, \ldots, e_m \rangle = \mathbb{N}^m$ the following result holds.

Theorem 4.1. *For $\mathcal{S} = \mathbb{N}^m$, the Möbius function is given by*

$$\mu_{\mathbb{N}^m}(x) = \begin{cases} (-1)^{|A|} & \textit{if } x = \sum_{i \in A} e_i \textit{ for some } A \subset \{1, \ldots, m\}, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. We observe that

$$\mathcal{H}_{\mathbb{N}^m}(\mathbf{t}) = \sum_{b \in \mathbb{N}^m} \mathbf{t}^b = \frac{1}{(1 - t_1) \cdots (1 - t_m)}.$$

Therefore, by Theorem 3.3 we derive that

$$\mathcal{G}_{\mathbb{N}^m}(\mathbf{t}) = (1-t_1)\cdots(1-t_m) = \sum_{A\subset\{1,\ldots,m\}} (-1)^{|A|} \prod_{i\in A} t_i = \sum_{A\subset\{1,\ldots,m\}} (-1)^{|A|}\, \mathbf{t}^{\sum_{i\in A} e_i},$$

and the result follows at once. □

4.2. **Semigroups with a unique Betti element.**

A semigroup $\mathcal{S} \subset \mathbb{N}^m$ is said to be a semigroup with a *unique Betti element* $b \in \mathbb{N}^m$ if its corresponding toric ideal is generated by a set of $\mathcal{S}$-homogeneous polynomials of $\mathcal{S}$-degree b. These semigroups were studied in detail by García-Sánchez, Ojeda and Rosales in [8]. In particular they prove in [8, Corollary 10] that these are complete intersection semigroups.

Theorem 4.2. *Let $\mathcal{S} = \langle a_1,\ldots,a_n\rangle \subset \mathbb{N}^m$ be a semigroup with a unique Betti element $b \in \mathbb{N}^m$ and denote by d the dimension of the $\mathbb{Q}$-vector space generated by $a_1,\ldots,a_n$. Then,*

$$\mu_{\mathcal{S}}(x) = \sum_{j=1}^{t} (-1)^{|A_j|} \binom{k_{A_j}+n-d-1}{k_{A_j}},$$

where $\{A_1,\ldots,A_t\} = \{A \subset \{1,\ldots,n\} \mid \exists\, k_A \in \mathbb{N}$ such that $x - \sum_{i\in A} a_i = k_A\, b\}$.

Proof. By (4) we have that

$$\mathcal{H}_{\mathcal{S}}(\mathbf{t}) = \frac{(1-\mathbf{t}^b)^{n-d}}{\prod_{i=1}^n (1-\mathbf{t}^{a_i})} = \frac{(1-\mathbf{t}^b)^{n-d}}{\sum_{A\subset\{1,\ldots,m\}} (-1)^{|A|}\, \mathbf{t}^{\sum_{i\in A} a_i}}.$$

Thus, applying Corollary 3.4, we have that for all $x \in \mathbb{Z}^m$

$$\mu_{\mathcal{S}}(x) = \sum_{A\subset\{1,\ldots,m\}} (-1)^{|A|}\, d_B\left(x - \sum_{i\in A} a_i\right),$$

where B is the $(n-d)$-uple $(b,\ldots,b)$. The equality

$$d_B(y) = \begin{cases} \binom{k+n-d-1}{k} & \textit{if } y = kb \textit{ with } k\in\mathbb{N}, \\ 0 & \textit{otherwise}, \end{cases}$$

for all $y \in \mathbb{Z}^m$, completes the proof. □

In the particular case when $\mathcal{S} = \langle a_1,\ldots,a_n\rangle \subset \mathbb{N}$ is a numerical semigroup with a unique Betti element $b \in \mathbb{N}$, it is proved in [8] that there exist pairwise relatively prime different integers $b_1,\ldots,b_n \geqslant 2$ such that $a_i := \prod_{j\neq i} b_j$, for all $i \in \{1,\ldots,n\}$, and $b = \prod_{i=1}^n b_i$. In this case Theorem 4.2 can be refined as follows.

