# A hybrid high-order locking-free method for linear elasticity on general meshes

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**Abstract**

We develop an arbitrary-order locking-free method for linear elasticity on general (polyhedral, possibly nonconforming) meshes without nodal unknowns. The key idea is to reconstruct the relevant differential operators in terms of the (generalized) degrees of freedom by solving an inexpensive local problem inside each element. The symmetric gradient and the divergence operators are reconstructed separately. The divergence operator satisfies a commuting diagram property, yielding robustness in the quasi-incompressible limit. Locking-free error estimates are derived for the energy norm and for the $L^2$-norm of the displacement, with optimal convergence rates for smooth solutions. The theoretical results are confirmed numerically, and the CPU cost is evaluated on both standard and general polygonal meshes.

**Keywords** Linear elasticity, general meshes, arbitrary order, locking-free methods

## 1 Introduction

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, denote a bounded connected polygonal domain. We consider the isotropic linear elasticity problem

$$\begin{aligned} -\nabla\cdot\underline{\underline{\sigma}} &= \underline{f} && \text{in } \Omega \\ \underline{\underline{\sigma}} &= 2\mu\nabla_{\mathrm{s}}\underline{u} + \lambda(\nabla\cdot\underline{u})\underline{\underline{I}}_d && \text{in } \Omega, \\ \underline{u} &= \underline{0} && \text{on } \partial\Omega, \end{aligned} \tag{1}$$

with $\mu > 0$ and $\lambda \geqslant 0$ scalar Lamé coefficients and $\nabla_{\mathrm{s}}$ denoting the symmetric part of the gradient operator (in short, the symmetric gradient) applied to vector-valued fields. We consider homogeneous Dirichlet boundary conditions on the displacement for simplicity. More general boundary conditions can be handled with straightforward modifications. For $X \subset \overline{\Omega}$, we denote by $(\cdot,\cdot)_X$ and $\|\cdot\|_X$ respectively the standard inner product and norm of $L^2(X)$, with the convention that the index is omitted if $X = \Omega$. A similar notation is used for $L^2(X)^d$

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and $L^2(X)^{d\times d}$. With $\underline{f} \in L^2(\Omega)^d$, the weak formulation of problem (1) consists in finding $\underline{u} \in \underline{U}_0 := H_0^1(\Omega)^d$ such that, for all $\underline{v} \in \underline{U}_0$,

$$(2\mu\nabla_{\rm s}\underline{u}, \nabla_{\rm s}\underline{v}) + (\lambda\nabla\cdot\underline{u}, \nabla\cdot\underline{v}) = (\underline{f}, \underline{v}). \tag{2}$$

It is known that the accurate approximation of problem (2) in the incompressible limit $\lambda \to +\infty$ requires the discrete space to be able to accurately represent nontrivial divergence-free vector-valued fields. When considering primal approximations where the displacement field is the sole unknown, this essentially amounts to having at hand a divergence operator that satisfies a suitable commuting diagram property. Combined with the regularity result on the displacement field and its divergence (see Eq. (50) below), the commuting diagram property allows one to prove error estimates uniform with respect to $\lambda$. In the work of Brenner and Sung [6], this property is obtained for the pure-displacement problem using the nonconforming element of Crouzeix and Raviart [7]. Therein, stability hinges on the use of the Navier–Cauchy formulation. When considering more general boundary conditions, one possibility to restore stability is to penalize the jumps of the discrete displacement field across interfaces in a least-squares fashion, as do Hansbo and Larson [16].

All of the above-mentioned approaches apply to standard simplicial or parallelepipedal meshes. In recent years, a large effort has been devoted to the development and analysis of discretization methods that apply to more general meshes possibly featuring polygonal or polyhedral elements and nonconforming interfaces. In this context, we can mention the lowest-order method of Beirão da Veiga, Gyrya, Lipnikov, and Manzini [4] and the work of Di Pietro and Lemaire [12], which introduces a generalization of the lowest-order Crouzeix–Raviart space and, correspondingly, of the methods of [6, 16]. Closer to the classical finite element spirit is the work of Tabarraei and Sukumar [22], where Lagrangian shape functions for convex polygonal elements are constructed. More recently, arbitrary-order methods on general polygonal or polyhedral meshes have received an increasing attention. We cite here, in particular, the work of Beirão da Veiga, Brezzi, and Marini [3], where the authors introduce a Virtual Element method for planar linear elasticity featuring vertex, edge, and element unknowns. The adjective virtual refers to the fact that the resulting discretization can be interpreted as a Finite Element method where the basis functions are not explicitly known. The incompressible limit is dealt with by introducing a suitable projection of the divergence.

In this work, we propose a different approach to designing an arbitrary-order method for quasi-incompressible linear elasticity on general meshes in space dimension $d \in \{2, 3\}$ based on the ideas of [10]. For a fixed polynomial degree $k \geqslant 1$, we select as degrees of freedom (DOFs) vector-valued polynomials at mesh elements and faces up to degree $k$. The associated interpolation (reduction) operator maps displacements to their moments up to degree $k$ at mesh elements and faces. The definition of the method then proceeds in three steps: (i) we reconstruct a discrete symmetric gradient operator of order $k$. This requires the solution of an inexpensive local pure-traction problem inside each element; (ii) we reconstruct a discrete divergence operator that satisfies by construction the commuting diagram property (this is the main purpose of reconstructing the divergence separately from the symmetric gradient); (iii) we devise a least-squares stabilization that weakly enforces the matching of element- and face-based DOFs; this stabilization remains local to one element (unlike that of [12]) and at the same time ensures coercivity while providing optimal approximation properties for smooth solutions. An important feature of the proposed method is the absence of nodal unknowns, which yields a compact stencil (especially in three dimensions) and simplifies data exchange in parallel implementations. This, together with the fact that the stabilization is local to one element, classically allows for an efficient implementation where the global problem is solved in terms of face unknowns only. Another salient feature of the method is the simultaneous use of element- and face-based DOFs, which are connected through the local reconstructions of differential operators and the stabilization term. More precisely, element-based DOFs can be regarded as intermediate variables used in the local reconstruction problems which do not appear (after static condensation) in the global system. Face-based DOFs establish inter-element connections at interfaces and can be used to strongly enforce essential boundary conditions at boundary faces.

The paper is organized as follows. In Section 2, we recall the definition of admissible mesh sequences in the spirit of [9] as well as some useful results, and we define the interpolation operator. In Section 3, we introduce the symmetric gradient and divergence reconstruction operators mapping DOFs to polynomial functions, and identify the key properties of these operators when combined with the interpolation operator. In Section 4, we define the stabilized discrete bilinear form, state the discrete problem, and show its well-posedness. In Section 5, we perform the error analysis of the method and prove locking-free error estimates with convergence rate of order $(k+1)$ in the energy norm and $(k+2)$ in the $L^2$-norm for smooth solutions. In Section 6, we discuss implementation aspects, present numerical examples on polygonal meshes confirming our theoretical results, and evaluate the efficiency of the proposed method in terms of CPU cost.

# 2 Setting

## 2.1 Admissible mesh sequences

In this section we briefly recall the notion of admissible mesh sequence of [9, Chapter 1]. Let $\mathcal{H} \subset \mathbb{R}_*^+$ denote a countable set of meshsizes having 0 as its unique accumulation point. We focus on sequences $(\mathcal{T}_h)_{h\in\mathcal{H}}$ where, for all $h \in \mathcal{H}$, $\mathcal{T}_h$ is a finite collection of nonempty disjoint open polyhedra (called elements or cells) $\mathcal{T}_h = \{T\}$ such that $\overline{\Omega} = \bigcup_{T\in\mathcal{T}_h} \overline{T}$ and $h = \max_{T\in\mathcal{T}_h} h_T$ ($h_T$ stands for the diameter of $T$). A hyperplanar closed connected subset $F$ of $\overline{\Omega}$ is called a face if it has positive $(d-1)$-dimensional measure and (i) either there exist $T_1, T_2 \in \mathcal{T}_h$ such that $F = \partial T_1 \cap \partial T_2$ (and $F$ is an interface) or (ii) there exists $T \in \mathcal{T}_h$ such that $F = \partial T \cap \partial\Omega$ (and $F$ is a boundary face). The set of interfaces is denoted by $\mathcal{F}_h^{\mathrm{i}}$, the set of boundary faces by $\mathcal{F}_h^{\mathrm{b}}$, and we let $\mathcal{F}_h := \mathcal{F}_h^{\mathrm{i}} \cup \mathcal{F}_h^{\mathrm{b}}$. The diameter of a face $F \in \mathcal{F}_h$ is denoted by $h_F$. For all $T \in \mathcal{T}_h$, $\mathcal{F}_T := \{F \in \mathcal{F}_h \mid F \subset \partial T\}$ denotes the set of faces lying on the boundary of $T$ and, for all $F \in \mathcal{F}_T$, we denote by $\underline{n}_{TF}$ the normal to $F$ pointing out of $T$. The $m$-dimensional Lebesgue measure, $m \in \{0 \ldots d\}$, is denoted by $|\cdot|_m$.

**Definition 1** (Admissible mesh sequence)**.** *We say that the mesh sequence $(\mathcal{T}_h)_{h\in\mathcal{H}}$ is admissible if, for all $h \in \mathcal{H}$, $\mathcal{T}_h$ admits a matching simplicial submesh $\mathfrak{T}_h$ and there exists a real number $\varrho > 0$ (the mesh regularity parameter) independent of $h$ such that the following conditions hold: (i) for all $h \in \mathcal{H}$ and all simplex $S \in \mathfrak{T}_h$ of diameter $h_S$ and inradius $r_S$, $\varrho h_S \leqslant r_S$ (shape-regularity) and (ii) for all $h \in \mathcal{H}$, all $T \in \mathcal{T}_h$, and all $S \in \mathfrak{T}_h$ such that $S \subset T$, $\varrho h_T \leqslant h_S$ (contact-regularity).*

In what follows, we consider meshes belonging to an admissible mesh sequence. We recall two useful geometric results that hold in this framework. Owing to [9, Lemma 1.42], for all

$h \in \mathcal{H}$, all $T \in \mathcal{T}_h$, and all $F \in \mathcal{F}_T$, $h_F$ is comparable to $h_T$ in the sense that

$$\varrho^2 h_T \leqslant h_F \leqslant h_T. \tag{3}$$

Moreover, owing to [9, Lemma 1.41], there exists an integer $N_\partial \geqslant (d+1)$ (depending on $\varrho$) such that the maximum number of faces of one element is bounded,

$$\forall h \in \mathcal{H}, \qquad \max_{T \in \mathcal{T}_h} \operatorname{card}(\mathcal{F}_T) \leqslant N_\partial. \tag{4}$$

In what follows, we often abbreviate as $a \lesssim b$ the inequality $a \leqslant Cb$ with $C > 0$ independent of $h$, $\mu$, and $\lambda$, but possibly depending on the mesh-regularity parameter $\varrho$. Tracking the dependence on Lamé's parameters of the constant appearing in the inequalities allows one to investigate the properties of the method in the incompressible limit.

### 2.2 Basic results

In this section we recall some basic results on admissible mesh sequences. There exist real numbers $C_{\rm tr}$ and $C_{\rm tr,c}$ depending on $\varrho$ but independent of $h$ such that the following discrete trace inequalities hold for all $T \in \mathcal{T}_h$, cf. [9, Lemmata 1.46 and 1.49]:

$$\|v\|_F \leqslant C_{\rm tr} h_F^{-1/2} \|v\|_T \qquad \forall v \in \mathbb{P}_d^l(T),\ \forall F \in \mathcal{F}_T, \tag{5}$$

$$\|v\|_{\partial T} \leqslant C_{\rm tr,c} \left( h_T^{-1} \|v\|_T^2 + h_T \|\nabla v\|_T^2 \right)^{1/2} \qquad \forall v \in H^1(T), \tag{6}$$

where $\mathbb{P}_d^l(T)$ denotes the space spanned by the restrictions to $T$ of $d$-variate polynomials of total degree $\leqslant l$. Using [9, Lemma 1.40] together with the results of [14], it can be proved that there exists a real number $C_{\rm app}$ depending on $\varrho$ but independent of $h$ such that, for all $T \in \mathcal{T}_h$, denoting by $\pi_T^l$ the $L^2$-orthogonal projector on $\mathbb{P}_d^l(T)$, the following inequality holds: For all $T \in \mathcal{T}_h$, all $s \in \{1, \ldots, (l+1)\}$, and all $v \in H^s(T)$,

$$|v - \pi_T^l v|_{H^m(T)} + h_T^{1/2} |v - \pi_T^l v|_{H^m(\partial T)} \leqslant C_{\rm app} h_T^{s-m} |v|_{H^s(T)} \qquad \forall m \in \{0, \ldots, (s-1)\}. \tag{7}$$

The following Poincaré inequality is valid for all $T \in \mathcal{T}_h$ and all $v \in H^1(T)$ such that $\int_T v = 0$:

$$\|v\|_T \leqslant C_{\rm P} h_T \|\nabla v\|_T, \tag{8}$$

where $C_{\rm P} = \pi^{-1}$ for convex elements (cf. [2]), while, for more general element shapes, $C_{\rm P}$ can be estimated in terms of the mesh-regularity parameter $\varrho$. For all $T \in \mathcal{T}_h$, we set

$$\underline{\mathcal{U}}(T) := \left\{ \underline{v} \in H^1(T)^d \mid \int_T \underline{v} = \underline{0} \text{ and } \int_T \nabla_{\rm ss} \underline{v} = \underline{\underline{0}} \right\}, \tag{9}$$

where $\nabla_{\rm ss}$ denotes the skew-symmetric part of the gradient operator. The following second Korn inequality holds for all $T \in \mathcal{T}_h$ and all $\underline{v} \in \underline{\mathcal{U}}(T)$:

$$\|\nabla \underline{v}\|_T \leqslant C_{\rm K,2} \|\nabla_{\rm s} \underline{v}\|_T, \tag{10}$$

where $C_{\rm K,2} > 0$ is independent of $h$ and can be estimated in terms of the mesh-regularity parameter $\varrho$, cf., e.g., [20, 18, 19] for the two-dimensional case. Combining (8) and (10), we infer that, for all $T \in \mathcal{T}_h$ and all $\underline{v} \in \underline{\mathcal{U}}(T)$,

$$\|\underline{v}\|_T \leqslant C_{\rm K} h_T \|\nabla_{\rm s} \underline{v}\|_T, \tag{11}$$

with $C_{\rm K} = C_{\rm K,2} C_{\rm P}$. The bound (11) also holds if $\underline{v} \in H^1(T)^d$ and there exists $\Gamma \subset \partial T$ with $|\Gamma|_{d-1} \neq 0$ such that $\underline{v}_{|\Gamma} = \underline{0}$.

