The convergence Newton polygon of a p-adic differential equation IV : local and global index theorems

Jérôme Poineau and Andrea Pulita

Abstract

We deal with locally free $\mathscr{O}_X$-modules $\mathscr{F}$ with connection over a Berkovich curve X. As a main result we prove local and global criteria proving the finite dimensionality of the de Rham cohomology of $\mathscr{F}$. Together with a global index formula.

This is a first draft, we plan to improve the exposition, also by adding some examples and applications.

Version of September 17, 2013

Contents

2000 Mathematics Subject Classification Primary 12h25; Secondary 14G22

Keywords: p-adic differential equations, Berkovich spaces, Radius of convergence, Newton polygon, spectral radius, Skeleton, finiteness

Introduction

In this paper we prove the *finite dimensionality* of the de Rham cohomology of a large class of p-adic differential equations over Berkovich curves. Moreover, we obtain a *global index formula* relating the index of the differential equation to its *global irregularity*, that we define in term of the slopes of the global radii of convergence of the equation as defined in [Bal10], [Ked13], [Pul12], [PP12], [PP13].

Major contribution to the finiteness of the de Rham cohomology in this context are [Ado76], [Dwo82], [CM93], [CM97], [CM00], [CM01], [Rob75], [Rob76], [Rob84], [Rob85]. For references concerning rigid cohomology related to the present paper we refer to the introduction of [Ked06], and related bibliography.

As firstly observed by Bernard Dwork (cf. [Dwo82]), and largely exploited by Philippe Robba (cf. [Rob84]), the major tool for the study of p-adic differential equations is the radius of convergence of the differential equation. One of the crucial contributions of Robba has been to relate the index of a p-adic differential operator with rational coefficients to the slopes of the radius of convergence, as a function, by means of a Grothendieck-Ogg-Shafarevich formula (cf. [Rob84] and [Rob85]). The program indicated by Robba has been subsequently completed by Gilles Christol and Zoghman Mebkhout (cf. [CM93], [CM97], [CM00], [CM01]).

From the point of view of rigid cohomology, differential equations are a category of coefficients for a good cohomological theory of an algebraic variety of characteristic $p > 0$.

The approach of this paper (and also [Pul12], [PP12], [PP13]) is different. We deal with the de Rham cohomology of *any* locally free $\mathscr{O}_X$-module $\mathscr{F}$ endowed with a connection ∇, where X is a quasi-smooth K-analytic Berkovich curve. We call the couple $(\mathscr{F}, \nabla)$ a differential equation over X.

This category of differential equations is abelian, and it covers the class of differential equations studied in the rigid cohomology. Namely the equations coming from rigid cohomology are subjected to some conditions:

i) They always have *over-convergent coefficients*. In the sense of Berkovich this means that X is a compact curve embedded into a projective curve $\overline{X}$, and that $\mathscr{F}$ is defined over an unspecified open neighborhood U of X in $\overline{X}$.

ii) By the fact that they have a Frobenius action, they are subjected to certain *solvability* conditions about their radii of convergence at the boundary of X. This means that the radii of convergence of their solutions are all maximal at these points, and hence on the whole curve X. The only region where these radii are possibly not maximal is along $U - X$ i.e. along the "over-convergent boundary" of X.

We remove both these assumptions from the picture, and we work with general differential equations over X. These equations have a further geometrical datum which is (almost) trivial for equations coming from rigid cohomology: a controlling graph

$$\Gamma_S(\mathscr{F}) \subset X \tag{0.1}$$

inside the Berkovich curve X. Roughly speaking this is defined as the locus of points that do not belong to any open disk on which the radii of convergence are all constant functions.[1] In [Pul12] and [PP12] we have proved that $\Gamma_S(\mathscr{F})$ is a locally finite connected graph such that $X - \Gamma_S(\mathscr{F})$ is a disjoint union of open disks, and that the radii of convergence are all continuous functions on X having the property that they are constant on the connected components of $X - \Gamma_S(\mathscr{F})$.

As showed by Robba the variation of the radii (i.e. their slopes) along $\Gamma_S(\mathscr{F})$ is highly related to the dimension of the local and global de Rham cohomology of $\mathscr{F}$. In the case of differential equations coming from rigid cohomology the information is all contained at the "over-convergent boundary" of X. In the general case we have an extremely richer situation, because the graph $\Gamma_S(X)$, containing the information, is much more complex.

As firstly observed by Robba [Rob75] and Dwork-Robba [DR77], if we are in a neighborhood of a Berkovich point $x \in X$, then $\mathscr{F}$ splits locally at x, into a decomposition separating its solvable radii from the smaller one. In this decomposition Robba observed (cf. [Rob75]) that the local de Rham cohomology of the non solvable part is zero: "*non solvable differential equation locally do not have cohomology*".

On the other hand over-convergence is unavoidable because solvable equations over a curve with boundary usually have infinite dimensional de Rham cohomology. As an example the derivation acting on the ring of functions over a closed disk has infinite dimensional cokernel, since the formal primitive of a function fails to converge on the closed disk. But if one restricts the study to differential equations that are spectral non solvable at the boundary of X, then over-convergence is unnecessary to have finite dimensionality. So we allow the boundary in our setting under a non solvability assumption on it.

Now consider the equation $\mathscr{F}$ from a global point of view over the Berkovich curve X. Recall that to recover the over-convergence setting of rigid cohomology it is enough to work over an individual open neighborhood of X in $\overline{X}$. Even though its radii are not solvable at the boundary of X, its cohomology is highly non trivial. We prove in fact that its global index takes into account the local indexes of $\mathscr{F}$ at the boundary points of $\Gamma_S(\mathscr{F})$, and at the points of $\Gamma_S(\mathscr{F})$ where the radii have some breaks. We are hence induced to give the following

Definition 1 (cf. Def. 3.4.1). *We say that $\Gamma_S(\mathscr{F})$ is* essentially finite *if it contains only finitely many points x such that at least one of the following condition is realized:*

i) *some of the radii have a break at x,*
ii) *$x \in \partial X$,*
iii) *x is a point of positive genus,*
iv) *is a bifurcation point of $\Gamma_S(X)$.*

[1] In this introduction we remove from the picture the complications arising from the choice of the triangulation S. We only give the general image.

We then obtain the following result relating the essential finiteness of $\Gamma_S(\mathscr{F})$ to the finiteness of the dimension of the de Rham cohomology:

THEOREM 1 (cf. Thm. 3.4.5)**.** *Le X be a quasi-smooth K-analytic Berkovich curve, and let $\mathscr{F}$ be a differential equation over X. Assume that*

i) *$\mathscr{F}$ is free of Liouville numbers over X,*
ii) *$\Gamma_S(\mathscr{F})$ is essentially finite,*
iii) *The radii of $\mathscr{F}$ are spectral non solvable at the points of the boundary ∂X of X.*

Then the de Rham cohomology of $\mathscr{F}$ is finite dimensional.

The Liouville condition means that the restriction of $\mathscr{F}$ to all open annuli in X is free of Liouville numbers in the sense of Christol-Mebkhout (cf. [CM93], [CM97], ...). We prove that it is enough to test the Liouville condition on a locally finite family of annuli with skeletons in $\Gamma_S(\mathscr{F})$ (cf. Lemma 3.3.7).

We notice that if $\Gamma_S(\mathscr{F})$ is essentially finite then it is *topologically finite* (i.e. $\Gamma_S(\mathscr{F})$ is a finite union of intervals), but not necessarily *finite as a graph* (i.e. having a finite number of edges).

This assumption moreover implies that X is a curve with finite genus in the sense of Q. Liu [Liu87], so that X is either projective or quasi-Stein. The projective case is well known, so we focus on the quasi-Stein case.

THEOREM 2 (cf. Thm. 3.6.4)**.** *Under the assumptions of Theorem 1, if X is not projective, then the index of $\mathscr{F}$ is expressed by the following formula of Grothendieck-Ogg-Shafarevich type:*

$$\chi(X,\mathscr{F}) \;=\; \mathrm{rank}(\mathscr{F})\cdot\chi(X) - \mathrm{Irr}_X(\mathscr{F})\;. \tag{0.2}$$

In this formula we have

$$\chi(X) \;:=\; 2-2g(X)-N(X)\;. \tag{0.3}$$

where $g(X)$ is the genus of X in the sense of [Liu87], and $N(X)$ is a topological invariant of X which roughly represents the maximal number of germs of open segments in X that are not relatively compact in X. We call it the "*open boundary of X*" (see section 3.5).

The quantity $\mathrm{Irr}_X(\mathscr{F})$ represents the global irregularity of $\mathscr{F}$. It is given by a sum of local terms: part of these terms are the slopes of the radii at the open an closed boundaries of X, here noted by $\mathrm{seg}(S_{\mathscr{F}})$, in analogy with the index formula of Christol-Mebkhout. The other local terms are related to the number of segments of $\Gamma_S(\mathscr{F})$ that are incident upon the (closed) boundary of X (cf. Definition 3.6.2):

$$\mathrm{Irr}_X(\mathscr{F}) \;:=\; \Big(\sum_{x\in\partial X}\chi(x,S_{\mathscr{F}})\Big)\cdot\mathrm{rank}(\mathscr{F}) - \sum_{b\in\mathrm{seg}(S_{\mathscr{F}})}\partial_b H_{\emptyset,r}(-,\mathscr{F}_{|\mathfrak{R}_b})\;. \tag{0.4}$$

The proof of Theorem 1 goes as follows. From the essential finiteness of $\Gamma_S(\mathscr{F})$ we construct an *finite* open covering of X, where the cohomology if finite dimensional, then we inductively apply a Mayer-Vietoris lemma to deduce the finite dimensionality of the global de Rham cohomology.

The proof of Theorem 2 results from an analysis of the Čech resolution of $\mathscr{F}$, of such a covering. The only terms of the covering that contribute to the index are those open subsets that are small neighborhoods of the points having at least one of the properties listed in Definition 1.

If X is compact, or more generally if X is relatively compact into a larger curve on which $\mathscr{F}$ is defined, the essential finiteness of $\Gamma_S(\mathscr{F})$ is automatic (as a consequence of [Pul12], and [PP12]).

In particular this is the case of rigid cohomology, by the over-convergence assumption. So we have a large class of equations that fulfill the theorem.

In the general case, there are several differential equations for which $\Gamma_S(\mathscr{F})$ is not essentially finite. The radii of these equations presents infinitely many breaks as one approaches the "*open boundary*" of X. These equations do not admits a *finite* covering on which the cohomology is finite, so a straightforward application of Mayer-Vietoris Lemma is not possible.

As a converse of Theorem 1 we provide criteria to prove that their cohomology is actually *infinite dimensional*.

THEOREM 3 (cf. Thm. 3.7.4)**.** *Assume that X is not projective, that it has finite genus $g(X)$, and that it admits a weak triangulation S whose skeleton Γ_S is topologically finite (cf. Def. 1.1.2). Let $\mathscr{F}$ be a differential equation free of Liouville numbers over X, with no solvable radii at the boundary ∂X of X. The following conditions are equivalent:*

i) $\Gamma_S(\mathscr{F})$ *is essentially finite;*
ii) *the de Rham cohomology of $\mathscr{F}$ is finite dimensional;*
iii) *for all germ of segment b at the open boundary of X, the radii of $\mathscr{F}$ have a finite number of breaks along b.*

The proof is a limit process coming from [CM00]. We approach X by a countable sequence $X_1 \subseteq X_2 \subseteq \cdots \subseteq X$ of affinoid domains in X, and we prove a limit formula

$$\chi(X,\mathscr{F}) \;=\; \lim_n \chi(X_n,\mathscr{F}_{|X_n})\;. \tag{0.5}$$

More precisely we have

$$\mathrm{H}^i_{\mathrm{dR}}(X,\mathscr{F}) \;=\; \varprojlim_n \mathrm{H}^i_{\mathrm{dR}}(X_n,\mathscr{F}_{|X_n}) \tag{0.6}$$

and the maps $\mathrm{H}^i_{\mathrm{dR}}(X,\mathscr{F}) \to \mathrm{H}^i_{\mathrm{dR}}(X_n,\mathscr{F}_{|X_n})$ are all surjective for all n large enough. So the de Rham cohomology of X is finite dimensional if and only if the sequence of dimensions of $\mathrm{H}^i_{\mathrm{dR}}(X_n,\mathscr{F}_{|X_n})$ stabilizes for all n large enough.

The proof of these results (even locally around a point) are not straightforward consequence of the finite dimensionality of rigid cohomology. Indeed essential ingredients are the finiteness results of [Pul12], [PP12], and also the decomposition results of [PP13].

Structure of the paper.

In section 1, we provide basic definitions about curves, and radii of convergence.

In section 2 we deal with the local cohomology of a differential equation at a Berkovich point. We generalize several local results to the non solvable case. The finiteness of $\Gamma_S(\mathscr{F})$ is systematically employed to reduce to the case of classical rigid cohomology over a tube $V_S(x,\mathscr{F})$ canonically attached to $(x,\mathscr{F})$, which is roughly the union of all the disks in X, with boundary point x, on which the radii of $\mathscr{F}$ are all constant.

In section 2 we also deal with the technical problem of *super-harmonicity* of the partial heights of the convergence Newton polygon. We know since [Pul12] and [PP13], that there are potentially a locally finite set $\mathscr{C}_{S,r}(\mathscr{F})$ of pathological points in X such that *super-harmonicity* fails. The super-harmonicity at these points is important because it improves the *global bound* on the size of $\Gamma_S(\mathscr{F})$, obtained in [PP13, Cor. 7.2.5] (cf. Cor. 2.8.8). We here obtain such a super-harmonicity at the pathological points under some technical assumptions. More precisely we have the following result, whose central point of the proof is Dwork's dual theory (cohomology with support):

Theorem 4 (cf. Thm. 2.8.6)**.** *Assume that the residual field of K has characteristic $p>0$. Let $\mathscr{F}$ be a differential equation over X or rank r. Let $x\in\mathscr{C}_{S,r}(\mathscr{F})$, let D_x be the closed disk in $X-\Gamma_S$ with boundary x, and let $V=V_S(x,\mathscr{F})$. Assume that*

i) *the canonical inclusion $\mathrm{H}^0_{\mathrm{dR}}(D_x^\dagger,\mathscr{F})\subseteq\mathrm{H}^0_{\mathrm{dR}}(V^\dagger,\mathscr{F})$ is an equality;*

ii) *the radii of $\mathscr{F}$ are compatible with duals;*

iii) *$\mathscr{F}$ is free of Liouville numbers at x (cf. Def. 2.7.11).*

Then for all $i=1,\ldots,r$ the partial height $H_{S,i}(-,\mathscr{F})$ is super-harmonic at x.

In section 3 we deal with global cohomology.

Note. This is a first draft. It appears clearly that several cohomological notions (as cohomology with supports, and its duality with the cohomology without supports) should follow from the results of this paper. We plan to add some parts with those applications.

Acknowledgments. This work finds its genesis in some discussions we had with Francesco Baldassarri, who was expecting the existence of a link between the finiteness of the controlling graphs and the finite dimensionality of the de Rham cohomology. We thank him heartily for sharing these inspiring ideas. We also thank Yves André, Gilles Christol, Richard Crew, Kiran S. Kedlaya, Adriano Marmora, Nicola Mazzari, Zoghman Mebkhout, Bertrand Toën, and Nobuo Tsuzuki for helpful discussions.

1. Definitions and notations

In this section we give definitions and notations that are used in the sequel of the paper. All definitions comes from [PP13], we only provide the essential vocabulary.

Let K be an ultrametric complete valued field of characteristic 0. Let p be the characteristic of its residue field k (either 0 or a prime number). We denote by $\widehat{K^{\mathrm{alg}}}$ the completion of an algebraic closure K^{alg} of K.

Setting 1.0.1. *Let X be a quasi-smooth K-analytic curve endowed with a weak triangulation S as in [PP12]. Without loss of generality from now on we assume that X is connected.*

By a *differential equation* or *differential module* over X we mean a coherent $\mathscr{O}_X$-module $\mathscr{F}$ endowed with an (integrable) connection ∇. $\mathscr{F}$ is automatically a locally free $\mathscr{O}_X$-module of finite rank. The category of locally free of finite rank $\mathscr{O}_X$-modules with connection is an abelian category. We refer to [PP13] for the main properties of such objects.

1.1 Curves

Here we introduce some definitions and notations that will be frequently used in the paper.

Notation 1.1.1. *Let $\mathbb{A}^{1,\mathrm{an}}_K$ be the Berkovich affine line with coordinate T. Let L be a complete valued extension of K and $c\in L$. We set*

$$D^+_L(c,R) = \left\{x\in\mathbb{A}^{1,\mathrm{an}}_L \,\middle|\, |(T-c)(x)|\leqslant R\right\}, \qquad R\geqslant 0 \tag{1.1}$$

$$D^-_L(c,R) = \left\{x\in\mathbb{A}^{1,\mathrm{an}}_L \,\middle|\, |(T-c)(x)|< R\right\}, \qquad R> 0 \tag{1.2}$$

$$C^+_L(c;R_1,R_2) = \left\{x\in\mathbb{A}^{1,\mathrm{an}}_L \,\middle|\, R_1\leqslant|(T-c)(x)|\leqslant R_2\right\}, \qquad R_2\geqslant R_1\geqslant 0 \tag{1.3}$$

$$C^-_L(c;R_1,R_2) = \left\{x\in\mathbb{A}^{1,\mathrm{an}}_L \,\middle|\, R_1<|(T-c)(x)|< R_2\right\}. \qquad R_2> R_1> 0 \tag{1.4}$$

If $D \subseteq \mathbb{A}_L^{1,\mathrm{an}}$ is a disk, we denote by $\mathscr{O}(D)$ the ring of *analytic* functions on D. If $D = D_L^-(c, R)$, then

$$\mathscr{O}(D) \;:=\; \Big\{\sum_{n\geqslant 0} a_n(T-c)^n,\; a_n \in L,\; \lim_n |a_n|\rho^n = 0,\; \forall\, \rho < R\Big\}\,. \tag{1.5}$$

1.1.1 *Branches, germ of segments and sections.* Let X be a curve as in Setting 1.0.1. The Berkovich space X is naturally endowed with a graph structure (cf. [Duc]). By a *closed segment* $[x, y] \subset X$ we mean the image in X of an injective continuous path $[0, 1] \to X$ with initial point x and end point y. We also call segments the images $]x, y[$, $]x, y]$, $[x, y[$ of $]0, 1[$, $]0, 1]$, $[0, 1[$ respectively. In this case we say that the segments are open, or semi-open. By convention a segment is never reduced to a point nor to the empty set.

A *germ of segment* b out of $x \in X$ is an equivalence class of open segments $]x, y[$ given by $]x, y_1[\sim]x, y_2[$ if and only if there exists z such that $\emptyset \neq]x, z[\subseteq]x, y_1[\cap]x, y_2[$. By abuse we often write $b =]x, y[$ instead of $]x, y[\in b$.

A *section of a germ of segment* b out of x in X is a connected open subset of U of X containing such an annulus C (resp. $D - \{x\}$), if x is of type 2 or 3 (resp. 1 or 4), and such that x belongs to the closure of U in X, but not to U itself.

We refer to [Duc] for the definition of a branch. Roughly speaking a branch out of x corresponds to a direction out of x. Germs of segments out of x correspond bijectively to branches out of x. They will often be denoted by the same symbol b. By a section of the branch b, we mean a section of the corresponding germ of segment.

1.1.2 *Slopes.* Let b be a germ of segment out of $x \in X$, let $]x, y[$ be a representative of b, and let $F : [x, y] \to \mathbb{R}$ be a continuous function. If it has a meaning, we use the symbol $\partial_b F(x)$ to indicate the log-*slope at x of F along b*. We refer to [PP13, 1.1.4] for the definition.

1.1.3 *Graphs.* The reader can find in [Duc, 1.3.1] the definition of a graph. A single point and the empty set are graphs. We say that a germ of segment b out of x *belongs to a graph* $\Gamma \subseteq X$ if b is represented by a segment $]x, y[\subset \Gamma$. A point $x \in \Gamma$ is a *bifurcation point* if there is more than two germs of segments out of x belonging to Γ. A point $x \in \Gamma$ is an *end point of* Γ if there is no open segments $]z, y[$ in X such that $x \in]z, y[\subset \Gamma$.

In this paper by a *locally finite graph* we mean a closed connected subset $\Gamma \subset X$ such that

i) Each point $x \in \Gamma$ admits a neighborhood U in X such that $\Gamma \cap U$ is a finite union of segments and points;

ii) $X - \Gamma$ is a disjoint union of virtual open disks.

All the graphs of this paper are locally finite.

Definition 1.1.2. *Let $\Gamma \subseteq X$ be a locally finite graph.*

We say that Γ is topologically finite *if Γ is homeomorphic to a finite union of real intervals.*

We say that Γ is finite as a graph *if it has a finite number of edges.*

A graph which is finite as a graph, is also topologically finite. The converse is not true. A locally finite graph which is topologically finite is not necessarily finite as a graph (e.g. an interval, having an infinite number of vertex and edges).

1.1.4 *Weak triangulations.* Following [Duc, Section 4] and [PP12, Section 2.1], a weak triangulation of X is a locally finite subset $S \subset X$, formed by points of type 2, and 3, such that $X - S$ is a disjoint union of virtual open annuli and virtual open disks. The skeleton Γ_S of S is then the union of S and the skeletons of the open annuli that are connected components of $X - S$.

Remark that the empty set is a triangulation of a virtual open annulus or virtual open disk.

1.1.5 *Star-shaped neighborhoods.* A connected open neighborhood (affinoid neighborhood V) U of a point $x \in X$ of type 2 or 3 will be called *star-shaped* if $\{x\}$ (resp. $\{x\} \cup \partial V$) is a weak triangulation of U (resp. V).

1.1.6 *quasi-Stein spaces.* Let us recall the definition of a quasi-Stein space (see [Kie67, Definition 2.3]).

Definition 1.1.3 (quasi-Stein)**.** *A k-analytic space X is said to be quasi-Stein if there exists a countable admissible covering $(X_n)_{n\geqslant 0}$ of X such that*

i) *$X_n \subseteq X_{n+1}$;*
ii) *X_n is an affinoid domain of X_{n+1};*
iii) *the map $\mathscr{O}(X_{n+1}) \to \mathscr{O}(X_n)$ has dense image.*

Let us now recall the main properties of those spaces (see [Kie67, Satz 2.4]).

Theorem 1.1.4. *Let X be a k-analytic quasi-Stein space. Let $\mathscr{F}$ be a coherent sheaf on X.*

i) *For every $q \geqslant 1$, we have $H^q(X, \mathscr{F}) = 0$.*
ii) *For every $x \in X$, the stalk $\mathscr{F}_x$ is generated by $\mathscr{F}(X)$ as an $\mathscr{O}_{X,x}$-module.*

Corollary 1.1.5. *Let X be a k-analytic quasi-Stein space. The functor $\mathscr{F} \mapsto \Gamma(X, \mathscr{F})$ from locally free sheaves of $\mathscr{O}_X$-modules of finite rank to projective $\mathscr{O}(X)$-modules of finite rank is an equivalence of categories preserving the rank.*

Proof. Starting from a projective $\mathscr{O}_X$-module M of rank n, we may associate to it a locally free sheaf $\widetilde{M}$ of rank n. This defines a functor that is a quasi-inverse to that of the statement. The proof relies on the second statement of Theorem 1.1.4. □

Remark 1.1.6. *We recall that by a result of Q.Liu [Liu87] every curve with finite genus over a spherically complete base field is either projective, or quasi-Stein.*

Remark 1.1.7. *As it will appear useful later we recall the following evidences:*

i) *The map $\mathscr{O}(X_{n+1}) \to \mathscr{O}(X_n)$ is injective by analytic continuation,*
ii) *The map $\mathscr{O}(X_{n+1}) \to \mathscr{O}(X_n)$ is K-linear and continuous, hence also* uniformly continuous*.*
iii) *Each $\mathscr{O}(X_n)$ is Banach, hence a Fréchet space. Moreover $\mathscr{O}(X)$ is also Fréchet since by definition of the Berkovich sheaf $\mathscr{O}_X$ one has*

$$\mathscr{O}(X) \;=\; \varprojlim_n \mathscr{O}(X_n)\;. \tag{1.6}$$

So $\mathscr{O}(X)$ is projective limit of a countable family of Banach spaces, hence Fréchet.