Corollary 4.3. *Let $\mathcal{S} = \langle a_1,\ldots,a_n\rangle \subset \mathbb{N}$ be a numerical semigroup with a unique Betti element $b \in \mathbb{N}$. Then,*

$$\mu_{\mathcal{S}}(x) = \begin{cases} (-1)^{|A|}\binom{k+n-2}{k} & \textit{if } x = \sum_{i\in A} a_i + kb \textit{ for some } A \subset \{1,\ldots,n\}, k\in\mathbb{N}, \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. As $d = 1$, by Theorem 4.2 it only suffices to prove that for every $A_1, A_2 \subset \{1,\ldots,n\}$, if b divides $\sum_{i\in A_1} a_i - \sum_{i\in A_2} a_i$, then $A_1 = A_2$. For we use [12, Theorem 2.9 and Remark 2.10] (see also [8, Example 12]), which asserts that if we take $b_1,\ldots,b_n \geqslant 2$ such that $a_i = \prod_{j\neq i} b_j$, then the toric ideal associated to $\mathcal{S}$ is $I_{\mathcal{S}} = (f_2,\ldots,f_n)$, where $f_i := x_1^{b_1} - x_i^{b_i}$ for all $i \in \{2,\ldots,n\}$. Assume that there exist $A_1, A_2 \subset \{1,\ldots,n\}$,

$A_1 \neq A_2$ and that $\sum_{i\in A_1} a_i - \sum_{i\in A_2} a_i = kb$ for some $k \in \mathbb{N}$. Thus, the binomial $g := \prod_{i\in A_1} x_i - x_1^{b_1 k} \prod_{i\in A_2} x_i \neq 0$ belongs to $I_{\mathcal{S}}$ and so it can be written as a combination of $f_2, \ldots, f_n$. However, this is a contradiction because $x_i^{b_i}$ does not divide $\prod_{j\in A_1} x_j$ for all $i \in \{1, \ldots, n\}$. □

As a direct consequence of this result we recover Dedden's classical result.

Corollary 4.4. *Let $a, b \in \mathbb{Z}^+$ be relatively prime integers and consider $\mathcal{S} := \langle a, b\rangle$. Then,*

$$\mu_{\mathcal{S}}(x) = \begin{cases} 1 & \text{if } x \geqslant 0 \text{ and } x \equiv 0 \text{ or } a+b \pmod{ab}, \\ -1 & \text{if } x \geqslant 0 \text{ and } x \equiv a \text{ or } b \ (mod\ ab), \\ 0 & \text{otherwise.} \end{cases}$$

4.3. **Three generated complete intersection numerical semigroups.**

We shall study the case when $\mathcal{S}$ is a complete intersection numerical semigroup minimally generated by the set $\{a_1, a_2, a_3\}$. In this setting we aim at providing a semi-explicit formula for $\mu_{\mathcal{S}}$. This objective is achieved with Theorem 4.5. We include two proofs of Theorem 4.5. In the first one, we prove the theorem as a consequence of Theorem 3.3. In the second one we prove the same result by means of Theorem 3.2, for a sake of brevity we include this second proof as an appendix.

Many authors have studied three generated numerical semigroups and one can find several (equivalent) characterizations of the complete intersection property for them. In particular, Herzog proved in [10] that $\mathcal{S}$ is a complete intersection if and only if $\gcd\{a_i, a_j\}\, a_k \in \langle a_i, a_j\rangle$ with $\{i, j, k\} = \{1, 2, 3\}$; from now on we assume that $\gcd\{a_2, a_3\}\, a_1 \in \langle a_2, a_3\rangle$ and denote $d := \gcd\{a_2, a_3\}$. Note that $d \geqslant 2$, otherwise $a_1 \in \langle a_2, a_3\rangle$, which contradicts the minimality of $\{a_1, a_2, a_3\}$. In [10], the author also proved that if we take $\gamma_2, \gamma_3 \in \mathbb{N}$ such that $da_1 = \gamma_2 a_2 + \gamma_3 a_3$, then

$$I_{\mathcal{S}} = (x_1^d - x_2^{\gamma_2} x_3^{\gamma_3},\ x_2^{a_3/d} - x_3^{a_2/d}), \tag{5}$$

whose $\mathcal{S}$-degrees are da_1 and a_2a_3/d, respectively.