# 3 Discretization

## 3.1 Degrees of freedom

Let a polynomial degree $k \geqslant 0$ be fixed. For all $T \in \mathcal{T}_h$, we define the local space of DOFs as

$$\underline{\mathsf{U}}_T^k := \mathbb{P}_d^k(T)^d \times \left\{ \bigtimes_{F \in \mathcal{F}_T} \underline{\mathsf{U}}_F^k \right\}, \qquad \underline{\mathsf{U}}_F^k := \mathbb{P}_{d-1}^k(F)^d. \tag{12}$$

Let $\underline{U}(T) := H^1(T)^d$. The local interpolation operator $I_T^k : \underline{U}(T) \to \underline{\mathsf{U}}_T^k$ is such that, for all $\underline{v} \in \underline{U}(T)$,

$$I_T^k \underline{v} = (\pi_T^k \underline{v}, (\pi_F^k \underline{v})_{F \in \mathcal{F}_T}). \tag{13}$$

The global space is obtained by patching interface DOFs as follows:

$$\underline{\mathsf{U}}_h^k := \left\{ \bigtimes_{T \in \mathcal{T}_h} \mathbb{P}_d^k(T)^d \right\} \times \left\{ \bigtimes_{F \in \mathcal{F}_h} \underline{\mathsf{U}}_F^k \right\}. \tag{14}$$

For all $T \in \mathcal{T}_h$, we denote by $\mathsf{L}_T$ the restriction operator that maps the global DOFs in $\underline{\mathsf{U}}_h^k$ to the corresponding local DOFs in $\underline{\mathsf{U}}_T^k$. Letting $\underline{U} := H^1(\Omega)^d$, the global interpolation operator $I_h^k : \underline{U} \to \underline{\mathsf{U}}_h^k$ is such that, for all $\underline{v} \in \underline{U}$, $I_h^k \underline{v} = ((\pi_T^k \underline{v})_{T \in \mathcal{T}_h}, (\pi_F^k \underline{v})_{F \in \mathcal{F}_h})$, so that

$$\mathsf{L}_T I_h^k \underline{v} = I_T^k (\underline{v}_{|T}) \qquad \forall v \in \underline{U}. \tag{15}$$

The homogeneous Dirichlet boundary condition on the displacement can be enforced explicitly on the DOFs attached to faces. We set

$$\underline{\mathsf{U}}_{T,0}^k := \left\{ \underline{\mathsf{v}} = (\underline{\mathsf{v}}_T, (\underline{\mathsf{v}}_F)_{F \in \mathcal{F}_T}) \in \underline{\mathsf{U}}_T^k \mid \underline{\mathsf{v}}_F \equiv \underline{0} \ \forall F \in \mathcal{F}_h^{\mathrm{b}} \right\}. \tag{16}$$

Letting $\underline{U}_0(T) := \left\{ \underline{v} \in \underline{U}(T) \mid \underline{v}_{|F} \equiv 0 \ \forall F \in \mathcal{F}_T \cap \mathcal{F}_h^{\mathrm{b}} \right\}$, we observe that the restriction of the interpolation operator $I_T^k$ to $\underline{U}_0(T)$ maps onto $\underline{\mathsf{U}}_{T,0}^k$. Finally, we define the global space with homogeneous Dirichlet boundary condition as

$$\underline{\mathsf{U}}_{h,0}^k := \left\{ \underline{\mathsf{v}}_h = ((\underline{\mathsf{v}}_T)_{T \in \mathcal{T}_h}, (\underline{\mathsf{v}}_F)_{F \in \mathcal{F}_h}) \in \underline{\mathsf{U}}_h^k \mid \underline{\mathsf{v}}_F \equiv \underline{0} \ \forall F \in \mathcal{F}_h^{\mathrm{b}} \right\}, \tag{17}$$

and observe that the restriction of $I_h^k$ to $\underline{U}_0$ maps onto $\underline{\mathsf{U}}_{h,0}^k$.

## 3.2 Reconstructions of differential operators

In this section we define local and global symmetric gradient and divergence reconstructions based on the DOF spaces (12) and (14). The reason for defining a discrete divergence operator independently is that we aim at satisfying the usual commuting diagram property in order to achieve robustness in the quasi-incompressible limit.

### 3.2.1 Symmetric gradient

Let an element $T \in \mathcal{T}_h$ be fixed and set, for a polynomial degree $l \geqslant 1$,

$$\underline{\mathcal{U}}_T^l := \mathbb{P}_d^l(T)^d \cap \underline{\mathcal{U}}(T), \tag{18}$$

with $\underline{\mathcal{U}}(T)$ defined by (9). The local discrete symmetric gradient operator $\underline{\underline{E}}_T^k : \underline{\mathsf{U}}_T^k \to \nabla_{\rm s}\underline{\mathcal{U}}_T^{k+1}$ is such that, for all $\underline{\mathsf{v}} = (\underline{\mathsf{v}}_T, (\underline{\mathsf{v}}_F)_{F\in\mathcal{F}_T}) \in \underline{\mathsf{U}}_T^k$ and all $\underline{w} \in \underline{\mathcal{U}}_T^{k+1}$,

$$(\underline{\underline{E}}_T^k\underline{\mathsf{v}}, \nabla_{\rm s}\underline{w})_T := (\nabla_{\rm s}\underline{\mathsf{v}}_T, \nabla_{\rm s}\underline{w})_T + \sum_{F\in\mathcal{F}_T} (\underline{\mathsf{v}}_F - \underline{\mathsf{v}}_T, \nabla_{\rm s}\underline{w}\underline{n}_{TF})_F \tag{19a}$$

$$= -(\underline{\mathsf{v}}_T, \nabla\cdot\nabla_{\rm s}\underline{w})_T + \sum_{F\in\mathcal{F}_T} (\underline{\mathsf{v}}_F, \nabla_{\rm s}\underline{w}\underline{n}_{TF})_F. \tag{19b}$$

Observing that, by definition, there exists $\underline{v} \in \underline{\mathcal{U}}_T^{k+1}$ such that $\underline{\underline{E}}_T^k\underline{\mathsf{v}} = \nabla_{\rm s}\underline{v}$, computing $\underline{\underline{E}}_T^k\underline{\mathsf{v}}$ amounts to solving the following pure-traction problem in $\underline{v} \in \underline{\mathcal{U}}_T^{k+1}$: For all $w \in \underline{\mathcal{U}}_T^{k+1}$,

$$(\nabla_{\rm s}\underline{v}, \nabla_{\rm s}\underline{w})_T = (\nabla_{\rm s}\underline{\mathsf{v}}_T, \nabla_{\rm s}\underline{w})_T + \sum_{F\in\mathcal{F}_T} (\underline{\mathsf{v}}_F - \underline{\mathsf{v}}_T, \nabla_{\rm s}\underline{w}\underline{n}_{TF})_F. \tag{20}$$

Problem (20) is well-posed as a result of (10)-(11).

**Remark 2** (Compatibility condition)**.** *Let*

$$\mathrm{RM} := \left\{\underline{\alpha} + \underline{\underline{\beta}}\underline{x} \mid \underline{\alpha} \in \mathbb{R}^d, \underline{\underline{\beta}} \in \mathbb{R}^{d\times d}, \underline{\underline{\beta}} + \underline{\underline{\beta}}^{\rm T} = \underline{\underline{0}}\right\} \subset [\mathbb{P}_d^1]^d, \tag{21}$$

*denote the space of rigid-body motions;* RM *is a three-dimensional space for* $d = 2$ *and a six-dimensional space for* $d = 3$*. For an element* $T \in \mathcal{T}_h$*, denote by* $\mathrm{RM}(T)$ *the space spanned by the restrictions of functions in* RM *to* $T$*. Combining the following classical facts:*

$$\ker(\nabla_{\rm s}) = \mathrm{RM} \quad \textit{and} \quad \mathbb{P}_d^{k+1}(T)^d = \underline{\mathcal{U}}_T^{k+1} \oplus \mathrm{RM}(T), \tag{22}$$

*we infer that* (19) *and* (20) *hold in fact for all* $\underline{w} \in \mathbb{P}_d^{k+1}(T)^d$*. This can be interpreted as the right-hand side of the linear system* (20) *satisfying the classical compatibility condition for the pure-traction problem.*

For all $T \in \mathcal{T}_h$, we define the local displacement reconstruction operator $\underline{r}_T^k : \underline{\mathsf{U}}_T^k \to \mathbb{P}_d^{k+1}(T)^d$ such that, for all $\underline{\mathsf{v}} \in \underline{\mathsf{U}}_T^k$,

$$\nabla_{\rm s}\underline{r}_T^k\underline{\mathsf{v}} := \underline{\underline{E}}_T^k\underline{\mathsf{v}}, \qquad \int_T \underline{r}_T^k\underline{\mathsf{v}} = \int_T \underline{\mathsf{v}}_T, \qquad \sum_{F\in\mathcal{F}_T}\int_F \underline{\underline{N}}_{TF}(\underline{r}_T^k\underline{\mathsf{v}}) = \sum_{F\in\mathcal{F}_T}\int_F \underline{\underline{N}}_{TF}(\underline{\mathsf{v}}_F), \tag{23}$$

where, for all $F \in \mathcal{F}_T$, $\underline{\underline{N}}_{TF} \in \mathbb{R}^{d\times d}$ is such that $\underline{\underline{N}}_{TF}\underline{a} = \frac{1}{2}(\underline{n}_{TF}\otimes\underline{a} - \underline{a}\otimes\underline{n}_{TF})$ for all $\underline{a} \in \mathbb{R}^d$. Note that $\int_T \nabla_{\rm ss}\underline{w} = \sum_{F\in\mathcal{F}_T}\int_F \underline{\underline{N}}_{TF}\underline{w}_{|F}$ for all $\underline{w} \in \underline{U}(T)$. Equation (23) means that $\underline{r}_T^k\underline{\mathsf{v}}$ is the sum of the function $\underline{v} \in \underline{\mathcal{U}}_T^{k+1}$ solving (20) and a rigid-body motion fixed by the two rightmost (closure) conditions in (23). The choice in (23) is motivated by the fact that $\underline{r}_T^k\underline{I}_T^k\underline{v} - \underline{v} \in \underline{\mathcal{U}}(T)$ for all $\underline{v} \in \underline{U}(T)$; indeed, recalling the definition (13) of $\underline{I}_T^k$, we infer that $\int_T \underline{r}_T^k\underline{I}_T^k\underline{v} = \int_T \pi_T^k\underline{v} = \int_T \underline{v}$, and since faces are hyperplanar, $\int_T \nabla_{\rm ss}\underline{r}_T^k\underline{I}_T^k\underline{v} = \sum_{F\in\mathcal{F}_T}\int_F \underline{\underline{N}}_{TF}\pi_F^k\underline{v} = \sum_{F\in\mathcal{F}_T}\int_F \underline{\underline{N}}_{TF}\underline{v} = \int_T \nabla_{\rm ss}\underline{v}$.

We now study the approximation properties of the operator $\underline{r}_T^k\underline{I}_T^k : \underline{U}(T) \to \mathbb{P}_d^{k+1}(T)^d$ on smooth functions.