1.1.7 *Open boundary of X.*

Definition 1.1.8 (Open pseudo-annulus)**.** *We say that the curve X is an open* pseudo-annulus *if it satisfies the following properties:*

i) *X has no boundary,*

ii) *X has no points of positive genus,*

iii) *the set of points that have no neighborhood isomorphic to a disk is an open interval* Γ

For any minimal weak triangulation S *of* X*, we have* $\Gamma_S = \Gamma$*. We call* Γ *the skeleton of* X*.*

Remark 1.1.9. *By [Liu87, Prop. 3.2], if* K *is non trivially valued, algebraically closed, and spherically complete, such a pseudo-annulus* X *can be embedded into the affine line. So* X *is either an open annulus, or* $X = Y - \{y\}$*, where* Y *is either the affine line or a disk, and* $y \in Y$ *is a rational point.*

Without using the result of [Liu87], we can however prove that a pseudo-annulus is quasi-Stein.

Namely let Γ *be the skeleton of* X*. It is homeomorphic to an open interval. Consider a non empty triangulation* S *of* X *such that* $\Gamma_S = \Gamma$*. Consider the canonical retraction* $r\colon X \to \Gamma$*.*

Let I *be a closed interval inside* Γ*. It contains a finite subset* T *of* S*. By [Duc, 5.2.2.3],* $S \setminus T$ *is still a triangulation of* X*. We deduce that* I *is an interval inside the skeleton of an annulus, hence* $r^{-1}(I)$ *is a closed annulus.*

From the previous argument, it follows that X *may be written as an increasing union of closed annuli. We deduce that* X *is quasi-Stein.*

We now define the open boundary of X. Consider the set $\mathcal{I}$ of maps $]0,1[\to X$ such that :

i) the map is an homeomorphism of $]0,1[$ onto its image I;

ii) I is the skeleton of a pseudo-annulus X' such that $X - X'$ is connected;

iii) I is not relatively compact in X.

We define an equivalence relation $\sim$ in $\mathcal{I}$ by saying that two maps are equivalent if the intersection of their images $I_1 \cap I_2$ is again the image of a map in $\mathcal{I}$.

Definition 1.1.10 (Open boundary)**.** *We call open boundary of* X *the set*

$$\partial^o X \ := \ (\mathcal{I}/\sim)\ . \tag{1.7}$$

of equivalence classes by the above equivalence relation. We call germ of segment at the open boundary of X *an element of* $(\mathcal{I}/\sim)$*. If the cardinality of the open boundary is finite we denote it by*

$$N(X)\ . \tag{1.8}$$

As an example, if X is an annulus $N(X) = 2$, if it is a disk $N(X) = 1$.

Lemma 1.1.11. *If the curve* X *admits a weak triangulation* S *such that* Γ_S *is* topologically finite*, then the cardinality of the open boundary is finite.*

Proof. If we express Γ_S as a finite union of real intervals, then there are a finite number of such intervals that are not relatively compact in X (i.e. their closure in X is not compact). Up to subdivide each interval into two sub-intervals, we can assume that each non relatively compact interval I of that union is *semi-open*. Under this convention, one sees that the interior of a non relatively compact interval of that union is the skeleton I of a well defined pseudo-annulus in X, defining an element of the open boundary of X. This correspondence is clearly a bijection with the open boundary of X, which is then finite. □

1.1.8 *Euler characteristic of X.* We now define the Euler characteristic $\chi(X)$ of X. Recall that for all point $x \in X$ of type 2, we denote by $g(x)$ the genus of the residual curve at x (cf. [PP13, 6.2.12], [Duc, 4.2.11.1]). If x is a point of type 3 or 4, its genus is 0 by definition (recall that a point of type 3 (resp. 4) always admits an open neighborhood in X isomorphic to an open annulus (resp. disk)). Notice that $\chi(X)$ is insensitive to scalar extension of K, so points of type 4 do not play any role here.

The genus of an arbitrary quasi-smooth k-analytic curve has been defined by Q. Liu in [Liu87, Définitions 1.4 et 1.5]. We state it in different terms, which are more suitable to our setting.

Definition 1.1.12. *If X is k-affinoid, and if S is a finite weak triangulation of X, we define the topological Euler characteristic of X as*

$$\chi_{top}(X) \ := \ \mathrm{Card}(S) - E_S \ , \tag{1.9}$$

where E_S is the number of edges of Γ_S and S the set of its vertexes. We also define the genus of X as

$$g(X) \ := \ \sum_{x \in S} g(x) + 1 - \chi_{top}(X) \ . \tag{1.10}$$

Those quantities do not depend on the choice of the triangulation S.

In general, we set

$$\chi_{top}(X) \ := \ \sup\{\chi_{top}(Y) \mid Y \text{ affinoid domain of } X\} \tag{1.11}$$

and

$$g(X) \ := \ \sup\{g(Y) \mid Y \text{ affinoid domain of } X\} \ . \tag{1.12}$$

We say that the curve X has finite genus if $g(X) < +\infty$.

If X is an analytic domain of an algebraic curve, then its genus is finite. Moreover if the curve X is projective and geometrically connected, by [Liu87, Proposition 1.5], the quantity $g(X)$ coincides with its genus.

Definition 1.1.13. *Let S be a weak triangulation of X. For all $x \in S$ we denote by $N_S(x)$ the number of germ of segments out of x belonging to Γ_S, and*

$$\chi(x, S) \ := \ 2 - 2g(x) - N_S(X) \ . \tag{1.13}$$

Lemma 1.1.14 (Compare with [PP13, Lemma 7.2.1])**.** *Let S be a non-empty weak triangulation of X. Assume that the skeleton Γ_S is* topologically *finite, the genus $g(X)$ is finite, and that there are a finite number of points of positive genus in X. Then the number*

$$\chi(X) \ := \ \lim_{\substack{S' \subseteq S \\ S' \text{ finite}}} \ \sum_{x \in S'} \chi(x, S') \tag{1.14}$$

is finite and independent of S, and it is equal to

$$\chi(X) \ = \ 2 - 2g(X) - N(X) \ . \tag{1.15}$$

Proof. Let $b_1, \dots, b_m$ be the germs of segment at the open boundary of X. We construct a sequence of affinoid domains $(Y_n)_n$ is X such that

i) $X = \cup_n Y_n$;
ii) for all n we have $X - Y_n = \cup_{i=1}^m A_{n,i}$, where, for all i, $A_{n,i}$ is a pseudo-annulus corresponding to b_i;
iii) $S_n := S \cap Y_n$ is a weak triangulation of Y_n.

One sees that for all n we have $g(Y_n) = g(X)$, since the genus of a point close to the open boundary of X is always 0.

Let E_{S_n} be the number of edges of $\Gamma_{S_n} \subset Y_n$ between points of S_n. We have

$$\sum_{x\in S_n}(2-2g(x)-N_S(x)) = -2\sum_{x\in S_n} g(x) + 2\,\mathrm{Card}(S_n) - 2E_{S_n} - N(X). \tag{1.16}$$

By (1.10) we conclude. □

1.2 Radii

In this section we recall the definition of the radii of convergence of a differential equation over X following [Pul12], [PP12] and [PP13]. The properties listed here are extracted from [PP13], where one can find a more extensive treatment.

Without explicit mention of the contrary, we assume everywhere that the curve X is endowed with a weak triangulation S.

Definition 1.2.1. *Let $x \in X$. The map $\mathscr{M}(\mathscr{H}(x)) \to X$ lifts canonically to a map $\mathscr{M}(\mathscr{H}(x)) \to X_{\mathscr{H}(x)}$ by the universal property of the Cartesian diagram $X_{\mathscr{H}(x)}/\mathscr{H}(x) \to X/K$. We denote by $t_x \in X_{\mathscr{H}(x)}$ the $\mathscr{H}(x)$-rational point so obtained.*

We fix once for all a large complete valued field extension Ω/K containing (isometrically) the fields $\mathscr{H}(x)$, for all $x \in X$ (this is possible by [PP13,]). We denote by $D(x) \in X_\Omega$ the maximal open disk centered at t_x whose image in X is reduced to x.

The weak triangulation S of X can be canonically lifted into a weak triangulation of X_Ω, still called S, (cf. [PP12]), and we denote by $D(x,S) \subseteq X_\Omega$ the maximal disk centered at t_x such that $D(x,S) \cap S = \emptyset$.

Choose an isomorphism $D(x,S) \xrightarrow{\sim} D^-_\Omega(0,R)$ sending t_x at 0. By a result of M. Lazard (cf. [Laz62] and [Chr12, Ch.II, Section 4.4]) the restriction $\widetilde{\mathscr{F}}$ of $\mathscr{F}$ to $D^-_\Omega(0,R)$ is free of rank $r = \mathrm{rank}(\mathscr{F}_x)$. Denote by

$$\mathcal{R}^{\widetilde{\mathscr{F}}}_{S,i}(x) \;>\; 0 \tag{1.17}$$

the radius of the maximal open disk centered at 0 and contained in $D^-_\Omega(0,R)$ on which the connection of $\widetilde{\mathscr{F}}$ admits at least $r-i+1$ horizontal sections that are linearly independent over Ω.

Definition 1.2.2 (Multiradius)**.** *We call multiradius of $\mathscr{F}$ at x the tuple*

$$\boldsymbol{\mathcal{R}}_S(x,\mathscr{F}) \;:=\; (\mathcal{R}_{S,1}(x,\mathscr{F}),\ldots,\mathcal{R}_{S,r}(x,\mathscr{F})) \tag{1.18}$$

where, for every i, one has $\mathcal{R}_{S,i}(x,\mathscr{F}) \;:=\; \mathcal{R}^{\widetilde{\mathscr{F}}}_{S,i}(x)/R \;\in]0,1]$.

The definition only depends on x and $(\mathscr{F},\nabla)$. Each $\mathcal{R}_{S,i}(x,\mathscr{F})$ is the inverse of the *modulus* (cf. [PP13, 1.1.2]) of a well defined sub-disk $D_{S,i}(x,\mathscr{F}) \subseteq D(x,S)$, centered at t_x:

$$\emptyset \neq D_{S,1}(x,\mathscr{F}) \;\subseteq\; D_{S,2}(x,\mathscr{F}) \;\subseteq\; \cdots \;\subseteq\; D_{S,r}(x,\mathscr{F}) \;\subseteq\; D(x,S)\;. \tag{1.19}$$

Remark 1.2.3. *Let S, S' be two weak triangulations of X. If $\Gamma_S = \Gamma_{S'}$, then $\boldsymbol{\mathcal{R}}_S(-,\mathscr{F}) = \boldsymbol{\mathcal{R}}_{S'}(-,\mathscr{F})$. Indeed the disk $D(x,S)$ only depends on Γ_S.*

Definition 1.2.4. *Let $i \in \{1,\ldots,r\}$. We say that the index i* separates the radii of $\mathscr{F}$ at $x \in X$ *if either $i = 1$ or if $\mathcal{R}_{S,i-1}(x,\mathscr{F}) < \mathcal{R}_{S,i}(x,\mathscr{F})$. We say that i* separates the radii of $\mathscr{F}$ *if it separates the radii of $\mathscr{F}$ at all $x \in X$.*

We say that the i-th radius $\mathcal{R}_{S,i}(x,\mathscr{F})$ is

$$\begin{cases} \textit{spectral} & \text{if} \quad D_{S,i}(x,\mathscr{F}) \subseteq D(x)\ , \\ \textit{solvable} & \text{if} \quad D_{S,i}(x,\mathscr{F}) = D(x)\ , \\ \textit{over-solvable} & \text{if} \quad D_{S,i}(x,\mathscr{F}) \supset D(x)\ . \end{cases} \tag{1.20}$$

Solvable radii are spectral by definition. We also say that *the index* i is spectral, solvable, over-solvable.

Remark 1.2.5. *For a radius $\mathcal{R}_{S,i}(x,\mathscr{F})$ being spectral non solvable is an intrinsic property, since it does not depend on the choice of S (cf. [PP13, 2.8.1]). The same is true for being "solvable or oversolvable", but the fact of being over-solvable highly depends on S. Over- solvable radii can be solvable with respect to another triangulation.*

Definition 1.2.6. *We denote by $0 \leqslant i_x^{\mathrm{sp}} \leqslant i_x^{\mathrm{sol}} \leqslant r$ the indexes such that*

i) *$\mathcal{R}_{S,i}(x,\mathscr{F})$ is spectral non solvable for $i \leqslant i_x^{\mathrm{sp}}$,*

ii) *$\mathcal{R}_{S,i}(x,\mathscr{F})$ is solvable for $i_x^{\mathrm{sp}} < i \leqslant i_x^{\mathrm{sol}}$,*

iii) *$\mathcal{R}_{S,i}(x,\mathscr{F})$ is over-solvable for $i_x^{\mathrm{sol}} < i$.*

We call i_x^{sp} and i_x^{sol} the spectral *and* over-solvable cutoffs *respectively.*

If $i_x^{\mathrm{sol}} = 0$ (resp. $i_x^{\mathrm{sol}} = r$), then all the radii are over-solvable (resp. spectral). If $i_x^{\mathrm{sp}} = 0$ (resp. $i_x^{\mathrm{sp}} = r$), then all the radii are solvable or over-solvable (resp. spectral non solvable). If $i_x^{\mathrm{sp}} = i_x^{\mathrm{sol}}$, then $\mathscr{F}$ has no solvable radii.

Definition 1.2.7 (Convergence Newton polygon)**.** *We call* convergence Newton polygon *of $\mathscr{F}$ at $x \in X$ the epigraph of the unique continuous convex function $h_x : [-\infty, r[\to \mathbb{R}_{\geqslant 0}$ satisfying*

i) *$h_x(0) = 0$, and $h_x(i) - h_x(i-1) = -\log(\mathcal{R}_{S,r-i+1}(x,\mathscr{F}))$, for all $i = 1,\dots,r$;*

ii) *For all $i = 1,\dots,r$ the function h_x is affine over $[i-1,i]$, and constant on $]-\infty,0]$.*

In other words it is the polygon whose slopes are $-\log\mathcal{R}_{S,r}(x,\mathscr{F}) \leqslant \cdots \leqslant -\log\mathcal{R}_{S,1}(x,\mathscr{F})$. For all $i = 1,\dots,r$ the numbers $h_x(i) = \sum_{j=r-i+1}^{r} -\log\mathcal{R}_{S,j}(x,\mathscr{F})$ are called the partial heights of the polygon.

The following definition is convenient for technical reasons concerning the super-harmonicity properties. It agrees with the conventions of [Pul12, Section 4.3].

Definition 1.2.8 (Reversed Newton polygon)**.** *We call* reversed convergence Newton polygon *of $\mathscr{F}$ the polygon whose slopes are $\log\mathcal{R}_{S,1}(x,\mathscr{F}) \leqslant \cdots \leqslant \log\mathcal{R}_{S,r}(x,\mathscr{F})$.*

Let $i \leqslant r = \mathrm{rank}(\mathscr{F})$. We call i-th partial height *of $\mathscr{F}$ the function*

$$H_{S,i}(x,\mathscr{F}) \ := \prod_{j=1,\dots,i} \mathcal{R}_{S,j}(x,\mathscr{F})\ . \tag{1.21}$$

With the notations of Def. 1.2.7 one has $\ln(H_{S,i}(x,\mathscr{F})) = h_x(r-i+1) - h_x(r)$.

Definition 1.2.9 (Vertex free of solvability)**.** *We say that $i = 1,\dots,r$ is a* vertex *at x, of the reversed convergence Newton polygon, if $i = r$, or if $i+1$ separates the radii at x. We say that i is a vertex* free of solvability *at x if i moreover none of the indexes $j \in \{1,\dots,i\}$ is solvable at x.*

Proposition 1.2.10 (integrality of the partial heights)**.** *Let $x \in X$, and let b be a germ of segment* *out of x. For all $i = 1, \ldots, r$, the slopes of $H_{S,i}(-, \mathscr{F})$ along b belong to the set*

$$\mathbb{Z} \cup \frac{1}{2}\mathbb{Z} \cup \cdots \cup \frac{1}{r}\mathbb{Z} . \tag{1.22}$$

Moreover if $i = r$ or if $\mathcal{R}_{S,i}(x, \mathscr{F}) < \mathcal{R}_{S,i+1}(x, \mathscr{F})$ (i.e. if i is a vertex of the reversed polygon), then

$$\partial_b H_{S,i}(x, \mathscr{F}) \in \mathbb{Z} . \qquad \square \tag{1.23}$$

1.3 Controlling graphs

In [Pul12] and [PP12] we obtained the following result.

Theorem 1.3.1 ([Pul12],[PP12])**.** *For all $i = 1, \ldots, r$ the functions $x \mapsto \mathcal{R}_{S,i}(x, \mathscr{F})$ are continuous. Moreover there exists a locally finite graph $\Gamma \subseteq X$ such that for all i the radius $\mathcal{R}_{S,i}(-, \mathscr{F})$ is constant on every connected components of $X - \Gamma$.*

The curve $X - \Gamma_S$ is disjoint union of virtual open disks.

Definition 1.3.2. *Let $\mathcal{T}$ be a set, and let $f : X \to \mathcal{T}$ be a function. We call S-*controlling graph *(or S-*skeleton*) of f the set $\Gamma_S(f)$ of points $x \in X$ that admit no neighborhoods[2] D in X such that*

i) *D is a virtual disk;*
ii) *f is constant on D;*
iii) *$D \cap \Gamma_S = \emptyset$ (or equivalently $D \cap S = \emptyset$).*

In particular $\Gamma_S \subseteq \Gamma_S(f)$.

Remark 1.3.3. *The graph $\Gamma_S(f)$ is different from the locus defined as the complement of the union of the open subsets of X on which f is constant. Indeed f can be constant along some segments in $\Gamma_S(f)$, and hence on the corresponding annulus in X. This is because the definition involves only disks on which f is constant, and not arbitrary subsets.*

We denote by $\Gamma_{S,i}(\mathscr{F})$ the controlling graph of the function $\mathcal{R}_{S,i}(-, \mathscr{F})$. By definition

$$\Gamma_S \subseteq \Gamma_{S,i}(\mathscr{F}) . \tag{1.24}$$

Hence $X - \Gamma_{S,i}(\mathscr{F})$ is a disjoint union of virtual open disks. If $X = D$ is a virtual open disk with empty weak triangulation, and if $\mathcal{R}_{S,i}(-, \mathscr{F})$ is constant on D, then $\Gamma_{S,i}(\mathscr{F}) = \Gamma_S = \emptyset$. In all other cases $\Gamma_{S,i}(\mathscr{F})$ is non empty. The *controlling graph* $\Gamma_S(\mathscr{F})$ of $(\mathscr{F}, \nabla)$ is by definition the union of all the $\Gamma_{S,i}(\mathscr{F})$:

$$\Gamma_S(\mathscr{F}) := \bigcup_{i=1}^{r} \Gamma_{S,i}(\mathscr{F}) . \tag{1.25}$$

One has $\Gamma_S(\mathscr{F}) = \emptyset$ if and only if $X = D$ is a virtual disk with empty weak triangulation, and $\boldsymbol{\mathcal{R}}_S(-, \mathscr{F})$ is a constant function on D. An operative description of the controlling graphs is given in [PP13, sections 6 and 7] .

Proposition 1.3.4 ([PP12, (2.3.1)], [PP13, 2.7.1])**.** *Let S, S' be two triangulations such that $\Gamma_S \subseteq \Gamma_{S'}$. Then for all $i = 1, \ldots, r$ one has*

$$\mathcal{R}_{S',i}(x, \mathscr{F}) = \min\Big(1 , f_{S,S'}(x) \cdot \mathcal{R}_{S,i}(x, \mathscr{F}) \Big) , \tag{1.26}$$

where $f_{S,S'} : X \to [1, +\infty[$ is the function associating to x the modulus $f_{S,S'}(x) \geqslant 1$ of the inclusion of disks $D(x, S') \subseteq D(x, S)$. $\square$

[2] Note that $D(x)$ is not a neighborhood of x in X.

Proposition 1.3.5 ([PP12, 3.3.1],[PP13, 2.7.2]). *Let S, S' be two triangulations such that $\Gamma_S \subseteq \Gamma_{S'}$. Then*

$$\Gamma_{S',i}(\mathscr{F}) \;=\; \Gamma_{S'} \cup \Gamma_{S,i}(\mathscr{F})\;. \qquad \square \tag{1.27}$$

Proposition 1.3.6. *Let $Y \subseteq X$ be an analytic domain. Let S_X and S_Y be triangulations of X and Y respectively such that $(\Gamma_{S_X} \cap Y) \subseteq \Gamma_{S_Y}$. If $y \in Y$, then for all $i = 1, \ldots, r$ we have*

$$\mathcal{R}_{S_Y,i}(y, \mathscr{F}_{|Y}) \;=\; \min\Big(\,1\;,\; f_{S_X,S_Y}(y)\cdot \mathcal{R}_{S_X,i}(y,\mathscr{F})\,\Big)\;, \tag{1.28}$$

where $f_{S_X,S_Y} : Y \to [1,+\infty[$ is the function associating to $y \in Y$ the modulus $f_{S_X,S_Y}(y) \geqslant 1$ of the inclusion $D(y, S_Y) \subseteq D(y, S_X)$. Hence

$$\Gamma_{S_Y,i}(\mathscr{F}_{|Y}) \;=\; (\Gamma_{S_X,i}(\mathscr{F}) \cap Y) \cup \Gamma_{S_Y}\;. \qquad \square \tag{1.29}$$

1.4 Weak super-harmonicity of $H_{S,i}(x, \mathscr{F})$.

Definition 1.4.1. *We define inductively a sequence of locally finite sets*

$$\mathscr{C}_{S,1}(\mathscr{F}) \;\subseteq\; \ldots \;\subseteq\; \mathscr{C}_{S,r}(\mathscr{F}) \;\subseteq\; X \tag{1.30}$$

as follows. Let $\aleph_1 := \emptyset$, and for $2 \leqslant i \leqslant r$ let $\aleph_i$ be the locally finite set of points $x \in X - \Gamma_S$ satisfying

i) *$\mathcal{R}_{S,i}(-, \mathscr{F})$ is solvable at x;*
ii) *x is an end point of $\Gamma_{S,i}(\mathscr{F})$;*
iii) *$x \in \Big(\cup_{j=1,\ldots,i}\Gamma_{S,j}(\mathscr{F})\Big) \cap \Gamma_{S,i}(\mathscr{F}) \cap \Gamma_S(H_{S,i}(-, \mathscr{F}))$.*[3]

Define

$$\mathscr{C}_{S,i}(\mathscr{F}) \;:=\; \bigcup_{j=1,\ldots,i} \aleph_j\;. \tag{1.31}$$

In the sequel if no confusion is possible we write $\mathscr{C}_{S,i} := \mathscr{C}_{S,i}(\mathscr{F})$ for short.