For every $x \in \mathbb{Z}$ there exists a unique $\alpha(x) \in \{0, \ldots, d-1\}$ such that $\alpha(x)a_1 \equiv x \pmod{d}$. It is easy to check that for every $x, y \in \mathbb{Z}$,

$$\alpha(x-y) = \begin{cases} \alpha(x) - \alpha(y) & \text{if } \alpha(x) \geqslant \alpha(y), \\ \\ d + \alpha(x) - \alpha(y) & \text{otherwise}. \end{cases} \tag{6}$$

Theorem 4.5. *Let $\mathcal{S} = \langle a_1, a_2, a_3\rangle$ be a numerical semigroup such that $da_1 \in \langle a_2, a_3\rangle$ where $d := \gcd\{a_2, a_3\}$. For all $x \in \mathbb{Z}$, we have*

$$\mu_{\mathcal{S}}(x) = 0$$

if $\alpha(x) \geqslant 2$, or

$$\mu_{\mathcal{S}}(x) = (-1)^{\alpha} \left(d_B(x') - d_B(x' - a_2) - d_B(x' - a_3) + d_B(x' - a_2 - a_3)\right)$$

otherwise, where $x' := x - \alpha(x)a_1$ and $B := (da_1, a_2\, a_3/d)$.

First proof. By (5) $I_{\mathcal{S}}$ is generated by two $\mathcal{S}$-homogeneous polynomials whose $\mathcal{S}$-degrees are da_1 and a_2a_3/d. Thus, applying (4) we obtain

$$\begin{aligned} \mathcal{H}_{\mathcal{S}}(t) &= \frac{\left(1 - t^{da_1}\right)\left(1 - t^{a_2a_3/d}\right)}{(1 - t^{a_1})(1 - t^{a_2})(1 - t^{a_3})} \\ &= \frac{\left(1 - t^{da_1}\right)\left(1 - t^{a_2a_3/d}\right)}{1 - t^{a_1} - t^{a_2} - t^{a_3} + t^{a_1+a_2} + t^{a_1+a_3} + t^{a_2+a_3} - t^{a_1+a_2+a_3}}. \end{aligned}$$

Hence, from Corollary 3.4, we have

$$\mu_{\mathcal{S}}(x) = \begin{array}{l} d_B(x) - d_B(x-a_1) - d_B(x-a_2) - d_B(x-a_3) + d_B(x-(a_1+a_2)) + \\ + d_B(x-(a_1+a_3)) + d_B(x-(a_2+a_3)) - d_B(x-(a_1+a_2+a_3)), \end{array} \tag{7}$$

for all integers x, where $B := (da_1, a_2a_3/d)$. Since $\alpha(da_1) = \alpha(a_2a_3/d) = 0$, it follows that $\alpha(y) = 0$ if $y \in \langle da_1, a_2a_3/d\rangle$. As a consequence of this, $d_B(y) = 0$ whenever $\alpha(y) \neq 0$.

We denote $C := \{0, a_1, a_2, a_3, a_1+a_2, a_2+a_3, a_3+a_1, a_1+a_2+a_3\}$ and observe that $\alpha(y) \in \{0,1\}$ for all $y \in C$.

We distinguish three different cases upon the value of $\alpha := \alpha(x)$, for $x \in \mathbb{Z}$.

Case 1. $\alpha \geqslant 2$.
We deduce that $\alpha(x-y) = \alpha(x) - \alpha(y) \neq 0$ and $d_B(x-y) = 0$, for all $y \in C$. Therefore, using (7), $\mu_{\mathcal{S}}(x) = 0$.

Case 2. $\alpha = 1$.
We deduce that $\alpha(x-y) \neq 0$ and $d_B(x-y) = 0$ for all $y \in \{0, a_2, a_3, a_2+a_3\}$. Therefore, using (7), we obtain that

$$\mu_{\mathcal{S}}(x) = -d_B(x-a_1) + d_B(x-a_1-a_2) + d_B(x-a_1-a_3) - d_B(x-a_1-a_2-a_3).$$

Case 3. $\alpha = 0$.
Since $d \geqslant 2$, we deduce that $\alpha(x-y) \neq 0$ and $d_B(x-y) = 0$ for all $y \in \{a_1, a_1+a_2, a_1+a_3, a_1+a_2+a_3\}$. Therefore, using (7), we obtain that

$$\mu_{\mathcal{S}}(x) = d_B(x) - d_B(x-a_2) - d_B(x-a_3) + d_B(x-a_2-a_3).$$

This completes the proof. □

Theorem 4.5 yields an algorithm for computing $\mu_{\mathcal{S}}(x)$ for all $x \in \mathbb{Z}$ which relies on the computation of four denumerants of the form $d_B(y)$, where $B = (da_1, a_2a_3/d)$. It is worth to mention that in [16, Section 4.4] there are several results and methods to compute this type of denumerants.