**Lemma 3** (Approximation properties for $\underline{r}_T^k\underline{I}_T^k$)**.** *Let* $k \geqslant 0$ *and* $T \in \mathcal{T}_h$*. There exists a real number* $C > 0$ *depending on* $\varrho$ *and* $k$*, but independent of* $h$*, such that, for all* $\underline{v} \in H^{k+2}(T)^d$*,*

$$\|\underline{r}_T^k\underline{I}_T^k\underline{v} - \underline{v}\|_T + h_T^{1/2}\|\underline{r}_T^k\underline{I}_T^k\underline{v} - \underline{v}\|_{\partial T} + h_T\|\nabla_{\rm s}(\underline{r}_T^k\underline{I}_T^k\underline{v} - \underline{v})\|_T + h_T^{3/2}\|\nabla_{\rm s}(\underline{r}_T^k\underline{I}_T^k\underline{v} - \underline{v})\|_{\partial T} \leqslant Ch_T^{k+2}\|\underline{v}\|_{H^{k+2}(T)^d}. \tag{24}$$

*Proof.* Let $\underline{v} \in H^{k+2}(T)^d$. Using the definitions (23) of $\underline{r}_T^k$, (19b) of $\underline{\underline{E}}_T^k$, and (13) of $I_T^k$, we infer that, for all $\underline{w} \in \mathbb{P}_d^{k+1}(T)^d$ (cf. Remark 2),

$$\begin{aligned}(\nabla_{\rm s}\underline{r}_T^k I_T^k \underline{v}, \nabla_{\rm s}\underline{w})_T = (\underline{\underline{E}}_T^k I_T^k \underline{v}, \nabla_{\rm s}\underline{w})_T &= -(\pi_T^k \underline{v}, \nabla{\cdot}\nabla_{\rm s}\underline{w})_T + \sum_{F\in\mathcal{F}_T} (\pi_F^k \underline{v}, \nabla_{\rm s}\underline{w}\underline{n}_{TF})_F \\ &= -(\underline{v}, \nabla{\cdot}\nabla_{\rm s}\underline{w})_T + \sum_{F\in\mathcal{F}_T} (\underline{v}, \nabla_{\rm s}\underline{w}\underline{n}_{TF})_F,\end{aligned}$$

since $\nabla{\cdot}\nabla_{\rm s}\underline{w} \in \mathbb{P}_d^{k-1}(T)^d \subset \mathbb{P}_d^k(T)^d$ and $(\nabla_{\rm s}\underline{w})_{|F}\underline{n}_{TF} \in \mathbb{P}_{d-1}^k(F)^d$. Integrating by parts the right-hand side yields

$$(\nabla_{\rm s}\underline{r}_T^k I_T^k \underline{v} - \nabla_{\rm s}\underline{v}, \nabla_{\rm s}\underline{w})_T = 0 \qquad \forall \underline{w} \in \mathbb{P}_d^{k+1}(T)^d. \tag{25}$$

The orthogonality condition (25) implies that

$$\|\nabla_{\rm s}(\underline{r}_T^k I_T^k \underline{v} - \underline{v})\|_T = \inf_{\underline{w}\in\mathbb{P}_d^{k+1}(T)^d} \|\nabla_{\rm s}(\underline{w} - \underline{v})\|_T \lesssim h_T^{k+1} \|\underline{v}\|_{H^{k+2}(T)^d}, \tag{26}$$

where the last inequality follows from the approximation property (7) of $\pi_T^{k+1}$ (with $s = k+2$ and $m = 1$). Using Korn's inequality (11) (since $(\underline{r}_T^k I_T^k \underline{v} - \underline{v}) \in \underline{\mathcal{U}}(T)$), together with (26), we infer that

$$\|\underline{r}_T^k I_T^k \underline{v} - \underline{v}\|_T \lesssim h_T \|\nabla_{\rm s}(\underline{r}_T^k I_T^k \underline{v} - \underline{v})\|_T \lesssim h_T^{k+2} \|\underline{v}\|_{H^{k+2}(T)^d}. \tag{27}$$

Using the continuous trace inequality (6) followed by Korn's inequality (10), together with (26) and (27), we infer that

$$h_T \|\underline{r}_T^k I_T^k \underline{v} - \underline{v}\|_{\partial T}^2 \lesssim \|\underline{r}_T^k I_T^k \underline{v} - \underline{v}\|_T^2 + h_T^2 \|\nabla_{\rm s}(\underline{r}_T^k I_T^k \underline{v} - \underline{v})\|_T^2 \lesssim h_T^{2(k+1)} \|\underline{v}\|_{H^{k+2}(T)^d}^2. \tag{28}$$

Finally, the bound on $h_T^{3/2} \|\nabla_{\rm s}(\underline{r}_T^k I_T^k \underline{v} - \underline{v})\|_{\partial T}$ is obtained by introducing $\pm\pi_T^k \nabla_{\rm s}\underline{v}$ inside the norm, using the triangle inequality, and concluding with the approximation property (7) of $\pi_T^k$ (applied component-wise to $\nabla_{\rm s}\underline{v}$ with $s = k+1$ and $m = 0$), the discrete trace inequality (5), the bound (4) on $\mathrm{card}(\mathcal{F}_T)$, the mesh regularity property (3), the fact that $\nabla_{\rm s}\underline{r}_T^k I_T^k \underline{v} \in \mathbb{P}_d^k(T)^{d\times d}$ so that $\|\pi_T^k \nabla_{\rm s}\underline{v} - \nabla_{\rm s}\underline{r}_T^k I_T^k \underline{v}\|_T \le \|\nabla_{\rm s}(\underline{v} - \underline{r}_T^k I_T^k \underline{v})\|_T$, and (26). □

The global discrete symmetric gradient operator $\underline{\underline{E}}_h^k : \underline{\mathsf{U}}_h^k \to \times_{T\in\mathcal{T}_h} \nabla_{\rm s}\underline{\mathcal{U}}_T^{k+1}$ is assembled element-wise, so that, for all $\underline{\mathsf{v}}_h \in \underline{\mathsf{U}}_h^k$,

$$\underline{\underline{E}}_h^k \underline{\mathsf{v}}_{h|T} = \underline{\underline{E}}_T^k \mathsf{L}_T \underline{\mathsf{v}}_h \qquad \forall T \in \mathcal{T}_h. \tag{29}$$

### 3.2.2 Divergence

For an element $T \in \mathcal{T}_h$, the local discrete divergence operator $D_T^k : \underline{\mathsf{U}}_T^k \to \mathbb{P}_d^k(T)$ is such that, for all $\underline{\mathsf{v}} = (\underline{\mathsf{v}}_T, (\underline{\mathsf{v}}_F)_{F\in\mathcal{F}_T}) \in \underline{\mathsf{U}}_T^k$ and all $q \in \mathbb{P}_d^k(T)$,

$$(D_T^k \underline{\mathsf{v}}, q)_T := -(\underline{\mathsf{v}}_T, \nabla q)_T + \sum_{F\in\mathcal{F}_T} (\underline{\mathsf{v}}_F{\cdot}\underline{n}_{TF}, q)_F \tag{30a}$$

$$= (\nabla{\cdot}\underline{\mathsf{v}}_T, q)_T + \sum_{F\in\mathcal{F}_T} ((\underline{\mathsf{v}}_F - \underline{\mathsf{v}}_T){\cdot}\underline{n}_{TF}, q)_F. \tag{30b}$$

The corresponding global discrete divergence operator $D_h^k : \underline{\mathsf{U}}_h^k \to \mathbb{P}_d^k(\mathcal{T}_h) := \times_{T\in\mathcal{T}_h} \mathbb{P}_d^k(T)$ is assembled element-wise, so that, for all $\underline{\mathsf{v}}_h \in \underline{\mathsf{U}}_h^k$,

$$D_h^k \underline{\mathsf{v}}_{h|T} = D_T^k \mathsf{L}_T \underline{\mathsf{v}}_h \qquad \forall T \in \mathcal{T}_h. \tag{31}$$

A key point in our analysis is the following commuting diagram property.

**Proposition 4** (Commuting property for discrete divergence operator)**.** *The following diagrams commute:*

$$\begin{array}{ccc} \underline{U}(T) & \xrightarrow{\nabla\cdot} & L^2(T) \\ I_T^k \big\downarrow & & \big\downarrow \pi_T^k \\ \underline{\mathsf{U}}_T^k & \xrightarrow{D_T^k} & \mathbb{P}_d^k(T) \end{array} \qquad\qquad \begin{array}{ccc} \underline{U}_0 & \xrightarrow{\nabla\cdot} & L_0^2(\Omega) \\ I_h^k \big\downarrow & & \big\downarrow \pi_h^k \\ \underline{\mathsf{U}}_{h,0}^k & \xrightarrow{D_h^k} & \mathbb{P}_{d,0}^k(\mathcal{T}_h) \end{array}$$

*where* $L_0^2(\Omega) := \left\{ v \in L^2(\Omega) \mid \int_\Omega v = 0 \right\}$, $\mathbb{P}_{d,0}^k(\mathcal{T}_h) := \mathbb{P}_d^k(\mathcal{T}_h) \cap L_0^2(\Omega)$, *and* $\pi_h^k$ *denotes the global version of the local* $L^2$*-projector* $\pi_T^k$.

*Proof.* Consider the local commuting diagram. Let $T \in \mathcal{T}_h$, let $\underline{v} \in \underline{U}(T)$, and set $\underline{\mathsf{v}} := I_T^k \underline{v}$. We infer that, for all $q \in \mathbb{P}_d^k(T)$,

$$\begin{aligned} (\pi_T^k(\nabla\cdot\underline{v}), q)_T = (\nabla\cdot\underline{v}, q)_T &= -(\nabla q, \underline{v})_T + \sum_{F\in\mathcal{F}_T} (q, \underline{v}\cdot\underline{n}_{TF})_F \\ &= -(\nabla q, \pi_T^k \underline{v})_T + \sum_{F\in\mathcal{F}_T} (q, \pi_F^k \underline{v}\cdot\underline{n}_{TF})_F \\ &= -(\nabla q, \underline{\mathsf{v}}_T)_T + \sum_{F\in\mathcal{F}_T} (q, \underline{\mathsf{v}}_F\cdot\underline{n}_{TF})_F = (D_T^k \underline{\mathsf{v}}, q)_T, \end{aligned}$$

where we have used that $\nabla q \in \mathbb{P}_d^{k-1}(T)^d \subset \mathbb{P}_d^k(T)^d$, $q_{|F} \in \mathbb{P}_{d-1}^k(F)$, and (30a). For the global commuting diagram, we use that, for all $T \in \mathcal{T}_h$, $(D_h^k \underline{\mathsf{v}}_h)_{|T} = D_T^k(\mathsf{L}_T \underline{\mathsf{v}}_h)$ for all $\underline{\mathsf{v}}_h \in \underline{\mathsf{U}}_h^k$ owing to (31), $\mathsf{L}_T(I_h^k \underline{v}) = I_T^k(\underline{v}_{|T})$ for all $\underline{v} \in \underline{U}$ owing to (15), whence

$$(D_h^k I_h^k \underline{v})_{|T} = D_T^k(I_T^k \underline{v}_{|T}) = \pi_T^k(\nabla\cdot(\underline{v}_{|T})) = (\pi_h^k(\nabla\cdot\underline{v}))_{|T},$$

owing to the local commuting diagram and the fact that the projection $\pi_h^k$ acts locally. Finally, restricting $\underline{v}$ to $\underline{U}_0$, we obtain $\int_\Omega \nabla\cdot\underline{v} = 0$, and summing (30a) tested with $q \equiv 1$ over all mesh elements yields $\int_\Omega D_h^k I_h^k \underline{v} = 0$ since the homogeneous Dirichlet boundary condition is incorporated in $\underline{\mathsf{U}}_{h,0}^k$ for all $F \in \mathcal{F}_h^{\mathrm{b}}$. □

