UNIVERSAL POSITIVE MASS THEOREMS

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ABSTRACT. In this paper, we recast Witten's proof of the positive mass theorem in a more abstract setting. This enables us to prove that it is not specific of the spinor bundle nor of its Dirac operator, and we show that any choice of an irreducible natural bundle and a very large choice of first-order operators lead to a positive mass theorem along the same lines if the necessary curvature conditions are satisfied.

INTRODUCTION

Mass is the most fundamental invariant of asymptotically flat manifolds. Originally defined by physicists in General Relativity, it has played a leading role in many mathematical contexts such as conformal geometry [15] or 3-dimensional Riemannian geometry [1, 4]. The most important feature of mass is its positivity in the presence of nonnegative scalar curvature and the subsequent rigidity statement (zero mass implies flatness), proved first by Schoen and Yau in dimensions between 3 and 7 with the help of minimal surfaces [20].

In 1981, Witten introduced a striking new method to prove positive mass theorems, relying on the use of the spinor bundle and the Lichnerowicz-Schrödinger formula [22] for the Dirac operator. Its efficiency made it usable in a variety of other contexts. A lot of generalizations of mass have been introduced in the recent years, and almost all statements of their positivity were proved along these lines, see [8, 9, 16, 17, 18] for instance.

The goal of this short paper is to elucidate further the underlying idea of this approach. Our main result is that any argument similar to that of Witten but applied to any other natural bundle and operator giving rise to a Bochner-type formula *always* leads to a resul of the same flavour, *i.e.* a positive mass theorem. We however believe that the main interest of the present paper lies in the method rather than in the positivity statements themselves, which may (and indeed do) lack of geometric motivation. Our hope is that the analysis performed here for the mass may be found useful for proving positivity of newly defined asymptotic invariants.

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1. Mass and Witten's proof of its positivity

Definition 1.1. An asymptotically flat manifold is a complete Riemannian manifold (M, g) of dimension $n \geqslant 3$ such that there exists a diffeomorphism Φ from the complement of a compact set K in M into the complement of a ball B in $\mathbb{R}^n$ (called a chart at infinity), such that, in these coordinates,

$$|g_{ij} - \delta_{ij}| = O(r^{-\tau}), \quad |\partial_k g_{ij}| = O(r^{-\tau-1}), \quad |\partial_k \partial_{_{e}ll} g_{ij}| = O(r^{-\tau-2}).$$

Definition 1.2. If $\tau > \frac{n-2}{2}$ and the scalar curvature is integrable, the quantity

$$m(g) = \lim_{r\to\infty} \int_{S_r} (\operatorname{div}_0 g - d\operatorname{tr}_0 g)(\nu)\, d\mathrm{vol}_{s_r} \tag{1.1}$$

(where ν denotes the field of outer unit normals to the coordinate spheres S_r and the subscript $\cdot_0$ refers to the euclidean metric in the given chart at infinity) exists and is independent of the chart chosen around infinity [3]. The number $m(g)$ is called the *mass* of the asymptotically flat manifold (M, g).

The positive mass theorem states that if the scalar curvature is nonnegative, then its mass is nonnegative, and it vanishes only if it is isometric to the euclidean space. Witten's approach for its proof can be described as follows: given any asymptotically flat manifold, one finds (by an analytical method in Witten's case, but this may not be an obligation) a spinor field ϕ satisfying

$$\mathfrak{D}\phi = 0, \quad \phi \to_\infty \phi_0 \tag{1.2}$$

where $\mathfrak{D}$ is the Dirac operator and ϕ_0 is any constant spinor on $\mathbb{R}^n$ [3, 19, 22]. The Lichnerowicz-Schrödinger formula then relates the Dirac laplacian $\mathfrak{D}^*\mathfrak{D}$ to the rough laplacian $\nabla^*\nabla$ and the scalar curvature:

$$\nabla^*\nabla + \frac{1}{4}\operatorname{Scal} - \mathfrak{D}^*\mathfrak{D} = 0. \tag{1.3}$$

After an integration over domains bounded by coordinates spheres S_r and an integration by parts, one gets

$$\int_M |\nabla\phi|^2 + \frac{1}{4}\operatorname{Scal}|\phi|^2 - |\mathfrak{D}\phi|^2 = \lim_{r\to\infty}\int_{S_r} b(\phi) \tag{1.4}$$

where b stands here for a boundary term. Analysis of the equations (1.2) above yields that all integrals converge, and moreover (this is the core of Witten's approach) the boundary-at-infinity contribution is given by

$$\lim_{r\to\infty}\int_{S_r} b(\phi) = \frac{1}{4}\lim_{r\to\infty}\int_{S_r}(\operatorname{div}_0 g - d\operatorname{tr}_0 g)(\nu)\,|\phi_0|^2\, d\mathrm{vol}_{s_r} = \frac{1}{4}|\phi_0|^2\, m(g).$$

This proves the positive mass theorem as the left hand side in (1.4) is nonnegative since ϕ is a solution of the Dirac equation.

This idea has been re-used in numerous cases. A specific example which deserves to be quoted is that of Bartnik [3], who showed that the Dirac operator on spinors could be replaced by the operator $d + \delta$ on 1-forms, proving that the mass of any nonnegatively Ricci curved manifold is nonnegative. The key element is the Bochner formula and the mass appears again in the boundary contribution. Another occurrence can be found in Jammes [14] with 2-forms in dimension 4 (interestingly enough, Jammes does not need analysis to solve the equation similar to (1.2) that appears in his proof). The approach also is also the main ingredient to prove positivity of generalizations of mass [8, 9, 16, 17, 18].