Also note that Theorem 4.5 generalizes [4, Theorem 3], where the authors provide a semi-explicit formula for $\mathcal{S} = \langle 2q, 2q+e, 2q+2e\rangle$ where $q, e \in \mathbb{Z}^+$ and $\gcd\{2q, 2q+e, 2q+2e\} = 1$. It is easy to see that these semigroups are complete intersection, if suffices to set $a_1 := 2q+e$, $a_2 := 2q$, $a_3 := 2q+2e$ and observe that $\gcd\{a_2, a_3\}a_1 = 2a_1 = a_2 + a_3 \in \langle a_2, a_3\rangle$. For some other families of complete intersection numerical semigroups we refer the reader to [1, 18].

5. When is a poset equivalent to a semigroup poset?

Let $D = \{d_1, \dots, d_m\}$ be a finite set and let us consider the (locally finite) poset $\mathcal{P}$ of the multisets over D ordered by inclusion. For every $S, T \in \mathcal{P}$ such that $T \subset S$, it is well known that

$$\mu_{\mathcal{P}}(T,S) = \begin{cases} (-1)^{|T\setminus S|} & \textit{if } T \subset S \textit{ and } T \setminus S \textit{ is a set}, \\ 0 & \textit{otherwise}. \end{cases} \tag{8}$$

We consider the semigroup $\mathbb{N}^m = \langle e_1, \dots, e_m\rangle$ and the map $\psi : \mathcal{P} \to \mathbb{N}^m$ defined as $\psi(S) = (s_1, \dots, s_m)$ where s_i denotes the multiplicity of d_i in S for all $S \in \mathcal{P}$. If we consider the order in $\mathbb{N}^m$ induced by the semigroup $\mathbb{N}^m$ we have that $\alpha \leqslant_{\mathbb{N}^m} \beta \Longleftrightarrow \alpha_i \leqslant \beta_i$ for all $i \in \{1, \dots, n\}$ and ψ is an *order isomorphism*, i.e., an order preserving and order reflecting bijection. Thus, we can say that the poset of multisets of a finite set is a particular case of semigroup poset. This implies in particular that for all $S, T \in \mathcal{P}$ such that $T \subset S$, then $\mu_{\mathcal{P}}(T,S) = \mu_{\mathbb{N}^m}(\psi(T), \psi(S)) = \mu_{\mathbb{N}^m}(\psi(S) - \psi(T))$ and, by Theorem 4.1 we retrieve the formula (8).

It is worth pointing out that we can also use the tools of the previous sections to recover the arithmetic Möbius function, i.e., the Möbius function for the poset of integers ordered by divisibility. Recall that for all $a, b \in \mathbb{N}$ such that $a \mid b$, then

$$\mu(a,b) = \begin{cases} (-1)^r & \textit{if } b/a \textit{ is a product of } r \textit{ different prime numbers}, \\ 0 & \textit{otherwise}. \end{cases} \tag{9}$$

Indeed, for every $m \in \mathbb{Z}^+$, if we denote by $p_1, \ldots, p_m$ the first m prime numbers, then one can define an order isomorphism between $(\mathbb{N}^m, \leqslant_{\mathbb{N}^m})$ and the poset of nonnegative integers that can be written as a product of powers of $p_1, \ldots, p_m$, which we denote by $\mathbb{N}_m$ ordered by divisibility. Hence for every $a, b \in \mathbb{N}$ we take $m \in \mathbb{Z}^+$ such that $a, b \in \mathbb{N}_m$ and, hence, we can recover the formula (9) by means of the Möbius function of $\mathbb{N}^m$ given in Theorem 4.1.

In these two examples we have easily found an ad-hoc order isomorphism between the poset and the semigroup poset $(\mathbb{N}^m, \leqslant_{\mathbb{N}^m})$ which allows us to compute the Möbius function. In this section, we provide tools to do this systematically.

Let $(\mathcal{P}, \leqslant)$ be a locally finite poset. For every $x \in \mathcal{P}$ we denote by $\mathcal{P}_x := \{y \in \mathcal{P} \mid x \leqslant y\}$. We aim at studying when the restriction of the Möbius function $\mu(-, x) : \mathcal{P} \to \mathbb{Z}$ can be computed by means of the Möbius function of a pointed semigroup $\mathcal{S} \subset \mathbb{Z}^m$. Firstly, we observe that $\mu(-, x)$ is 0 for every y such that $x \not\leq y$, then it makes sense to study $\mu(-, x) : \mathcal{P}_x \to \mathbb{Z}$. Of course $\mu(-, x)$ can be studied by means of the Möbius function of a semigroup $\mathcal{S}$ if there exists an order isomorphism $\psi : (\mathcal{P}_x, \leqslant) \longrightarrow (\mathcal{S}, \leqslant_{\mathcal{S}})$. In this section we will characterize in Theorem 5.3 when there exists such an isomorphism in terms of the poset $\mathcal{P}_x$.