## 4 Discrete problem and well-posedness

In what follows, we assume that $k \geqslant 1$. The reason for not allowing $k = 0$ is related to the design of the stabilization bilinear form as discussed below. For all $T \in \mathcal{T}_h$, we define the local bilinear form on $\underline{\mathsf{U}}_T^k \times \underline{\mathsf{U}}_T^k$ such that, for all $\underline{\mathsf{u}}, \underline{\mathsf{v}} \in \underline{\mathsf{U}}_T^k$,

$$a_T(\underline{\mathsf{u}}, \underline{\mathsf{v}}) := 2\mu(\underline{\underline{E}}_T^k \underline{\mathsf{u}}, \underline{\underline{E}}_T^k \underline{\mathsf{v}})_T + \lambda(D_T^k \underline{\mathsf{u}}, D_T^k \underline{\mathsf{v}}) + (2\mu) s_T(\underline{\mathsf{u}}, \underline{\mathsf{v}}), \tag{32}$$

with stabilization bilinear form

$$s_T(\underline{\mathsf{u}},\underline{\mathsf{v}}) := \sum_{F\in\mathcal{F}_T} h_F^{-1}(\pi_F^k(\underline{R}_T^k\underline{\mathsf{u}} - \underline{\mathsf{u}}_F), \pi_F^k(\underline{R}_T^k\underline{\mathsf{v}} - \underline{\mathsf{v}}_F))_F, \tag{33}$$

where, for all $T \in \mathcal{T}_h$, we have introduced the local displacement reconstruction operator $\underline{R}_T^k : \underline{\mathsf{U}}_T^k \to \mathbb{P}_d^{k+1}(T)^d$ such that, for all $\underline{\mathsf{v}} = (\underline{\mathsf{v}}_T, (\underline{\mathsf{v}}_F)_{F\in\mathcal{F}_T}) \in \underline{\mathsf{U}}_T^k$,

$$\underline{R}_T^k\underline{\mathsf{v}} := \underline{\mathsf{v}}_T + (\underline{r}_T^k\underline{\mathsf{v}} - \pi_T^k\underline{r}_T^k\underline{\mathsf{v}}), \tag{34}$$

and $\underline{r}_T^k$ is defined by (23). The term in parentheses can be interpreted as a higher-order correction of the element unknown $\underline{\mathsf{v}}_T$ derived from the discrete symmetric gradient reconstruction operator (19) (note that this correction is independent of the closure relations in (23) since $k \geqslant 1$). The stability analysis hinges on the following local discrete strain (semi-)norms on $\underline{\mathsf{U}}_T^k$: For all $\underline{\mathsf{v}} \in \underline{\mathsf{U}}_T^k$,

$$\|\underline{\mathsf{v}}\|_{\varepsilon,T}^2 := \|\nabla_{\mathrm{s}}\underline{\mathsf{v}}_T\|_T^2 + |\underline{\mathsf{v}}|_{\varepsilon,\partial T}^2, \qquad \|\underline{\mathsf{v}}\|_{E,s,T}^2 := \|\underline{\underline{E}}_T^k\underline{\mathsf{v}}\|_T^2 + s_T(\underline{\mathsf{v}},\underline{\mathsf{v}}). \tag{35}$$

where $|\underline{\mathsf{v}}|_{\varepsilon,\partial T}^2 := \sum_{F\in\mathcal{F}_T} h_F^{-1}\|\underline{\mathsf{v}}_T - \underline{\mathsf{v}}_F\|_F^2$ (the subscript $\varepsilon$ refers to strain). The stabilization bilinear form $s_T$ defined by (33) introduces a least-squares penalty of the $L^2$-orthogonal projection on $\mathbb{P}_{d-1}^k(F)^d$ of the difference between $\underline{\mathsf{v}}_F$ and $(\underline{R}_T^k\underline{\mathsf{v}})_{|F}$. This penalty is tailored to ensure control on the $\|\cdot\|_{\varepsilon,T}$-norm as reflected by the first inequality in (40) below, and, at the same time, to achieve the same convergence order as the symmetric gradient and divergence reconstruction operators in the error analysis, see the bound (60) below.

The global bilinear forms $a_h$ and $s_h$ on $\underline{\mathsf{U}}_h^k \times \underline{\mathsf{U}}_h^k$ are assembled element-wise so that, for all $\underline{\mathsf{u}}_h, \underline{\mathsf{v}}_h \in \underline{\mathsf{U}}_h^k$,

$$a_h(\underline{\mathsf{u}}_h,\underline{\mathsf{v}}_h) := \sum_{T\in\mathcal{T}_h} a_T(\mathsf{L}_T\underline{\mathsf{u}}_h, \mathsf{L}_T\underline{\mathsf{v}}_h), \qquad s_h(\underline{\mathsf{u}}_h,\underline{\mathsf{v}}_h) := \sum_{T\in\mathcal{T}_h} s_T(\mathsf{L}_T\underline{\mathsf{u}}_h, \mathsf{L}_T\underline{\mathsf{v}}_h). \tag{36}$$

The loading term is discretized by means of the linear form $l_h$ on $\underline{\mathsf{U}}_h^k$ such that, for all $\underline{\mathsf{v}}_h \in \underline{\mathsf{U}}_h^k$,

$$l_h(\underline{\mathsf{v}}_h) := \sum_{T\in\mathcal{T}_h} l_T(\mathsf{L}_T\underline{\mathsf{v}}_h), \qquad l_T(\underline{\mathsf{v}}) := \int_T \underline{f}\cdot\underline{\mathsf{v}}_T. \tag{37}$$

The discrete problem reads: Find $\underline{\mathsf{u}}_h \in \underline{\mathsf{U}}_{h,0}^k$ such that, for all $\underline{\mathsf{v}}_h \in \underline{\mathsf{U}}_{h,0}^k$,

$$a_h(\underline{\mathsf{u}}_h,\underline{\mathsf{v}}_h) = l_h(\underline{\mathsf{v}}_h). \tag{38}$$

The global (semi-)norms are such that, for all $\underline{\mathsf{v}}_h \in \underline{\mathsf{U}}_h^k$,

$$\|\underline{\mathsf{v}}_h\|_{\varepsilon,h}^2 := \sum_{T\in\mathcal{T}_h} \|\mathsf{L}_T\underline{\mathsf{v}}_h\|_{\varepsilon,T}^2, \qquad \|\underline{\mathsf{v}}_h\|_{E,s,h}^2 := \sum_{T\in\mathcal{T}_h} \|\mathsf{L}_T\underline{\mathsf{v}}_h\|_{E,s,T}^2. \tag{39}$$

**Proposition 5** (Norm $\|\cdot\|_{\varepsilon,h}$)**.** *The map $\|\cdot\|_{\varepsilon,h}$ defined by (39) is a norm on $\underline{\mathsf{U}}_{h,0}^k$.*

*Proof.* It suffices to show that, for all $\underline{\mathsf{v}}_h \in \underline{\mathsf{U}}_{h,0}^k$, $\|\underline{\mathsf{v}}_h\|_{\varepsilon,h} = 0$ if and only if $\underline{\mathsf{v}}_T \equiv \underline{0}$ for all $T \in \mathcal{T}_h$ and $\underline{\mathsf{v}}_F \equiv \underline{0}$ for all $F \in \mathcal{F}_h$. We start by observing that $\|\underline{\mathsf{v}}_h\|_{\varepsilon,h} = 0$ means that

$$\forall T \in \mathcal{T}_h, \qquad \nabla_{\mathrm{s}}\underline{\mathsf{v}}_T \equiv \underline{\underline{0}} \quad \text{and} \quad \underline{\mathsf{v}}_T - \underline{\mathsf{v}}_F \equiv \underline{0} \quad \forall F \in \mathcal{F}_h.$$

For a boundary element $T \in \mathcal{T}_h$ with $F \in \mathcal{F}_T \cap \mathcal{F}_h^{\rm b}$, using $\underline{v}_F \equiv \underline{0}$ (cf. the definition (16) of $\underline{\mathsf{U}}_{T,0}^k$), we infer that $\underline{v}_{T|F} \equiv \underline{0}$ which, combined with $\nabla_{\rm s}\underline{v}_T \equiv \underline{\underline{0}}$ and Korn's inequality, implies $\underline{v}_T \equiv \underline{0}$ and, hence, $\underline{v}_{F'} \equiv \underline{0}$ for all $F' \in \mathcal{F}_T \backslash \{F\}$ since $\underline{v}_T - \underline{v}_{F'} \equiv \underline{0}$. Repeating the argument for the next layer of elements (and the corresponding faces), the result is proved iterating until all elements and faces have been visited. □

**Remark 6** (Bound on $\|\cdot\|$-norm)**.** *Using discrete Korn inequalities for piecewise smooth fields, see Duarte et al. [13] and Brenner [5], and a triangle inequality for the face term, we infer the following stronger result on the $\|\cdot\|_{\varepsilon,h}$-norm: For all $\underline{\mathsf{v}}_h \in \underline{\mathsf{U}}_{h,0}^k$, $\|\underline{\mathsf{v}}_h\|_{\varepsilon,h} \gtrsim C\|\underline{v}_h\|$ with real number $C > 0$ independent of $h$ and $\underline{v}_h$ reconstructed from the cell-wise DOFs of $\underline{\mathsf{v}}_h$ as $(\underline{v}_h)_{|T} = \underline{v}_T$ for all $T \in \mathcal{T}_h$.*

**Lemma 7** (Equivalence of the discrete strain norms)**.** *Let $k \geqslant 1$. There is a real number $\eta > 0$ independent of $h$, $\mu$, and $\lambda$ such that, for all $T \in \mathcal{T}_h$ and all $\underline{\mathsf{v}} \in \underline{\mathsf{U}}_T^k$,*

$$\eta\|\underline{\mathsf{v}}\|_{\varepsilon,T}^2 \leqslant \|\underline{\mathsf{v}}\|_{E,s,T}^2 \leqslant \eta^{-1}\|\underline{\mathsf{v}}\|_{\varepsilon,T}^2, \tag{40}$$

*and, for all $\underline{\mathsf{v}}_h \in \underline{\mathsf{U}}_h^k$,*

$$\eta\|\underline{\mathsf{v}}_h\|_{\varepsilon,h}^2 \leqslant \|\underline{\mathsf{v}}_h\|_{E,s,h}^2 \leqslant \eta^{-1}\|\underline{\mathsf{v}}_h\|_{\varepsilon,h}^2. \tag{41}$$

*Proof.* Let $T \in \mathcal{T}_h$ and let $\underline{\mathsf{v}} \in \underline{\mathsf{U}}_T^k$. We prove the first inequality in (40). Taking $\underline{w} = \underline{v}_T$ in (19a) (this is possible even if $\underline{v}_T \notin \underline{\mathcal{U}}_T^{k+1}$ in view of Remark 2), we infer that

$$\begin{aligned}\|\nabla_{\rm s}\underline{v}_T\|_T^2 &= (\underline{\underline{E}}_T^k\underline{\mathsf{v}}, \nabla_{\rm s}\underline{v}_T)_T + \sum_{F\in\mathcal{F}_T}(\underline{v}_T - \underline{v}_F, \nabla_{\rm s}\underline{v}_T\underline{n}_{TF})_F \\ &\leqslant \|\underline{\underline{E}}_T^k\underline{\mathsf{v}}\|_T^2 + \frac{1}{2}\|\nabla_{\rm s}\underline{v}_T\|_T^2 + N_\partial C_{\rm tr}^2|\underline{\mathsf{v}}|_{\varepsilon,\partial T}^2,\end{aligned} \tag{42}$$

where we have used the Cauchy–Schwarz and Young inequalities followed by the discrete trace inequality (5) for the last term on the right-hand side. As a result,

$$\|\nabla_{\rm s}\underline{v}_T\|_T^2 \lesssim \|\underline{\underline{E}}_T^k\underline{\mathsf{v}}\|_T^2 + |\underline{\mathsf{v}}|_{\varepsilon,\partial T}^2. \tag{43}$$

Additionally, for all $F \in \mathcal{F}_T$,

$$\begin{aligned}h_F^{-1/2}\|\underline{v}_F - \underline{v}_T\|_F &\leqslant h_F^{-1/2}\|\underline{v}_F - \pi_F^k\underline{R}_T^k\underline{\mathsf{v}}\|_F + h_F^{-1/2}\|\pi_F^k\underline{R}_T^k\underline{\mathsf{v}} - \underline{v}_T\|_F \\ &= h_F^{-1/2}\|\pi_F^k(\underline{v}_F - \underline{R}_T^k\underline{\mathsf{v}})\|_F + h_F^{-1/2}\|\pi_F^k(\underline{R}_T^k\underline{\mathsf{v}} - \underline{v}_T)\|_F \\ &= h_F^{-1/2}\|\pi_F^k(\underline{v}_F - \underline{R}_T^k\underline{\mathsf{v}})\|_F + h_F^{-1/2}\|\underline{r}_T^k\underline{\mathsf{v}} - \pi_T^k\underline{r}_T^k\underline{\mathsf{v}}\|_F\end{aligned} \tag{44}$$

where we have used the triangle inequality in the first line, the fact that $\underline{v}_F \in \mathbb{P}_{d-1}^k(F)^d$ and $\underline{v}_{T|F} \in \mathbb{P}_{d-1}^k(F)$ in the second line, and the fact that $\pi_F^k$ is a projector and the definition (34) of $\underline{R}_T^k$ in the third line. Using the discrete trace inequality (5), Korn's inequality (11) (since $\underline{r}_T^k\underline{\mathsf{v}} - \pi_T^k\underline{r}_T^k\underline{\mathsf{v}} \in \underline{\mathcal{U}}_T^{k+1}$ for $k \geqslant 1$), the mesh regularity property (3), and recalling the definition (23) of $\underline{r}_T^k$, the last term on the right-hand side can be estimated as

$$h_F^{-1/2}\|\underline{r}_T^k\underline{\mathsf{v}} - \pi_T^k\underline{r}_T^k\underline{\mathsf{v}}\|_F \leqslant \varrho^{-2}C_{\rm tr}C_{\rm K}\|\underline{\underline{E}}_T^k\underline{\mathsf{v}}\|_T,$$

so that

$$h_F^{-1/2}\|\underline{v}_F - \underline{v}_T\|_F \lesssim h_F^{-1/2}\|\pi_F^k(\underline{v}_F - \underline{R}_T^k\underline{\mathsf{v}})\|_F + \|\underline{\underline{E}}_T^k\underline{\mathsf{v}}\|_T. \tag{45}$$