We refer to [PP13, 6.2.5] for the definition of $dd^c H_{S,i}(x, \mathscr{F})$.

Theorem 1.4.2 (weak super-harmonicity). *Let $x \in X$. If it is of type 2, assume that it satisfies the condition (TR) of [PP13, 6.2.17]. Let $i \in \{1, \ldots, r\}$.*

i) *If $x \in \Gamma_S \cap \mathrm{Int}(X)$, then*

$$dd^c H_{S,i}(x, \mathscr{F}) \;\leqslant\; (2g(x) - 2 + N_S(x)) \cdot \min(i, i_x^{\mathrm{sp}}). \tag{1.32}$$

ii) *If $x \notin (S \cup \mathscr{C}_{S,i})$, then*

$$dd^c H_{S,i}(x, \mathscr{F}) \;\leqslant\; 0. \tag{1.33}$$

Moreover equalities hold in (1.32) and (1.33), if i is a vertex free of solvability at x (cf. Def. 1.2.9).

2. Local measure of the irregularity at a Berkovich point

In [PP13, Rk. 5.5.4 and section 5.7] we have seen that the Christol-Mebkhout Newton polygon of a *solvable* differential equation over the Robba ring equals the derived of the convergence Newton polygon. In particular the irregularity of such an equation (which is defined as the height of the Christol-Mebkout Newton polygon) coincides with the slope of the height $H_{S,r}(-, \mathscr{F})$ of the reversed convergence Newton polygon.

[3] Here $\Gamma_S(H_{S,i}(-, \mathscr{F}))$ have been defined in Def. 1.3.2.

The Grothendieck-Ogg-Shafarevich formula for an over-convergent isocrystal (cf. [Rob84], [Rob84], [CM97], [CM00], [CM01]) describes the index as the sum of the Euler characteristic of the variety multiplied by the rank of the crystal, plus the sum of the irregularities of the equations at the singularities of the isocrystal. In the sense of Berkovich the sum of the irregularities is the negative of a certain Laplacian of $H_{S,r}(-,\mathscr{F})$ (after a convenient localization, cf. Thm. 2.7.13). The negative of the Laplacian

$$-\,dd^c H_{S,r}(x,\mathscr{F}) \tag{2.1}$$

is then related to the *local measure of the irregularity at* x.

It seems to us important to prove that the sum of the local irregularities at x is a non negative integer, that is the function $H_{S,r}(x,\mathscr{F})$ is super harmonic at x:

$$-\,dd^c H_{S,r}(x,\mathscr{F}) \;\geqslant\; 0\,. \tag{2.2}$$

The fact that this number is an integer is clear from Proposition 1.2.10, and it is supposed to be related to the local ramification filtration of the Tannakian group attached to the fiber functor at x.[4] This is one of the motivations of this section. The other motivation is to slight extend the result of Christol and Mebkhout (cf. [CM00], [CM01]) about the finite dimensionality of the de Rham cohomology, in order to cover the case of non solvable differential modules. In section 3 in fact we obtain a global finite dimensionality result together with a global Grothendieck-Ogg-Shafarevich formula (i.e. index formula) that constitutes a global analogue of Robba's (local) measure of the irregularity (cf. [Rob84], [Rob85]).

Points that belongs to the boundary of X behave as if some branches out of them were missing. In this case $-dd^c H_{S,r}(x,\mathscr{F})$ does not contain "*the entire information*". So we bound our study to the points of $\mathrm{Int}(X)$.

On the other hand we have already proved in Theorem 1.4.2 that we have super-harmonicity outside a locally finite subset of X which is included in the union of S, ∂X, and the set $\mathscr{C}_{S,r}(\mathscr{F})$ (cf. Definition 1.4.1). In particular if none of the indexes $i = 1,\dots,r$ is solvable at $x \in \mathrm{Int}(X)$ we have

$$dd^c H_{S,r}(x,\mathscr{F}) \;=\; 0\,. \tag{2.3}$$

The default of harmonicity comes then from the solvability. Theorem 1.4.2 shows that we do not have super-harmonicity at the points of S. It remains to study the situation at a point of $\mathscr{C}_{S,r}(\mathscr{F})$. Such a point does not belongs to Γ_S, hence it always have a maximal disk as an open neighborhood, so we are reduced to working in the affine line.

In this section we deal with that problem by considering the local cohomological situation underling to the formula given in Theorem 1.4.2. We translate in our context the index formula of Robba [Rob85] and Christol-Mebkhout [CM01], and relate it to the Laplacian $dd^c H_{S,r}(x,\mathscr{F})$. A straightforward application of these result shows that, locally around x, the formula of Theorem 1.4.2 comes from a local Grothendieck-Ogg-Shafarevich formula under certain classical assumptions concerning Liouville numbers.

Surprisingly enough, the classical conditions of non Liouville exponents, and the local Grothendieck-Ogg-Shafarevich formula, are not enough to obtain super-harmonicity (2.2) at the points of $\mathscr{C}_{S,r}(\mathscr{F})$ (see Thm. 2.8.6). We also need some extra conditions. In particular we need a compatibility condition between the radii of $\mathscr{F}$ and of its dual, in analogy with the condition of [PP13, 5.4.1, 5.4.3], where an analogous condition implies a direct sum decomposition.

[4] Note that such a filtration only exists locally (i.e. for the Tannakian group of the category of differential equations over $\mathscr{O}_{X,x}$), since a filtration of the global Tannakian group would imply a global decomposition theorem separating the radii at x which is false.

Finally we notice that the cohomological results of this section are not entirely straightforward consequence of the index results of Christol and Mebkhout. Indeed the finiteness of the radii (cf. [Pul12], [PP12], Theorem 1.3.1) is used to guarantee that the number of singular directions of $\mathscr{F}$ out of x is not infinite (cf. Definition 2.2.2). And the decomposition theorems of [PP13], in their local form, are a crucial step.

2.1 Tubes and over-convergence.

In this section we give some definitions imitating the rigid cohomology setting.

Definition 2.1.1 (Elementary tube)**.** *Let $x \in X$. An* elementary tube *V centered at x is one of the two kind of domains:*

i) *V is an affinoid domain of X, containing x, such that $V - \{x\}$ is a disjoint union of open disks.*

ii) *$V = X$, and $V - \{x\}$ is a disjoint union of open disks.*

Remark 2.1.2. *The usual tubes of rigid cohomology are more complex domains. We here restrict the definition because this corresponds to our needing.*

Remark 2.1.3. *If x is a point of type 1 or 4, there is no elementary tubes centered at x, because there are no disks with boundary x.*

If x has type 3, an elementary tube V centered at x is either reduced to $\{x\}$, or a closed disk having x at its boundary, or $X = \mathbb{P}^{1,\mathrm{an}}_K$ and $V = X$.

If x is of type 2, then V is always an affinoid domain of X, except in the individual case where $x \in \mathrm{Int}(X)$ and V contains all germ of segments out of x. In this case $V = X$, because X is connected.

Definition 2.1.4 (Singular directions)**.** *Let $x \in X$, and let V be an elementary tube centered at x. There are a finite number of branches $b_1, \ldots, b_n$ out of x that do not intersect V. We call them* singular directions *of x with respect to V:*

$$\mathrm{Sing}(x, V) \;:=\; \{b_1, \ldots, b_n\}\;. \tag{2.4}$$

If K is algebraically closed we set

$$N_V(x) \;:=\; n \;=\; \mathrm{Card}(\mathrm{Sing}(x, V))\;. \tag{2.5}$$

If K is general we set

$$N_V(x) \;:=\; N_{V_{\widehat{K^{\mathrm{alg}}}}}(x)\;. \tag{2.6}$$

Definition 2.1.5 (Over-convergent functions)**.** *Let V be an elementary tube centered at $x \in X$. We set*

$$\mathscr{O}^\dagger_X(V) \;:=\; \bigcup_{V \subset U} \mathscr{O}(U)\;, \tag{2.7}$$

where U runs in the family of all neighborhoods of V in X. We often write $\mathscr{O}^\dagger(V) := \mathscr{O}^\dagger_X(V)$ if no confusion is possible.

Remark 2.1.6. *If $V = X$ one has $\mathscr{O}^\dagger(V) = \mathscr{O}(V)$. The word* over-convergent *loose then its meaning. The definition is completely satisfactory only if $x \in \mathrm{Int}(X)$. In rigid cohomology this is often fulfilled by embedding V into a projective curve $\overline{X}$ and considering $\mathscr{O}^\dagger_{\overline{X}}(V)$ instead of $\mathscr{O}^\dagger_X(V)$.*

Remark 2.1.7. *Let V be an elementary tube centered at $x \in X$. A basis of neighborhoods of V is* *formed by star-shaped neighborhoods of x in X containing V (cf. 1.1.5). Such a neighborhood can be written as*

$$U := \Big(\bigcup_{b\in\mathrm{Sing}(x,V)} C_b\Big)\bigcup V \tag{2.8}$$

where C_b is a virtual open annulus, which is a section of b (cf. section 1.1.1).

Definition 2.1.8 (Basic neighborhoods)**.** *We call* basic neighborhood *of the elementary tube V, any neighborhood of V of the form (2.8).*

Definition 2.1.9. *If b is a branch out of $x \in X$, and if the connected component of $X - \{x\}$ containing b is a virtual open disk, then we denote it by D_b. If $x \notin \Gamma_S$ we denote by*

$$D_x \tag{2.9}$$

the closed disk with boundary x.

Remark 2.1.10. *Let V be an elementary tube centered at x. If M is a differential module over $\mathscr{O}^\dagger(V)$, then it is defined over some basic neighborhood U of V. Since $\{x\}$ is always a weak triangulation of U, we may consider the radii*

$$\mathcal{R}_{\{x\},i}(y,\mathrm{M})\;, \quad y \in U\;. \tag{2.10}$$

In the sequel M will often be the restriction to $\mathscr{O}^\dagger(V)$ of a global differential equation over X, and we will compare the global radii over X with the radii $\mathcal{R}_{\{x\},i}(y,\mathrm{M})$ (cf. proof of Thm. 2.8.2).

Definition 2.1.11 (Euler characteristic)**.** *Let $x \in \mathrm{Int}(X)$ be a point of type 2, and let V be an elementary tube centered at x. If K is algebraically closed, and if $b_1,\ldots,b_{N_V(x)}$ are the singular directions of V, we denote by*

$$\chi(V^\dagger) \;:=\; \chi(\mathcal{C}_x - \{b_1,\ldots,b_{N_V(x)}\}) \;=\; 2-2g(x)-N_V(x) \tag{2.11}$$

the Euler-Poincaré characteristic of $\mathcal{C}_x - \{b_1,\ldots,b_{N_V(x)}\}$.

If K is general, we set

$$\chi(V^\dagger) \;:=\; \chi(V^\dagger_{\widehat{K^{\mathrm{alg}}}})\;. \tag{2.12}$$

Remark 2.1.12 (Mittag-Leffler decomposition)**.** *Assume that K is algebraically closed. Let X be the affine line, $x := x_{0,1}$, and $D_x := D^+_K(0,1)$ be the closed unit disk. Let $V \subset X$ be an elementary tube centered at x. Let b_∞ be the direction out of x not in D_x (and hence not in V).*

Each function of $\mathscr{O}^\dagger(V)$ can be written uniquely as (cf. [Chr12, Section 3.1])

$$f_{b_\infty}(T) + \sum_{b\in\mathrm{Sing}(x,V)-\{b_\infty\}} f_b(T)\;, \tag{2.13}$$

where

- i) $f_{b_\infty}(T) = \sum_{k\geqslant 0} a_{k,b_\infty}T^k$ *converges for $|T| \leqslant 1+\varepsilon$ for some $\varepsilon > 0$;*
- ii) *For all $b \in \mathrm{Sing}(x,V) - \{b_\infty\}$, $f_b(T) = \sum_{k\geqslant 1} a_{k,b}(T-c_b)^{-k}$ converges for $|T-c_b| > 1-\varepsilon$ for some unspecified $\varepsilon > 0$, and c_b is such that the open disk D_b centered at c_b with boundary x, contains the branch b.*

The same happens for any $f \in \mathscr{O}(V)$ with the difference that f_∞ converges for $|T| \leqslant 1$, and f_b converges for $|T-c_b| \geqslant 1$.

So we have

$$\mathscr{O}^\dagger(V) \;=\; \mathscr{O}^\dagger(D_x)\oplus\Big(\bigoplus_{b\in\mathrm{Sing}(x,V),b\neq b_\infty}(T-c_b)^{-1}\cdot\mathscr{O}^\dagger(\mathbb{P}^{1,\mathrm{an}}_K-D_b)\Big)\;, \tag{2.14}$$

$$\mathscr{O}(V) \;=\; \mathscr{O}(D_x)\oplus\Big(\bigoplus_{b\in\mathrm{Sing}(x,V),b\neq b_\infty}(T-c_b)^{-1}\cdot\mathscr{O}(\mathbb{P}^{1,\mathrm{an}}_K-D_b)\Big)\;. \tag{2.15}$$

2.2 Canonical local decomposition.

Proposition 2.2.1. *Let $\mathscr{F}$ be a differential equation over X, and let $x\in X$ be a point of type 2 or 3. The union of $\{x\}$ with all virtual open disks D, with boundary x, such that $D\cap\Gamma_S(\mathscr{F})=\emptyset$, is an elementary tube centered at x.*

Proof. This follows immediately from the locally finiteness of the graph $\Gamma_S(\mathscr{F})$ (cf. Thm. 1.3.1). □

Definition 2.2.2 (Canonical tube and singular directions)**.** *Let $\mathscr{F}$ be a differential equation over X, and let $x\in X$ be a point of type 2 or 3. We denote by*

$$V_S(x,\mathscr{F}) \tag{2.16}$$

the elementary tube centered at x defined in Proposition 2.2.1. We set

$$\mathrm{Sing}(x,\mathscr{F}) \;:=\; \mathrm{Sing}(x,V_S(x,\mathscr{F}))\;. \tag{2.17}$$

The following proposition is a direct consequence of the global decomposition theorems [PP13, 5.4.3, 5.4.10, 5.6.14].

Proposition 2.2.3 (Canonical local decomposition)**.** *Let $x\in X$ be a point of type 2 or 3. There exists a basic neigborhood U of $V_S(x,\mathscr{F})$ in X, such that the restriction $\mathscr{F}_{|U}$ of $\mathscr{F}$ to U admits a direct sum decomposition*

$$\mathscr{F}_{|U} \;:=\; \oplus_{0<\rho\leqslant 1}\mathscr{F}^{\rho}_{|U} \tag{2.18}$$

where $\mathscr{F}^{\rho}_{|U}$ takes in account the radii of $\mathscr{F}$ whose value at x is ρ.

In particular we have a decomposition

$$\mathscr{F}_{|U} \;=\; \mathscr{F}^{\geqslant\mathrm{sol}}_{|U}\oplus\mathscr{F}^{<\mathrm{sol}}_{|U}\;, \tag{2.19}$$

where $\mathscr{F}^{<\mathrm{sol}}_{|U}$ takes in account the radii of $\mathscr{F}$ that are spectral non solvable at x, and $\mathscr{F}^{\geqslant\mathrm{sol}}_{|U}$ the larger radii.[5] *We call (2.19) the* canonical local decomposition *of $\mathscr{F}$.* □

2.3 Over-convergent isocrystals.

We continue our analogy with rigid cohomology.

Definition 2.3.1 (Over-convergent isocrystals)**.** *Let V be an elementary tube centered at a point $x\in X$ of type 2 or 3. An over-convergent isocrystal over V is a differential module M over $\mathscr{O}^\dagger(V)$ such that*

$$\mathcal{R}_{\{x\},1}(x,\mathrm{M}) \;=\; 1\;. \tag{2.20}$$

Remark 2.3.2. *Let M be an over-convergent isocrystal over an elementary tube V centered at x. By [PP13, 6.1.4,ii)] it follows that the radii $\mathcal{R}_{\{x\},i}(-,\mathrm{M})$ are all constant functions over V with value 1. In particular M is trivial over each disk in $V-\{x\}$. So if M is considered as a differential*

[5] Spectral radii are invariant by localization, and by changing of the weak triangulation (cf. Remark 1.2.5), so we do not need here to specify the triangulation.

module over a basic neighborhood U *of* V*, then the branches out of* x *that belong to* $\Gamma_{\{x\},i}(\mathrm{M})$ *are exactly those of* $\mathrm{Sing}(x,V)$*. And if* U *is small enough, one also has*

$$\Gamma_{\{x\},i}(\mathrm{M}) = \Gamma_U\ , \quad \textit{for all } i = 1,\ldots,r = \mathrm{rank}(\mathrm{M})\ . \tag{2.21}$$

Remark 2.3.3. *Let* $\mathscr{F}$ *be a differential equation over* X*. If* x *is a point of type* 2 *or* 3*, then* $V = V_S(x,\mathscr{F})$ *is an elementary tube centered at* x*, and the module* $\mathscr{F}^{\geqslant \mathrm{sol}}_{|V^\dagger}$ *of Proposition 2.2.3, is naturally an over-convergent isocrystal over* V*.*

Remark 2.3.4. *If* M *is an over-convergent isocrystal over an elementary tube* V*, then*

$$V_{\{x\}}(x,\mathrm{M}) \ = \ V\ , \qquad \mathrm{Sing}(x,\mathrm{M}) := \mathrm{Sing}(x,V)\ . \tag{2.22}$$

Definition 2.3.5. *Let* $\mathscr{F}$ *be a differential equation over* X*, and let* $x \in X$ *be a point of type* 2 *or* 3*. If* K *is algebraically closed, we denote by*

$$N_S(x,\mathscr{F}) \ := \ \mathrm{Card}(\mathrm{Sing}(x,\mathscr{F})) \tag{2.23}$$

the number of singular directions of $V_S(x,\mathscr{F})$*. Namely*

$$N_S(x,\mathscr{F}) \ = \ \begin{cases} \textit{number of branches out of } x \textit{ belonging to } \Gamma_S(\mathscr{F}) & \textit{if} \quad x \in \Gamma_S(\mathscr{F}) \\ 1 & \textit{if} \quad x \notin \Gamma_S(\mathscr{F}). \end{cases} \tag{2.24}$$

If K *is general, we denote by* $N_S(x,\mathscr{F})$ *the number of singular directions of* $V_S(x,\mathscr{F})_{\widehat{K^{\mathrm{alg}}}}$*.*

Remark 2.3.6. *Assume that* K *is algebraically closed. Let* $x \in \mathrm{Int}(X)$ *be a point of type* 2*. Let* $\mathscr{C}_x - \{b_1,\ldots,b_{N_S(x,\mathscr{F})}\}$ *be the residual curve of* $V_S(x,\mathscr{F})$*. Hence*

$$\chi(V_S(x,\mathscr{F})^\dagger) \ = \ \begin{cases} 2-2g(x)-N_S(x,\mathscr{F}) & \textit{if} \quad x \in \Gamma_S(\mathscr{F}) \\ 1 & \textit{if} \quad x \notin \Gamma_S(\mathscr{F}). \end{cases} \tag{2.25}$$

2.4 De Rham cohomology and index (first properties).

Definition 2.4.1 (De Rham cohomology and Index)**.** *Let* A *be a* K*-algebra. Let* Ω^1_A *be an locally free* A*-module of rank one with a derivation* $d : A \to \Omega^1_A$*, with kernel* K*. Let* $\nabla : \mathrm{M} \to \mathrm{M} \otimes_A \Omega^1_A$ *be a differential module, that we often identify to a differential operator* $\nabla : \mathrm{M} \to \mathrm{M}$*. If the kernel and the cokernel of* ∇ *are finite dimensional* K*-vector spaces, we say that* M has a finite index *and we set*

$$\mathrm{H}^0_{\mathrm{dR}}(A,\mathrm{M}) \ := \ \mathrm{Ker}(\nabla), \tag{2.26}$$
$$\mathrm{H}^1_{\mathrm{dR}}(A,\mathrm{M}) \ := \ \mathrm{Coker}(\nabla), \tag{2.27}$$
$$\chi(\mathrm{M},A) \ := \ \dim_K \mathrm{H}^0_{\mathrm{dR}}(A,\mathrm{M}) - \dim_K \mathrm{H}^1_{\mathrm{dR}}(A,\mathrm{M})\ . \tag{2.28}$$

The index *of* M *is by definition the difference* $\chi(A,\mathrm{M})$*.*

If $Y \subseteq X$ *is a quasi-Stein analytic domain, if* M *is an* $\mathscr{O}(Y)$ *module, and if the index exists, we write*

$$\mathrm{H}^i_{\mathrm{dR}}(Y,\mathrm{M}) \ := \ \mathrm{H}^i_{\mathrm{dR}}(\mathscr{O}(Y),\mathrm{M})\ , \qquad \chi(Y,\mathrm{M}) \ := \ \chi(\mathscr{O}(Y),\mathrm{M})\ . \tag{2.29}$$

If V *is an elementary tube around* x*, if* M *is a differential module over* $\mathscr{O}^\dagger(V)$*, and if the index exists, we set*

$$\mathrm{H}^i_{\mathrm{dR}}(V^\dagger,\mathrm{M}) \ := \ \mathrm{H}^i_{\mathrm{dR}}(\mathscr{O}^\dagger(V),\mathrm{M})\ , \qquad \chi(V^\dagger,\mathrm{M}) \ := \ \chi(\mathscr{O}^\dagger(V),\mathrm{M})\ . \tag{2.30}$$

We denote by $h^i(\cdot,\cdot)$ *the dimension as a* K*-vector space of* $H^i(\cdot,\cdot)$*.*

Let $\mathscr{F}$ be a differential equation over the curve X. We call $\mathcal{E}(\mathscr{F})^\cdot : (\cdots \to 0 \to \mathscr{F} \xrightarrow{\nabla} \mathscr{F} \otimes \Omega^1_X \to 0 \to \cdots)$ the complex of coherent sheaves of groups, where $\mathscr{F}$ is placed at the degree 0.

The cohomology of $\mathscr{F}$ (resp. the hypercohomology of $\mathcal{E}(\mathscr{F})^{\cdot}$) will be denoted by $H^i(\cdot,\mathscr{F})$ (resp $\mathbb{H}^i(\cdot,\mathcal{E}(\mathscr{F})^{\cdot})$).

Definition 2.4.2. *The de Rham cohomology groups* $\mathrm{H}^i_{\mathrm{dR}}(X,\mathscr{F})$ *of* $\mathscr{F}$ *are by definition the hypercohomology groups* $\mathbb{H}^i(X,\mathcal{E}(\mathscr{F})^{\cdot})$ *of the complex* $\mathcal{E}(\mathscr{F})^{\cdot}$.

If X is a quasi-Stein space definitions 2.4.1 and 2.4.2 agrees : $\mathrm{H}_{\mathrm{dR}}(X,\mathscr{F}) = \mathrm{H}_{\mathrm{dR}}(\mathscr{O}(X),\mathscr{F}(X))$. The following proposition together with Corollary 2.4.6 provide conditions to have zero index in the spectral non solvable case.