The most striking fact in Witten's approach is the appearance of the mass as the boundary contribution in the Lichnerowicz formula for spinors. It is even more surprising that it appears also when other bundles are chosen. One may then wonder whether there is a simple way to express the form of the boundary contribution and for which bundles it can be related to the mass. The goal of this short paper is to analyze this problem. As the arguments below will show, it turns out that once the context is correctly set, the boundary contribution can *always* be connected to the mass, and an explicit formula can be given in terms of representation-theoretic data of the bundle involved without too specific computations.

2. General setting

Let (M, g) be a Riemannian manifold of dimension $n \geqslant 3$ and decay order $\tau > \frac{n-2}{2}$ such that mass is defined. We now consider a natural bundle E on M issued from an *irreducible* representation $(\mathbb{V}, \rho)$ of the special orthogonal group $\mathrm{SO}(n)$ or its universal covering $\mathrm{Spin}(n)$. Thus $E = \mathcal{P} \times_\rho \mathbb{V}$ where $\mathcal{P}$ is either the frame bundle or the spin frame bundle, the manifold being of course assumed to be spin in the latter case.

To give an explicit description of our results, we shall recall a few classical facts of representation theory of the Lie algebra $\mathfrak{so}(n)$. From now on, we shall freely identify elements of $\mathbb{R}^n$ and $(\mathbb{R}^n)^*$, *i.e.* forms and vectors ; this simplification will be used repeatedly and without notice in the sequel of the paper. Thus, for any pair of vectors u_1 and u_2, we denote by $u_1 \wedge u_2$ the element of $\mathfrak{so}(n)$ given by $X \mapsto \langle u_1, X\rangle u_2 - \langle u_2, X\rangle u_1$.

Let $\{e_j\}_{1\leqslant j\leqslant n}$ be an orthonormal basis of $\mathbb{R}^n$ and $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{so}(n)$ generated by $\epsilon_k = e_{2k-1} \wedge e_{2k}$ where k runs from 1 to $m = \left[\frac{n}{2}\right]$. Any representation $\mathbb{V}$ of $\mathfrak{so}(n)$ may be split into eigenspaces for the action of $i\mathfrak{h}$ (whose elements are all simultaneously diagonalizable), which are labelled by elements of $(i\mathfrak{h})^*$ called weights. All the weights appearing for a given representation may be written in coordinates relative to the basis $\{\mu_j\}_{1\leqslant j\leqslant m}$ of $(i\mathfrak{h})^*$ defined by $\mu_j(i\epsilon_k) = \delta_{jk}$, and the largest weight for the lexicographic order is called the *dominant weight* of the representation. The main classification result of the theory then states that representations $\mathbb{V}$ are in one-to-one correspondence with their dominant weights, that are m-uplets $(\rho^1, \cdots, \rho^m)$ in $\left(\frac{1}{2}\mathbb{Z}\right)^m$ such that

$$\rho^1 \geqslant \cdots \geqslant \rho^m \geqslant 0 \text{ if } n = 2m+1, \quad \text{and} \quad \rho^1 \geqslant \cdots \geqslant |\rho^m| \text{ if } n = 2m. \tag{2.5}$$

The tensor product $\mathbb{R}^n \otimes \mathbb{V}$ splits under the action of $\mathfrak{so}(n)$ into N irreducible components:

$$\mathbb{R}^n \otimes \mathbb{V} = \otimes_{j=1}^N \mathbb{W}_j;$$

the representation of $\mathfrak{so}(n)$ on $\mathbb{W}_j$ will be denoted by λ_j. To know which representations appear in the tensor product $\mathbb{R}^n \otimes \mathbb{V}$, one may use the following rule: a weight λ appears as the dominant weight of a summand $\mathbb{W}$ in $\mathbb{R}^n \otimes \mathbb{V}$ iff. it is dominant, *i.e.* it satisfies the conditions given by (2.5), and $\lambda = \rho \pm \mu_i$ for some i, or if $\lambda = \rho$ in the case n is odd and $\rho^m > 0$ [11].

At the bundle level, one obtains the corresponding splitting

$$T^*M \otimes E = \otimes_{j=1}^N F_j.$$

Letting $\Pi_j : \mathbb{R}^n \otimes \mathbb{V} \to \mathbb{W}_j$ be the projection onto the j-th summand (or the analogous map at the bundle level), we shall follow [13] and denote by p_j the *generalized Clifford action* $p_j(X)\sigma = \Pi_j(X \otimes \sigma)$. Each projection induces a natural first-order operator $P_j = \Pi_j \circ \nabla$ (where ∇ is the Levi-Civita connection) sending sections of E into sections of F_j, known as a *Stein-Weiss operator*. The principal symbol of P_j is $p_j(\xi)$ and that of $(P_j)^* P_j$ is $p_j(\xi)^* p_j(\xi)$.

We shall also need the so-called *conformal weight operator*, which plays an important role in conformal geometry, *e.g.* as a simple mean to determine the conformal weights that make the Stein-Weiss operators P_j conformally covariant [10]. It is usually described as the operator $B : \mathbb{R}^n \otimes \mathbb{V} \to \mathbb{R}^n \otimes \mathbb{V}$ defined by

$$B(\alpha \otimes v) = \sum_{i=1}^n e_i \otimes \rho(e_i \wedge \alpha) v,$$

and, as such, is equivariant under the action of $\mathfrak{so}(n)$. The eigenvalues of B are real, hence, by Schur's Lemma, it is a constant on each summand appearing in the decomposition $\mathbb{R}^n \otimes \mathbb{V}$ into irreducibles. For any summand $\mathbb{W}$ induced by a representation λ of $\mathfrak{so}(n)$, the corresponding eigenvalue is

$$w(\lambda, \rho) = \frac{1}{2} \left(c(\lambda) - c(\rho) - c(\tau) \right),$$

where $c(\cdot)$ denotes the Casimir number of a representation of $\mathfrak{so}(n)$ [7, 12]. Note that we define here for any representation ρ its Casimir number $c(\rho) = \langle \rho + \delta, \rho + \delta \rangle - \langle \delta, \delta \rangle$, where δ is the half-sum of the roots, *i.e.* $\delta_k = \frac{n-2k}{2}$. Hence, $c(\tau) = n - 1$ for the standard representation τ on $\mathbb{R}^n$.