The poset $\mathcal{P}_x$ is said to be *autoequivalent* if and only if for all $y \in \mathcal{P}_x$ there exists an order isomorphism $g_y : \mathcal{P}_x \longrightarrow \mathcal{P}_y$ such that $g_y \circ g_z = g_z \circ g_y$ for all $y, z \in \mathcal{P}_x$ and g_x is the identity. For all $y \in \mathcal{P}_x$ we denote by $l_1(y) := \{z \in \mathcal{P} \mid \text{it does not exist } u \in \mathcal{P} \text{ such that } y \lneq u \lneq z\}$. Whenever $\mathcal{P}_x$ is autoequivalent with isomorphisms $\{g_y\}_{x \leqslant y}$ and $l_1(x)$ is a finite set of n elements, we associate to $\mathcal{P}$ a subgroup $L_{\mathcal{P}} \subset \mathbb{Z}^n$ in the following way.

Denote $l_1(x) = \{x_1, \ldots, x_n\} \subset \mathcal{P}$ and consider the map

$$f : \mathbb{N}^n \longrightarrow \mathcal{P}$$

defined as $f(0, \ldots, 0) = x$, and for all $\alpha \in \mathbb{N}^n$ and all $i \in \{1, \ldots, n\}$, then $f(\alpha + e_i) = g_{x_i}(f(\alpha))$, where $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{N}^m$. In particular, $f(e_i) = g_{x_i}(f(0)) = g_{x_i}(x) = x_i$ for all $i \in \{1, \ldots, n\}$.

Lemma 5.1. *f is well defined and is surjective.*

Proof. Suppose that $\alpha + e_i = \beta + e_j$, then we set $\gamma := \alpha - e_j = \beta - e_i \in \mathbb{N}^n$. Thus, $f(\alpha + e_i) = g_{x_i}(f(\alpha)) = g_{x_i}(g_{x_j}(f(\gamma))) = g_{x_j}(g_{x_i}(f(\gamma))) = g_{x_j}(f(\beta)) = f(\beta + e_j)$ and f is well defined.

Take $y \in \mathcal{P}_x$. If $y = x$, then $y = f(0)$. If $y \neq x$, then there exists $z \in \mathcal{P}_x$ such that $y \in l_1(z)$; therefore $y = g_z(x_j)$ for some $j \in \{1, \ldots, n\}$. We claim that if $z = f(\alpha)$, then $y = f(\alpha + e_j)$. Indeed, $f(\alpha + e_j) = g_{x_j}(f(\alpha)) = g_{x_j}(z) = g_{x_j}(g_z(x)) = g_z(g_{x_j}(x)) = g_z(x_j) = y$. □

Now we set $L_{\mathcal{P}} := \{\alpha - \beta \in \mathbb{Z}^n \mid f(\alpha) = f(\beta)\}$.

Lemma 5.2. *$L_{\mathcal{P}}$ is a subgroup of $\mathbb{Z}^n$.*

Proof. If $\gamma \in L_{\mathcal{P}}$, then $-\gamma \in L_{\mathcal{P}}$ clearly. Moreover, if $\gamma_1, \gamma_2 \in L_{\mathcal{P}}$, then $\gamma_1 + \gamma_2 \in L_{\mathcal{P}}$. Indeed, take $\alpha, \alpha', \beta, \beta' \in \mathbb{N}^m$ such that $f(\alpha) = f(\alpha')$, $\gamma_1 = \alpha - \alpha'$, $f(\beta) = f(\beta')$ and $\gamma_2 = \beta - \beta'$. Then $f(\alpha + \beta) = f(\alpha' + \beta) = f(\alpha' + \beta')$ and we are done. □

For every subgroup $L \subset \mathbb{Z}^n$ we define the *saturation* of L as the group

$$\mathrm{Sat}(L) := \{\gamma \in \mathbb{Z}^n \mid \text{ exists } d \in \mathbb{Z}^+ \text{ such that } d\gamma \in L\}.$$