Proposition 2.4.3. *Assume that the curve* X *is a quasi-Stein space. Let* $\mathscr{F}$ *be a differential equation over* X*. Assume that we are in one of the following two situations*

Situation 1*:*

i) $\Gamma_S \neq \emptyset$ *(i.e.* X *is not a virtual open disk with empty weak triangulation).*

ii) $\Gamma_S(\mathscr{F}) = \Gamma_S$*;*

iii) *all the radii of* $\mathscr{F}$ *are spectral non solvable at each point* $x \in \Gamma_S(\mathscr{F})$*.*

Situation 2*:*

i) $X = D$ *is a virtual open disk with empty triangulation,*

ii) *Let* I *be the germ of segment at the boundary of the disk* D*. Then the radii of* $\mathscr{F}$ *are all constant and spectral non solvable over* I*.*

Then

$$\mathrm{H}^0_{\mathrm{dR}}(X,\mathscr{F}) \;=\; \mathrm{H}^1_{\mathrm{dR}}(X,\mathscr{F}) \;=\; \chi(X,\mathscr{F}) \;=\; 0\,. \tag{2.31}$$

Proof. A global solution of $\mathscr{F}$ produces a solution converging on some maximal disk $D(x,S)$, and hence a solvable or over-solvable radius. So $\mathrm{H}^0_{\mathrm{dR}}(\mathscr{O}(X),\mathscr{F}) = 0$. By the equivalence with global sections we have the equality $\mathrm{H}^1_{\mathrm{dR}}(X\mathscr{F}) = \mathrm{Ext}^1(\mathscr{F}^*(X),\mathscr{O}(X))$ (cf. [Ked10, 5.3.3]). We prove that $\mathrm{Ext}^1(\mathscr{F}^*(X),\mathscr{O}(X)) = 0$ by proving that any sequence

$$0 \to \mathscr{O}(X) \to E(X) \to \mathscr{F}^*(X) \to 0 \tag{2.32}$$

splits.

In the situation 1, condition iii) implies that the radii of $\mathscr{F}$ and of $\mathscr{F}^*$ coincide by [PP13, Prop. 6.3.2 and Thm. 5.4.1], while condition ii) implies $\Gamma_{S,1}(\mathscr{F}) \cup \cdots \cup \Gamma_{S,i-1}(\mathscr{F}) \subseteq \Gamma_{S,i}(\mathscr{F})$ for all i. So the sequence (2.32) fulfills the assumptions of the decomposition Theorem [PP13, 5.4.10], hence it splits.

In the situation 2, by [Ked10, 12.4.1] together with the concavity property of the radii [PP13, Remark 6.1.3] (cf. also [Pul12, Prop. 7.5, Lemma 7.7]), the radii of $\mathscr{F}$ are stable by duality and constant on D. By [PP13, Prop. 2.9.5] the radii of E are the union of those of $\mathscr{F}^*$ and of $\mathscr{O}$, in particular they are all constant. So by the decomposition Theorem [PP13, 5.4.10] the sequence splits. □

Corollary 2.4.4. *Assume that the curve* X *is quasi-Stein. Let* $\mathscr{F}$ *be a differential equation over* X*. Assume that* i *is an index separating the radii of* $\mathscr{F}$ *over* X*, and that* $\mathscr{F}_{<i}$ *satisfies the assumptions of Proposition 2.4.3. Then* $\mathscr{F}$ *has a finite index over* $\mathscr{O}(X)$ *if and only if* $\mathscr{F}_{\geqslant i}$ *has a finite index, and we have*

$$\mathrm{H}^0_{\mathrm{dR}}(X,\mathscr{F}) = \mathrm{H}^0_{\mathrm{dR}}(X,\mathscr{F}_{\geqslant i})\,,\quad \mathrm{H}^1_{\mathrm{dR}}(X,\mathscr{F}) = \mathrm{H}^1_{\mathrm{dR}}(X,\mathscr{F}_{\geqslant i})\,,\quad \chi(X,\mathscr{F}) = \chi(X,\mathscr{F}_{\geqslant i})\,. \tag{2.33}$$

Proof. Write the snake diagram of ∇ acting on the sequence $0 \to \mathscr{F}_{\geqslant i} \to \mathscr{F} \to \mathscr{F}_{<i} \to 0$. □

Corollary 2.4.5. *Let V be an elementary tube centered at x, and let M be a differential module over $\mathscr{O}^\dagger(V)$. Assume that*

i) *the radii $\mathcal{R}_{\{x\},i}(-,\mathrm{M})$ are all spectral and non solvable at x;*
ii) *the radii of M are all constant functions over V (i.e. if $\Gamma_{\{x\}}(\mathrm{M}_{|V}) = \{x\}$).*

Then

$$\mathrm{H}^0_{\mathrm{dR}}(V^\dagger,\mathrm{M}) \;=\; \mathrm{H}^1_{\mathrm{dR}}(V^\dagger,\mathrm{M}) \;=\; \chi(V^\dagger,\mathrm{M}) \;=\; 0\,. \qquad \square \tag{2.34}$$

Corollary 2.4.6. *Let x be a point of type 2 or 3. Let $\mathscr{F}$ be a differential equation over X, and let $V := V_S(x,\mathscr{F})$. Let $\mathscr{F}_{|V^\dagger} = \mathscr{F}^{<\mathrm{sol}}_{|V^\dagger} \oplus \mathscr{F}^{\geqslant\mathrm{sol}}_{|V^\dagger}$ be the canonical local decomposition of Proposition 2.2.3, with $V = V_S(x,\mathscr{F})$. Then*

$$\mathrm{H}^0_{\mathrm{dR}}(V^\dagger,\mathscr{F}^{<\mathrm{sol}}_{|V^\dagger}) \;=\; \mathrm{H}^1_{\mathrm{dR}}(V^\dagger,\mathscr{F}^{<\mathrm{sol}}_{|V^\dagger}) \;=\; \chi(V^\dagger,\mathscr{F}^{<\mathrm{sol}}_{|V^\dagger}) \;=\; 0\,. \tag{2.35}$$

Moreover $\mathscr{F}_{V^\dagger}$ has a finite index if and only if $\mathscr{F}^{\geqslant\mathrm{sol}}_{V^\dagger}$ has a finite index, and in this case for $i=0,1$ we have

$$\mathrm{H}^i_{\mathrm{dR}}(V^\dagger,\mathscr{F}_{|V^\dagger}) \;=\; \mathrm{H}^i_{\mathrm{dR}}(V^\dagger,\mathscr{F}^{\geqslant\mathrm{sol}}_{|V^\dagger})\,, \qquad \chi(V^\dagger,\mathscr{F}_{|V^\dagger}) \;=\; \chi(V^\dagger,\mathscr{F}^{\geqslant\mathrm{sol}}_{|V^\dagger})\,. \qquad \square \tag{2.36}$$

Corollary 2.4.7. *Let $\mathscr{F}$ be a differential equation over X. Let $D \subseteq X$ be a virtual open disk such that $\Gamma_S(\mathscr{F}) \cap D = \emptyset$. Then $\mathscr{F}$ has a finite index on D and $\mathrm{H}^1_{\mathrm{dR}}(D,\mathscr{F}) = 0$.*

Proof. The radii of $\mathscr{F}$ are all constant over D. By corollary 2.4.4, the cohomology of $\mathscr{F}_{|D}$ equals that of its trivial sub-module. Since $d : \mathscr{O}(D) \to \mathscr{O}(D)$ is surjective, its connection has zero cokernel. $\square$

2.5 Equations with log-affine radii over annuli

Let $C := C^-_K(0;r_1,r_2)$ be a virtual open annulus with empty weak triangulation. Let $\mathscr{F}$ be a differential equation over C of rank r.

Definition 2.5.1 (Robba property)**.** *We say that the index $i \in \{1,\ldots,r\}$ satisfies the* Robba property *if $\mathcal{R}_{\emptyset,i}(x_{0,\rho},\mathscr{F}) = 1$, for all $\rho \in]r_1,r_2[$.*

Proposition 2.5.2 (Existence of the Robba part)**.** *Let $\mathscr{F}$ be a differential equation over C of rank r. Let $i \in \{1,\ldots,r\}$ be the smallest index satisfying the Robba property. Assume that i separates the radii of $\mathscr{F}$ at the points of the segment $]x_{0,r_1},x_{0,r_2}[$. Then i separates the radii of $\mathscr{F}$ globally over C. In particular, by [PP13, Thm.5.3.1], $\mathscr{F}$ admits a sub-module $\mathscr{F}_{\geqslant i} \subseteq \mathscr{F}$.*

Proof. By [PP13, 6.1.4] one has $\mathcal{R}_{\emptyset,i}(x,\mathscr{F}) = 1$, for all $x \in C$. One has $\mathcal{R}_{\emptyset,i-1}(x,\mathscr{F}) < 1$ for all $x \in C$. Indeed if $\mathcal{R}_{\emptyset,i-1}(x,\mathscr{F}) = 1$, then the function $\mathcal{R}_{\emptyset,i-1}(-,\mathscr{F})$ is constant on the maximal disk $D(x,S)$ containing x. By continuity if y is the boundary of D we have $\mathcal{R}_{\emptyset,i-1}(y,\mathscr{F}) = 1$. Since $y = x_{0,\rho}$ for some $\rho \in]r_1,r_2[$ we have a contradiction. $\square$

Definition 2.5.3. *If $\mathscr{F}$ fulfills the assumptions of Proposition 2.5.2 we say that $\mathscr{F}$ admits a Robba part, and we call the Robba part of $\mathscr{F}$ the sub-module*

$$\mathscr{F}^{\mathrm{Robba}} \;:=\; \mathscr{F}_{\geqslant i}\,. \tag{2.37}$$

Corollary 2.5.4. *Assume that the radii $\{\mathcal{R}_{\emptyset,i}(-,\mathscr{F})\}_i$ of $\mathscr{F}$ are all* log*-affine on the segment $]x_{0,r_1},x_{0,r_2}[$. Then $\mathscr{F}^{\mathrm{Robba}}$ is a direct summand of $\mathscr{F}$, and both the kernel and cokernel of the connection of $\mathscr{F}/\mathscr{F}^{\mathrm{Robba}}$ are zero. In particular $\mathscr{F}$ has a finite index if and only if $\mathscr{F}^{\mathrm{Robba}}$ has a* *finite index. Moreover for $i = 0, 1$ we have*

$$\mathrm{H}^i_{\mathrm{dR}}(C, \mathscr{F}^{\mathrm{Robba}}) = \mathrm{H}^i_{\mathrm{dR}}(C, \mathscr{F}) \;, \quad \chi(C, \mathscr{F}^{\mathrm{Robba}}) = \chi(C, \mathscr{F}) \;. \tag{2.38}$$

Proof. Apply Corollary 2.4.4 to the sequence $0 \to \mathscr{F}^{\mathrm{Robba}} \to \mathscr{F} \to \mathscr{F}/\mathscr{F}^{\mathrm{Robba}} \to 0$. The affinity assumption on the radii implies that $\Gamma_\emptyset(\mathscr{F}/\mathscr{F}^{\mathrm{Robba}}) = \Gamma_C =]x_{0,r_1}, x_{0,r_2}[$ (cf. [PP13, 5.5.1]). □

Remark 2.5.5. *We will prove in Corollary 3.7.1 that if the radii of $\mathscr{F}$ are not* log*-affine over $]x_{0,r_1}, x_{0,r_2}[$, the cohomology of $\mathscr{F}/\mathscr{F}^{\mathrm{Robba}}$ is possibly not zero. Moreover (under some conditions of non Liouville along the controlling graphs of $\mathscr{F}$) the cohomology is infinite dimensional if the concave function $\rho \mapsto H_{\emptyset,r}(x_{0,\rho}, \mathscr{F})$ has a finite number of slopes for $x_{0,\rho}$ approaching the boundary of the annulus.*

We provide here a first definition of equations free of Liouville numbers, which holds for equations having log-affine radii along the skeleton of an annulus. This is the first of three definitions, the second (cf. Def. 2.7.11) is given for differential modules over elementary tubes (it is a local definition). Finally the the global definition (cf. Def. 3.3.6, and Lemma 3.3.7) will involve the controlling graphs.

Definition 2.5.6 (Equations free of Liouville numbers)**.** *Let $\mathscr{F}$ be a differential module over C whose radii are all* log*-affine along Γ_C. We say that $\mathscr{F}$ is* free of Liouville numbers *if the modules $\mathscr{F}^{\mathrm{Robba}}$ and* End$(\mathscr{F}^{\mathrm{Robba}})$ *over C satisfy the property* (**NL**) *of [CM01].*

A differential module over C is called *unipotent* if it can be obtained as successive extension of the trivial equation. An unipotent equation coincides with its Robba part, and its index is 0.[6] Each differential equation over C admits a largest unipotent submodule.

Theorem 2.5.7 (Christol-Mebkhout's index theorem for annuli)**.** *Assume that*

i) *the radii $\{\mathcal{R}_{\emptyset,i}(-, \mathscr{F})\}_i$ of $\mathscr{F}$ are all* log*-affine on the segment $]x_{0,r_1}, x_{0,r_2}[$,*

ii) *$\mathscr{F}$ is free of Liouville numbers.*

Then $\mathscr{F}$ has a finite index. Moreover if $\mathscr{U}$ denotes its largest unipotent submodule, then for all $i = 0, 1$ one has

$$\mathrm{H}^i_{\mathrm{dR}}(C, \mathscr{F}) \;=\; \mathrm{H}^i_{\mathrm{dR}}(C, \mathscr{U}) \;, \qquad \chi(C, \mathscr{F}) \;=\; \chi(C, \mathscr{U}) \;=\; 0 \;. \tag{2.39}$$

In particular if $C' \subseteq C$ is an open sub-annulus such that $\Gamma_{C'} \subseteq \Gamma_C$, then for all $i = 0, 1$ one has

$$\mathrm{H}^i_{\mathrm{dR}}(C, \mathscr{F}) \;=\; \mathrm{H}^i_{\mathrm{dR}}(C', \mathscr{F}) \;, \qquad \chi(C, \mathscr{F}) \;=\; \chi(C', \mathscr{F}) \;. \tag{2.40}$$

Proof. By Corollary 2.5.4 we can assume $\mathscr{F} = \mathscr{F}^{\mathrm{Robba}}$. The claim is then proved in [CM02, Thm. 12.1] (see also [CM97] and [CM00]), if the residual chracteristic of K is $p > 0$, and in [Ked13, 3.7.6] in the general case. □

Lemma 2.5.8. *Assume that $\mathscr{F}$ has* log*-affine radii along Γ_C. If $C' \subseteq C$ is a sub-annulus such that $\Gamma_{C'} \subseteq \Gamma_C$, and if $\mathscr{F}$ is a differential equation over C, then $\mathscr{F}$ is free of Liouville numbers if and only if $\mathscr{F}_{|C'}$ is free of Liouville numbers.*

Proof. This follows in fact from the definition of the exponents [CM97] or [Ked13, Thm.3.4.16]. □

2.6 Localization of the partial heights to an elementary tube.

We here investigate the behavior of the Laplacian by localization to an elementary tube.

[6] Indeed by [Ked10, 5.3.3] one sees that an unipotent module over a disk is always trivial, since the derivation is surjective.

Lemma 2.6.1. *Let $\mathscr{F}$ be a differential equation over X of rank r. Let $x \in X$ be a point of type 2, 3, or 4. Let b be a branch out of x, and let C_b be an open annulus which is a section of b. Then*

i) *If $b \in \Gamma_S$, then $\partial_b H_{\emptyset,r}(x, \mathscr{F}_{|C_b}) = \partial_b H_{S,r}(x, \mathscr{F})$;*

ii) *If b is contained in a disk D_b with boundary x such that $D_b \cap \Gamma_S = \emptyset$,[7] then*

$$\partial_b H_{\emptyset,r}(x, \mathscr{F}_{|C_b}) = \partial_b H_{S,r}(x, \mathscr{F}) - h^0(\mathscr{F}, D_b) + r\ ; \tag{2.41}$$

iii) *Let $x \notin \Gamma_S$. Let $D_x \subseteq X - \Gamma_S$ be the closed disk with boundary x, and let b_∞ is the direction out of x not in D_x. Then*

$$\partial_{b_\infty} H_{\emptyset,r}(x, \mathscr{F}_{|C_b}) = \partial_{b_\infty} H_{S,r}(x, \mathscr{F}) + h^0(\mathscr{F}, D_x^\dagger) - r\ . \tag{2.42}$$

Proof. All the statement are deduced from Proposition 1.3.6. i) is immediate.

ii) The restriction from X to D_b leave unchanged the slopes of the radii along a germ of open segment $]y, x[$ representing b. While the restriction to C_b adds $+1$ to the slopes of the radii corresponding to the indexes i such that $\mathcal{R}_{S,i}(-, \mathscr{F})$ is spectral all over $]y, x[$, and it leaves unchanged the slopes of the radii coming from the solutions of $\mathscr{F}$ on D_b (i.e. the radii $\mathcal{R}_{S,i}(-, \mathscr{F})$ that are over-solvable over $]y, x[$).

iii) As above restriction from X to $D_x^\dagger$ leave unchanged the slopes of the radii along b_∞. Now since b_∞ is directed in the opposite direction with respect to the others branches out of x, the restriction to C_b adds -1 to the slopes of the radii corresponding to the indexes i such that $\mathcal{R}_{S,i}(-, \mathscr{F})$ is spectral at x, and it leaves unchanged the slopes of the radii that are over-solvable at x. □

Proposition 2.6.2. *Let $\mathscr{F}$ be a differential equation over X of rank r. Let x be a point of type 2 or 3, and let $V := V_S(x, \mathscr{F})$. Then*

i) *If $x \notin \Gamma_S$. Let $D_x \subseteq X - \Gamma_S$ be the closed disk with boundary x, and let b_∞ is the direction out of x not in D_x. Then*

$$dd^c H_{\{x\},r}(x, \mathscr{F}_{|V^\dagger}) \;=\; dd^c H_{S,r}(x, \mathscr{F}) - r \cdot \chi(V_S(x, \mathscr{F})^\dagger) + h^0(\mathscr{F}, D_x^\dagger) - \sum_{\substack{b \in \mathrm{Sing}(x, \mathscr{F}) \\ b \neq b_\infty}} h^0(\mathscr{F}, D_b) \tag{2.43}$$

ii) *If $x \in \Gamma_S$, then*

$$dd^c H_{\{x\},r}(x, \mathscr{F}_{|V^\dagger}) \;=\; dd^c H_{S,r}(x, \mathscr{F}) + r \cdot (N_S(x, \mathscr{F}) - N_S(x)) - \sum_{\substack{b \in \mathrm{Sing}(x, \mathscr{F}) \\ b \notin \Gamma_S}} h^0(\mathscr{F}, D_b) \tag{2.44}$$

Proof. This is a direct consequence of Corollary 2.4.6 and Lemma 2.6.1. □

2.7 Local Grothendieck-Ogg-Shafarevich formula, following [CM01].

In this section we apply the index results of [CM01] to obtain finite dimensionality of local de Rham cohomology around a point of type 2 or 3.

In sections 2.7.1, 2.7.2, 2.7.3, we consider possibly non solvable differential equations over the Robba ring, with log-affine radii, with a regard to the Dwork dual theory applied to equations over the open unit disk. In section 2.7.4 we consider possibly non solvable differential equations over an over-convergent elementary tube centered at a point of X.

The results of this section are proved under the following assumption:

[7] Here x can be a point of Γ_S or not.

Hypothesis 2.7.1. *K is algebraically closed and spherically complete, with a residual field $\widetilde{K}$ of characteristic $p > 0$. In alternative, we assume that K is an unspecified finite extension of a discretely valued field with residual field of characteristic $p > 0$. This means that all the claims hold, up to enlarge K and replace it by a convenient unspecified finite extension of it.*

In the statements concerning super-harmonicity we will be able to remove in part such an assumption because super-harmonicity is insensitive to extensions of the ground field K. So we only assume that the residual field of K has characteristic p. Part of the material holds without such an assumption, so in the sequel we explicitly mention if this assumption is necessary or not.

Remark 2.7.2. *The results claimed here can certainly be extended to the case in which the residual field $\widetilde{K}$ is of characteristic 0. In fact this case is easier. The difference between characteristic p and 0 only appears when one uses Frobenius techniques. These techniques are used to reduce the value of the radii, in order to make them explicitly intelligible in a cyclic basis by the result of Young [You92]. If the residual characteristic is 0, the radii are either solvable or they are already "small", and there is no need of Frobenius (cf. [Pul13]). In fact, in this case, the bound that prescribes if a radius is small is given by $\lim_n |n!|_0^{1/n} = 1$, where $|.|_0$ is the trivial absolute value on K.*

2.7.1 *Equations over the Robba ring.* The Robba ring is defined as

$$\mathfrak{R} \;=\; \cup_{r<1}\mathscr{O}(C_K^-(0;r,1)) \;; \tag{2.45}$$

Let M be a differential module over $\mathfrak{R}$ of rank r. By Prop. 2.5.2 we know that every differential module M over the Robba ring admits a Robba part $\mathrm{M}^{\mathrm{Robba}}$.[8] Moreover Thm. 2.5.7 asserts that if M is free of Liouville numbers, and if the radii of M are all log-affine, then it has finite index over $\mathfrak{R}$ and

$$\chi(\mathfrak{R},\mathrm{M}) \;=\; 0\;. \tag{2.46}$$

We now extend the Christol-Mebkhout definition of irregularity to non solvable modules. This the first of three definitions of irregularity. This is the first of three definitions of irregularity, the second (cf. Def. 2.7.10) is given for differential modules over elementary tubes (it is a local definition). Finally the the global definition (cf. Def. 3.6.2) will involve the controlling graphs.

Definition 2.7.3 (Irregularity over the Robba ring)**.** *Let* M *be a differential module over $\mathfrak{R}$ such that the radii of* M *are all* log*-affine over $]x_{0,1-\varepsilon},x_{0,1}[$ for some $\varepsilon > 0$. We define the irregularity of* M *as*

$$\mathrm{Irr}(\mathrm{M}) \;:=\; -\partial_b H_{\emptyset,r}(x,\mathrm{M})\;, \tag{2.47}$$

where b is the germ of segment defined by $]x_{0,1-\varepsilon},x_{0,1}[$ oriented as inside $D_K^-(0,1)$.

Remark 2.7.4. *Since* M *is not necessarily a solvable module over $\mathfrak{R}$ in the sense of Christol-Mebkhout,*[9] *then the irregularity* Irr(M) *can be negative. Indeed we will see that it represents a certain non trivial index.*

2.7.2 *Generalized index.*

Definition 2.7.5 (Generalized index)**.** *Let A, B, C be K-vector spaces such that $B = A \oplus C$. Denote*

[8] Let i be the smallest index satisfying the Robba property (cf. Def. 2.5.1). Up to restrict ε, i separates the radii over $]x_{0,1-\varepsilon},x_{0,1}[$ so we can apply Prop. 2.5.2.
[9] A differential module over $\mathfrak{R}$ is solvable module in the terminology of Christol-Mebkhout if $\lim_{\rho\to 1^-}\mathcal{R}_{\emptyset,1}(x_{0,\rho},\mathrm{M}) = 1$.

by

$$A \xrightarrow{\gamma_+} B \xrightarrow{\gamma^+} A\;, \qquad C \xrightarrow{\gamma_-} B \xrightarrow{\gamma^-} C\;. \tag{2.48}$$

the canonical projections and inclusions. Let $u : B^n \to B^n$ *be a linear map. We denote by* $u_A : A^n \to A^n$ *and by* $u_C : C^n \to C^n$ *the following endomorphisms*

$$u_A \;:=\; \gamma^+ \circ u \circ \gamma_+\;, \qquad u_C \;:=\; \gamma^- \circ u \circ \gamma_-\;. \tag{2.49}$$

We define then the generalized index of u *as*

$$\chi^{\mathrm{gen}}(A,u) \;:=\; \dim_K \mathrm{Ker}(u_A) - \dim_K \mathrm{Coker}(u_A)\;, \tag{2.50}$$

$$\chi^{\mathrm{gen}}(C,u) \;:=\; \dim_K \mathrm{Ker}(u_C) - \dim_K \mathrm{Coker}(u_C)\;. \tag{2.51}$$

We say that u *has a generalized index on* A *(resp.* C*) if* $\mathrm{Ker}(u_A)$ *and* $\mathrm{Coker}(u_A)$ *(resp.* $\mathrm{Ker}(u_C)$ *and* $\mathrm{Coker}(u_C)$*) are finite dimensional.*

Let $D := D^-_K(0,1)$ be the open unit disk. Denote by D_∞ the open disk which is the complement in $\mathbb{P}^1_K$ of the closed unit disk $D := D^+_K(0,1)$.