The main interest of conformal weights is that they are easy to compute from the knowledge of the dominant weight of $\mathbb{V}$: indeed, one always has $w(\lambda,\rho) = 1 + \rho_i - i$ if, for some i, $\lambda = \rho + \mu_i$, $w(\lambda,\rho) = 1 - n - \rho_i + i$ if $\lambda = \rho - \mu_i$, and $w(\lambda,\rho) = \frac{1-n}{2}$ if $\lambda = \rho$.

The last ingredient is a Bochner-Weitzenböck formula, equivalently a choice of coefficients $(a_j)_{1\leqslant j\leqslant N}$ such that the subsequent linear combination of the operators $(P_j)^*P_j$ is a zeroth-order operator, that is a curvature term:

$$\sum_{j=1}^{N} a_j\,(P_j)^*P_j - \mathcal{R} = 0. \tag{2.6}$$

Weitzenböck formulas are classified. Indeed, finding them is a purely (but somehow tricky) algebraic problem, which reduces to finding coefficients $(a_j)_{1\leqslant j\leqslant N}$ such that the principal symbol of the operator $\sum_{j=1}^{N} a_j\,(P_j)^*P_j$ has vanishing second-order part. It is known that there are $\left[\frac{N}{2}\right]$ linearly independent such formulas [6], and they can be explicitly obtained through Vander-Monde systems [13] or recursive formulas [21].

Assumption 2.1. From now on, coefficients (a_j) are chosen such that they give rise to a Weitzenböck formula (2.6).

Once integrated, using $P_+ = \sum_{a_j>0} \sqrt{a_j}\,P_j$ and $P_- = \sum_{a_j<0} \sqrt{-a_j}\,P_j$ and assuming that all integral terms converge, formula (2.6) leads to an expression similar to (1.4):

$$\int_M |P_-\sigma|^2 + \langle\sigma,\mathcal{R}\sigma\rangle - |P_+\sigma|^2 = \lim_{r\to\infty}\int_{S_r} b(\sigma) \tag{2.7}$$

where b stands again for the boundary contribution. This can be used towards a positive mass theorem if

(1) the boundary term can be related to the mass;
(2) the curvature term is nonnegative in the geometric context at hand;
(3) a section σ of E can be found such that $P_+\sigma = 0$ and all integrals and limits make sense.

Our goal from now on is to give an answer to the question raised by (1). We shall consider here the case where the section σ is asymptotic to a constant section σ_0 of E, and we prove below that the boundary contribution only depends on σ_0 in this case. Its analysis is then turned into a purely algebraic problem, which we shall study in the next sections. We shall not discuss (2) and (3): the validity of (2) of course depends on the context at hand, and an answer to (3) is usually obtained through the resolution of a PDE problem (although this may be useless in some cases, as [14] shows).

To be more precise in our analysis of the question raised by (1), we shall now need a bit more of notation. We define a self-adjoint map H of TM by

$$\langle HX, HY\rangle_g = \langle X, Y\rangle_0 \quad \forall\, X, Y \in TM,$$

where the latter is the euclidean metric on $M \setminus K$ identified to $\mathbb{R}^n \setminus B$. This enables us to transfer sections of bundles over $\mathbb{R}^n \setminus B$ to bundles over $M \setminus K$ (or *vice-versa*) in a metric preserving way : this is of course obvious in the tensor case, and the spinor and mixed cases may be treated as in [5]. Thus it makes sense to speak of *constant sections* of the bundle E w.r.t. to the flat connection over a neighbourhood of infinity. Equivalently, one may transfer the flat connection ∇^0 itself to a flat metric connection (but with torsion) on the (spinor) frame bundle $\mathcal{P}$ or, conversely, the connection ∇ induced by g as a connection (again non torsion-free) on the trivial bundle $(\mathbb{R}^n \setminus B) \times \mathbb{V}$ which is compatible with the euclidean metric. As a result, if $\{e_i\}_{1\leqslant i\leqslant n}$ is a (direct) orthonomal basis of $\mathbb{R}^n$, ω is the connection 1-form of ∇ in the frame $\{He_i\}_{1\leqslant i\leqslant n}$, and σ_0 is a constant section of E (w.r.t. to the flat connection), then $\nabla^0\sigma_0 = 0$ and $\nabla\sigma_0 = \rho(\omega)\sigma_0$.

We now proceed to the computation of the boundary contribution. Letting $A = \sum_j a_j \Pi_j$, our first key result is:

Lemma 2.2. *Let σ_0 be an element of $\mathbb{V}$, seen as a constant section of E over $\mathbb{R}^n$. If σ is a section of E such that $|\sigma - \sigma_0| = O(r^{-a})$ and $\nabla\sigma = O(r^{-a-1})$ for some $a > \frac{n-2}{2}$, then the integrals in (2.7) converge and moreover*

$$\lim_{r\to\infty} \int_{S_r} b(\sigma) \;=\; -\lim_{r\to\infty} \int_{S_r} \langle \nu \otimes \sigma_0, A(\rho(\omega)\sigma_0)\rangle \, d\mathrm{vol}_{S_r} \,.$$

The value $a = \frac{n-2}{2}$ is the lower bound of the necessary decay rates for the following arguments to hold true. It is moreover remarkable that this decay rate is the same as that one can usually infer from a PDE analysis of the equations

$$P_+\sigma = 0, \quad \sigma \to_\infty \sigma_0,$$

in weighted functional spaces such as in [3, 18, 19].