Squaring (45), summing over $F \in \mathcal{F}_T$, and using the bound (4) on card($\mathcal{F}_T$) leads to

$$|\underline{\mathsf{v}}|^2_{\varepsilon,\partial T} \lesssim s_T(\underline{\mathsf{v}},\underline{\mathsf{v}}) + \|\underline{\underline{E}}^k_T \underline{\mathsf{v}}\|^2_T. \tag{46}$$

The first inequality in (40) immediately follows from (43) and (46). The proof of the second inequality in (40) uses similar arguments and is omitted for the sake of brevity. Finally, (41) follows from (40) by summing over $T \in \mathcal{T}_h$. □

**Corollary 8** (Well-posedness of (38)). *Let $k \geqslant 1$. For all $\underline{\mathsf{v}}_h \in \underline{\mathsf{U}}^k_h$, the following inequality holds:*

$$(2\mu\eta)\|\underline{\mathsf{v}}_h\|^2_{\varepsilon,h} \leqslant 2\mu\|\underline{\mathsf{v}}_h\|^2_{E,s,h} + \lambda\|D^k_h \underline{\mathsf{v}}_h\|^2 = a_h(\underline{\mathsf{v}}_h,\underline{\mathsf{v}}_h). \tag{47}$$

*As a consequence, problem* (38) *is well-posed.*

*Proof.* Inequality (47) is a straightforward consequence of (36) together with the first inequality in (41). The well-posedness of (38) then follows from the Lax–Milgram Lemma since $\|\cdot\|_{\varepsilon,h}$ is a norm on $\underline{\mathsf{U}}^k_{h,0}$ owing to Proposition 5. □

# 5 Error analysis

## 5.1 Basic error estimate

We bound the error in the energy norm such that, for all $\underline{\mathsf{v}}_h \in \underline{\mathsf{U}}^k_h$,

$$\|\underline{\mathsf{v}}_h\|^2_{\mathrm{en},h} := 2\mu\|\underline{\mathsf{v}}_h\|^2_{E,s,h} + \lambda\|D^k_h \underline{\mathsf{v}}_h\|^2 = a_h(\underline{\mathsf{v}}_h,\underline{\mathsf{v}}_h). \tag{48}$$

Owing to (47), this readily implies an error bound in the $\|\cdot\|_{\varepsilon,h}$-norm as well.

**Theorem 9** (Convergence). *Let $k \geqslant 1$. Let $\underline{u} \in \underline{U}_0$ and $\underline{\mathsf{u}}_h \in \underline{\mathsf{U}}^k_{h,0}$ denote the unique solutions to* (2) *and* (38)*, respectively. Set $\underline{\widehat{\mathsf{u}}}_h := I^k_h \underline{u} \in \underline{\mathsf{U}}^k_{h,0}$, with $I^k_h$ defined by* (15)*. Assume the additional regularity $\underline{u} \in H^{k+2}(\mathcal{T}_h)^d$ and $\underline{\underline{\sigma}} \in H^{k+1}(\mathcal{T}_h)^{d\times d}$. Then, there exists a real number $C > 0$ independent of $h$, $\mu$, and $\lambda$, such that*

$$(2\mu)^{1/2}\|\underline{\mathsf{u}}_h - \underline{\widehat{\mathsf{u}}}_h\|_{\mathrm{en},h} \leqslant Ch^{k+1}\left(2\mu\|\underline{u}\|_{H^{k+2}(\mathcal{T}_h)^d} + \|\underline{\underline{\sigma}}\|_{H^{k+1}(\mathcal{T}_h)^{d\times d}}\right). \tag{49}$$

**Remark 10** (Locking-free estimate). *For $d = 2$ and $\Omega$ convex, the fact that the error estimate* (49) *for $k = 0$ is uniform in $\lambda$ follows from the regularity estimate*

$$\mu\|\underline{u}\|_{H^2(\Omega)^d} + \lambda\|\nabla\cdot\underline{u}\|_{H^1(\Omega)} \leqslant C_\mu\|\underline{f}\|, \tag{50}$$

*where $C_\mu > 0$ denotes a real number depending on $\Omega$ and $\mu$ but independent of $\lambda$. A proof of* (50) *can be found, e.g., in [6]. More generally, for $k \geqslant 1$, we need $\underline{f} \in H^k(\Omega)^d$ and the regularity shift $\mu\|\underline{u}\|_{H^{k+2}(\Omega)^d} + \lambda\|\nabla\cdot\underline{u}\|_{H^{k+1}(\Omega)} \leqslant C_\mu\|\underline{f}\|_{H^k(\Omega)^d}$.*

*Proof.* We define, for all $\underline{\mathsf{v}}_h \in \underline{\mathsf{U}}^k_{h,0}$, the discrete stress (dual) norm

$$\|\underline{\mathsf{v}}_h\|_{S,h} := \sup_{\underline{\mathsf{w}}_h \in \underline{\mathsf{U}}^k_{h,0}, \|\underline{\mathsf{w}}_h\|_{E,s,h}=1} a_h(\underline{\mathsf{v}}_h,\underline{\mathsf{w}}_h). \tag{51}$$

Since $\|\underline{\mathsf{v}}_h\|^2_{\mathrm{en},h} = 2\mu\|\underline{\mathsf{v}}_h\|^2_{E,s,h} + \lambda\|D^k_h \underline{\mathsf{v}}_h\|^2 = a_h(\underline{\mathsf{v}}_h,\underline{\mathsf{v}}_h) \leqslant \|\underline{\mathsf{v}}_h\|_{S,h}\|\underline{\mathsf{v}}_h\|_{E,s,h}$, we infer that

$$(2\mu)^{1/2}\|\underline{\mathsf{v}}_h\|_{\mathrm{en},h} \leqslant \|\underline{\mathsf{v}}_h\|_{S,h}.$$

Observing that $\underline{\mathsf{u}}_h - \widehat{\underline{\mathsf{u}}}_h \in \underline{\mathsf{U}}_{h,0}^k$ yields $(2\mu)^{1/2}\|\underline{\mathsf{u}}_h - \widehat{\underline{\mathsf{u}}}_h\|_{\mathrm{en},h} \leqslant \|\underline{\mathsf{u}}_h - \widehat{\underline{\mathsf{u}}}_h\|_{S,h}$, and we need to bound $\|\underline{\mathsf{u}}_h - \widehat{\underline{\mathsf{u}}}_h\|_{S,h}$. Using (38) yields

$$\|\underline{\mathsf{u}}_h - \widehat{\underline{\mathsf{u}}}_h\|_{S,h} = \sup_{\underline{\mathsf{v}}_h \in \underline{\mathsf{U}}_{h,0}^k, \|\underline{\mathsf{v}}_h\|_{E,s,h}=1} \mathcal{E}_h(\underline{\mathsf{v}}_h), \tag{52}$$

with consistency error $\mathcal{E}_h(\underline{\mathsf{v}}_h) := l_h(\underline{\mathsf{v}}_h) - a_h(\widehat{\underline{\mathsf{u}}}_h, \underline{\mathsf{v}}_h)$. We bound $\mathcal{E}_h(\underline{\mathsf{v}}_h)$ for a generic $\underline{\mathsf{v}}_h \in \underline{\mathsf{U}}_{h,0}^k$. Recalling that $\underline{f} = -\nabla\cdot\underline{\underline{\sigma}}$ a.e. in $\Omega$, and integrating by parts element-wise, we infer that

$$l_h(\underline{\mathsf{v}}_h) = \sum_{T\in\mathcal{T}_h} \left\{ 2\mu(\nabla_{\mathrm{s}}\underline{u}, \nabla_{\mathrm{s}}\underline{\mathsf{v}}_T)_T + \lambda(\nabla\cdot\underline{u}, \nabla\cdot\underline{\mathsf{v}}_T)_T - \sum_{F\in\mathcal{F}_T} (\underline{\underline{\sigma}}\underline{n}_{TF}, \underline{\mathsf{v}}_T - \underline{\mathsf{v}}_F)_F \right\}, \tag{53}$$

where we have used the continuity of the normal stress component at interfaces together with $\underline{\mathsf{v}}_F \equiv \underline{0}$ for all $F \in \mathcal{F}_h^{\mathrm{b}}$ to infer that $\sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T}(\underline{\underline{\sigma}}\underline{n}_{TF}, \underline{\mathsf{v}}_F)_F = 0$. Taking $\underline{w} = \underline{\breve{u}}_T := \underline{r}_T^k \mathsf{L}_T \widehat{\underline{\mathsf{u}}}_h = \underline{r}_T^k \underline{I}_T^k(\underline{u}_{|T})$ in the definition (19a) of $\underline{\underline{E}}_h^k \mathsf{L}_T \underline{\mathsf{v}}_h$ for all $T \in \mathcal{T}_h$ (cf. Remark 2), and using the definitions (29) of $\underline{\underline{E}}_h^k$ and (23) of $\underline{r}_T^k$, we infer that

$$2\mu(\underline{\underline{E}}_h^k\widehat{\underline{\mathsf{u}}}_h, \underline{\underline{E}}_h^k\underline{\mathsf{v}}_h) = \sum_{T\in\mathcal{T}_h} \left\{ 2\mu(\nabla_{\mathrm{s}}\underline{\breve{u}}_T, \nabla_{\mathrm{s}}\underline{\mathsf{v}}_T)_T + \sum_{F\in\mathcal{F}_T} 2\mu(\nabla_{\mathrm{s}}\underline{\breve{u}}_T\underline{n}_{TF}, \underline{\mathsf{v}}_F - \underline{\mathsf{v}}_T)_F \right\}. \tag{54}$$

Using the definition (31) of $D_h^k$ and taking $q := \pi_T^k(\nabla\cdot\underline{u})$ in the definition (30b) of $D_h^k \mathsf{L}_T \underline{\mathsf{v}}_h$ for all $T \in \mathcal{T}_h$, and recalling the local commuting diagram property for $\underline{I}_T^k$, we infer that

$$\lambda(D_h^k\widehat{\underline{\mathsf{u}}}_h, D_h^k\underline{\mathsf{v}}_h) = \sum_{T\in\mathcal{T}_h} \left\{ \lambda(\nabla\cdot\underline{u}, \nabla\cdot\underline{\mathsf{v}}_T)_T + \sum_{F\in\mathcal{F}_T} \lambda(\pi_T^k(\nabla\cdot\underline{u}), (\underline{\mathsf{v}}_F - \underline{\mathsf{v}}_T)\cdot\underline{n}_{TF})_F \right\}. \tag{55}$$

Using (53)–(55) to replace the corresponding terms in the expression of $\mathcal{E}_h(\underline{\mathsf{v}}_h)$, we infer that

$$\begin{aligned}\mathcal{E}_h(\underline{\mathsf{v}}_h) = \sum_{T\in\mathcal{T}_h} 2\mu \left\{ (\nabla_{\mathrm{s}}(\underline{u} - \underline{\breve{u}}_T), \nabla_{\mathrm{s}}\underline{\mathsf{v}}_T)_T + \sum_{F\in\mathcal{F}_T} (\nabla_{\mathrm{s}}(\underline{u} - \underline{\breve{u}}_T)\underline{n}_{TF}, \underline{\mathsf{v}}_F - \underline{\mathsf{v}}_T)_F \right\} \\ - \sum_{F\in\mathcal{F}_T} \lambda((\nabla\cdot\underline{u} - \pi_T^k(\nabla\cdot\underline{u}))\underline{n}_{TF}, \underline{\mathsf{v}}_T - \underline{\mathsf{v}}_F)_F - (2\mu)s_h(\widehat{\underline{\mathsf{u}}}_h, \underline{\mathsf{v}}_h) := \mathfrak{T}_1 + \mathfrak{T}_2 + \mathfrak{T}_3.\end{aligned} \tag{56}$$

To estimate $\mathfrak{T}_1$, we use the Cauchy–Schwarz inequality, the approximation property (24) of $\underline{r}_T^k \underline{I}_T^k$, the mesh regularity properties (3) and (4), and the first inequality in (40) to infer that

$$|\mathfrak{T}_1| \lesssim 2\mu h^{k+1}\|\underline{u}\|_{H^{k+2}(\mathcal{T}_h)^d}\|\underline{\mathsf{v}}_h\|_{\varepsilon,h} \lesssim 2\mu h^{k+1}\|\underline{u}\|_{H^{k+2}(\mathcal{T}_h)^d}\|\underline{\mathsf{v}}_h\|_{E,s,h}. \tag{57}$$

Proceeding similarly for $\mathfrak{T}_2$ using the approximation property (7) of $\pi_T^k$ yields

$$|\mathfrak{T}_2| \lesssim \lambda h^{k+1}\|\nabla\cdot\underline{u}\|_{H^{k+1}(\mathcal{T}_h)}\|\underline{\mathsf{v}}_h\|_{E,s,h}. \tag{58}$$