Consider the sequence

$$0 \to \mathscr{O}(D) \to \mathfrak{R} \to \mathcal{H}^\dagger \to 0 \tag{2.52}$$

where

$$\mathcal{H}^\dagger := T^{-1}\mathscr{O}^\dagger(D_\infty) \;:=\; \{\sum_{n\leqslant -1} a_n T^n\;,\; a_n \in K\;,\; \lim_n |a_n|\rho^n = 0\;,\; \text{for some unspecified } \rho < 1\,\}\;. \tag{2.53}$$

The elements of $\mathcal{H}^\dagger$ can be seen as analytic over-convergent functions over D_∞ whose value at ∞ is zero. It is also convenient to imagine its elements as *microfunctions* as explained in [Cre12].

Theorem 2.7.6 ([CM00, 8.2-4])**.** *Assume that* K *is spherically complete. Let* $u : \mathfrak{R}^n \to \mathfrak{R}^n$ *be a* $n\times n$ *matrix with entries in* $\mathfrak{R}\langle d/dT\rangle$*. Then* u *has a finite index if and only if* $u_{\mathscr{O}(D)}$ *and* $u_{\mathcal{H}^\dagger}$ *both have a finite index. In this case one has*

$$\chi(u,\mathfrak{R}) \;=\; \chi^{\mathrm{gen}}(\mathscr{O}(D),u) + \chi^{\mathrm{gen}}(\mathcal{H}^\dagger,u)\;. \qquad \square \tag{2.54}$$

As a direct consequence of (2.46) we have

Proposition 2.7.7. *Let* M *be a differential module over* $\mathfrak{R}$ *which is free of Liouville numbers, and whose radii are all* log*-affine over* $]x_{0,1-\varepsilon}, x_{0,1}[$*. Then* $\nabla : \mathrm{M} \to \mathrm{M}$ *has a finite generalized indexes and one has*

$$\chi^{\mathrm{gen}}(\mathscr{O}(D),\nabla) \;=\; -\chi^{\mathrm{gen}}(\mathcal{H}^\dagger,\nabla)\;. \tag{2.55}$$

If moreover there exists a differential module M_0 *over* $\mathscr{O}(D)$ *such that* $\mathrm{M} = \mathrm{M}_0 \otimes_{\mathscr{O}(D)} \mathfrak{R}$*,* $\nabla = \nabla_0 \otimes 1 + 1 \otimes d/dT$*, then* ∇_0 *has a finite index over* $\mathscr{O}(D)$ *and one has*

$$\chi^{\mathrm{gen}}(\mathscr{O}(D),\nabla) \;=\; \chi(\mathscr{O}(D),\nabla_0)\;. \tag{2.56}$$

Proof. The first part is immediate from (2.46) and Theorem 2.7.6. For the second it is enough to observe that ∇_0 coincides with the truncated map $\nabla_{\mathscr{O}(D)}$ obtained from ∇ as per (2.49). $\square$

2.7.3 *Dwork dual theory.* Let M_0 be a differential module over $\mathscr{O}(D)$, let $\mathrm{M} := \mathrm{M}_0 \otimes_{\mathscr{O}(D)} \mathfrak{R}$ and $\mathrm{M}_\infty := \mathrm{M}_0 \otimes_{\mathscr{O}(D)} \mathcal{H}^\dagger$. We have an exact sequence :

$$0 \to \mathrm{M}_0 \to \mathrm{M} \to \mathrm{M}_\infty \to 0\;. \tag{2.57}$$

The connection of M commutes by construction with that of M_0, and induces an operator $\Delta_\infty : M_\infty \to M_\infty$ which is far from being a connection (it is in fact the dual map of a connection).

As firstly observed by Dwork the K-vector space M_∞ is the *topological* dual vector space of M_0^* in the sense of [Rob84, 8.2], [Cre98, Section 5], [CM00, 2.1]. This is a particular case of cohomology with support [Chi90], [Ked06]. This holds over any ultrametric complete valued field K (cf. [Chr12, Thm. 5.7]).

More precisely for all $\varepsilon > 0$ the K-vector space $\mathscr{O}(C_K^-(0; 1-\varepsilon, 1))$ is of Fréchet type (a space of type $\mathcal{F}$) that is a complete locally convex metric space. The ring $\mathfrak{R}$ is a $\mathcal{LF}$-space i.e. a separated locally convex space which is inductive limit of a countable family of vector spaces of Fréchet type. The space $\mathcal{H}^\dagger$ is also $\mathcal{LF}$, and the Robba ring is the topological direct sum of $\mathscr{O}(D)$ and $\mathcal{H}^\dagger$.

The Robba ring is the dual of itself by the perfect pairing $\langle .,. \rangle : \mathfrak{R} \times \mathfrak{R} \to K$ defined by

$$\langle f, g \rangle \ := \ \mathrm{Res}(f \cdot g) \ . \tag{2.58}$$

In this duality $\mathscr{O}(D)$ is the dual of $\mathcal{H}^\dagger$. We can extend this definition to a free differential module by choosing a basis of M (i.e. an isomorphism $M \xrightarrow{\sim} \mathfrak{R}^r$).

Theorem 2.7.8 ([Rob84, 8.2])**.** *Let* M *be a free differential module over* $\mathfrak{R}$*. The adjoint endomorphism of* $\nabla : M \to M$ *under* $\langle .,. \rangle$ *is* $-\nabla^* : M^* \to M^*$*. Moreover if* M *comes by scalar extension from a differential module* M_0 *over* $\mathscr{O}(D)$*, then the adjoint of* $\Delta_\infty : M_\infty \to M_\infty$ *is* $-\nabla_0^* : M_0^* \to M_0^*$*. In particular* $H^1_{dR}(\mathscr{O}(D), M_0^*)$ *is finite dimensional, and we have*

$$\dim \mathrm{Ker}(\Delta_\infty) \ = \ \dim \mathrm{Coker}(\nabla_0^*) \tag{2.59}$$

$$\dim \mathrm{Coker}(\Delta_\infty) \ = \ \dim \mathrm{Ker}(\nabla_0^*) \ . \qquad \square \tag{2.60}$$

Proof. The original statement of Robba was stated under the *assumption* that $H^1_{dR}(\mathscr{O}(D), M_0^*)$ is finite dimensional, but this is true by Christol-Mebkhout's Proposition 2.7.7. □

2.7.4 *Local cohomology over a tube.*

Definition 2.7.9 (Robba ring at a branch)**.** *Let* x *be a point of type* 2*, and lat* b *be a branch out of* x*. We call the Robba at* b *the ring*

$$\mathfrak{R}_b \ = \ \bigcup_C \mathscr{O}(C) \tag{2.61}$$

where C *runs in the family of open virtual annuli that are sections of* b *(cf. section 1.1.1).*

If M *is a differential module over* $\mathscr{O}_{X,x}$*, we denote by* M_b *the restriction of* M *to* $\mathfrak{R}_b$*.*

Definition 2.7.10. *Let* $x \in \mathrm{Int}(X)$ *be a point of type* 2*. Let* b *be a branch out of* x*. Let* M *be a differential module over* $\mathscr{O}_{X,x}$*. We define the irregularity of* M *at* b *as*

$$\mathrm{Irr}_b(M) \ := \ \mathrm{Irr}(M_b) \ = \ -\partial_b H_{\emptyset,r}(x, M_b) \ , \tag{2.62}$$

where M_b *is the restriction of* M *to* $\mathfrak{R}_b$*, and* b *is directed as outside* x*.*

Definition 2.7.11 (Equation free of Liouville numbers)**.** *Let* $x \in X$ *be a point of type* 2*. Let* $\mathscr{F}$ *be a differential equation over* X*, or over an elementary tube* V *centered at* x*. We say that* $\mathscr{F}$ *is* free of Liouville numbers *at* x *if for all branch* $b \in \mathrm{Sing}(x, V)$ *out of* x *the module* $\mathscr{F}_b$ *is free of Liouville numbers (cf. Def. 2.5.6).*

If $x \in X$ *is a point of type* 3 *or* 4*, we say that* $\mathscr{F}$ *is* free of Liouville numbers *at* x *if after scalar extension to* X_Ω *the equation* $\mathscr{F}_\Omega$ *is free of Liouville numbers at the peaked point* $\sigma_\Omega(x) \in X_\Omega$ *of [PP13, 2.1.2] (cf. [PP12]).*

Remark 2.7.12. *If $b \notin \mathrm{Sing}(x, \mathscr{F})$, then $\mathscr{F}_b$ is automatically free of Liouville numbers. Indeed the radii are all constant over D_b, so the solvable part $\mathscr{F}_b^{\geqslant \mathrm{sol}}$ of $\mathscr{F}_b$ is trivial.*

Theorem 2.7.13 ([CM01, Thm. 5.0-10])**.** *Assume that K satisfies Hypothesis 2.7.1. Let $x \in \mathrm{Int}(X)$ be a point of type 2. Let M be an over-convergent isocrystal over an elementary tube V centered at x, which is free of Liouville numbers. Then the kernel and the cokernel of $\nabla : \mathrm{M} \to \mathrm{M}$ are finite dimensional over K, and one has the Grothendieck-Ogg-Shafarevich formula:*

$$\chi(V^\dagger, \mathrm{M}) \;=\; \mathrm{rank}(\mathrm{M}) \cdot \chi(V^\dagger) - \sum_{b \in \mathrm{Sing}(x,V)} \mathrm{Irr}_b(\mathrm{M}) \;. \qquad \square \tag{2.63}$$

Remark 2.7.14. *The sum of the irregularities appearing in (2.67) can also be written as*

$$\chi(V^\dagger, \mathrm{M}) \;=\; \mathrm{rank}(\mathrm{M}) \cdot \chi(V^\dagger) + dd^c H_{\{x\},r}(x, \mathrm{M}) \;. \tag{2.64}$$

Indeed, by Remark 2.3.2, the non singular branches out of x do not contribute to the Laplacian.

Lemma 2.7.15. *Under the assumptions of Theorem 2.7.13, there exists a basic neighborhood U of V such that $\chi(U, \mathrm{M}) = \chi(V, \mathrm{M})$.*

Proof. Choose U small enough in order that $C := U - V$ is a disjoint union of open annuli on which all the radii are log-affine. For all basic neighborhood $V \subset U' \subseteq U$ we set $C' := U' - V$. Then $U = U' \cup C$, and $U' \cap C = C'$. All these open subsets are quasi-Stein, and we can consider the Mayer-Vietoris sequence (cf. section 3.2)

$$0 \to \mathrm{H}^0_{\mathrm{dR}}(U, \mathscr{F}) \to \mathrm{H}^0_{\mathrm{dR}}(U', \mathscr{F}) \oplus \mathrm{H}^0_{\mathrm{dR}}(C, \mathscr{F}) \to \mathrm{H}^0_{\mathrm{dR}}(C', \mathscr{F}) \to \tag{2.65}$$

$$\to \mathrm{H}^1_{\mathrm{dR}}(U, \mathscr{F}) \to \mathrm{H}^1_{\mathrm{dR}}(U', \mathscr{F}) \oplus \mathrm{H}^1_{\mathrm{dR}}(C, \mathscr{F}) \to \mathrm{H}^1_{\mathrm{dR}}(C', \mathscr{F}) \to 0 \;. \tag{2.66}$$

Now by Theorem 2.5.7, for $i = 0, 1$, we have isomorphisms $\mathrm{H}^i_{\mathrm{dR}}(C, \mathscr{F}) \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{dR}}(C', \mathscr{F})$. Hence also $\mathrm{H}^i_{\mathrm{dR}}(U, \mathscr{F}) \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{dR}}(U', \mathscr{F})$. Since de Rham cohomology commutes with inductive limits we also have $\mathrm{H}^i_{\mathrm{dR}}(U, \mathscr{F}) \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{dR}}(U', \mathscr{F}) \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{dR}}(V^\dagger, \mathscr{F})$. $\square$

The following proposition shows that, up to replace V by $V_S(x, \mathscr{F})$, the G.O.S formula holds for general (possible non solvable) differential modules over $\mathscr{O}^\dagger(V)$.

Proposition 2.7.16. *Assume that K satisfies Hypothesis 2.7.1. Let $x \in \mathrm{Int}(X)$ be a point of type 2. Let M be a differential module of rank r over an elementary tube V centered at x, such that*

i) *$\Gamma_{\{x\}}(\mathrm{M}) = \Gamma_U$ for some basic neighborhoods U of V (i.e. the radii $\mathcal{R}_{\{x\},i}(x, \mathrm{M})$ are all constant functions on V);*

ii) *M is free of Liouville Numbers at x.*

Then the kernel and the cokernel of $\nabla : \mathrm{M} \to \mathrm{M}$ are finite dimensional over K, and one has the Grothendieck-Ogg-Shafarevich formula:

$$\chi(V^\dagger, \mathrm{M}) \;=\; r \cdot \chi(V^\dagger) - \sum_{b \in \mathrm{Sing}(x,V)} \mathrm{Irr}_b(\mathrm{M}) \;=\; r \cdot \chi(V^\dagger) + dd^c H_{\{x\},r}(x, \mathrm{M}) \;. \tag{2.67}$$

Proof. By Proposition 2.2.3, condition i) guarantee that M splits over V as $0 \to \mathrm{M}^{\geqslant \mathrm{sol}} \to \mathrm{M} \to \mathrm{M}^{<\mathrm{sol}} \to 0$. Let $r^{\geqslant s}$, r, $r^{<s}$ be the ranks of $\mathrm{M}^{\geqslant \mathrm{sol}}$, M, $\mathrm{M}^{<\mathrm{sol}}$ respectively. By Proposition 2.4.5 one has $\chi(\mathrm{M}^{<\mathrm{sol}}, V^\dagger) = 0$, so by (2.64) one has

$$\chi(V^\dagger, \mathrm{M}) \;=\; \chi(V^\dagger, \mathrm{M}^{\geqslant \mathrm{sol}}) \;=\; r^{\geqslant s} \chi(V^\dagger) + dd^c H_{\{x\},r^{\geqslant s}}(x, \mathrm{M}^{\geqslant \mathrm{sol}}) \;. \tag{2.68}$$

On the other hand by Theorem 1.4.2 one has $dd^c H_{\{x\},r^{<s}}(x, \mathrm{M}^{<\mathrm{sol}}) = -r^{<s} \cdot \chi(V^\dagger)$. Since $dd^c H_{\{x\},r}(x, \mathrm{M}) = dd^c H_{\{x\},r^{<s}}(x, \mathrm{M}^{<\mathrm{sol}}) + dd^c H_{\{x\},r^{\geqslant s}}(x, \mathrm{M}^{\geqslant \mathrm{sol}})$ the claim follows. $\square$

2.8 Applications to super-harmonicity

In this section we apply the above index results to prove a super-harmonicity statement of the partial heights $H_{S,i}(-,\mathscr{F})$ of $\mathscr{F}$. We mainly focus on points $x \notin \Gamma_S$, because we can not hope to have super-harmonicity at the points of S.

Definition 2.8.1. *For every $i \in \{1,\dots,r\}$, set*

$$\mathscr{E}_{S,i}(\mathscr{F}) \ := \ \{x \in X \mid dd^c H_{S,i}(x,\mathscr{F}) > 0\}\ . \tag{2.69}$$

In the sequel, if no confusion is possible, we write $\mathscr{E}_{S,i} := \mathscr{E}_{S,i}(\mathscr{F})$ for short.

Proposition 2.8.2. *Assume that K satisfies Hypothesis 2.7.1. Let $\mathscr{F}$ be a differential equation over X or rank r. Let $x \in \mathrm{Int}(X)$ be a point of type 2, and let $V := V_S(x,\mathscr{F})$. Assume that $\mathscr{F}$ is free of Liouville numbers at x. Then the following hold:*

i) *If $x \notin \Gamma_S$, and if D_x denotes the closed disk in $X - \Gamma_S$ with boundary x, then*

$$dd^c H_{S,r}(x,\mathscr{F}) \ = \ \chi(V^\dagger,\mathscr{F}) - h^0(D_x^\dagger,\mathscr{F}) + \sum_{\substack{b\in \mathrm{Sing}(x,\mathscr{F})\\ b\neq b_\infty}} h^0(D_b,\mathscr{F})\ , \tag{2.70}$$

where b_∞ is the direction out of x that do not belongs to the closed disk with boundary x.

ii) *If $x \in \Gamma_S$, then*

$$dd^c H_{S,r}(x,\mathscr{F}) \ = \ \chi(V^\dagger,\mathscr{F}) - r\cdot\chi(x,S) + \sum_{\substack{b\in \mathrm{Sing}(x,\mathscr{F})\\ b\notin \Gamma_S}} h^0(D_b,\mathscr{F})\ , \tag{2.71}$$

where $\chi(x,S) := -(2g(x) - 2 + N_S(x))$ (cf. Def. 1.1.13).

Proof. The statement is a direct consequence of Propositions 2.6.2, and 2.7.16. □

Proposition 2.8.3. *Assume that the residual field of K has characteristic $p > 0$. Let $\mathscr{F}$ be a differential equation over X or rank r. Let $x \notin \Gamma_S$. Assume that $\mathscr{F}$ is free of Liouville numbers at x, and that the radii $\{\mathcal{R}_{S,i}(x,\mathscr{F})\}_{i=1,\dots,r}$ are all solvable or over-solvable at x. Then $H_{S,r}(-,\mathscr{F})$ is super-harmonic at x, i.e.*

$$x \notin \mathscr{E}_{S,r}\ . \tag{2.72}$$

Proof. We can assume that K is algebraically closed and spherically complete. Let $D_x \subset X$ be the closed disk whose boundary is x.

Since the first radius has the concavity property (cf. [PP13, point iv) of Remark 6.1.3]), then $\mathscr{F}$ is trivial over every open disk with boundary x, and x is an end point of $\Gamma_S(\mathscr{F})$. So $V_S(x,\mathscr{F}) = D_x$, and (2.70) gives

$$dd^c H_{S,r}(x,\mathscr{F}) \ = \ -h^1(D_x^\dagger,\mathscr{F}) \ \leqslant \ 0\ . \tag{2.73}$$

□

Proposition 2.8.4. *Assume that the residual field of K has characteristic $p > 0$. Let $\mathscr{F}$ be a differential equation over X or rank r. Let $x \notin \Gamma_S$. Assume that*

i) *$\mathscr{F}$ is free of Liouville numbers at x,*

ii) *i separates the radii of $\mathscr{F}$ at x,*

iii) *i is solvable at x.*

iv) *x is an end point of $\Gamma_{S,i}(\mathscr{F})$.*

Then $H_{S,i}(-,\mathscr{F})$ and $H_{S,r}(-,\mathscr{F})$ are both super-harmonic at x, i.e.

$$x \notin (\mathscr{E}_{S,i} \cup \mathscr{E}_{S,r}) \ . \tag{2.74}$$

Proof. We can assume that K is algebraically closed and spherically complete. Let D_x be the closed disk with boundary x. We claim that i separates the radii of $\mathscr{F}$ over D_x. Indeed assume, by contrapositive, that for some $y \in D_x$ one has $\mathcal{R}_{S,j}(y,\mathscr{F}) = \mathcal{R}_{S,i}(y,\mathscr{F}) = \mathcal{R}_{S,i}(x,\mathscr{F})$. Since $\mathcal{R}_{S,i}(-,\mathscr{F})$ is constant on the connected component D' of $D_x - \{x\}$ containing y, then $D' = D_{S,i}(y,\mathscr{F}) = D_{S,j}(y,\mathscr{F}) \subseteq D^c_{S,j}(y,\mathscr{F})$ by [PP13, (2.27)]. This implies that $\mathcal{R}_{S,j}(y,\mathscr{F}) = \mathcal{R}_{S,i}(y,\mathscr{F})$ on the whole connected component of $D_x - \{x\}$ containing y. This is absurd because i separates the radii at x. So i separates the radii on the whole D_x.

By continuity i separates the radii over $D^\dagger_x$, and we have a decomposition $0 \to (\mathscr{F}_{|D^\dagger_x})_{\geqslant i} \to \mathscr{F}_{|D^\dagger_x} \to (\mathscr{F}_{|D^\dagger_x})_{<i} \to 0$. One sees that $D_x = V_\emptyset(x,(\mathscr{F}_{|D^\dagger_x})_{\geqslant i})$, hence as in (2.73) one has

$$dd^c H_{\emptyset,r-i+1}(x,(\mathscr{F}_{|D^\dagger_x})_{\geqslant i}) \;=\; -h^1(D^\dagger_x,(\mathscr{F}_{|D^\dagger_x})_{\geqslant i}) \;\leqslant\; 0\,, \tag{2.75}$$

where $r-i+1 = \mathrm{rank}(\mathscr{F}_{|D^\dagger_x})_{\geqslant i}$. Now write

$$dd^c H_{S,i}(x,\mathscr{F}) \;=\; dd^c H_{S,i-1}(x,\mathscr{F}) + dd^c \mathcal{R}_{S,i}(x,\mathscr{F})\ . \tag{2.76}$$

We observe that $dd^c H_{S,i-1}(x,\mathscr{F}) = 0$ by Theorem 1.4.2. Moreover

$$dd^c \mathcal{R}_{S,i}(x,\mathscr{F}) \;=\; dd^c \mathcal{R}_{\emptyset,i}(x,\mathscr{F}_{|D^\dagger_x}) \;=\; dd^c \mathcal{R}_{\emptyset,1}(x,(\mathscr{F}_{|D^\dagger_x})_{\geqslant i}) \;\leqslant\; 0\ . \tag{2.77}$$

This proves that $x \notin \mathscr{E}_{S,i}$. Finally from (2.75) we obtain

$$dd^c H_{S,r}(x,\mathscr{F}) \;=\; dd^c H_{S,i-1}(x,\mathscr{F}) + \sum_{j=i}^{r} dd^c \mathcal{R}_{S,j}(x,\mathscr{F}) \tag{2.78}$$

$$= \sum_{j=i}^{r} dd^c \mathcal{R}_{\emptyset,j}(x,\mathscr{F}_{|D^\dagger_x}) \;=\; \sum_{j=1}^{r-i+1} dd^c \mathcal{R}_{\emptyset,j}(x,(\mathscr{F}_{|D^\dagger_x})_{\geqslant i}) \tag{2.79}$$

$$= dd^c H_{\emptyset,r-i+1}(x,(\mathscr{F}_{|D^\dagger_x})_{\geqslant i}) \;=\; -h^1(D^\dagger_x,(\mathscr{F}_{|D^\dagger_x})_{\geqslant i}) \;\leqslant\; 0\ . \tag{2.80}$$

This means that $x \notin \mathscr{E}_{S,r}$. □

Corollary 2.8.5. *Assume that the residual field of K has characteristic $p > 0$. Let $\mathscr{F}$ be a differential equation over X or rank $r = 2$. Let $x \notin \Gamma_S$. If $\mathscr{F}$ is free of Liouville numbers at each point of $\mathscr{C}_{S,2}(\mathscr{F})$, then*

$$\mathscr{E}_{S,1} \cup \mathscr{E}_{S,2} \;\subseteq\; S\ . \tag{2.81}$$

Proof. $\mathscr{E}_{S,1} \subseteq S$ because $\mathcal{R}_{S,1}(-,\mathscr{F})$ has the concavity property of [PP13, point iv) of Remark 6.1.3]. Now $(\mathscr{E}_{S,2} - S) \subseteq \mathscr{C}_{S,2}$, and by definition $i = 2$ is solvable at the points $x \in \mathscr{C}_{S,2}$. If $i = 1$ is solvable too at x, then apply Proposition 2.8.3. If i separates the radii at x, then apply Proposition 2.8.4. □

The following Theorem provides the full super-harmonicity under some quite strong assumptions. Recall that we already have the super harmonicity of $H_{S,i}(-,\mathscr{F})$ outside $S \cup \mathscr{C}_{S,i}(\mathscr{F})$. If $x \in S$ we can not have super-harmonicity.