Proof. – From 2.7, one computes on a bounded domain D:

$$\int_D |P_+\sigma|^2 - |P_-\sigma|^2 \;=\; \sum_{j=1}^N a_j \int_D |\Pi_j(\nabla\sigma)|^2 \;=\; \sum_{j=1}^N a_j \int_D \langle \nabla\sigma, \Pi_j(\nabla\sigma)\rangle.$$

It is thus enough to manage the integration by part for a single summand corresponding to an index j in $\{1, ..., N\}$:

$$\begin{aligned}\int_D \langle \nabla\sigma, \Pi_j(\nabla\sigma)\rangle &= \int_D \mathrm{tr}_g \left[\nabla\left(\langle \sigma, \Pi_j(\nabla\sigma)\rangle\right)\right] - \langle \sigma, \mathrm{tr}_g\left[\nabla\left(\Pi_j(\nabla\sigma)\right)\right]\rangle \\ &= \int_{\partial D} \langle \sigma, \Pi_j(\nabla\sigma)\rangle(\nu) - \int_D \langle \sigma, \mathrm{tr}_g\left[\nabla\left(\Pi_j(\nabla\sigma)\right)\right]\rangle \\ &= \int_{\partial D} \langle \nu\otimes\sigma, \Pi_j(\nabla\sigma)\rangle + \int_D \langle \sigma, (P_j)^* P_j(\sigma)\rangle.\end{aligned}$$

Hence,

$$\int_D |P_+\sigma|^2 - |P_-\sigma|^2 = \int_{\partial D} \langle \nu\otimes\sigma, A(\nabla\sigma)\rangle + \int_D \langle \sigma, \mathcal{R}\sigma\rangle.$$

Applying this to any domain enclosed by a coordinated sphere S_r, one eventually gets:

$$\int_{S_r} b(\sigma) = -\int_{S_r} \langle \nu\otimes\sigma, A(\nabla\sigma)\rangle \, d\mathrm{vol}_{S_r}.$$

We now show that the limit as $r\to\infty$ only depends on σ_0. We again argue first with a single projection. On a fixed sphere S, one write $\sigma = \sigma_0+\sigma_1$, and

$$\begin{aligned}\int_S \langle \nu\otimes\sigma, \Pi(\nabla\sigma)\rangle &= \int_S \langle \nu\otimes\sigma_0, \Pi(\nabla\sigma_0)\rangle + \int_S \langle \nu\otimes\sigma_0, \Pi(\nabla\sigma_1)\rangle \\ &\quad + \int_S \langle \nu\otimes\sigma_1, \Pi(\nabla\sigma_0)\rangle + \int_S \langle \nu\otimes\sigma_1, \Pi(\nabla\sigma_1)\rangle.\end{aligned}$$

When $r\to\infty$, the assumptions on σ_1 imply that the last two terms vanish since $\tau > \frac{n-2}{2}$ as their integrands are $O(r^{-\eta})$ with $\eta > n-1$. As regards the second term, one notices that (denoting with a bold dot an absent variable)

$$\begin{aligned}\langle \bullet\otimes\sigma_0, \Pi(\nabla\sigma_1)\rangle &= \sum_{j=1}^N \langle \Pi(\bullet\otimes\sigma_0), \varepsilon_j\otimes\nabla_{\varepsilon_j}\sigma_1\rangle \\ &= \sum_{j=1}^N \nabla_{\varepsilon_j}\left(\langle \Pi(\bullet\otimes\sigma_0), \varepsilon_j\otimes\sigma_1\rangle\right) \\ &\qquad - \sum_{j=1}^N \langle \Pi(\bullet\otimes\nabla_{\varepsilon_j}\sigma_0), \varepsilon_j\otimes\sigma_1\rangle,\end{aligned}$$

where $\{\varepsilon_i\}_{1\leqslant i\leqslant N} = \{He_i\}_{1\leqslant i\leqslant N}$ is the g-orthonormal frame of TM deduced from a (constant) Euclidean basis of $\mathbb{R}^n$. After integrating on S_r and letting r tend to ∞, the same decay considerations as above show that the very last term does not contribute. Moreover, the first one is a divergence, *i.e.* is of the form $*d*\beta$ where β is a 2-form on M and $*$ is the (n-dimensional) Hodge operator. Thus, on any closed manifold S with outer unit normal ν,

$$\int_S (*\,d*\beta)(\nu)\, d\mathrm{vol}_S = \int_S d*\beta = 0.$$

This shows that the only non-zero contribution in the limit comes from the first of the four terms above, as expected. □

3. Main statement

In the context of the previous section, let $(\sigma_\kappa)_{1\leqslant\kappa\leqslant\dim\mathbb{V}}$ be an orthonormal basis of $\mathbb{V}$, and $\mathbf{a} = (a_j)$ be a choice of coefficients satisfying Assumption 2.1. Our main result may then be stated as follows

Theorem 3.1. *There is a constant $\mu(\mathbf{a})$ such that*

$$-\sum_{\kappa=1}^{\dim\mathbb{V}} \lim_{r\to\infty} \int_{S_r} \langle\, \nu\otimes\sigma_\kappa\,,\, A(\rho(\omega)\sigma_\kappa)\,\rangle_0 \, d\mathrm{vol}_{S_r} \;=\; \mu(\mathbf{a})\, m(g).$$

Moreover, $\mu(\mathbf{a})$ only depends on **a** *and representation-theoretic data of $\mathbb{V}$:*

$$\mu(\mathbf{a}) \;=\; -\sum_{j=1}^{N} a_j\, \frac{(\dim\mathbb{W}_j)\, w(\lambda_j,\rho)}{2n(n-1)}\,,$$

where $w(\lambda_j,\rho)$ denotes the conformal weight *of the summand $\mathbb{W}_j$ in $\mathbb{R}^n\otimes\mathbb{V}$.*