To estimate $\mathfrak{T}_3$, we observe that, for all $T \in \mathcal{T}_h$ and all $F \in \mathcal{F}_T$,

$$\begin{aligned} h_F^{-1/2}\|\pi_F^k(\underline{R}_T^k\mathsf{L}_T\widehat{\underline{\mathsf{u}}}_h - \widehat{\underline{\mathsf{u}}}_F)\|_F &\leqslant h_F^{-1/2}\|\underline{R}_T^k\mathsf{L}_T\widehat{\underline{\mathsf{u}}}_h - \underline{u}\|_F \\ &= h_F^{-1/2}\|(\underline{\breve{u}}_T - \underline{u}) - \pi_T^k(\underline{\breve{u}}_T - \underline{u})\|_F \\ &\lesssim h_F^{-1/2}\|\underline{\breve{u}}_T - \underline{u}\|_F + C_{\mathrm{tr}}h_F^{-1}\|\underline{\breve{u}}_T - \underline{u}\|_T \\ &\lesssim h_T^{k+1}\|\underline{u}\|_{H^{k+2}(T)^d}, \end{aligned} \tag{59}$$

where we have used the fact that $\pi_F^k$ is a projector and the definition of $\widehat{\mathsf{u}}_F$ in the first line, the definition (34) of $\underline{R}_T^k$ and that of $\widehat{\mathsf{u}}_T$ in the second line, the triangle inequality, the discrete trace inequality (5), and the fact that $\pi_T^k$ is a projector in the third line, and the approximation property (24) of $\underline{r}_T^k I_T^k$ and the mesh regularity property (3) in the fourth line. Hence, recalling the definition (36) of $s_h$ and using the above inequality together with the bound (4) on card($\mathcal{F}_T$), we infer that

$$s_h(\widehat{\underline{\mathsf{u}}}_h, \widehat{\underline{\mathsf{u}}}_h) = \sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T} h_F^{-1}\|\pi_F^k(\underline{R}_T^k \mathsf{L}_T\widehat{\underline{\mathsf{u}}}_h - \underline{\mathsf{u}}_F)\|_F^2 \lesssim h^{2(k+1)}\|\underline{u}\|_{H^{k+2}(\mathcal{T}_h)^d}^2. \tag{60}$$

Using the symmetry and positivity of $s_h$, the bound (60), and the fact that $s_h(\underline{\mathsf{v}}_h, \underline{\mathsf{v}}_h)^{1/2} \leqslant \|\underline{\mathsf{v}}_h\|_{E,s,h}$, we infer that

$$|\mathfrak{T}_3| \leqslant (2\mu) s_h(\widehat{\underline{\mathsf{u}}}_h, \widehat{\underline{\mathsf{u}}}_h)^{1/2} s_h(\underline{\mathsf{v}}_h, \underline{\mathsf{v}}_h)^{1/2} \lesssim 2\mu h^{k+1}\|\underline{u}\|_{H^{k+2}(\mathcal{T}_h)^d}\|\underline{\mathsf{v}}_h\|_{E,s,h}. \tag{61}$$

The estimate (49) follows using inequalities (57), (58), and (61) to estimate the right-hand side of (56), and using the resulting bound in (52). □

### 5.2 $L^2$-error estimate for the displacement

In this section, we bound the $\|\cdot\|$-norm of the displacement error. We assume elliptic regularity in the following form: For all $\underline{g} \in L^2(\Omega)^d$, the unique solution of

$$\begin{aligned} -\nabla\cdot\underline{\underline{\varsigma}} &= \underline{g} && \text{in } \Omega, \\ \underline{\underline{\varsigma}} &= 2\mu\nabla_{\mathrm{s}}\underline{z} + \lambda(\nabla\cdot\underline{z})\underline{\underline{I}}_d && \text{in } \Omega, \\ \underline{z} &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{62}$$

satisfies the a priori estimate

$$\mu\|\underline{z}\|_{H^2(\Omega)^d} + \lambda\|\nabla\cdot\underline{z}\|_{H^1(\Omega)} \leqslant C_\mu\|\underline{g}\|. \tag{63}$$

**Theorem 11** ($L^2$-error estimate for the displacement)**.** *Under the assumptions of Theorem 9 and the elliptic regularity assumption* (63)*, the following holds:*

$$\|\underline{e}_h\| \leqslant Ch^{k+2}\left(\|\underline{u}\|_{H^{k+2}(\mathcal{T}_h)^d} + \|\underline{\underline{\sigma}}\|_{H^{k+1}(\mathcal{T}_h)^{d\times d}}\right), \tag{64}$$

*with displacement error $\underline{e}_h \in \mathbb{P}_d^k(\mathcal{T}_h)^d$ such that $\underline{e}_{h|T} := \underline{\mathsf{u}}_T - \pi_T^k\underline{u}$ for all $T \in \mathcal{T}_h$, and where $C > 0$ is a real number depending on $\Omega$, $\mu$, and $\varrho$, but independent of $\lambda$ and $h$.*

*Proof.* In the proof, we abbreviate by $a \lesssim b$ the inequality $a \leqslant Cb$ with real number $C > 0$ independent of $h$ and $\lambda$, but possibly depending on $\mu$. Since (48) and (41) imply that $\|\cdot\|_{\mathrm{en},h} \geqslant (2\mu)^{1/2}\|\cdot\|_{E,s,h}$ and $\|\cdot\|_{E,s,h} \geqslant \eta^{1/2}\|\cdot\|_{\varepsilon,h}$, respectively, we infer from the error estimate (49) that, with $\underline{\mathsf{e}}_h := \underline{\mathsf{u}}_h - \widehat{\underline{\mathsf{u}}}_h \in \underline{\mathsf{U}}_{h,0}^k$ and $B(\underline{u}, \underline{\underline{\sigma}}, k) := \|\underline{u}\|_{H^{k+2}(\mathcal{T}_h)^d} + \|\underline{\underline{\sigma}}\|_{H^{k+1}(\mathcal{T}_h)^{d\times d}}$,

$$\|\underline{\mathsf{e}}_h\|_{\varepsilon,h} + s_h(\underline{\mathsf{e}}_h, \underline{\mathsf{e}}_h)^{1/2} \lesssim h^{k+1}B(\underline{u}, \underline{\underline{\sigma}}, k). \tag{65}$$

Consider the auxiliary problem (62) with $\underline{g} := \underline{e}_h$. Integrating by parts element-wise, we infer that

$$\|\underline{e}_h\|^2 = -\sum_{T\in\mathcal{T}_h}(\underline{\mathsf{e}}_T, \nabla\cdot\underline{\underline{\varsigma}})_T = \sum_{T\in\mathcal{T}_h}\left[(\nabla_{\mathrm{s}}\underline{\mathsf{e}}_T, \underline{\underline{\varsigma}})_T + \sum_{F\in\mathcal{F}_T}(\underline{\mathsf{e}}_F - \underline{\mathsf{e}}_T, \underline{\underline{\varsigma}}\underline{n}_{TF})_F\right], \tag{66}$$

where we have used the continuity of the normal component of $\underline{\underline{\varsigma}}$ across interfaces together with the fact that $\underline{\mathsf{e}}_F \equiv \underline{0}$ for all $F \in \mathcal{F}_h^{\mathrm{b}}$ to infer that $\sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T}(\underline{\mathsf{e}}_F, \underline{\underline{\varsigma}}\underline{n}_{TF})_F = 0$. Let $\widehat{\underline{\mathsf{z}}}_h := I_h^k\underline{z}$ and observe that, for all $T \in \mathcal{T}_h$,

$$\|\nabla_{\mathrm{s}}\underline{z} - \underline{\underline{E}}_h^k\mathsf{L}_T\widehat{\underline{\mathsf{z}}}_h\|_T + h_T^{1/2}\|\nabla_{\mathrm{s}}\underline{z} - \underline{\underline{E}}_h^k\mathsf{L}_T\widehat{\underline{\mathsf{z}}}_h\|_{\partial T} \lesssim h_T\|\underline{z}\|_{H^2(T)^d}, \tag{67a}$$

$$\|\nabla{\cdot}\underline{z} - D_h^k\mathsf{L}_T\widehat{\underline{\mathsf{z}}}_h\|_T + h_T^{1/2}\|\nabla{\cdot}\underline{z} - D_h^k\mathsf{L}_T\widehat{\underline{\mathsf{z}}}_h\|_{\partial T} \lesssim h_T\|\nabla{\cdot}\underline{z}\|_{H^1(T)}, \tag{67b}$$

$$s_T(\mathsf{L}_T\widehat{\underline{\mathsf{z}}}_h, \mathsf{L}_T\widehat{\underline{\mathsf{z}}}_h)^{1/2} \lesssim h_T\|\underline{z}\|_{H^2(T)^d}. \tag{67c}$$

Estimate (67a) is proved as in the proof of Lemma 3; estimate (67b) results from the local commuting diagram of Proposition 4; estimate (67c) is proved as in the proof of Theorem 9. Since $a_h(\underline{\mathsf{e}}_h, \widehat{\underline{\mathsf{z}}}_h) = \mathcal{E}_h(\widehat{\underline{\mathsf{z}}}_h)$ with $\mathcal{E}_h(\widehat{\underline{\mathsf{z}}}_h) = l_h(\widehat{\underline{\mathsf{z}}}_h) - a_h(\widehat{\underline{\mathsf{u}}}_h, \widehat{\underline{\mathsf{z}}}_h)$, we decompose the error as follows:

$$\|\underline{e}_h\|^2 = \left\{\sum_{T\in\mathcal{T}_h}\left[(\nabla_{\mathrm{s}}\underline{\mathsf{e}}_T, \underline{\underline{\varsigma}})_T + \sum_{F\in\mathcal{F}_T}(\underline{\mathsf{e}}_F - \underline{\mathsf{e}}_T, \underline{\underline{\varsigma}}\underline{n}_{TF})_F\right] - a_h(\underline{\mathsf{e}}_h, \widehat{\underline{\mathsf{z}}}_h)\right\} + \mathcal{E}_h(\widehat{\underline{\mathsf{z}}}_h) := \mathfrak{T}_1 + \mathfrak{T}_2. \tag{68}$$

For all $T \in \mathcal{T}_h$, using the definition (19a) of $\underline{\underline{E}}_h^k\mathsf{L}_T\underline{\mathsf{e}}_h$ with $\nabla_{\mathrm{s}}\underline{w} = \underline{\underline{E}}_T^k\mathsf{L}_T\widehat{\underline{\mathsf{z}}}_h$ and the definition (30b) of $D_h^k\mathsf{L}_T\underline{\mathsf{e}}_h$ with $q = D_T^k\mathsf{L}_T\widehat{\underline{\mathsf{z}}}_h$, we infer that

$$a_h(\underline{\mathsf{e}}_h, \widehat{\underline{\mathsf{z}}}_h) = \sum_{T\in\mathcal{T}_h}\left\{(\nabla_{\mathrm{s}}\underline{\mathsf{e}}_T, \underline{\underline{S}}_T^k)_T + \sum_{F\in\mathcal{F}_T}(\underline{\mathsf{e}}_F - \underline{\mathsf{e}}_T, \underline{\underline{S}}_T^k\underline{n}_{TF})_F\right\} + (2\mu)s_h(\underline{\mathsf{e}}_h, \widehat{\underline{\mathsf{z}}}_h), \tag{69}$$

with $\underline{\underline{S}}_T^k := 2\mu\underline{\underline{E}}_T^k\mathsf{L}_T\widehat{\underline{\mathsf{z}}}_h + \lambda D_T^k\mathsf{L}_T\widehat{\underline{\mathsf{z}}}_h\underline{\underline{I}}_d$. Plugging this expression into $\mathfrak{T}_1$, multiple uses of the Cauchy–Schwarz inequality together with mesh regularity properties (3) and (4) yield, with $\delta_T(\underline{z}) := \underline{\underline{\varsigma}} - \underline{\underline{S}}_T^k\mathsf{L}_T\widehat{\underline{\mathsf{z}}}_h$,

$$|\mathfrak{T}_1| \leqslant \left\{\|\underline{\mathsf{e}}_h\|_{\varepsilon,h}^2 + s_h(\underline{\mathsf{e}}_h, \underline{\mathsf{e}}_h)\right\}^{1/2} \times \left\{\sum_{T\in\mathcal{T}_h}\left\{\|\delta_T(\underline{z})\|_T^2 + h_T\|\delta_T(\underline{z})\|_{\partial T}^2\right\} + (2\mu)^2 s_h(\widehat{\underline{\mathsf{z}}}_h, \widehat{\underline{\mathsf{z}}}_h)\right\}^{1/2}.$$

Owing to the estimate (65), the approximation properties (67), and the regularity estimate (63), we infer that

$$|\mathfrak{T}_1| \lesssim h^{k+2}B(\underline{u}, \underline{\underline{\sigma}}, k)\left(\|\underline{z}\|_{H^2(\Omega)^d} + \lambda\|\nabla{\cdot}\underline{z}\|_{H^1(\Omega)}\right) \lesssim h^{k+2}B(\underline{u}, \underline{\underline{\sigma}}, k)\|\underline{e}_h\|. \tag{70}$$