Theorem 5.3. *Let $\mathcal{P}$ be a locally finite poset and $x \in \mathcal{P}$. Then, $(\mathcal{P}_x, \leqslant)$ is isomorphic to $(\mathcal{S}, \leqslant_{\mathcal{S}})$ for some (pointed) semigroup $\mathcal{S} \subset \mathbb{Z}^n \iff \mathcal{P}_x$ is autoequivalent, $l_1(x)$ is finite and $L_{\mathcal{P}} = \mathrm{Sat}(L_{\mathcal{P}})$.*

Proof. ($\Rightarrow$) Let $\mathcal{S} \subset \mathbb{Z}^m$ be a (pointed) semigroup and denote by $\{a_1, \ldots, a_n\}$ its unique minimal set of generators. Assume that $\psi : \mathcal{P}_x \to \mathcal{S}$ is an order isomorphism; let us prove that $\mathcal{P}_x$ is autoequivalent, that $l_1(x) = n$ and that $L_{\mathcal{P}} = \mathrm{Sat}(L_{\mathcal{P}})$. Firstly, we observe that setting $x_i := \psi^{-1}(a_i)$, then $l_1(x) = \{x_1, \ldots, x_n\}$. Now, for every $y \in \mathcal{P}_x$ we set

$$\begin{array}{rccl} g_y : & \mathcal{P}_x & \longrightarrow & \mathcal{P}_y \\ & z & \longmapsto & \psi^{-1}(\psi(z) + \psi(y)), \end{array}$$

then it is straightforward to check that g_y is an order isomorphism. Moreover, g_x is the identity map in $\mathcal{P}_x$ and $g_y \circ g_z = g_z \circ g_y$ for all $y, z \in \mathcal{P}_x$.

Now we take $f : \mathbb{N}^m \to \mathcal{P}_x$ the map associated to $\{g_y\}_{y \leqslant x}$, i.e., $f(0) = x$ and if $f(\alpha) = y$, then $f(\alpha + e_j) = g_{x_j}(f(\alpha))$. We claim that $\psi(f(\alpha)) = \sum \alpha_i a_i \in \mathcal{S}$ for all $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. Indeed, $\psi(f(0)) = \psi(x) = 0$ and if we assume that $\psi(f(\alpha)) = \sum \alpha_i a_i$ for some $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^m$, then $\psi(f(\alpha + e_j)) = \psi(g_{x_j}(\alpha)) = \psi(z) + \psi(x_j) = \sum \alpha_i a_i + a_j$, as desired.

Since $L_{\mathcal{P}} \subset \mathrm{Sat}(L_{\mathcal{P}})$ by definition, let us prove that $\mathrm{Sat}(L_{\mathcal{P}}) \subset L_{\mathcal{P}}$. We take $\gamma \in \mathrm{Sat}(L_{\mathcal{P}})$, then $d\gamma \in L_{\mathcal{P}}$ for some $d \in \mathbb{Z}^+$. This means that there exist $\alpha, \beta \in \mathbb{N}^n$ such that $f(\alpha) = f(\beta)$ and $d\gamma = \alpha - \beta$. Hence we have that $\sum \alpha_i a_i = \psi(f(\alpha)) = \psi(f(\beta)) = \sum \beta_i a_i$; which implies that $\sum \gamma_i a_i = 1/d \left(\sum (\alpha_i - \beta_i) a_i\right) = 0$. Thus, if we take $\alpha', \beta' \in \mathbb{N}^m$ such that $\gamma = \alpha' - \beta'$, then $\psi(f(\alpha')) = \psi(f(\beta'))$ and, whence, $f(\alpha') = f(\beta')$ and $\gamma \in L_{\mathcal{P}}$.

($\Leftarrow$) Since $L_{\mathcal{P}} = \mathrm{Sat}(L_{\mathcal{P}})$, we have that $\mathbb{Z}^n / L_{\mathcal{P}}$ is a torsion free group; hence there exists a group isomorphism $\rho : \mathbb{Z}^n / L_{\mathcal{P}} \to \mathbb{Z}^m$, where $m = n - \mathrm{rk}(L_{\mathcal{P}})$. We denote $a_i := \rho(e_i + L_{\mathcal{P}})$ for all $i \in \{1, \ldots, n\}$ and set $\mathcal{S} := \langle a_1, \ldots, a_n \rangle \subset \mathbb{Z}^m$. We claim that $(\mathcal{P}_x, \leqslant)$ and $(\mathcal{S}, \leqslant_{\mathcal{S}})$ are isomorphic. More precisely, it is straightforward to check that the map