The content of this theorem is the following: although it is not always true that the mass is related as the boundary-at-infinity contributions in (2.7) for a single section σ, it always appears when we sum the formula over a basis of constant sections. Coming back to (2.7), and assuming that one may find for each κ a solution $\bar{\sigma}_\kappa$ of

$$P_+\bar{\sigma}_\kappa = 0, \quad \bar{\sigma}_\kappa \to_\infty \bar{\sigma}_\kappa,$$

this means that

$$\mu(\mathbf{a})\, m(g) \;=\; \int_M \sum_{\kappa=1}^{\dim\mathbb{V}} |P_-\bar{\sigma}_\kappa|^2 + \int_M \sum_{\kappa=1}^{\dim\mathbb{V}} \langle\bar{\sigma}_\kappa, \mathcal{R}\bar{\sigma}_\kappa\rangle.$$

Thus, assuming that the curvature operator $\mathcal{R}$ is non-negative, we obtain a *positive mass theorem* if $\mu(\mathbf{a})$ is positive, and a *negative mass theorem* if $\mu(\mathbf{a})$ is negative.

The proof of the Theorem in divided into two parts. The first one is very simple: one notices that the map β_A from $\mathbb{R}^n\otimes\mathfrak{so}(n)$ into $\mathbb{R}^n$ defined by

$$\omega \mapsto \beta_A(\omega) \;=\; -\sum_{\kappa=1}^{\dim\mathbb{V}} \langle\, \cdot\otimes\sigma_\kappa, A(\rho(\omega)\sigma_\kappa)\,\rangle_0$$

is equivariant under the action of $\mathfrak{so}(n)$. Thus Schur's Lemma implies that it is always a multiple of the projection onto the (unique) factor $\mathbb{R}^n$ appearing in the splitting of $\mathbb{R}^n\otimes\mathfrak{so}(n)$ into irreducible summands (it is a simple computation, which we shall omit, to show that there is indeed a single occurrence of the standard representation in this tensor product). Moreover, the computation of this map is well-known in the case $\mathbb{V}$ is the spin representation and **a** is the set of coefficients leading to the classical Lichnerowicz-Schrödinger formula. Indeed, we know that in this case the limit at infinity of the sum of the boundary contributions over an orthonormal basis of constant spinors in $\mathbb{R}^n$ equals a (precisely known) multiple of

$$\lim_{r\to\infty} \int_{S_r} (\mathrm{div}_0\, g - d(\mathrm{tr}_0\, g))\,(\nu_r)\, d\mathrm{vol}_{S_r}\,.$$

Thus it must be so for any choice of representation $\mathbb{V}$ and this proves the existence of the constants $\mu(\mathbf{a})$.

Computing the constants now involves a slightly more thorough study. As $\mu(\mathbf{a})$ is linear in **a**, it is enough to compute the constant when A is a single projection on an irreducible summand $\mathbb{W}$ in $\mathbb{R}^n \otimes \mathbb{V}$. This will be done in the next section

4. Computation for a single projection

We fix here a given factor $\mathbb{W}$ appearing in the decomposition of $\mathbb{R}^n \otimes \mathbb{V}$ into irreducibles, and let $A = \Pi$ the projection from $\mathbb{R}^n \otimes \mathbb{V}$ onto $\mathbb{W}$ (seen as a subspace of $\mathbb{R}^n \otimes \mathbb{V}$). Thus, the map to be considered is $\beta = \beta_\Pi$ defined by

$$\beta(\omega) \;=\; -\sum_{\kappa=1}^{\dim \mathbb{V}} \langle\, \cdot \otimes \sigma_\kappa, \Pi(\rho(\omega)\sigma_\kappa)\,\rangle_0$$

where $(\sigma_\kappa)_{1\leqslant\kappa\leqslant\dim\mathbb{V}}$ is any orthonormal basis of $\mathbb{V}$. To compute this, one may restrict β to the subfactor $\mathbb{R}^n$ in $\mathbb{R}^n \otimes \mathfrak{so}(n)$. The equivariant injection of the former into the latter may be explicitly described as follows: for any 1-form α, one lets $i(\alpha)$ be the 2-form with values into $\mathbb{R}^n$, *i.e.* an element of $\mathbb{R}^n \otimes \mathfrak{so}(n)$, defined by

$$i(\alpha)\,(X,Y) \;=\; (\alpha \wedge I)\,(X,Y) \;=\; \alpha(X)Y - \alpha(Y)X,$$

and our goal is now to compute $\beta \circ i$.

As a first step, we study $\rho(\alpha \wedge I)$ for an arbitrary 1-form α (recall that we freely identify elements of $\mathbb{R}^n$ and $(\mathbb{R}^n)^*$, *i.e.* forms and vectors). By definition, for any vector Z in $\mathbb{R}^n$,

$$\begin{aligned}(\alpha \wedge I)(Z) &= \sum_{i,j} \Big(\alpha(e_i)\langle e_j, Z\rangle - \alpha(e_j)\langle e_i, Z\rangle\Big)\, e_i \otimes e_j \\ &= \sum_{i<j} \Big(\alpha(e_i)\langle e_j, Z\rangle - \alpha(e_j)\langle e_i, Z\rangle\Big)\, e_i \wedge e_j.\end{aligned}$$

or equivalently,

$$(\alpha \wedge I) = \sum_{i<j} e_i \wedge e_j \otimes \Big(\alpha(e_i)e_j - \alpha(e_j)e_i\Big)\,.$$

This leads eventually to:

$$\begin{aligned}\rho(\alpha\wedge I) &= \sum_{i<j}\rho(e_i\wedge e_j)\otimes\Big(\alpha(e_i)e_j-\alpha(e_j)e_i\Big)\\ &= \sum_{i<j}\alpha(e_i)\rho(e_i\wedge e_j)\otimes e_j-\alpha(e_j)\rho(e_i\wedge e_j)e_i\\ &= \sum_{i,j}\alpha(e_i)\rho(e_i\wedge e_j)\otimes e_j\\ &= \sum_{j}\rho(\alpha\wedge e_j)\otimes e_j.\end{aligned}$$

Thus, the operator $\rho(\alpha\wedge I)$ is nothing but the opposite of the conformal weight operator!