Consider now $\mathfrak{T}_2$. Adding $(\underline{\underline{\sigma}}, \nabla_{\mathrm{s}}\underline{z}) - (\underline{f}, \underline{z}) = 0$ and since $l_h(\widehat{\underline{\mathsf{z}}}_h) = (\underline{f}, \pi_h^k\underline{z}) = (\pi_h^k\underline{f}, \underline{z})$, we infer that

$$\begin{aligned}\mathcal{E}_h(\widehat{\underline{\mathsf{z}}}_h) &= (\pi_h^k\underline{f} - \underline{f}, \underline{z}) + \left\{(\underline{\underline{\sigma}}, \nabla_{\mathrm{s}}\underline{z}) - (2\mu)(\underline{\underline{E}}_h^k\widehat{\underline{\mathsf{u}}}_h, \underline{\underline{E}}_T^k\widehat{\underline{\mathsf{z}}}_h) - \lambda(D_h^k\widehat{\underline{\mathsf{u}}}_h, D_h^k\widehat{\underline{\mathsf{z}}}_h)\right\} - (2\mu)s_h(\widehat{\underline{\mathsf{u}}}_h, \widehat{\underline{\mathsf{z}}}_h)\\ &:= \mathfrak{T}_{2,1} + \mathfrak{T}_{2,2} + \mathfrak{T}_{2,3}.\end{aligned} \tag{71}$$

Since $k \geqslant 1$, we infer that

$$|\mathfrak{T}_{2,1}| = |(\pi_h^k\underline{f} - \underline{f}, \underline{z} - \pi_h^1\underline{z})| \lesssim h^{k+2}\|\underline{f}\|_{H^k(\Omega)^d}\|\underline{z}\|_{H^2(\Omega)^d}. \tag{72}$$

Using the orthogonality relation (25) and the right commuting diagram of Proposition 4 implying that $D_h^k\widehat{\underline{\mathsf{z}}}_h = \pi_h^k(\nabla{\cdot}\underline{z})$, we infer that

$$\mathfrak{T}_{2,2} = (2\mu)(\nabla_{\mathrm{s}}\underline{u} - \underline{\underline{E}}_h^k\widehat{\underline{\mathsf{u}}}_h, \nabla_{\mathrm{s}}\underline{z} - \underline{\underline{E}}_h^k\widehat{\underline{\mathsf{z}}}_h) + \lambda(\nabla{\cdot}\underline{u} - D_h^k\widehat{\underline{\mathsf{u}}}_h, \nabla{\cdot}\underline{z} - D_h^k\widehat{\underline{\mathsf{z}}}_h).$$

Hence, using the approximation properties (24) and (7) of $\underline{r}_T^k I_T^k$ and $\pi_T^k$, respectively, to bound the terms with $\underline{u}$ and using (67a)-(67b) to bound the terms with $\underline{z}$ leads to

$$|\mathfrak{T}_{2,2}| \lesssim h^{k+2} B(\underline{u}, \underline{\underline{\sigma}}, k) \left( \|\underline{z}\|_{H^2(\Omega)^d} + \lambda \|\nabla \cdot \underline{z}\|_{H^1(\Omega)} \right). \tag{73}$$

Finally, using the symmetry and positivity of $s_h$, the bound (60) on $s_h(\widehat{\underline{\mathsf{u}}}_h, \widehat{\underline{\mathsf{u}}}_h)$, and the bound (67c) on $s_h(\widehat{\underline{\mathsf{z}}}_h, \widehat{\underline{\mathsf{z}}}_h)$, we infer that

$$s_h(\widehat{\underline{\mathsf{u}}}_h, \widehat{\underline{\mathsf{z}}}_h) \leqslant s_h(\widehat{\underline{\mathsf{u}}}_h, \widehat{\underline{\mathsf{u}}}_h)^{1/2} s_h(\widehat{\underline{\mathsf{z}}}_h, \widehat{\underline{\mathsf{z}}}_h)^{1/2} \lesssim h^{k+2} B(\underline{u}, \underline{\underline{\sigma}}, k) \|\underline{z}\|_{H^2(\Omega)^d}. \tag{74}$$

Using (72)–(74) and the regularity estimate (63), we obtain

$$|\mathfrak{T}_2| \lesssim h^{k+2} B(\underline{u}, \underline{\underline{\sigma}}, k) \|\underline{e}_h\|. \tag{75}$$

The estimate (64) follows using (70) and (75) to bound the right-hand side of (68). □

# 6 Implementation and numerical study

In this section we discuss some implementation aspects and present a numerical study to assess the theoretical results. The numerical efficiency in terms of CPU cost is also evaluated.

## 6.1 Implementation

An essential step in the implementation consists in selecting a basis for each of the polynomial spaces that appear in the construction. Let $T \in \mathcal{T}_h$ and denote by $\underline{x}_T$ a point with respect to which $T$ is star-shaped (in the numerical tests, the barycenter of $T$ was used). As a basis for $\mathbb{P}_d^l(T)$, $l \in \{k, k+1\}$, we take, letting $A^l := \{\underline{\alpha} = (\alpha_i)_{1 \leqslant i \leqslant d} \in \mathbb{N}^d \mid \|\underline{\alpha}\|_{\ell^1} \leqslant l\}$,

$$\mathcal{B}_T^l := \left\{ \prod_{i=1}^d \xi_{T,i}^{\alpha_i} \,\middle|\, \underline{\alpha} \in A^l, \quad \xi_{T,i} := \frac{x_i - x_{T,i}}{h_T} \quad \forall 1 \leqslant i \leqslant d \right\}, \tag{76}$$

i.e., the basis $\mathcal{B}_T^l$ is spanned by monomials in the translated and scaled coordinate variables $(\xi_{T,i})_{1 \leqslant i \leqslant d}$. A basis for the polynomial space of vector-valued functions $\mathbb{P}_d^l(T)^d$ is then obtained by the Cartesian product of $\mathcal{B}_T^l$. Similarly, for all $F \in \mathcal{F}_h$, we can define for $\mathbb{P}_{d-1}^k(F)$ a basis $\mathcal{B}_F^k$ of monomials with respect to a local frame scaled using the face diameter and a point with respect to which $F$ is star-shaped. A basis for $\mathbb{P}_{d-1}^k(F)^d$ is obtained by Cartesian product.

Equation (76) defines hierarchical bases, so that we can construct and evaluate $\mathcal{B}_T^{k+1}$ (required to solve (20)) at quadrature nodes and obtain $\mathcal{B}_T^k$ (used to solve (30)) by simply discarding the higher-order functions. The constraints in (18) are accounted for in problem (20) as follows: the zero-average condition for the displacement is replaced by the requirement that functions vanish at $\underline{x}_T$, and the latter condition is incorporated by simply discarding the constant function in $\mathcal{B}_T^{k+1}$; the zero-average condition for the skew-symmetric part of the gradient is enforced using a Lagrange multiplier (which is scalar-valued for $d = 2$ and $\mathbb{R}^3$-valued for $d = 3$). Moreover, the homogeneous Dirichlet boundary condition is also enforced by means of a Lagrange multiplier in $\mathbb{P}_{d-1}^k(F)^d$ for all $F \in \mathcal{F}_h^{\mathrm{b}}$.

Concerning numerical integration, in the two-dimensional case we can exploit the decomposition of elements into triangles and use standard quadrature rules. In our implementation, we have used the quadrature rules available in GetFem++ [21]. In the three-dimensional case, this is also possible provided the faces of the elements are triangles or quadrangles yielding pyramidal subelements for which standard cubature rules are available. If this is not the case, a simplicial decomposition of the element can be considered, usually implying an increase in the number of quadrature nodes. Similarly, numerically integrating on the mesh faces is straightforward in two space dimensions and for elements with triangular or quadrangular faces in three space dimensions. For more general polygonal faces in three space dimensions, triangulating the face may be required.

### 6.2 Numerical study

To numerically validate the estimates of Theorems 9 and 11, we solve the two-dimensional, pure-displacement problem with $\mu = 1$, $\lambda \in \{1, 1000\}$, displacement $\underline{u} = (u_1, u_2)$ such that

$$u_1 = \sin(\pi x_1)\sin(\pi x_2) + \frac{1}{2\lambda}x_1, \qquad u_2 = \cos(\pi x_1)\cos(\pi x_2) + \frac{1}{2\lambda}x_2, \tag{77}$$

and load $\underline{f} = (f_1, f_2)$ such that

$$f_1 = 2\pi^2 \sin(\pi x_1)\sin(\pi x_2), \qquad f_2 = 2\pi^2 \cos(\pi x_1)\cos(\pi x_2).$$

The solution (77) has vanishing divergence in the limit $\lambda \to +\infty$. This, together with the fact that $\underline{f}$ does not depend on $\lambda$, make this test case suitable to check numerically that the estimates (49) and (64) are indeed uniform in $\lambda$. We consider the three families of meshes depicted in Figure 1: the matching triangular and Kershaw mesh families of [17] and the (predominantly) hexagonal mesh family considered in [12, Section 4.2.3]. The stress error is measured in the energy norm defined by (48). For the displacement, the error is the one appearing on the left-hand side of (64). The convergence rates displayed in Figure 2 and 3 for $\lambda = 1$ and $\lambda = 1000$, respectively, are in agreement with the theoretical predictions. The slight superconvergence observed for the Kershaw mesh family is due to the fact that the mesh regularity increases when refining. As predicted, the error does not depend on $\lambda$.

To check the performance of the proposed method in terms of CPU time, we have instrumented our code in the spirit of [11] to separately measure (i) the *assembly time* $\tau_{\text{ass}}$, accounting for the construction of the local contributions to the bilinear form $a_h$ (cf. (36)), the local elimination of cell unknowns, and the assembly into a global matrix; (ii) the *solution time* $\tau_{\text{sol}}$ corresponding to the solution of the global linear system. Local computations are based on the linear algebra facilities provided by the Eigen3 library [15]. The linear systems corresponding to problems (20), (30), and to the $L^2$-orthogonal projectors $\pi_T^k$ and $\pi_F^k$ are solved using the robust Cholesky factorization available in Eigen3. The global system (involving face unknowns only) is solved using SuperLU [8] through the PETSc 3.4 interface [1]. The tests have been run sequentially on a laptop computer powered by an Intel Core i7-3520 CPU running at 2.90 GHz and equipped with 8Gb of RAM.

To check how the more elaborate local computations (with respect, e.g., to standard finite elements) affect the overall CPU cost, we plot in Figure 4 the ratio $\tau_{\text{ass}}/\tau_{\text{sol}}$ as a function of $\text{card}(\mathcal{F}_h)$. We consider the triangular and hexagonal mesh families for which $N_\partial$ (cf. (4)) is respectively the smallest and the largest. It can be observed that, when refining the mesh, the ratio $\tau_{\text{ass}}/\tau_{\text{sol}}$ rapidly decreases as a result of having (approximately) $\tau_{\text{ass}} \propto \text{card}(\mathcal{F}_h)$ and $\tau_{\text{sol}} \propto \text{card}(\mathcal{F}_h)^{3/2}$. This means that, in large test cases, the local computations can be expected to have a negligible impact on the global CPU time.

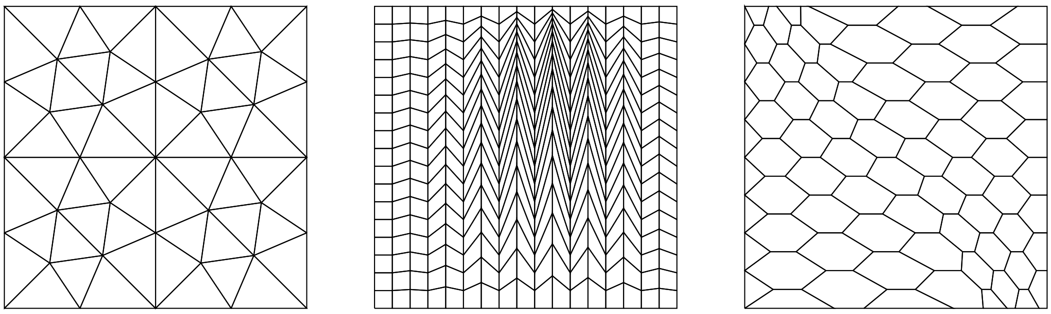

Figure 1: Triangular, Kershaw, and hexagonal meshes for the numerical example of Section 6

Figures 5 and 6 depict the stress and displacement errors as a function of the total CPU time $\tau_{\text{tot}} := \tau_{\text{ass}} + \tau_{\text{sol}}$. This representation is included to provide a fair basis of future comparison with other methods. As expected from the regularity of the exact solution (77), the highest-order computation provides in all the cases the best precision for a given CPU time as well as the largest reduction rate for the error.