Theorem 2.8.6. *Assume that the residual field of K has characteristic $p > 0$. Let $\mathscr{F}$ be a differential equation over X or rank r. Let $x \in \mathscr{C}_{S,r}(\mathscr{F})$, let D_x be the closed disk in $X - \Gamma_S$ with boundary x, and let $V = V_S(x,\mathscr{F})$. Assume that*

i) *the canonical inclusion $\mathrm{H}^0_{\mathrm{dR}}(D^\dagger_x,\mathscr{F}) \subseteq \mathrm{H}^0_{\mathrm{dR}}(V^\dagger,\mathscr{F})$ is an equality;*

ii) *for all i one has $\mathcal{R}_{S,i}(-,\mathscr{F}^*) = \mathcal{R}_{S,i}(-,\mathscr{F})$;*

iii) *$\mathscr{F}$ is free of Liouville numbers at x (cf. Def. 2.7.11).*

Then for all $i = 1,\ldots,r$ the partial height $H_{S,i}(-,\mathscr{F})$ is super-harmonic at x.

Proof. We can assume that K is algebraically closed and spherically complete. We firstly prove the claim for the vertexes of the reversed convergence Newton polygon, and then deduce the claim for the other vertexes by *interpolation*.

Vertexes. Let i be a vertex of the reversed convergence Newton polygon (cf. Def. 1.2.9).

By Theorem 1.4.2 if i is spectral non solvable at x, or if $x \notin \mathscr{C}_{S,i}(\mathscr{F})$, then $dd^c H_{S,i}(x,\mathscr{F}) \leqslant 0$, and and we are done. So we can assume that $x \in \mathscr{C}_{S,r}(\mathscr{F})$, that i is solvable or over-solvable at x, and that there exists at least an index $j \leqslant i$ which is solvable at x (see the definition of $\mathscr{C}_{S,r}$).

Since i is a vertex, then $i_x^{\mathrm{sol}} \leqslant i$. If $i_x^{\mathrm{sol}} < i$, we have a decomposition

$$0 \to (\mathscr{F}_{|D_x^\dagger})_{\geqslant i_x^{\mathrm{sol}}+1} \to \mathscr{F}_{|D_x^\dagger} \to (\mathscr{F}_{|D_x^\dagger})_{< i_x^{\mathrm{sol}}+1} \to 0 \tag{2.82}$$

satisfying for all $i_x^{\mathrm{sol}} \leqslant j \leqslant i$

$$dd^c H_{S,j}(x,\mathscr{F}) \;=\; dd^c H_{\emptyset,j}(x,\mathscr{F}_{|D_x^\dagger}) \;=\; dd^c H_{\emptyset,i_x^{\mathrm{sol}}}(x,\mathscr{F}_{|D_x^\dagger}) \;=\; dd^c H_{\emptyset,i_x^{\mathrm{sol}}}(x,(\mathscr{F}_{|D_x^\dagger})_{<i_x^{\mathrm{sol}}+1})\;. \tag{2.83}$$

Indeed over-solvable radii does not contribute to the Laplacian. So we can assume moreover that $X = D_x^\dagger$, $S = \emptyset$, and $i = i_x^{\mathrm{sol}} = r$.

Let $V := V_S(x,\mathscr{F})$, and let $b_\infty \in \mathrm{Sing}(x,V)$ be the direction not belonging to D_x. Now we consider the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{O}^\dagger(D_x) & \longrightarrow & \mathscr{O}^\dagger(V) & \longrightarrow & \oplus_{b\in\mathcal{S}}\mathcal{H}_b^\dagger & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \oplus_{b\in\mathcal{S}}\mathscr{O}(D_b) & \longrightarrow & \oplus_{b\in\mathcal{S}}\mathfrak{R}_b & \longrightarrow & \oplus_{b\in\mathcal{S}}\mathcal{H}_b^\dagger & \longrightarrow & 0 \end{array} \tag{2.84}$$

where $\mathcal{S} := \mathrm{Sing}(x,V) - \{b_\infty\}$, and $\mathcal{H}_b^\dagger := (T-c_b)^{-1}\mathscr{O}^\dagger(\mathbb{P}_K^{1,\mathrm{an}} - D_b)$ as in (2.52). From the first exact sequence we obtain, by the snake diagram, the long exact sequence

$$0 \to \mathrm{H}^0_{\mathrm{dR}}(D_x^\dagger,\mathscr{F}) \xrightarrow{\sim} \mathrm{H}^0_{\mathrm{dR}}(V^\dagger,\mathscr{F}) \to \mathrm{Ker}(\Delta_\infty) \to \mathrm{H}^1_{\mathrm{dR}}(D_x^\dagger,\mathscr{F}) \to \mathrm{H}^1_{\mathrm{dR}}(V^\dagger,\mathscr{F}) \to \mathrm{Coker}(\Delta_\infty) \to 0 \tag{2.85}$$

where Δ_∞ is the endomorphism of $\oplus_{b\in\mathcal{S}}\mathscr{F}\otimes_{\mathscr{O}^\dagger(D_x)}\mathcal{H}_b^\dagger$ induced by the connection of $\mathscr{F}$ (cf. Thm. 2.7.8). The vector spaces are all finite dimensional by the Christol-Mebkhout's results of sections 2.7.1, 2.7.2, 2.7.3. By the assumption i) the first arrow of (2.85) is a bijection, so that

$$\dim \mathrm{Ker}(\Delta_\infty) \;\leqslant\; h^1(D_x^\dagger,\mathscr{F})\;. \tag{2.86}$$

We then have

$$h^1(V^\dagger,\mathscr{F}) \;=\; h^1(D_x^\dagger,\mathscr{F}) - \dim\mathrm{Ker}(\Delta_\infty) + \dim\mathrm{Coker}(\Delta_\infty)\;. \tag{2.87}$$

By duality one has $\dim\mathrm{Coker}(\Delta_\infty) = \sum_{b\in\mathcal{S}} h^0(D_b,\mathscr{F}^*)$ (cf. Thm. 2.7.8). Equation (2.70) then becomes

$$dd^c H_{S,r}(x,\mathscr{F}) \;=\; -h^1(\mathscr{F},V^\dagger) + \sum_{b\in\mathcal{S}} h^0(\mathscr{F},D_b) \tag{2.88}$$

$$\;=\; -h^1(D_x^\dagger,\mathscr{F}) + \dim\mathrm{Ker}(\Delta_\infty) - \sum_{b\in\mathcal{S}} h^0(D_b,\mathscr{F}^*) + \sum_{b\in\mathcal{S}} h^0(D_b,\mathscr{F}) \tag{2.89}$$

The assumption of compatibility with the dual gives $\sum_{b\in\mathcal{S}} h^0(D_b\mathscr{F}^*) = \sum_{b\in\mathcal{S}} h^0(D_b,\mathscr{F})$, so $dd^c H_{S,r}(x,\mathscr{F}) \leqslant 0$ by (2.86).

Other indexes. If i is not a vertex the claim follows by interpolation. Namely we proceed as follows

Lemma 2.8.7. *Let $f, g : X \to \mathbb{R}$ be two functions. Assume that*

i) $dd^c f(x) \leqslant 0$,

ii) *$g \leqslant f$ along all germ of segment out of x,*

iii) $g(x) = f(x)$,

then $dd^c g(x) \leqslant 0$. □

Let k, j, where $k < i < j$, be the vertexes that are closest to i. Then $H_{S,k}(-, \mathscr{F})$ and $H_{S,j}(-, \mathscr{F})$ are both super-harmonic at x. For all $i \in \{1, \ldots, r\}$ and all $y \in X$, let $v_i(y) := \log H_{S,i}(y, \mathscr{F})$. Consider the function

$$f(y) \ := \ v_k(y) + (i-k) \cdot [\frac{v_j(y) - v_k(y)}{j-k}] \ . \tag{2.90}$$

This function f is also super-harmonic at x since for $a = \frac{i-k}{j-k}$ we have $0 < a < 1$ and $f(y) = a \cdot v_j(y) + (1-a)v_k(y)$, so $dd^c f(x) = a \cdot dd^c v_j(x) + (1-a)dd^c v_k(x)$. Hence

$$\min(dd^c v_j(x), dd^c v_k(x)) \ \leqslant \ dd^c f(x) \ \leqslant \ \max(dd^c v_j(x), dd^c v_k(x)) \leqslant 0 \ . \tag{2.91}$$

So the function f is super-harmonic at x. Moreover $v_i \leqslant f$ by convexity of the reversed Newton polygon of $\mathscr{F}$, and $v_i(x) = f(x)$. Hence $v_i = \log H_{S,i}(-, \mathscr{F})$ is super-harmonic by Lemma 2.8.7. □

Corollary 2.8.8. *If $\mathscr{F}$ satisfies the conditions of Theorem 2.8.6 at all points of $\mathscr{C}_{S,r}$, we have*

$$\mathscr{E}_{S,i} \subseteq S \ , \qquad \textit{for all } i = 1, \ldots, r \ . \tag{2.92}$$

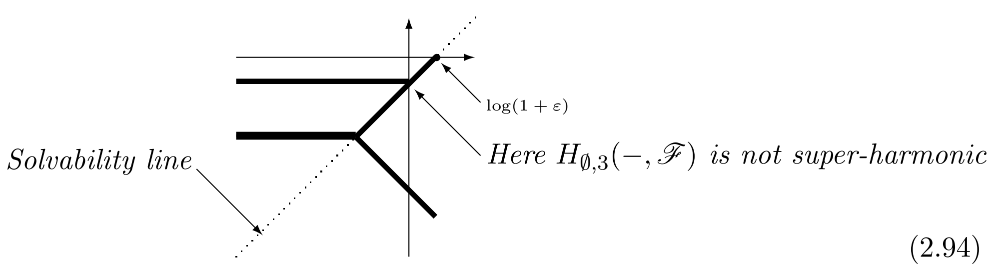
In this case if X is moreover a smooth geometrically connected projective curve, then, by [PP13, Cor. 7.2.5], the weighted number of edges of $\Gamma''_{S,j}(\mathscr{F})$ is at most

$$E_S + 2r(g-1)j(j+1) \ , \tag{2.93}$$

where E_S is the weighted number of edges of Γ_S, and g is the genus of X. □

Remark 2.8.9. *A general super-harmonicity statement seems to need new methods permitting to control the* "solvable breaks of the radii going towards the over-solvable side" *at the points of $\mathscr{C}_{S,i}(\mathscr{F})$. As we have seen in the proof of Thm. 2.8.6 the problem is localized at the point $x = x_{0,1}$ of the over-convergent open unit disk $D_x^\dagger$ with empty weak triangulation. The following picture express the typical example of the* "pathological" *situation that we have in mind. We are unable to prove in full generality that such kind pathologies do not arises.*

Here is the picture: $r = \mathrm{rank}(\mathscr{F}) = 3$, $x = x_{0,1}$, $D_x^\dagger = D^+(0,1)^\dagger$:



(2.94)

The picture express the functions $\widetilde{\rho} \mapsto \log \mathcal{R}_{\emptyset,i}(x_{0,\exp(\widetilde{\rho})}, \mathscr{F})$, where $\widetilde{\rho} = \log(\rho) \in]-\infty, \log(1+\varepsilon)[$. At the left hand side of the solvability line the radii are all over- solvable, hence constant. At the right hand side of the line the radii are spectral non solvable.

3. Global measure of the irregularity.

In this section we study the global de Rham cohomology of $\mathscr{F}$. In particular we are interested in proving its finite dimensionality, and establish its index formula. For this we introduce a canonical weak triangulation $S_{\mathscr{F}}$, and a canonical covering of X, built from the controlling graphs, which permit to compute the de Rham cohomology via the Čech complex. The fact that there exists a *finite* canonical covering is encoded in the structure of the controlling graph $\Gamma_S(\mathscr{F})$. We call this property *essential finiteness* of $\Gamma_S(\mathscr{F})$. As explained in the introduction the essential finiteness of $\Gamma_S(\mathscr{F})$ will imply the finite dimensionality of the de Rham cohomology, and the index formula.

In the case where the graph $\gamma_S(\mathscr{F})$ do not have that property, but the curve is "*approachable*" by a countable family of quasi-Stein sub-spaces X_n, where the controlling graph is finite, then we are able to obtain a limit formula expressing the global cohomology as a limit of the equations over the X_n. In this case the process proves that we have finite dimensionality of the global cohomology over X *if and only if* the sequence of indexes over the X_n are constant for all n large enough. And we are able to show that if the cohomology is finite dimensional the the controlling graph is essentially finite, which constitutes a converse of the above statement.

All along section 3 we assume that K satisfies hypothesis 2.7.1.

3.1 The Christol-Mebkhout limit formula.

In this section we show that if X is a quasi-Stein curve that can be *approached* by a family of quasi-Stein curves $(X_\alpha)_\alpha$, then the de Rham cohomology of a differential equation $\mathscr{F}$ can be recovered as the limit of the de Rham cohomologies of its restrictions to X_α. The fundamental assumption here is the finite dimensionality of the cohomology of $\mathscr{F}$ over the X_α's, indeed the fact that X_α is Fréchet implies that $\mathrm{H}^1_{\mathrm{dR}}(X_\alpha,\mathscr{F})$ is separated, which the crucial step.

This technique have been introduced by Christol-Mebkhout over annuli (cf. [CM00]), and it essentially follows from [Gro61, Chap.0, 13.2.4].

Assume that the connected curve X is a quasi-Stein space. Assume that there exists a filtering ordered set Λ admitting a *countable* subset which is cofinal in Λ, and an inductive limit of connected curves[10] $(X_\alpha)_{\alpha\in\Lambda}$ $\{X_\alpha\}_{\alpha\in\Lambda}$ such that :

i) for all α, X_α is a quasi-Stein space. In particular $\mathscr{O}(X_\alpha)$ is a Fréchet space;
ii) there exists $\beta\in\Lambda$ such that for all $\alpha\geqslant\alpha'\geqslant\beta$, the restriction map $\mathscr{O}(X_\alpha)\to\mathscr{O}(X_{\alpha'})$ is *injective*, with *dense image*, and *uniformly continuous* with respect to the metric structures that make them Fréchet spaces;
iii) one has $\mathscr{O}(X)=\varprojlim_\alpha\mathscr{O}(X_\alpha)$ as locally convex vector spaces.

Remark 3.1.1. *This is the typical situation satisfied by a quasi-Stein space (cf. Def. 1.1.6).*

In the following theorem we compute $\mathrm{H}^1_{\mathrm{dR}}(X,\mathscr{F})$ as the limit of $\mathrm{H}^1_{\mathrm{dR}}(X_\alpha,\mathscr{F})$.

Theorem 3.1.2. *Let X and $\{X_\alpha\}_\alpha$ as above. Let $\mathscr{F}$ be a differential equation over X. Denote by $\mathscr{F}_\alpha:=\mathscr{F}_{|X_\alpha}$. Assume that there exists $\beta\in\Lambda$ such that for all $\alpha\geqslant\beta$ the de Rham cohomology $\mathrm{H}^1_{\mathrm{dR}}(X_\alpha,\mathscr{F}_\alpha)$ of $\mathscr{F}_\alpha$ is finite dimensional. Then*

i) *The space*

$$\mathrm{H}^0_{\mathrm{dR}}(X,\mathscr{F})\;=\;\varprojlim_{\alpha}\mathrm{H}^0_{\mathrm{dR}}(X_\alpha,\mathscr{F}_\alpha)\tag{3.1}$$

[10] Recall that the word curve here, and everywhere in the paper, means a quasi-smooth K-analytic curve.

is finite dimensional, and there exists $\beta' \in \Lambda$ *such that for all* $\alpha \geqslant \alpha' \geqslant \beta'$ *the map* $\mathrm{H}^0_{\mathrm{dR}}(X_\alpha, \mathscr{F}_\alpha) \to \mathrm{H}^0_{\mathrm{dR}}(X_{\alpha'}, \mathscr{F}_{\alpha'})$ *is an strict isomorphism;*

ii) *For all* $\alpha \in \Lambda$*, the quotient topology of* $\mathrm{H}^1_{\mathrm{dR}}(X_\alpha, \mathscr{F}_\alpha)$ *induced by the projection* $\mathscr{F}_\alpha(X_\alpha) \xrightarrow{\nabla} \mathscr{F}_\alpha(X_\alpha) \to \mathrm{H}^1_{\mathrm{dR}}(X_\alpha, \mathscr{F}_\alpha)$ *is separated;*

iii) *there exists* $\beta \in \Lambda$ *such that for all* $\alpha \geqslant \alpha' \geqslant \beta$ *the map* $\mathrm{H}^1_{\mathrm{dR}}(X_\alpha, \mathscr{F}_\alpha) \to \mathrm{H}^1_{\mathrm{dR}}(X_{\alpha'}, \mathscr{F}_{\alpha'})$ *is surjective;*

iv) *One has*

$$\mathrm{H}^1_{\mathrm{dR}}(X, \mathscr{F}) \;=\; \varprojlim_{\alpha} \mathrm{H}^1_{\mathrm{dR}}(X_\alpha, \mathscr{F}_\alpha)\;, \tag{3.2}$$

and the canonical maps $\mathrm{H}^1_{\mathrm{dR}}(X, \mathscr{F}) \to \mathrm{H}^1_{\mathrm{dR}}(X_\alpha, \mathscr{F}_\alpha)$ *are surjective.*

In particular $\mathrm{H}^1_{\mathrm{dR}}(X, \mathscr{F})$ *is finite dimensional if and only if the sequence of the dimensions* $\dim \mathrm{H}^1_{\mathrm{dR}}(X_\alpha, \mathscr{F}_\alpha)$ *(or equivalently the indexes* $\chi(X_\alpha, \mathscr{F}_\alpha)$*) is constant for all* α *large enough, and we have*

$$\chi(X, \mathscr{F}) \;=\; \lim_{\alpha} \chi(X_\alpha, \mathscr{F}_\alpha)\;. \tag{3.3}$$

Proof. i) Projective limits commute with kernels. The dimension of each kernel is as usual bounded by $\dim \mathscr{F}$. For $\alpha \geqslant \alpha'$ large enough the map $\mathscr{O}(X_\alpha) \to \mathscr{O}(X_{\alpha'})$ is injective, so the $\mathrm{H}^0_{\mathrm{dR}}(X_\alpha, \mathscr{F}_\alpha) \to \mathrm{H}^0_{\mathrm{dR}}(X_{\alpha'}, \mathscr{F}_{\alpha'})$ are injective too. The sequence of the dimensions of the spaces $\{\mathrm{H}^0_{\mathrm{dR}}(X_\alpha, \mathscr{F}_\alpha)\}_\alpha$ is then decreasing, and hence the maps are isomorphisms for all α large enough.

ii) The topology of $\mathrm{H}^1_{\mathrm{dR}}(X_\alpha, \mathscr{F}_\alpha)$ is separated because $\mathscr{O}(X_\alpha)$ is Fréchet, and Fréchet spaces have the Banach's property [CM95, Theorem 4].

iii) For all $\alpha \geqslant \alpha'$ large enough the restrictions $\mathscr{O}(X_\alpha) \to \mathscr{O}(X_{\alpha'})$ have dense image, then the induced map $\mathrm{H}^1_{\mathrm{dR}}(X_\alpha, \mathscr{F}_\alpha) \to \mathrm{H}^1_{\mathrm{dR}}(X_{\alpha'}, \mathscr{F}_{\alpha'})$ also have dense image. Since these are separated finite dimensional spaces, then the restriction is surjective.

iv) We then have a system of exact sequences $0 \to \mathrm{H}^0_{\mathrm{dR}}(X_\alpha, \mathscr{F}_\alpha) \to \mathscr{F}_\alpha(X_\alpha) \xrightarrow{\nabla} \mathscr{F}_\alpha(X_\alpha) \to \mathrm{H}^1_{\mathrm{dR}}(X_\alpha, \mathscr{F}_\alpha) \to 0$. Let A_α be the image of ∇. We have a system of exact sequences

$$0 \to A_\alpha \to \mathscr{F}_\alpha(X_\alpha) \to \mathrm{H}^1_{\mathrm{dR}}(X_\alpha, \mathscr{F}_\alpha) \to 0\;. \tag{3.4}$$

Since $\mathrm{H}^1_{\mathrm{dR}}(X_\alpha, \mathscr{F}_\alpha)$ is separated, A_α is closed in the Fréchet space $\mathscr{F}_\alpha(X_\alpha)$, and the connection ∇ is a strict map by [Bou87, I, par. 3, N.3, p. I.19, Cor.3]. Now since, for all $\alpha \geqslant \alpha'$ large enough, the map $\mathscr{F}_\alpha(X_\alpha) \to \mathscr{F}_{\alpha'}(X_{\alpha'})$ is dense, then so does $A_\alpha \to A_{\alpha'}$, because $\nabla : \mathscr{F}(X_\alpha) \to A_\alpha$ is a strict surjective morphism. This proves that the system $\{A_\alpha\}_\alpha$ verifies the Mittag-Leffler condition (ML') for projective systems of complete metric spaces with dense image (cf. [Gro61, Chap.0, 13.2.4]).