These remarks reduce our problem to computing, for any α in $\mathbb{R}^n$,

$$\beta\circ i(\alpha) \;=\; w(\lambda,\rho)\sum_{\kappa=1}^{\dim\mathbb{V}}\langle\,\cdot\otimes\sigma_\kappa,\Pi(\alpha\otimes\sigma_\kappa)\,\rangle_0\,.$$

But the map

$$\alpha\otimes X\in\mathbb{R}^n\otimes\mathbb{R}^n\;\mapsto\;\sum_{\kappa=1}^{\dim\mathbb{V}}\langle\,X\otimes\sigma_\kappa,\Pi(\alpha\otimes\sigma_\kappa)\,\rangle_0$$

is invariant under the action of $\mathfrak{so}(n)$ on $\mathbb{R}^n\otimes\mathbb{R}^n$, thus must be a constant multiple of the trivial contraction $\alpha\otimes X\mapsto\alpha(X)$. It then suffices to compute

$$\sum_{i=1}^{n}\sum_{\kappa=1}^{\dim\mathbb{V}}\langle\,e_i\otimes\sigma_\kappa,\Pi(e_i\otimes\sigma_\kappa)\,\rangle_0\,.$$

This is of course the trace of matrix representing $\Pi:\mathbb{R}^n\otimes\mathbb{V}\to\mathbb{R}^n\otimes\mathbb{V}$ in the orthonormal basis $\{e_i\otimes\sigma_\kappa\}_{i,\kappa}$. As Π is an orthogonal projection operator, this trace equals the rank of Π, *i.e.* the dimension of its image $\mathbb{W}$. Thus, for any α,

$$\beta\circ i(\alpha) \;=\; w(\lambda,\rho)\sum_{\kappa=1}^{\dim\mathbb{V}}\langle\,\cdot\otimes\sigma_\kappa,\Pi(\alpha\otimes\sigma_\kappa)\,\rangle_0 \;=\; \frac{(\dim\mathbb{W})\,w(\lambda,\rho)}{n}\,\alpha\,.$$

Letting π denote the orthogonal projection from $\mathbb{R}^n\otimes\mathfrak{so}(n)$ onto the unique summand $\mathbb{R}^n$ appearing in its decomposition into irreducibles, this implies that

$$\beta \;=\; \frac{(\dim\mathbb{W})\,w(\lambda,\rho)}{n}\,\pi\,.$$

In the setting of Theorem 3.1, where we consider no more a single projection but a linear combination $A=\sum a_j\Pi_j$, one infers that

$$\beta_A \;=\; \left(\sum_{j=1}^{N}a_j\,\frac{(\dim\mathbb{W}_j)\,w(\lambda_j,\rho)}{n}\right)\pi$$

and it now remains to relate this to the mass.

Let now $\mathbb{V}$ be chosen, in odd dimensions, as the spinor bundle Σ, with associated representation ς, , and in even dimensions, as the positive spinor bundle Σ_+, with associated representation ς_+. In both cases, there are only two irreducible summands in the tensor product: one has $\mathbb{R}^n \otimes \Sigma = \mathbb{T} \oplus \Sigma$ in odd dimensions and $\mathbb{R}^n \otimes \Sigma_+ = \mathbb{T}_+ \oplus \Sigma_-$ in even dimensions, where $\mathbb{T}_{(+)}$ denotes the (half) twistor bundle and Σ_- is the negative half spinor bundle. To have uniform notations with respect to dimensions, we shall now write $\mathbb{R}^n \otimes \mathbb{V} = \mathbb{W}_1 \oplus \mathbb{W}_2$ where $\mathbb{W}_1$ is the relevant twistor representation and $\mathbb{W}_2$ the relevant spinor representation. The classical Lichnerowicz formula $\mathfrak{D}^*\mathfrak{D} - \nabla^*\nabla = \frac{1}{4}\,\mathrm{Scal}$ can then be rewritten as

$$(n-1)(P_2)^*P_2 - (P_1)^*P_1 = \tfrac{1}{4}\,\mathrm{Scal}\ .$$

(Note that one has to be careful with the choice of norms: here and everywhere else in the paper, each $\mathbb{W}_j$ is endowed with the norm induced by the product norm on $\mathbb{R}^n \otimes \mathbb{V}$; as a result, $|\Pi_2(\nabla\psi)|^2 = \frac{1}{n}|\mathfrak{D}\psi|^2$ for any spinor ψ.) Thus,

$$a_1 = -1, \quad a_2 = (n-1), \quad A = (n-1)\Pi_2 - \Pi_1$$

and Assumption 2.1 is satisfied. It now remains to re-interpret Witten's boundary term in our setting, but this is easily done. It has indeed been computed in [2] that for any j, k in $\{1, ..., n\}$ and any tangent vector X,

$$\begin{aligned}\omega_j^k(X) \;=\; & \frac{1}{2}\,\langle\, (\nabla^0_X H)e_j - (\nabla^0_{He_j}H)H^{-1}X\,,\, He_k\,\rangle \\ & - \frac{1}{2}\,\langle\, (\nabla^0_X H)e_k - (\nabla^0_{He_k}H)H^{-1}X\,,\, He_j\,\rangle \\ & - \frac{1}{2}\,\langle\, (\nabla^0_{He_j}H)e_k - (\nabla^0_{He_k}H)e_j\,,\, HX\,\rangle.\end{aligned}$$