$$\begin{array}{rccl} \psi : & \mathcal{P}_x & \longrightarrow & \mathcal{S} \\ & y & \longmapsto & \sum \alpha_i a_i, \; if \; f(\alpha) = y \end{array}$$

is an order isomorphism. □

In algebraic terms, the idea under ($\Leftarrow$) in Theorem 5.3 is that whenever $\mathcal{P}_x$ is autoequivalent and $l_1(x)$ is finite, the subgroup $L_{\mathcal{P}}$ defines a lattice ideal I. Moreover, $\mathcal{P}_x$ is isomorphic to a semigroup poset $(\mathcal{S}, \leqslant_{\mathcal{S}})$ if and only if the ideal I itself is the toric ideal of a semigroup $\mathcal{S}$, but this happens if and only if I is prime or, equivalently, if $L_{\mathcal{P}} = \mathrm{Sat}(L_{\mathcal{P}})$ (see, e.g., [6] or [11]).

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Appendix: Second proof of Theorem 4.5

In this appendix we give a second proof of Theorem 4.5 by using Theorem 3.2.

Proposition 5.4. *Let $\mathcal{S} = \langle a_1, a_2, a_3\rangle$ be a numerical semigroup such that $da_1 \in \langle a_2, a_3\rangle$ where $d := \gcd\{a_2, a_3\}$. Then,*
(a) $\mu_{\mathcal{S}}(x) = \mu_{\mathcal{S}}(x - da_1) + \mu_{\mathcal{S}}(x - a_2a_3/d) - \mu_{\mathcal{S}}(x - da_1 - a_2a_3/d)$ for all $x \in \mathbb{Z} \setminus C$, and

(b) $\displaystyle\sum_{i=0}^{d-1} \mu_{\mathcal{S}}(x - ia_1) = \sum_{i=0}^{d-1} \mu_{\mathcal{S}}(x - ia_1 - a_2a_3/d)$ for all $x \in \mathbb{Z} \setminus C'$,
where $C' = \{0, a_2, a_3, a_2 + a_3\}$ and $C = C' \cup \{0, a_1, a_1 + a_2, a_1 + a_3, a_1 + a_2 + a_3\}$.

Proof. By (5), $I_{\mathcal{S}}$ is generated by two $\mathcal{S}$-homogeneous polynomials whose $\mathcal{S}$-degrees are da_1 and a_2a_3/d. Applying (4) we get that

$$\mathcal{H}_{\mathcal{S}}(t) = \frac{(1 - t^{da_1})(1 - t^{a_2a_3/d})}{(1 - t^{a_1})(1 - t^{a_2})(1 - t^{a_3})} = \frac{1 - t^{da_1} - t^{a_2a_3/d} + t^{da_1+a_2a_3/d}}{(1 - t^{a_1})(1 - t^{a_2})(1 - t^{a_3})}.$$

A direct application of Theorem 3.2 proves (a). To prove (b) it suffices to observe that $(1 - t^{da_1})/(1 - t^{a_1}) = 1 + t + \cdots + t^{d-1}$, giving

$$\mathcal{H}_{\mathcal{S}}(t) = \frac{(1 - t^{da_1})(1 - t^{a_2a_3/d})}{(1 - t^{a_1})(1 - t^{a_2})(1 - t^{a_3})} = \frac{\sum_{i=0}^{d-1}(t^i - t^{i+(a_2a_3/d)})}{(1 - t^{a_2})(1 - t^{a_3})},$$

and apply again Theorem 3.2. □

Recall that for every $x \in \mathbb{Z}$ we denote by $\alpha(x)$ the only integer in $\{0, \ldots, d - 1\}$ such that $\alpha(x)a_1 \equiv x \pmod d$.

Second proof of Theorem 4.5. We set $C' := \{0, a_2, a_3, a_2 + a_3\}$ and $C := C' \cup \{a_1, a_1 + a_2, a_1 + a_3, a_1 + a_2 + a_3\}$. We observe that for all $y \in C$, $\alpha(y) \in \{0, 1\}$. Moreover $\alpha(y) = 0$ if $y \in C'$ and $\alpha(y) = 1$ if $y \in C \setminus C'$. Let x be an integer, we distinguish three different cases upon the value of $\alpha := \alpha(x)$.

Case 1. $\alpha \geqslant 2$.
We observe that $\alpha(x - \lambda\,(da_1) - \delta\,(a_2a_3/d)) = \alpha(x) \geqslant 2$ for all $\lambda, \delta \in \mathbb{N}$, which implies that $x - \lambda\,(da_1) - \delta\,(a_2a_3/d) \notin C$. By iteratively applying Proposition 5.4 (a) we get that $\mu_{\mathcal{S}}(x) = 0$.