So we can apply [Gro61, Chap.0, 13.2.2 and 13.2.4] to the system (3.4) to obtain (3.2). Moreover since, for all $\alpha \geqslant \alpha'$ large enough, the maps $\mathrm{H}^1_{\mathrm{dR}}(X_\alpha, \mathscr{F}_\alpha) \to \mathrm{H}^1_{\mathrm{dR}}(X_{\alpha'}, \mathscr{F}_{\alpha'})$ are surjective, then so does the maps $\mathrm{H}^1_{\mathrm{dR}}(X, \mathscr{F}) \to \mathrm{H}^1_{\mathrm{dR}}(X_\alpha, \mathscr{F}_\alpha)$, for all α large enough. □

Remark 3.1.3. *Of course the major application in the paper will be the case where* X *is quasi-Stein covered by a sequence* X_n *as in definition 1.1.6.*

3.2 Mayer-Vietoris.

Let $U, V \subseteq X$ be two open subsets. Recall that the de Rham cohomology of $\mathscr{F}$ is by definition the hypercohomology of the complex of sheaves $\mathcal{E}(\mathscr{F})^\cdot : \mathscr{F} \to \mathscr{F} \otimes \widehat{\Omega}^1_{X/K}$ in the sense of the theory of sheaves. We then have the Mayer-Vietoris long exact sequence

$$\cdots \to \mathbb{H}^{i-1}(U \cap V, \mathcal{E}^\cdot) \to \mathbb{H}^i(X, \mathcal{E}^\cdot) \to \mathbb{H}^i(U, \mathcal{E}^\cdot) \oplus \mathbb{H}^i(V, \mathcal{E}^\cdot) \to \mathbb{H}^i(V \cap U, \mathcal{E}^\cdot) \to \mathbb{H}^{i+1}(X, \mathcal{E}^\cdot) \to \cdots \tag{3.5}$$

Lemma 3.2.1. *Let $\mathcal{E}^{\cdot}$ be a complex of coherent $\mathscr{O}_X$-modules over X. Assume that for all $i \in \mathbb{Z}$ the spaces $\mathbb{H}^i(U, \mathcal{E}^{\cdot})$, $\mathbb{H}^i(V, \mathcal{E}^{\cdot})$, $\mathbb{H}^i(U \cap V, \mathcal{E}^{\cdot})$ are finite dimensional. Then $\mathbb{H}^i(X, \mathcal{E}^{\cdot})$ is finite dimensional.*

Proof. The term $\mathbb{H}^i(X, \mathcal{E})$ fits between two finite dimensional spaces in (3.5). □

From the lemma we deduce the following

Proposition 3.2.2. *Assume that X admits a finite covering by open subsets $U_1, \dots, U_n$ such that $\mathrm{H}^1_{\mathrm{dR}}(U_i, \mathscr{F})$ and $\{\mathrm{H}^1_{\mathrm{dR}}(U_{i_1} \cap \cdots \cap U_{i_n}, \mathscr{F})\}_{i_1,\dots,i_n \in \{1,\dots,n\}}$, are all finite dimensional (cf. Def. 2.4.2). Then the de Rham cohomology group $\mathrm{H}^1_{\mathrm{dR}}(X, \mathscr{F})$ is finite dimensional.* □

3.3 Canonical triangulation, canonical covering, and Liouville numbers

Since K is spherically complete, there are no points of type 4.

Definition 3.3.1 (Canonical triangulation)**.** *We call* canonical triangulation *of X relatively to S and $\mathscr{F}$ the minimal weak triangulation*

$$S_{\mathscr{F}} \tag{3.6}$$

of X such that

i) $\Gamma_{S_{\mathscr{F}}} = \Gamma_S(\mathscr{F})$,

ii) *over each edge I of $\Gamma_{S_{\mathscr{F}}}$ the radii are all* log-affine,

iii) *over each edge I each radius $\mathcal{R}_{S_{\mathscr{F}},i}(-, \mathscr{F})$ is either always solvable over I or never solvable over I,*

The vertexes of $\Gamma_{S_{\mathscr{F}}}$ are by definition the points of $S_{\mathscr{F}}$.

Remark 3.3.2. *The triangulation $S_{\mathscr{F}}$ coincides with that of [PP13, Remark 5.6.16] attached to the clean decomposition of $\mathscr{F}$. Notice also that ii) implies that*

ii′) *over each edge I one has, for all $i < j$, either $\mathcal{R}_{S_{\mathscr{F}},i}(x, \mathscr{F}) < \mathcal{R}_{S_{\mathscr{F}},j}(x, \mathscr{F})$ for all $x \in I$, or $\mathcal{R}_{S_{\mathscr{F}},i}(x, \mathscr{F}) = \mathcal{R}_{S_{\mathscr{F}},j}(x, \mathscr{F})$ for all $x \in I$.*

Remark 3.3.3. *By Proposition 1.3.5 we also have $\Gamma_{S_{\mathscr{F}}} = \Gamma_{S_{\mathscr{F}}}(\mathscr{F})$. Hence for all point $x \in X$ of type 2 or 3 we have*

$$V_S(x, \mathscr{F}) \;=\; V_{S_{\mathscr{F}}}(x, \mathscr{F})\;. \tag{3.7}$$

Definition 3.3.4 (Canonical covering)**.** *We say that a covering $\mathcal{U} = \{U_i\}_i$ of X is an S-*canonical covering for *$\mathscr{F}$ if:*

i) *There exists in $\mathcal{U}$ a family $\mathcal{A}(\mathcal{U}) \subseteq \mathcal{U}$ of open subsets such that:*
   (a) *Every element of $\mathcal{A}(\mathcal{U})$ is an open pseudo-annulus with skeleton in $\Gamma_S(\mathscr{F})$ (cf. Def. 1.1.8),*
   (b) *For all $U \in \mathcal{A}(\mathcal{U})$ that the radii of $\mathscr{F}$ are all* log-affine *on the skeleton of Γ_U,*
   (c) *the intersection of two distinct elements of $\mathcal{A}(\mathcal{U})$ is empty,*
   (d) *The set of points $\mathfrak{s}(\mathcal{U}) := \Gamma_S(\mathscr{F}) - \cup_{U \in \mathcal{A}(\mathcal{U})} \Gamma_U$ is locally finite in X contained in $S_{\mathscr{F}}$.*

ii) *For all $x \in \mathfrak{s}(\mathcal{U})$ there exists an unique open in $U_x \in \mathcal{U}$ containing x. Moreover U_x verifies:*
   (a) *U_x is a basic neighborhood of $V_{S_{\mathscr{F}}}(x, \mathscr{F})$ that does not contain any other point of $S_{\mathscr{F}}$;*
   (b) *If $x \in \mathrm{Int}(X)$, then U_x is small enough in order that the index of $\mathscr{F}_{|U_x}$ coincides with $\chi(\mathscr{F}, V_{S_{\mathscr{F}}}(x, \mathscr{F})^\dagger)$, and so it is given by (cf. Theorem 2.7.13 and Lemma 2.7.15)*

$$\chi(\mathscr{F}, U_x) \;=\; \mathrm{rank}(\mathrm{M}) \cdot \chi(V_{S_{\mathscr{F}}}(x, \mathscr{F})^\dagger) - \sum_{b \in \mathrm{Sing}(x, \mathscr{F})} \mathrm{Irr}_b(\mathscr{F})\;; \tag{3.8}$$

(c) *Assume that x lies in the boundary of X. Then U_x is a star-shaped open neighborhood of x in X, endowed with its canonical triangulation $S_{U_x} = \{x\}$, which is small enough in order that the radii of $\mathscr{F}_{|U_x}$ are all spectral non solvable at the skeleton $\Gamma_{\{x\}} \subseteq U$ of S_{U_x} in order to fulfill the assumptions of Proposition 2.4.3.*

It is understood that if one of the radii of $\mathscr{F}$ is not spectral non solvable at a point x of the boundary of X, then there are no canonical coverings for $\mathscr{F}$.

iii) *For all $U_i \neq U_j \in \mathcal{U}$ one has $U_i \cap U_j \neq U_i, U_j$ i.e. there are no repetitions in $\mathcal{U}$, and the intersection of three distinct elements of $\mathcal{U}$ is empty.*

Remark 3.3.5. *The existence of a canonical covering follows immediately from the local finiteness of the controlling graphs (cf. Thm. 1.3.1). It is clear that a canonical covering is locally finite.*

Definition 3.3.6 (Equations globally free of Liouville numbers)**.** *We say that $\mathscr{F}$ is free of Liouville numbers over X if its restriction to any annulus C in X is free of Liouville numbers (cf. Definition 2.5.6).*

The following Lemma asserts that the (**NL**) condition can be tested on a locally finite family of annuli, depending on $\mathscr{F}$, which is relatively small.

Lemma 3.3.7. *Let $\mathcal{U}$ be a canonical covering for $\mathscr{F}$. Assume that for all pseudo-annulus $U \in \mathcal{A}(\mathcal{U})$, there exists at least a germ of segment b in the skeleton Γ_U such that the restriction of $\mathscr{F}$ to the Robba ring $\mathfrak{R}_b$ is free of Liouville numbers in the sense of Definition 2.5.6. Then $\mathscr{F}$ is free of Liouville numbers over X.*

Proof. Let C be an annulus in X. By Lemma 2.5.8 we can restrict C if necessary. So we can assume either that $\Gamma_C \subset \Gamma_U$ for some pseudo-annulus $U \in \mathcal{A}(\mathcal{U})$, or that $\Gamma_C \cap \Gamma_{S_{\mathscr{F}}} = \emptyset$. If $\Gamma_C \subset \Gamma_U$, then Lemma 2.5.8 gives the result. Indeed, if $b_U \subset \Gamma_U$ is the germ of segment on which $\mathscr{F}$ is free of Liouville numbers, there exists an annulus C' in U such that $b, \Gamma_C \subseteq \Gamma_{C'} \subseteq \Gamma_U$. If $\Gamma_C \cap \Gamma_{S_{\mathscr{F}}} = \emptyset$, then C is contained in a disk on which the radii are all constant. Hence the Robba part of $\mathscr{F}_{|C}$ is trivial (cf. Proposition 2.4.3), and so $\mathscr{F}$ is free of Liouville numbers at C. □

As an application of Theorem 3.1.2 we have the following generalization of the Christol-Mebkhout index theorem over annuli (cf. Thm. 2.5.7):

Theorem 3.3.8. *Let X be a* pseudo-annulus*. Let $\mathscr{F}$ be a differential equation over X such that*

i) *The radii are all* log-*affine over the skeleton of X;*

ii) *$\mathscr{F}$ is free of Liouville numbers.*

Then $\mathscr{F}$ has finite dimensional de Rham cohomology. For $i = 0, 1$ the dimension of $\mathrm{H}^i_{\mathrm{dR}}(X, \mathscr{F})$ is bounded by the rank of $\mathscr{F}$, and we have $\chi(\mathscr{F}, X) = 0$.

Proof. By Remark 1.1.9, X is an increasing union of closed annuli. It may also been written as an increasing union of open annuli $(X_n)_{n\geqslant 0}$. Since X_n is an annulus, it is a Stein space and $\mathscr{O}(X_n)$ is Fréchet. Moreover, since $X_n \subseteq X_{n+1}$ is the inclusion of an annulus in a bigger one, the map $\mathscr{O}(X_{n+1}) \to \mathscr{O}(X_n)$ is injective, uniformly continuous and its image is dense.

So, by Theorem 3.1.2, we have $\mathrm{H}^i_{\mathrm{dR}}(X, \mathscr{F}) = \varprojlim_n \mathrm{H}^i_{\mathrm{dR}}(X_n, \mathscr{F}_n)$. Now by Theorem 2.5.7, for all n one has $\dim \mathrm{H}^0_{\mathrm{dR}}(X_n, \mathscr{F}_n) = \dim \mathrm{H}^1_{\mathrm{dR}}(X_n, \mathscr{F}_n)$, so the dimensions of $\mathrm{H}^i_{\mathrm{dR}}(X_n, \mathscr{F}_n)$ is bounded by the rank of $\mathscr{F}$ and it stabilizes for all n large enough. □

3.4 Global finite dimensionality of the de Rham cohomology I: Essentially finite controlling graphs

The following theorem relates the finiteness of the canonical triangulation $S_{\mathscr{F}}$ with the finite dimensionality of the de Rham cohomology.

Definition 3.4.1 (Essentially finite controlling graphs)**.** *Let $\mathscr{F}$ be a differential equation over X. We say that the controlling graph $\Gamma_S(\mathscr{F})$ is essentially finite if it contains only finitely many points x where at least one of the following conditions is satisfied:*

i) *$x \in \partial X$;*

ii) *$g(x) > 0$;*

iii) *x is a bifurcation point of $\Gamma_S(\mathscr{F})$;*

iv) *x admits an open annulus C as a neighborhood such that $x \in \Gamma_C \subseteq \Gamma_S(\mathscr{F})$, and some radius $\mathcal{R}_{S_{\mathscr{F}},i}(-,\mathscr{F})$ restricted to Γ_C has a break at x.*

Remark 3.4.2. *An essentially finite graph is topologically finite (cf. proof of Lemma 3.4.3), but not necessarily finite as a graph (since S can be infinite e.g. for a pseudo-annulus). Note that $\Gamma_S(\mathscr{F})$ can be topologically finite, without being essentially finite (e.g. an interval with an infinite number of point with non zero genus, or lying in the boundary), nor finite as a graph (e.g. the example of an open disk with a rational point removed does not admits any finite weak triangulation).*

From the definition we immediately have the following

Lemma 3.4.3. *$\Gamma_S(\mathscr{F})$ is essentially finite if and only if there exists a finite canonical covering for $\mathscr{F}$.*

Proof. Assume that $\Gamma_S(\mathscr{F})$ is essentially finite. Let $\mathfrak{s} \subseteq \Gamma_S(\mathscr{F})$ be the finite set of points verifying one of the conditions of Definition 3.4.1. By construction each connected component U of $X - \mathfrak{s}$ is either an open disk, or an open pseudo-annulus with skeleton included in $\Gamma_S(\mathscr{F})$. The number of such pseudo-annuli is finite because $\Gamma_S(\mathscr{F})$ is locally finite, and $\mathfrak{s}$ is finite. If U is a disk, the radii $\mathcal{R}_{S,i}(-,\mathscr{F})$ are all constant on it. In the other cases the radii $\mathcal{R}_{S,i}(-,\mathscr{F})$ are all log-linear along its skeleton, and Theorem 3.3.8 applies to U. Now we can cover each point of $x \in \mathfrak{s}$ by a star-shaped neighborhood of $V_S(x,\mathscr{F})$ as in point ii) of Definition 3.3.4, and we obtain a finite canonical covering for $\mathscr{F}$.

Conversely assume that $\mathscr{F}$ admits a finite canonical covering $\mathcal{U}$. The set of bad points of Definition 3.4.1 is included in $\mathfrak{s}(\mathcal{U})$ which is finite. So $\Gamma_S(\mathscr{F})$ is essentially finite. □

Lemma 3.4.4. *If $\Gamma_S(\mathscr{F})$ is essentially finite, then X is either projective or quasi-Stein.*

Proof. The graph Γ_S is topologically finite, and since $\Gamma_S(\mathscr{F})$ is essentially finite, there are a finite number of points in X verifying one of the conditions i), ii), iii) of Definition 3.4.1. It follows that X has finite genus. So by [Liu87] X is either quasi-Stein of projective. □

Theorem 3.4.5. *Let $\mathscr{F}$ be a differential equation over X. Assume that*

i) *the radii of $\mathscr{F}$ are all spectral non solvable at the points of the boundary of X,*

ii) *X admits a weak triangulation S such that $\Gamma_S(\mathscr{F})$ is essentially finite,*

iii) *$\mathscr{F}$ is free of Liouville numbers over X.*

Then the de Rham cohomology of $\mathscr{F}$ is finite dimensional.

Proof. Let $\mathcal{U}$ be a finite canonical covering for $\mathscr{F}$ (cf. Lemma 3.4.3). The theorem is an easy consequence of the Mayer-Vietoris sequence (cf. Prop. 3.2.2) since, by construction, $\mathcal{U}$ is formed by a finite number of opens on which the de Rham cohomology is finite dimensional. Namely

i) if U_i is a pseudo-annulus in $\mathcal{A}(\mathcal{U})$, then the finite dimensionality of $\mathrm{H}^1_{\mathrm{dR}}(U_i, \mathscr{F})$ follows from Christol-Mebkhout Theorem 3.3.8. This also gives the finite dimensionality of $\mathrm{H}^1_{\mathrm{dR}}(U_i \cap U_j, \mathscr{F})$, since $U_i \cap U_j$ is always a pseudo-annulus on which Theorem 3.3.8 applies.

ii) if U_i is a basic neighborhood of $V_{S_{\mathscr{F}}}(x, \mathscr{F})$, for some vertex $x \in \mathrm{Int}(X)$ of $\Gamma_{S_{\mathscr{F}}}$, then the finite dimensionality of $\mathrm{H}^1_{\mathrm{dR}}(U_i, \mathscr{F})$ follows from Proposition 2.7.16.

iii) if U_i is a basic neighborhood of $V_{S_{\mathscr{F}}}(x, \mathscr{F})$, for some vertex x of $\Gamma_{S_{\mathscr{F}}}$ that lies in the boundary of X, then the cohomology is zero by Proposition 2.4.3.

This proves the claim. □

The typical application of Theorem 3.4.5 is the following

Corollary 3.4.6. *Let X be any quasi-smooth K-analytic curve, and $\mathscr{F}$ be a differential equation over X. Let U be an analytic domain of X such that*

i) *U is relatively compact in X;*

ii) *The radii of $\mathscr{F}$ are all spectral non solvable at the boundary ∂U of U;*[11]

iii) *$\mathscr{F}$ is free of Liouville numbers over U.*

Then the de Rham cohomology of $\mathscr{F}_{|U}$ is finite dimensional.

Proof. It is enough to prove that U admits a weak triangulation S_U such that $\Gamma_{S_U}(\mathscr{F})$ is essentially finite. Let S be a weak triangulation of X. Since U is relatively compact, $\Gamma_S \cap U$ is topologically finite. Since $X - \Gamma_S$ is a disjoint union of disks D, and since U is an analytic domain in X, then there are a locally finite number of such disks such that $D \cap U \neq D$. Since U is relatively compact the number of such disks if actually finite, and one sees that there exists a weak triangulation S_U of U such that $\Gamma_S \cap U \subseteq \Gamma_{S_U}$. By [PP13, Prop. 2.8.2] one has $\Gamma_{S_U}(\mathscr{F}) = (\Gamma_S(\mathscr{F}) \cap U) \cup \Gamma_{S_U}$. Hence $\Gamma_{S_U}(\mathscr{F})$ is topologically finite, because $\Gamma_S(\mathscr{F})$ is locally finite, and U is relatively compact. Now the compactness implies that the radii of $\mathscr{F}$ have a finite number of breaks, because each radius has a finite number of breaks over a compact segment. It also implies that one has a finite number of points $x \in U$ such that $g(x) > 0$, since those points form a locally finite set included in S. Hence $\Gamma_{S_U}(\mathscr{F})$ is essentially finite. □

Remark 3.4.7. *Notice that, in general, U does not admit a* finite *weak triangulation (e.g. an open disk with a rational point removed). So in general $\Gamma_{S'_{\mathscr{F}}}$ always has an infinite number of edges. It is not finite* as a graph. *This is the reason of the introduction of pseudo-annuli in the picture.*

Recall that $(\mathscr{F}, \nabla)$ is said to be over-convergent on X if there exists a smooth K-analytic curve (with no boundary) X' and a connexion $(\mathscr{F}', \nabla')$ on X' such that X embeds into X' and $(\mathscr{F}', \nabla')$ restricts to $(\mathscr{F}, \nabla)$ on X. In this case, we define the over-convergent de Rham cohomology of $\mathscr{F}$ on X as the inductive limit of the de Rham cohomologies of $\mathscr{F}'$ on U, where U runs through the neighborhoods of X in X'.

From the previous result, we deduce the following

[11] Here the boundary of U is the absolute boundary $\partial U = \partial(U/K)$, and not the relative boundary $\partial(U/X)$ of U in X.

Corollary 3.4.8. *Assume that $\mathscr{F}$ is over-convergent and free of Liouville numbers over X (in the over-convergent directions too). Then the over-convergent de Rham cohomology of $\mathscr{F}$ is finite dimensional.*

Remark 3.4.9 (Essential triangulations and essential graphs)**.** *The result of this section suggest the following generalization of the notion of weak triangulation. A locally finite subset $\mathfrak{s}$ of X, formed by points of type 2 or 3, is an* essential triangulation *if $X - \mathfrak{s}$ is a disjoint union of virtual open disks, and open quasi-annuli. A weak triangulation of X is an essential triangulation. Denote by $\mathfrak{G}$ the union of $\mathfrak{s}$ with the skeletons of the pseudo-annuli that are connected components of $X - \mathfrak{s}$. Following [PP13, Appendix A], one sees that $\mathfrak{G}$ is a* weakly admissible graph *of X, so one is allowed to define the radii $\mathcal{R}_{\mathfrak{G},i}(x, \mathscr{F})$ with respect to $\mathfrak{G}$. Namely, roughly speaking, one defines $D(x, \mathfrak{G})$ as the largest open disk centered at x that do not encounter $\mathfrak{G}$, and we imitate the definition (1.2.2). The interest of such a definition is that a graph $\Gamma_S(\mathscr{F})$ is essentially finite if and only if there exists a finite essential triangulation such that $\Gamma_S(\mathscr{F}) = \mathfrak{G}$.*

3.5 Global measure of the irregularity for equations with finite controlling graphs

We here provide a definition of *global irregularity* of $\mathscr{F}$, together with a global form of Grothendieck-Ogg-Shafarevich formula.

Hypothesis 3.5.1. *In this section we assume that $(X, \mathscr{F})$ satisfies the assumptions of Theorem 3.4.5. In particular X admits a weak triangulation S such that $\Gamma_S(\mathscr{F})$ is* essentially finite.

Hypothesis 3.5.1 implies that the graph Γ_S is *topologically* finite. So by Lemma 1.1.11, the open boundary of X is finite. Moreover, by the assumptions of Definition 3.4.1, the curve X has *finite genus* in the sense of Definition 1.1.12. In particular it is either a projective curve, or a quasi-Stein curve (cf. Remark 1.1.6). We here focus on the case of a quasi-Stein curve.

3.6 Global irregularity

In this section we define the *global irregularity* os a differential equation with essentially finite graph.

3.6.1 *Essential segments of Γ_S.* If x is a point in the boundary ∂X of X, then we define

$$\mathrm{seg}(\Gamma_S, \partial X) \tag{3.9}$$

as the family formed by germ of segments out of $x \in \partial X$ belonging to Γ_S. In analogy with germs in the open boundary of X, we say that the elements of $\mathrm{seg}(\Gamma_S, \partial X)$ are *germs of segments of Γ_S at the closed boundary* of X.

Definition 3.6.1 (Essential germs of segments in Γ_S)**.** *We call* essential germs segments of Γ_S *the family of germ of segments*

$$\mathrm{seg}(S) \;:=\; \mathrm{seg}(\Gamma_S, \partial X) \cup \partial^o X\ . \tag{3.10}$$

3.6.2 *Global Irregularity.*

Definition 3.6.2 (Global Irregularity)**.** *Assume that $(X, \mathscr{F})$ satisfies the assumptions of Thm. 3.4.5. Let S be any* finite *weak triangulation of X, and let $S_{\mathscr{F}}$ be the corresponding canonical triangulation of $\mathscr{F}$. We define the* global irregularity *of $\mathscr{F}$ as*

$$\mathrm{Irr}_X(\mathscr{F}) \;:=\; \Big(\sum_{x\in\partial X} \chi(x, S_{\mathscr{F}})\Big)\cdot \mathrm{rank}(\mathscr{F}) - \sum_{b\in \mathrm{seg}(S_{\mathscr{F}})} \partial_b H_{\emptyset,r}(-, \mathscr{F}_{|\mathfrak{R}_b})\ , \tag{3.11}$$

where $\mathfrak{R}_b$ is the Robba ring defined by b, and b is oriented as outside X.[12]

Lemma 3.6.3. *The global irregularity $\mathrm{Irr}_X(\mathscr{F})$ is independent on the chosen finite weak triangulation S of X.*

Proof. Indeed let S' be another weak triangulation. Considering a triangulation S'' such that $\Gamma_S, \Gamma_{S'} \subseteq \Gamma_{S''}$, one sees that we can assume that $\Gamma_S \subseteq \Gamma_{S'}$. We know that $\Gamma_{S'_{\mathscr{F}}} = \Gamma_{S'} \cup \Gamma_{S_{\mathscr{F}}}$ as sets (cf. Proposition 1.3.5). We then can assume that $S_{\mathscr{F}} = S$. It is enough to prove that the addition of a point $s \in \Gamma_{S_{\mathscr{F}}}$ to $S_{\mathscr{F}}$, and the further addition of a new edge branched at a point of $S_{\mathscr{F}}$ do not change $\mathrm{Irr}_X(\mathscr{F})$. Since $\partial X \subset S_{\mathscr{F}}$, then $s \notin \partial X$, and the addition of s does not affect $\mathrm{Irr}_X(\mathscr{F})$ at all. The addition of a new segment branched at a point s of $S_{\mathscr{F}}$, produces a change in the sum if and only if $s \in \partial X$. In this case $\chi(s, S_{\mathscr{F}})$ and the number of essential segments increase of 1, so there is one more term $-\partial_{b_0} H_{\emptyset,r}(-, \mathscr{F}_{|\mathfrak{R}_b})$ corresponding to the new essential segment b at x. Now $-\partial_{b_0} H_{\emptyset,r}(-, \mathscr{F}_{|\mathfrak{R}_b}) = -r$ because all the radii $\mathcal{R}_{S_{\mathscr{F}},i}(x, \mathscr{F})$ are constant over the new branch b, and spectral non solvable at x. □

3.6.3 *Grothendieck-Ogg-Shafarevich formula.* The following theorem constitutes the analogue of the Grothendieck-Ogg-Shafarevich formula. Recall that by Lemma 3.4.4, under the assumptions of Theorem 3.4.5, X is either projective or quasi-Stein.