The main decay assumption $\tau > \frac{n-2}{2}$ moreover implies that when passing to the limit $r \to \infty$, all occurrences of H at zeroth-order may be replaced by the identity, $\langle\cdot,\cdot\rangle$ may by $\langle\cdot,\cdot\rangle_0$, and $\nabla^0 H$ may be replaced by $-\frac{1}{2}\nabla^0 g$ without harm. Thus,

$$\begin{aligned}\omega_j^k(X) \;=\; & \frac{1}{4}\,\langle\, (\nabla^0_X g)e_k - (\nabla^0_{e_k} g)X\,,\, e_j\,\rangle_0 \\ & - \frac{1}{2}\,\langle\, (\nabla^0_X g)e_j - (\nabla^0_{e_j} g)X\,,\, e_k\,\rangle_0 \\ & - \frac{1}{2}\,\langle\, (\nabla^0_{e_k} g)e_j - (\nabla^0_{e_j} g)e_k\,,\, X\,\rangle_0.\end{aligned}$$

Taking into account the symmetry of g, this reduces to

$$\omega_j^k(X) \;=\; \frac{1}{2}\left[\nabla^0_{e_j} g(X, e_k) \;-\; \nabla^0_{e_k} g(X, e_j)\right].$$

As explained above, thisc corresponds to an element of $(\mathbb{R}^n)^* \otimes \mathfrak{so}(n)$ and it enters the boundary term of the Weitzenböck formula only through its projection into the irreducible summand $(\mathbb{R}^n)^*$ which is obtained by taking a trace in the variables X and e_j. Thus the boundary contribution is a multiple of $(\mathrm{div}_0\, g - d\,\mathrm{tr}_0\, g)$. In Witten's case, the proportionality factor is known precisely (see section 1), and the integrand of the boundary term is

$$\beta_A(\omega) \;=\; \frac{\dim \Sigma_{(+)}}{4}\;(\mathrm{div}_0\, g - d\,\mathrm{tr}_0\, g)\,,$$

(the trace taken over an orthonormal basis of the (half) spinor bundle explains the appearance of $\dim \Sigma_{(+)}$ in this formula). Of course, this should be equal to the expression obtained before, *i.e.*

$$\beta_A(\omega) \;=\; \left(\frac{(n-1)}{n}\,(\dim \mathbb{W}_2)\, w(\lambda_2, \varsigma_{(+)}) \;-\; \frac{1}{n}\,(\dim \mathbb{W}_1)\, w(\lambda_1, \varsigma_{(+)})\right) \pi(\omega)\,.$$

One has $\dim \Sigma = 2^{\left[\frac{n}{2}\right]}$ in odd dimensions and $\dim \Sigma_+ = \dim \Sigma_- = 2^{\frac{n}{2}-1}$ in even dimensions. Thus,

$$\dim \mathbb{W}_1 = 2^{\left[\frac{n}{2}\right]-\delta}(n-1), \quad \dim \mathbb{W}_2 = 2^{\left[\frac{n}{2}\right]-\delta}$$

where $\delta = 0$ in odd dimensions and $\delta = 1$ in even dimensions. It is moreover easily computed that, irrespective of the parity of dimension,

$$w(\varsigma,\varsigma) = w(\varsigma_-, \varsigma_+) = -\frac{n-1}{2}, \quad w(\mathfrak{t},\varsigma) = w(\mathfrak{t}_+, \varsigma_+) = \frac{1}{2},$$

where $\mathfrak{t}_{(+)}$ is of course the (half) twistor representation. Thus,

$$\begin{aligned}
\beta_A(\omega) \;&=\; -\left(\frac{(n-1)^2}{n}\,(\dim \mathbb{W}_2) \;+\; \frac{1}{2n}\,(\dim \mathbb{W}_1)\right)\pi(\omega)\\
&=\; -\frac{(n-1)}{2}\, 2^{\left[\frac{n}{2}\right]-\delta}\,\pi(\omega)\\
&=\; -\frac{(n-1)}{2}\;(\dim \Sigma_{(+)})\;\pi(\omega).
\end{aligned}$$

One deduces immediately that

$$\pi(\omega) \;=\; -\frac{1}{2(n-1)}\;(\mathrm{div}_0\, g - d\,\mathrm{tr}_0\, g)\,.$$

Coming back to the case of a single projection Π onto a summand $\mathbb{W}$ of $\mathbb{R}^n \otimes \mathbb{V}$ with $\mathbb{V}$ being an arbitrary irreducible representation of $\mathfrak{so}(n)$, one concludes that

$$\beta(\omega) \;=\; -\frac{(\dim \mathbb{W})\, w(\lambda,\rho)}{2n(n-1)}\;(\mathrm{div}_0\, g - d\,\mathrm{tr}_0\, g)\,,$$

and this ends the proof of Theorem 3.1. □

5. Examples of applications

We give here an example where the ideas above can be applied. Of course it does not exhaust all possible applications, and we chose it only to illustrate the fact that Theorem 3.1 is easy to use.