Case 2. $\alpha = 1$.
By Proposition 5.4 (b), we have that

$$\sum_{i=0}^{d-1} \mu_{\mathcal{S}}(y - ia_1) = \sum_{i=0}^{d-1} \mu_{\mathcal{S}}(y - ia_1 - a_2a_3/d)$$

for all $y \notin C'$. Moreover, for $i \geqslant 2$, by *Case 1* we have that $\mu_{\mathcal{S}}(x - ia_1) = \mu_{\mathcal{S}}(x - ia_1 - a_2a_3/d) = 0$. This gives that

$$\mu_{\mathcal{S}}(y) + \mu_{\mathcal{S}}(y - a_1) = \mu_{\mathcal{S}}(y - a_2a_3/d) + \mu_{\mathcal{S}}(y - a_1 - a_2a_3/d)$$

for all $y \notin C'$. We set $\sigma(y) := \mu_{\mathcal{S}}(y) + \mu_{\mathcal{S}}(y - a_1)$ and have that

$$\sigma(y) = \sigma(y - a_2a_3/d) \tag{10}$$

for all $y \notin C'$. Moreover, we observe that $\mu_{\mathcal{S}}(x - \lambda a_2a_3/d) = 1$ for every $\lambda \in \mathbb{N}$; hence, $x - \lambda a_2a_3/d \notin C'$. Applying (10) iteratively, we get that $\sigma(x) = 0$ and $\mu_{\mathcal{S}}(x) = -\mu_{\mathcal{S}}(x - a_1)$.

Case 3. $\alpha = 0$.
We denote $\tau(y) := \mu_{\mathcal{S}}(y) - \mu_{\mathcal{S}}(y - da_1)$, then by Proposition 5.4 (a) we have that

$$\tau(y) = \tau(y - a_2a_3/d)\ if\ y \notin C. \tag{11}$$

We take $\lambda \in \mathbb{N}$ the minimum integer such that $x - \lambda a_2a_3/d < 0$ or $x - \lambda a_2a_3/d \in C$; an iterative application of (11) yields $\tau(x) = \tau(x - \lambda da_1)$, hence $\mu_{\mathcal{S}}(x) = \mu_{\mathcal{S}}(x - da_1) + \tau(x - \lambda da_1)$. Let us compute $\tau(x - \lambda da_1)$. If $x' := x - \lambda da_1 < 0$, then $\tau(x') = 0$. Since $\alpha(x') = \alpha(x) = 0$, we have that $x' \in C'$. A direct computation yields $\tau(0) = 1$, $\tau(a_2) = \tau(a_3) = -1$, $\tau(a_2 + a_3) = \mu_{\mathcal{S}}(da_1) - \mu_{\mathcal{S}}(0) = 2 - 1 = 1$ if $a_2 + a_3 = da_1$ or $\tau(a_2 + a_3) = \mu_{\mathcal{S}}(a_2 + a_3) - \mu_{\mathcal{S}}(a_2 + a_3 - da_1) = 1 - 0 = 1$ if $a_2 + a_3 \neq da_1$. Hence,

$$\mu_{\mathcal{S}}(x) = \mu_{\mathcal{S}}(x - da_1) + \begin{cases} 1 & \text{if } x \equiv 0 \pmod{a_2a_3/d}, \\ -1 & \text{if } x \equiv a_2 \pmod{a_2a_3/d}, \\ -1 & \text{if } x \equiv a_3 \pmod{a_2a_3/d}, \\ 1 & \text{if } x \equiv a_2 + a_3 \pmod{a_2a_3/d}, \\ 0 & \text{otherwise.} \end{cases}$$

This formula gives $\mu_{\mathcal{S}}(x)$ in terms of $\mu_{\mathcal{S}}(x - da_1)$ and the residue of x modulo a_2a_3/d. We iterate this argument to compute $\mu_{\mathcal{S}}(x - ida_1)$ for all $i \geqslant 1$ such that $x - ida_1 \geqslant 0$. Finally note that for $B = (da_1, a_2a_3/d)$, then $d_B(y)$ is the number of $i \in \mathbb{N}$ such that $y - i(da_1) \geqslant 0$ and $y - i(da_1) \equiv 0 \pmod{a_2a_3/d}$, hence the result is proved. ∎

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