Theorem 3.6.4. *Assume that X is not projective. Under the assumptions of Theorem 3.4.5 one has*

$$\chi(X, \mathscr{F}) \;=\; \mathrm{rank}(\mathscr{F}) \cdot \chi(X) - \mathrm{Irr}_X(\mathscr{F}) \;. \tag{3.12}$$

Proof. The de Rham cohomology equals the kernel and the cokernel of ∇ acting on $\mathscr{F}(X)$. Let $\mathcal{U} = \{U_1, \ldots, U_n\}$ be a finite canonical covering for $\mathscr{F}$ (cf. Lemma 3.4.3).

The assumptions of Theorem 3.4.4 imply that X has finite genus. Hence every U_i has finite genus too. Since X is not projective, U_i cannot be projective either, hence it is quasi-Stein by [Liu87].

Since the U_i are quasi-Stein spaces, the first Čech cohomology group $\check{H}^1(\mathcal{U}, \mathscr{F})$ coincides with the coherent cohomology group $H^1(X, \mathscr{F})$. Since X is quasi-Stein too, this last group is 0 and the Čech complex of $\mathcal{U}$ is a short exact sequence:

$$0 \to \mathscr{F}(X) \to \prod_{1 \leqslant i \leqslant n} \mathscr{F}(U_i) \to \prod_{1 \leqslant i < j \leqslant n} \mathscr{F}(U_i \cap U_j) \to 0 \;. \tag{3.13}$$

Indeed the intersection of three distinct opens is empty (cf. Def. 3.3.4). The derivation acts on these three terms, and we have a long exact sequence by the Snake diagram:

$$0 \to \mathrm{H}^0_{\mathrm{dR}}(X, \mathscr{F}) \to \prod_i \mathrm{H}^0_{\mathrm{dR}}(U_i, \mathscr{F}) \to \prod_{i<j} \mathrm{H}^0_{\mathrm{dR}}(U_i \cap U_j, \mathscr{F}) \to \tag{3.14}$$

$$\to \mathrm{H}^1_{\mathrm{dR}}(X, \mathscr{F}) \to \prod_i \mathrm{H}^1_{\mathrm{dR}}(U_i, \mathscr{F}) \to \prod_{i<j} \mathrm{H}^1_{\mathrm{dR}}(U_i \cap U_j, \mathscr{F}) \to 0 \tag{3.15}$$

Remark 3.6.5. *The finite dimensionality of $\mathrm{H}^1_{\mathrm{dR}}(X, \mathscr{F})$ then follows from that of each $\mathrm{H}^1_{\mathrm{dR}}(U_i, \mathscr{F})$, as in the proof of Theorem 3.4.5.*

The intersection of two opens in $\mathcal{U}$ is always an open pseudo-annulus verifying Theorem 3.3.8.

[12] If $b \in \mathrm{seg}(\Gamma_S, \partial X)$, and if $x \in \partial X$ is its boundary point, this means that b is oriented as inside x.

So it has zero index. Hence the term $\prod_{1\leqslant i<j\leqslant n}\mathscr{F}(U_i\cap U_j)$ of the sequence (3.13) has zero index. So

$$\chi(X,\mathscr{F}) \;=\; \sum_i \chi(U_i,\mathscr{F})\;. \tag{3.16}$$

Now if U_i is an open pseudo-annulus in $\mathcal{A}(\mathcal{U})$ (cf. Def. 3.3.4), then its index is zero again by Thm. 3.3.8. The same happens if U_i is an open neighborhood of a point of $\mathfrak{s}(\mathcal{U})$ at the boundary of X by Proposition 2.4.3.

So the sum can be considered over the set of open $U_x\in\mathcal{U}$ that contains some point $x\in\mathfrak{s}(\mathcal{U})\cap\mathrm{Int}(X)$. Let $V_{S_{\mathscr{F}}}(x,\mathscr{F})\subset U_x$ be the canonical elementary tube centered at x. Then index of U_x is given by

$$\begin{aligned}\chi(U_x,\mathscr{F}) \;&=\; \mathrm{rank}(\mathscr{F})\cdot\chi(V_{S_{\mathscr{F}}}(x,\mathscr{F})) - \sum_{b\in\mathrm{Sing}(x,\mathscr{F})}\mathrm{Irr}_b(\mathscr{F}) && (3.17)\\ &=\; \mathrm{rank}(\mathscr{F})\cdot\chi(x,S_{\mathscr{F}}) - \sum_{b\in\mathrm{Sing}(x,\mathscr{F})}\mathrm{Irr}_b(\mathscr{F})\;. && (3.18)\end{aligned}$$

If $r:=\mathrm{rank}(\mathscr{F})$, the sum (3.16) becomes:

$$\chi(X,\mathscr{F}) \;=\; \sum_{x\in\mathfrak{s}(\mathcal{U})-\partial X}\chi(U_x,\mathscr{F}) = \sum_{x\in\mathfrak{s}(\mathcal{U})-\partial X}\Big(r\cdot\chi(x,S_{\mathscr{F}}) - \sum_{b\in\mathrm{Sing}(x,\mathscr{F})}\mathrm{Irr}_b(\mathscr{F})\Big)\;. \tag{3.19}$$

If C is a *relatively compact* pseudo-annulus in $\mathcal{A}(\mathcal{U})$, such that both the points at its boundary lie in $\mathrm{Int}(X)$, then the irregularities $\mathrm{Irr}_b(\mathscr{F})$ of the two points at its boundary are equal and appear in both in (3.19), but with opposite sign (because of the orientation of b). This is because the radii are log-linear over Γ_C. The only irregularities $\mathrm{Irr}_b(\mathscr{F})$ that remains after cancellation are those relative to a b belonging to a pseudo-annulus which is either with a boundary in ∂X, or which is not relatively compact in X. So we have

$$\chi(X,\mathscr{F}) \;=\; r\chi(X) - r\Big(\sum_{x\in\partial X}\chi(x,S_{\mathscr{F}})\Big) + \sum_{b\in\mathrm{seg}(S_{\mathscr{F}})}\partial_b H_{S_{\mathscr{F}},r}(x,\mathscr{F})\;. \tag{3.20}$$

This proves the claim. □

Remark 3.6.6. *The proof of Theorem 3.6.4 provides another proof (perhaps more explicit) of Theorem 3.4.5. Indeed the fact that $\Gamma_S(\mathscr{F})$ is essentially finite implies that X is a curve of finite genus. Hence, by a result of Liu (cf. [Liu87]), X is either projective or quasi-Stein. Projective curves are classic, while the proof of Theorem 3.6.4 gives another proof of the finiteness in the quasi-Stein case.*

3.7 Global finite dimensionality of the cohomology II: infinite controlling graphs

In this section we investigate more general situations where the assumption of the essential finiteness of $\Gamma_{S_{\mathscr{F}}}$ is dropped. For this we use Christol-Mebkhout limit formula.

Corollary 3.7.1. *Let X be a quasi-Stein curve. Let X_n be the sequence of Definition 1.1.3. Let $\mathscr{F}_n:=\mathscr{F}_{|X_n}$. Assume that*

i) *$\mathscr{F}$ is free of Liouville number over X.*
ii) *for all n large enough X_n is a relatively compact in X*
iii) *for all n large enough, the radii of $\mathscr{F}_n$ are all spectral non solvable at the points x of the boundary of X_n;*[13]

[13] This is the *absolute* boundary ∂X_n of X_n, not the relative boundary $\partial(X_n/X)$.

Then, for all n large enough, Theorem 3.4.5 applies to $\mathscr{F}_n$, and $\mathrm{H}^1_{\mathrm{dR}}(X_n,\mathscr{F}_n)$ is finite dimensional and Theorem 3.1.2 applies. In particular $\mathrm{H}^1_{\mathrm{dR}}(X,\mathscr{F}) = \varprojlim_n \mathrm{H}^i_{\mathrm{dR}}(X_n,\mathscr{F}_n)$ is finite dimensional if and only if the sequence of the dimensions $\dim \mathrm{H}^1_{\mathrm{dR}}(X_n,\mathscr{F}_n)$ (or equivalently the sequence of indexes $\chi(X_n,\mathscr{F}_n)$) is constant for all n large enough, and we have

$$\chi(X,\mathscr{F}) \;=\; \lim_n \chi(X_n,\mathscr{F}_n)\;. \tag{3.21}$$

Proof. The assertions follows from Corollary 3.4.6 and Theorem 3.1.2. □

Typical examples are the following cases :

i) A pseudo-annulus (resp. disk) covered by a family of relatively compact sub-annuli (resp. disks),
ii) A curve of the form $X-\{x_1,\dots,x_n\}$, $n\geqslant 1$, where X is a smooth, connected, projective curve, and where $x_1,\dots,x_n$ are rational points.

The following corollary is useful if the X_n all have the same shape, for example, if we have an open disk (resp. annulus, punctured disk) covered by open sub-disks (resp. sub-annuli):

Corollary 3.7.2. *We preserve the assumptions of Corollary 3.7.1. Assume moreover that the sequence $\chi(X_n)$ is constant with value χ for all n large enough. In this case $H^1(X,\mathscr{F})$ is finite dimensional if and only if the sequence $\mathrm{Irr}_{X_n}(\mathscr{F}_n)$ is constant for all n large enough, and (3.21) becomes*

$$\chi(X,\mathscr{F}) \;=\; \mathrm{rank}(\mathscr{F})\cdot\chi - \lim_n \mathrm{Irr}_{X_n}(\mathscr{F}_n)\;. \tag{3.22}$$

□

Corollary 3.7.3 (pseudo-annuli)**.** *Let $\mathscr{F}$ be a differential equation over a pseudo-annulus X, which is free of Liouville numbers. Let x_n, y_n be two sequences of points in skeleton of X approaching the open boundary of X. Let C_n be the open annulus with skeleton $]x_n,y_n[$.*

Then the de Rham cohomology of $\mathscr{F}$ is finite dimensional if and only if *the sequence of slopes $\sigma_n^- := \partial_{b_n} H_{\emptyset,r}(x_n,\mathscr{F})$ and $\sigma_n^+ := \partial_{b'_n} H_{\emptyset,r}(y_n,\mathscr{F})$ are both constant for all n large enough, where $b_n\in C_n$ (resp. $b'_n\in C_n$) is a germ of segment out of $x_n\notin C_n$ (resp. $y_n\notin C_n$) and oriented as outside C_n (i.e. oriented as into x_n and y_n respectively). Moreover in this case we have*

$$\mathrm{Irr}_{X_n}(\mathscr{F}_n) \;=\; -(\sigma_n^- + \sigma_n^+) \;\geqslant\; 0\;, \tag{3.23}$$

and

$$\chi(X,\mathscr{F}) \;=\; \lim_n \chi(C_n,\mathscr{F}_n) \;=\; \lim_n(\sigma_n^- + \sigma_n^+) \;\neq\; -\infty\;. \tag{3.24}$$

Proof. Apply Corollaries 3.7.1 and 3.7.2 to $X=\cup_n C_n$. □

Corollary 3.7.3 is a particular case of the following Theorem 3.7.4 that constitutes a reciprocal of Theorem 3.4.5.

Theorem 3.7.4. *Let X be a quasi-Stein curve with finite genus $g(X)$ admitting a weak triangulation S whose skeleton Γ_S is topologically finite (cf. Def. 1.1.2). Let $\mathscr{F}$ be a differential equation free of Liouville numbers over X, with no solvable radii at the boundary ∂X of X. The following conditions are equivalent:*

i) *$\Gamma_S(\mathscr{F})$ is essentially finite;*
ii) *the de Rham cohomology of $\mathscr{F}$ is finite dimensional;*
iii) *for all germ of segment b at the open boundary of X, the radii of $\mathscr{F}$ have a finite number of breaks along b.*

Proof. Since Γ_S is topologically finite, the open boundary of X is finite. Let $b_1, \ldots, b_m$ be the germs of segments at the open boundary. For all i let $n \mapsto x_{n,i}$ be a strictly monotone sequence of points along b_i approaching the open boundary of X (i.e. going toward outside X). As a notation denote b_i by $b_i :=]x_{1,i}, x_{\infty,i}[$ so that $]x_{n,i}, x_{\infty,i}[$ is a strictly decreasing sequence of segments. Since X has finite genus, for n large enough $]x_{n,i}, x_{\infty,i}[$ is the skeleton of a well defined pseudo-annulus $C_{n,i} \subseteq X$. Let $A_{n,i}$ be the *semi-open* pseudo-annulus with skeleton is $[x_{n,i}, x_{\infty,i}[$.[14] Let

$$X_n \;:=\; X - (\cup_{i=1}^m A_{n,i}) \;. \tag{3.25}$$

Then for all n large enough we have

i) $\chi(X_n) = \chi(X)$,
ii) X_n is relatively compact in X,
iii) $\partial X_n = \partial X$,

Moreover, if S is any weak triangulation such that for all n one has $S \cap]x_{1,i}, x_{\infty,i}[= \{x_{n,i}\}_n$, then

i) for all n large enough $S_n := S \cap X_n$ is a weak triangulation of X and
$$\Gamma_{S_n} \;=\; \Gamma_S \cap X_n \;, \tag{3.26}$$
ii) $\Gamma_S(\mathscr{F}) = \bigcup_n \Gamma_{S_n}(\mathscr{F}_{|X_n})$.

It is then clear that $\Gamma_S(\mathscr{F})$ is essentially finite if and only if for all i the radii of $\mathscr{F}$ have a finite number of breaks along b_i.

Now let $U_n := \cup_i C_{n-1,i}$ so that $X = X_n \cup U_n$ is an open covering of X. The de Rham cohomologies of $\mathscr{F}_{|X_n}$ and of $\mathscr{F}_{|X_n \cap U_n}$ are both finite dimensional by Corollary 3.4.6. Moreover by Corollary 3.7.3 the de Rham cohomology of $\mathscr{F}_{|U_n}$ is finite dimensional if and only if for all i the radii of $\mathscr{F}$ have a finite number of breaks along b_i. The Mayer-Vietoris sequence of the covering $X_n \cup U_n$

$$\cdots \to \mathrm{H}^{i-1}_{\mathrm{dR}}(X_n \cap U_n, \mathscr{F}) \to \mathrm{H}^i_{\mathrm{dR}}(X, \mathscr{F}) \to \mathrm{H}^i_{\mathrm{dR}}(X_n, \mathscr{F}) \oplus \mathrm{H}^i_{\mathrm{dR}}(U_n, \mathscr{F}) \to \mathrm{H}^i_{\mathrm{dR}}(X_n \cap U_n, \mathscr{F}) \to \cdots \tag{3.27}$$

shows that the cohomology of $\mathscr{F}$ if finite dimensional if and only if so is that of $\mathscr{F}_{|U_n}$. □

Remark 3.7.5. *Hypothesis 2.7.1 can likely be removed from sections 2 and 3. Indeed it is used only in the following cases:*

i) *To fulfill the assumptions of Christol-Mebkhout index Thm. 2.7.13;*
ii) *To ensure that $\Gamma_S(\mathscr{F})$ has no points of type 4 in Definition 3.3.1;*
iii) *When one uses [Liu87] to prove that X is either pojective or quasi-Stein .*

The Christol-Mebkhout index result in the case of an annulus (cf. Thm. 2.5.7) have been recently generalized by Kedlaya [Ked13, Lemma 3.7.6] to an arbitrary base field, and it seems that similar techniques permit to remove the assumptions about K also in Thm. 2.7.13. On the other hand again in [Ked13] one proves that $\Gamma_S(\mathscr{F})$ has no points of type 4.

REFERENCES

Ado76 A. Adolphson, *An index theorem for p-adic differential operators*, Trans. Amer. Math. Soc. **216** (1976), 279–293. MR 0387284 (52 #8127)

Bal10 Francesco Baldassarri, *Continuity of the radius of convergence of differential equations on p-adic analytic curves*, Invent. Math. **182** (2010), no. 3, 513–584. MR 2737705 (2011m:12015)

[14]More precisely $A_{n,i}$ is the analytic domain of X defined as the union of the semi-open annuli with skeletons $[x_{n,i}, x_{m,i}[$, for all $m \to \infty$.

Bou87 N. Bourbaki, *Topological vector spaces. Chapters 1–5*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1987, Translated from the French by H. G. Eggleston and S. Madan. MR 910295 (88g:46002)

Chi90 Bruno Chiarellotto, *Duality in rigid analysis*, p-adic analysis (Trento, 1989), Lecture Notes in Math., vol. 1454, Springer, Berlin, 1990, pp. 142–172. MR 1094850 (92d:32015)

Chr12 Gilles Christol, *Le théorème de turritin p-adique. (book in preparation)*, 2012, http://www.math.jussieu.fr/~christol/courspdf.pdf.

CM93 G. Christol and Z. Mebkhout, *Sur le théorème de l'indice des équations différentielles p-adiques. I*, Ann. Inst. Fourier (Grenoble) **43** (1993), no. 5, 1545–1574. MR 1275209 (95j:12009)

CM95 Gilles Christol and Zoghman Mebkhout, *Topological p-adic vector spaces and index theory*, Proceedings of the 3rd International Conference on p-adic Functional Analysis (Aubière, 1994), vol. 2, 1995, pp. 93–98. MR 1342807 (96k:46142)

CM97 G. Christol and Z. Mebkhout, *Sur le théorème de l'indice des équations différentielles p-adiques. II*, Ann. of Math. (2) **146** (1997), no. 2, 345–410. MR 1477761 (99a:12009)

CM00 ———, *Sur le théorème de l'indice des équations différentielles p-adiques. III*, Ann. of Math. (2) **151** (2000), no. 2, 385–457. MR 1765703 (2001k:12014)

CM01 ———, *Sur le théorème de l'indice des équations différentielles p-adiques. IV*, Invent. Math. **143** (2001), no. 3, 629–672. MR 1817646 (2002d:12005)

CM02 Gilles Christol and Zoghman Mebkhout, *Équations différentielles p-adiques et coefficients p-adiques sur les courbes*, Astérisque (2002), no. 279, 125–183, Cohomologies p-adiques et applications arithmétiques, II. MR 1922830 (2003i:12014)

Cre98 Richard Crew, *Finiteness theorems for the cohomology of an overconvergent isocrystal on a curve*, Ann. Sci. École Norm. Sup. (4) **31** (1998), no. 6, 717–763. MR 1664230 (2000a:14023)

Cre12 ———, *Arithmetic $\mathscr{D}$-modules on the unit disk*, Compos. Math. **148** (2012), no. 1, 227–268. MR 2881315

DR77 B. Dwork and P. Robba, *On ordinary linear p-adic differential equations*, Trans. Amer. Math. Soc. **231** (1977), no. 1, 1–46. MR 0447247 (56 #5562)

Duc Antoine Ducros, *La structure des courbes analytiques*, http://www.math.jussieu.fr/~ducros/livre.html.

Dwo82 B. Dwork, *Lectures on p-adic differential equations*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science], vol. 253, Springer-Verlag, New York, 1982.

Gro61 A. Grothendieck, *Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I*, Inst. Hautes Études Sci. Publ. Math. (1961), no. 11, 167. MR 0217085 (36 #177c)

Ked06 Kiran S. Kedlaya, *Finiteness of rigid cohomology with coefficients*, Duke Math. J. **134** (2006), no. 1, 15–97. MR 2239343 (2007m:14021)

Ked10 ———, *p-adic differential equations*, Cambridge Studies in Advanced Mathematics, vol. 125, Cambridge Univ. Press, 2010.

Ked13 ———, *Local and global structure of connections on nonarchimedean curves*, arxiv, 2013, http://arxiv.org/abs/1301.6309, pp. 1–76.

Kie67 Reinhardt Kiehl, *Theorem A und Theorem B in der nichtarchimedischen Funktionentheorie*, Invent. Math. **2** (1967), 256–273. MR MR0210949 (35 #1834)

Laz62 Michel Lazard, *Les zéros des fonctions analytiques d'une variable sur un corps valué complet*, Inst. Hautes Études Sci. Publ. Math. (1962), no. 14, 47–75. MR 0152519 (27 #2497)

Liu87 Qing Liu, *Ouverts analytiques d'une courbe algébrique en géométrie rigide*, Ann. Inst. Fourier (Grenoble) **37** (1987), no. 3, 39–64. MR 916273 (89c:14032)

PP12 Andrea Pulita and Jérôme Poineau, *The convergence newton polygon of a p-adic differential equation ii : Continuity and finiteness on berkovich curves*, arxiv, 2012, http://arxiv.org/abs/1209.3663, pp. 1–16.

PP13 ———, *The convergence newton polygon of a p-adic differential equation iii : global decomposition and controlling graphs*, arxiv, 2013, http://arxiv.org/abs/1308.0859, pp. 1–81.

Pul12 Andrea Pulita, *The convergence newton polygon of a p-adic differential equation i : Affinoid domains of the berkovich affine line*, arxiv, 2012, http://arxiv.org/abs/1208.5850, pp. 1–44.

Pul13 ______, *An algorithm computing non-solvable spectral radii of p-adic differential equations*, C. R. Acad. Sci. Paris, Ser. I **351** (2013), no. 5-6, 167–171.

Rob75 P. Robba, *On the index of p-adic differential operators. I*, Ann. of Math. (2) **101** (1975), 280–316. MR 0364243 (51 #498)

Rob76 ______, *On the index of p-adic differential operators. II*, Duke Math. J. **43** (1976), no. 1, 19–31. MR 0389910 (52 #10739)

Rob84 P. Robba, *Indice d'un opérateur différentiel p-adique. III. application to twisted exponential sums*, Asterisque **119-120** (1984), 191–266.

Rob85 ______, *Indice d'un opérateur différentiel p-adique. IV. Cas des systèmes. Mesure de l'irrégularité dans un disque*, Ann. Inst. Fourier (Grenoble) **35** (1985), no. 2, 13–55.

You92 Paul Thomas Young, *Radii of convergence and index for p-adic differential operators*, Trans. Amer. Math. Soc. **333** (1992), no. 2, 769–785. MR 1066451 (92m:12015)

Jérôme Poineau jerome.poineau@math.unistra.fr
Institut de recherche mathématique avancée, 7, rue René Descartes, 67084 Strasbourg, France

Andrea Pulita pulita@math.univ-montp2.fr
Département de Mathématiques, Université de Montpellier II, CC051, Place Eugène Bataillon, F-34 095, Montpellier CEDEX 5.