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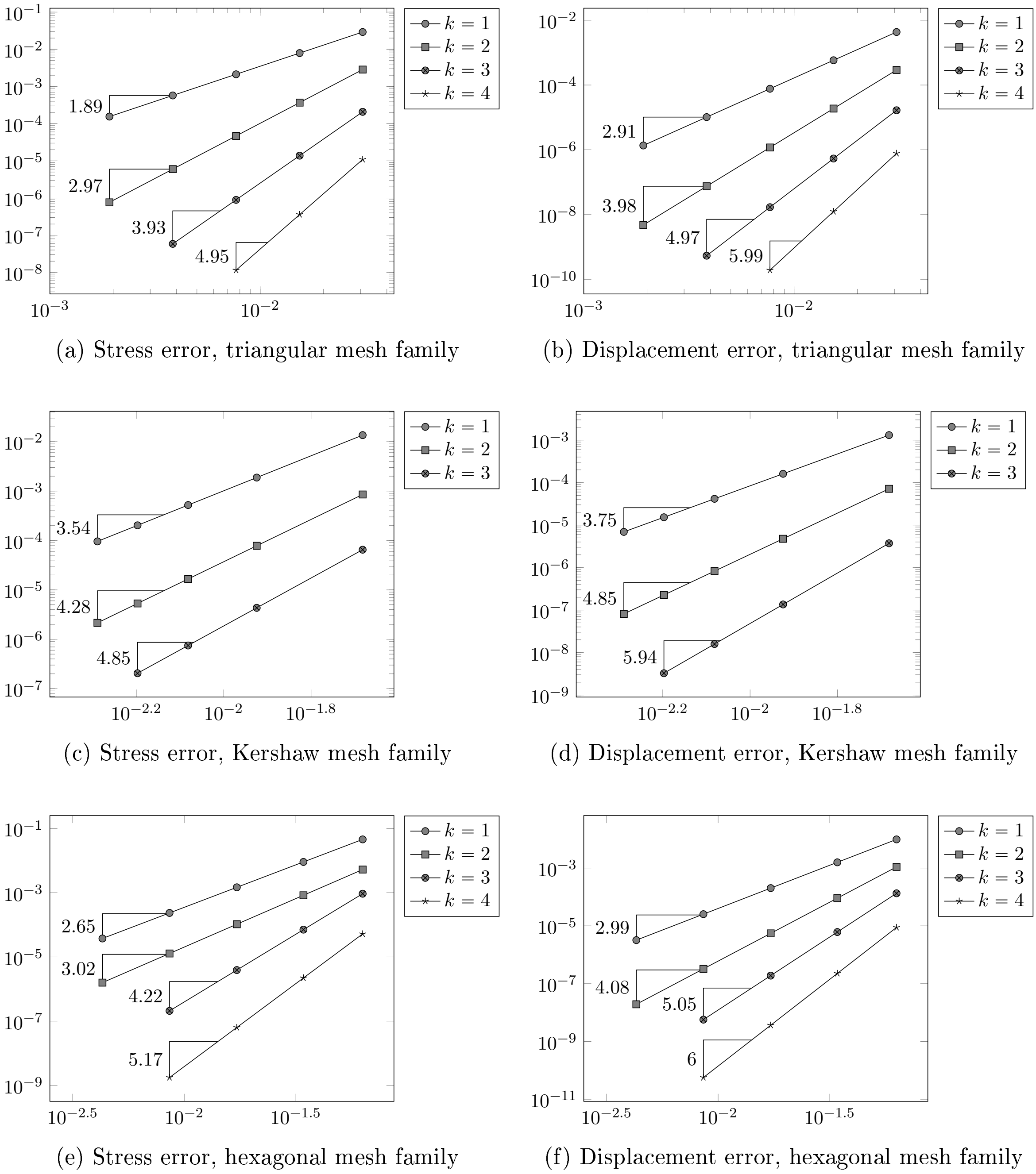



(a) Stress error, triangular mesh family

(b) Displacement error, triangular mesh family

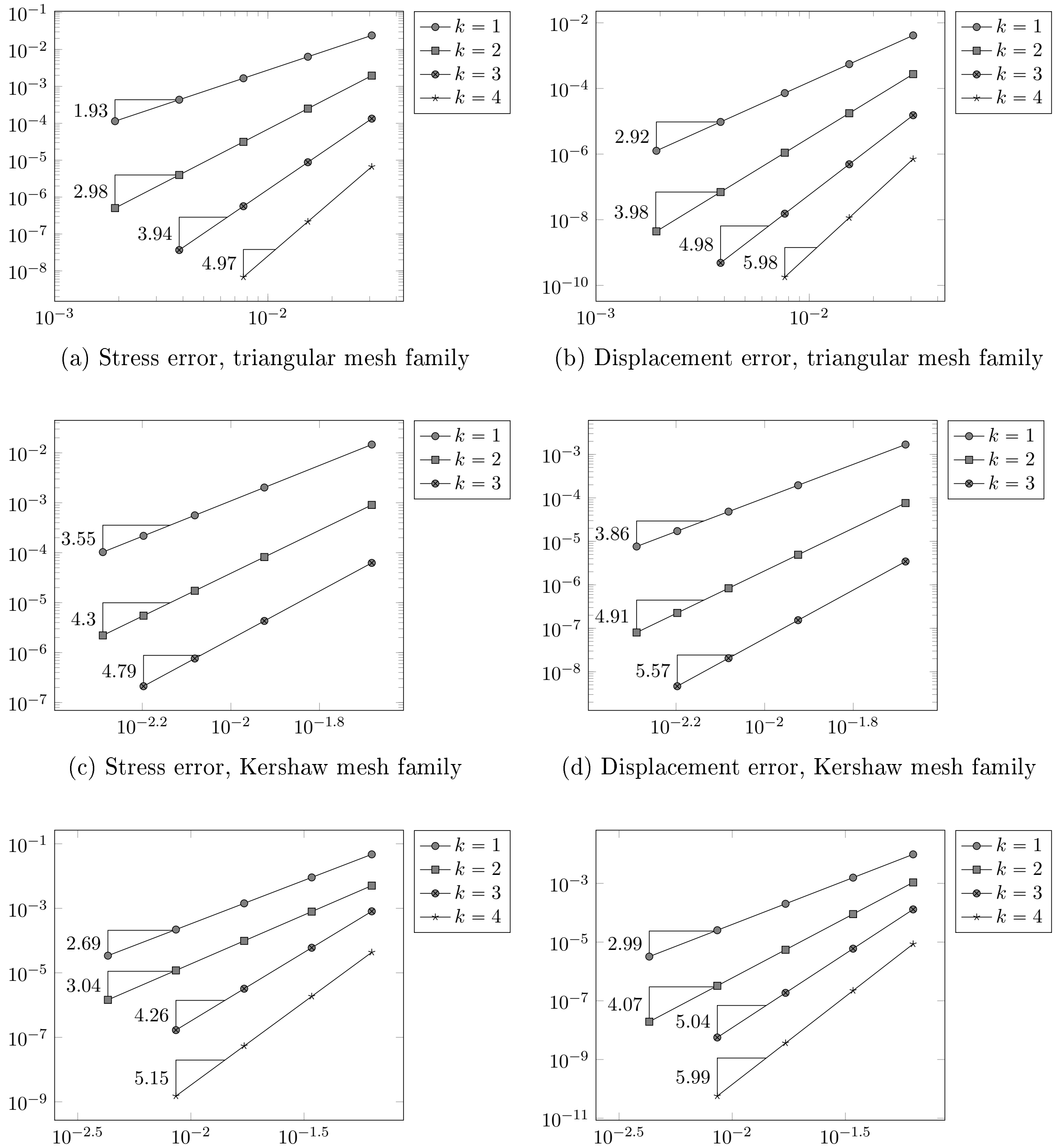
(c) Stress error, Kershaw mesh family

(d) Displacement error, Kershaw mesh family

(e) Stress error, hexagonal mesh family

(f) Displacement error, hexagonal mesh family

Figure 2: Errors vs. $h$ for $\lambda = 1$




(a) Stress error, triangular mesh family

(b) Displacement error, triangular mesh family

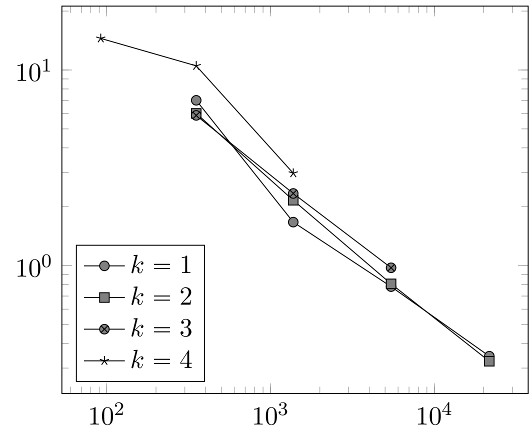
(c) Stress error, Kershaw mesh family

(d) Displacement error, Kershaw mesh family

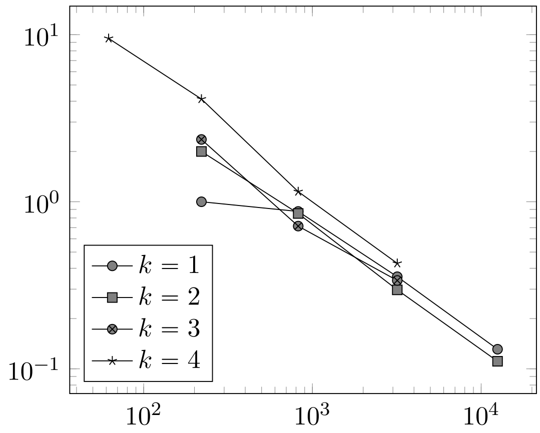
(e) Stress error, hexagonal mesh family

(f) Displacement error, hexagonal mesh family

Figure 3: Errors vs. $h$ for $\lambda = 10^3$

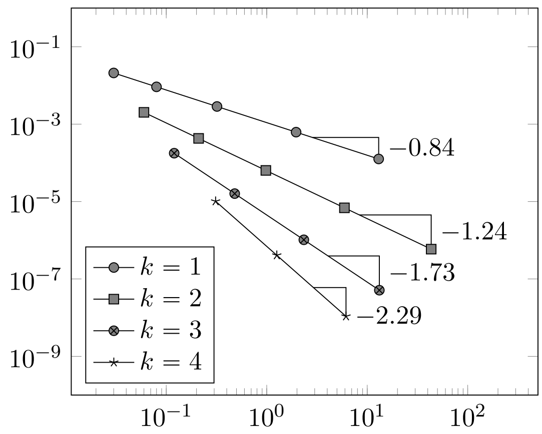



(a) Triangular mesh family

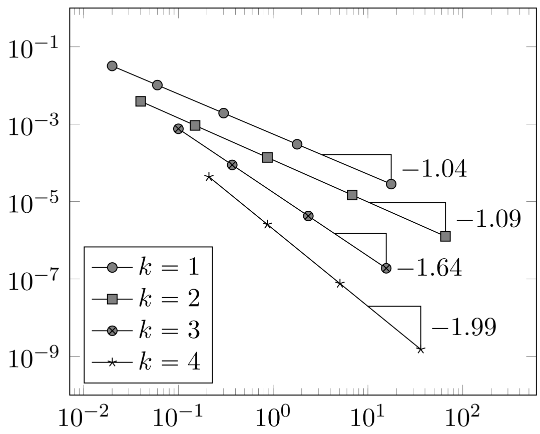



(b) Hexagonal mesh family

Figure 4: $\tau_{\text{ass}}/\tau_{\text{sol}}$ vs. $\text{card}(\mathcal{F}_h)$

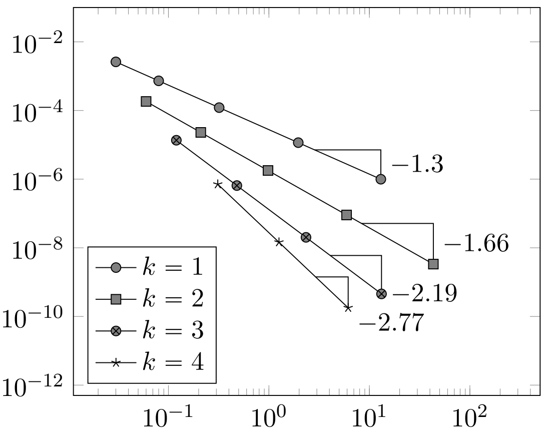



(a) Triangular mesh family

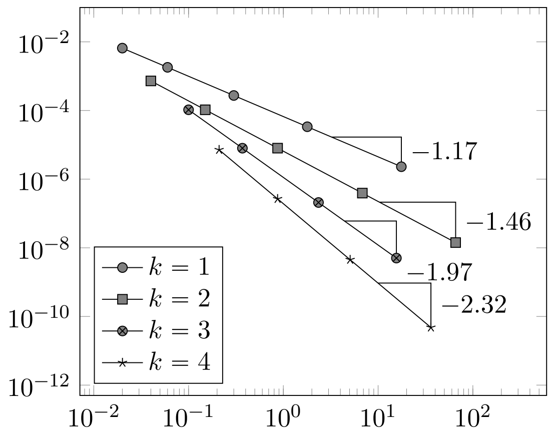



(b) Hexagonal mesh family

Figure 5: Stress error vs. $\tau_{\text{tot}}$ (s)




(a) Triangular mesh family




(b) Hexagonal mesh family

Figure 6: Displacement error vs. $\tau_{\text{tot}}$ (s)

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