5.1. **A universal positive mass theorem.** It has been remarked by Gauduchon [12] that choosing $a_j = w(\lambda_j, \rho)$ (with notations as in the previous sections) always leads to a Weitzenböck formula. As a matter of fact, a trivial computation shows that the principal symbol $\sigma_\xi\left(\sum w(\lambda_j,\rho)(P_j)^*P_j\right)$ vanishes. Indeed,

$$\left\langle \sigma_\xi\left(\sum w(\lambda_j,\rho)(P_j)^*P_j\right) v, w\right\rangle = \langle B(\xi\otimes v), \xi\otimes w\rangle = 0,$$

hence $\sum w(\lambda_j,\rho)(P_j)^*P_j$ is a zeroth-order operator. This formula is called the *universal Weitzenböck formula* in [21, Proposition 3.6, p. 519]. One may now apply our main result to this case, and conclude that there is always an underlying universal positive mass theorem:

Proposition 5.1. *Let E be the natural bundle over a complete asymptotically flat Riemannian manifold (M, g) induced from an irreducible representation $(\mathbb{V}, \rho)$ of $\mathfrak{so}(n)$. With the notations of section 2, assume that the curvature operator*

$$\mathcal{R} = \sum_{j=1}^{N} w(\lambda_j,\rho)(P_j)^*P_j$$

is nonpositive and that there exists a full set of solutions $\{\sigma_\kappa\}_{1\leqslant\kappa\leqslant\dim\mathbb{V}}$ of the equation

$$\sum_{w(\lambda_j,\rho)<0} w(\lambda_j,\rho)P_j\sigma_\kappa = 0$$

that are asymptotic to an orthonomal basis of $\mathbb{V}$ and satisfy the estimates of Lemma 2.2. Then one has

$$\int_M \sum_{\kappa=1}^{\dim\mathbb{V}} \sum_{w(\lambda_j,\rho)>0} w(\lambda_j,\rho)\,|P_j\sigma_\kappa|^2 - \langle\sigma_\kappa, \mathcal{R}\sigma_\kappa\rangle = \frac{c(\rho)}{n(n-1)}(\dim\mathbb{V})\, m(g),$$

where $c(\rho)$ is the Casimir operator of the representation ρ. Hence the mass $m(g)$ is nonnegative.

Proof. – The only missing point is the precise computation of the proportionality factor, which is *a priori* equal to

$$\frac{1}{2n(n-1)}\left(\sum_{j=1}^{N} w(\lambda_j,\rho)^2\ \dim\mathbb{W}_j\right).$$

The term inside brackets is obviously $\operatorname{tr} B^2$, and one may then apply [7, §4, p.230]. We denote as in this paper p-tr the partial trace of an operator from $\mathbb{R}^n\otimes\mathbb{V}$ to itself, *i.e.* the endomorphism of $\mathbb{V}$ obtained by taking the trace on the $\mathbb{R}^n$-factor. Then p-tr(B^2) is nothing but twice the Casimir operator $c(\rho)$ of $(\mathbb{V}, \rho)$ [7, Equation (4.1)], and

$$\sum_{j=1}^{N} w(\lambda_j,\rho)^2\ \dim\mathbb{W}_j \;=\; \operatorname{tr} B^2 \;=\; (\dim\mathbb{V})\ \text{p-tr}(B^2) \;=\; 2\,(\dim\mathbb{V})\,c(\rho),$$

which proves the desired formula □

5.2. **The case when $N = 2$.** From [7, 13], the number N of irreducible summands in $\mathbb{R}^n \otimes \mathbb{V}$ is $N = 2$, *i.e.* $\mathbb{R}^n \otimes \mathbb{V} = \mathbb{W}_1 \oplus \mathbb{W}_2$, when the dimension n is even and the representation's dominant weight is given by $\rho = (k, \dots, k, \pm k)$ with k an arbitrary non-zero integer or half-integer. (There is also a possibility when n is odd and $\rho = (\frac{1}{2}, \dots, \frac{1}{2})$, but this is nothing but the spin representation and we already know this case in detail hence we shall forget about it here.)

There is only one non trivial Weitzenböck formula relating the operators P_1 and P_2, which is the universal $w_1\,(P_1)^* P_1 + w_2\,(P_2)^* P_2 = \mathcal{R}$, and the values of the weights are $w_1 = k$ and $w_2 = 1 - \frac{n}{2} - k$. There are only two possible boundary terms, the first one being obtained by choosing $P_+ = P_1$ and $P_- = P_2$ (up to positive constants) and the second one by choosing the opposite. The proportionality factors in Proposition 5.1 are $\frac{\varepsilon}{n(n-1)}\,(\dim \mathbb{V}) c(\rho)$, where $\varepsilon = -1$ for the first choice and $\varepsilon = +1$ for the second.

Hence, we get a positive mass theorem in the latter case (provided that the adequate curvature condition holds and that there exists enough asymptotically constant elements in $\ker P_2$) and a negative one for the former case (provided that $\mathcal{R}$ is non-negative and that there exists enough asymptotically constant elements in $\ker P_1$). This is of course consistent with the classical spinorial case, as the reader has to keep in mind that $\mathcal{R} = -\frac{1}{4}\,\mathrm{Scal}$ when the Weitzenböck formula is written as the universal formula.

5.3. **The case of forms.** As a further example of application, one may look at the case of p-forms with $p < \frac{n}{2}$. One has in this case $\rho = (1, ..., 1, 0, ..., 0)$ where the number of 1's is equal to p, and the number of summands is equal to $N = 3$. (The case where $p = \frac{n}{2}$ in even dimensions, *i.e.* $\rho = (1, ..., 1, \pm 1)$ is special, as it has $N = 2$; it has thus been already treated.)

One may then apply the previous Proposition: the only possible Weitzenböck formula is the universal one and the conformal weights are $w_1 = 1$, $w_2 = -p$, and $w_3 = -(n-p)$. There are again two possible operators: either $P_+ = P_1$ (and, up to a constant, this is the conformal Killing operator), or $P_+ = \sqrt{-w_2} P_2 + \sqrt{-w_3} P_2$ (and, up to a constant again, this is the classical Hodge-de Rham operator $d + \delta$). The two possible boundary terms can be computed from Proposition 5.1 and this leads to a positive mass theorem for the latter and a negative one for the former (under the condition that $\mathcal{R}$ is non-positive in the first case and non-negative in the second).

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